# SUBNORMAL OPERATOR WITH A CYCLIC VECTOR 

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In this paper, we aim to characterize the non-trivial closed invariant subspace of a subnormal operator and to study the existence of such subspaces.

An operator $A$ acting on a Hilbert space $H$ is said to be subnormal if, on some space $K$ containing $H$, there exists a normal operator $B$ such that $B x=A x$ for every $x$ in $H$; then $B$ is called a normal extension of $A$.

The normal extension $B$, acting on $K$, of a subnormal operator $A$, acting on $H$, a subspace of $K$, is the minimal normal extension of $A$ if the smallest subspace of $K$ that contains $H$ and reduces $B$ is $K$ itself; Halmos has shown that any two minimal normal extensions are unitarily equivalent ([3]).

If $A$ is subnormal on $H$, we call that a vector $x$ is cyclic with respect to $A$ if the smallest subspace containing $x$ and invariant under $A$ is $H$; in this case we say that $H$ is cyclic with respect to $A$.

For our purpose, it is natural to assume that the subnormal operator $A$ on $H$ has a cyclic vector $x$; bacause, if $\vee\left\{A^{n} x ; n \geqq 0\right\}$ (which denotes the smallest closed subspace containing $A^{n} x ; n \geqq 0$ ) is properly included in $H$, then it is clearly a non-trivial closed invariant subspace of $A$.

Bram proved in [1] that when $A$ is normal and acts on $H$, the fact that $H$ is cyclic with respect to $A$ in the sense just defined is equivalent to the fact that $H$ is cyclic in the usual sense, i.e., that there exists $x$ in $H$ such that $H$ is the smallest closed subspace that contains $x$ and reduces $A$.

It is known that if $B$ is a normal operator on $K$ with a cyclic vector, then there exists a unitary mapping $U$ of $K$ onto a suitable function space $L^{2}(d \mu(\lambda) ; \sigma(B))$ such that $U B U^{-1}$ has the form of "multiplication by $\lambda$ " ([2]).

Applying this representation theorem of normal operators to the minimal normal extension $B$ on $K$ of a subnormal operator $A$ on $H$ with a cyclic vector, we can show that $H$ admits a representation relative to $A$ onto a subspace $H^{2}(d \mu(\lambda) ; \sigma(B))$ of $L^{2}(d \mu(\lambda) ; \sigma(B))$.

In the next section, we show this representation of a subnormal operator with a cyclic vector and using this, we give the sufficient conditions of the existence of non-trivial closed invariant subspaces of subnormal operators.

We state here a characterization of subnormal operators given by Halmos [3] and Bram [1] without the proof.

Theorem 1. (Halmos[3]) An operator $A$ on a Hilbert space $H$ is subnormal if and only if
(1) $\sum_{m, n=0}^{r}\left(A^{n} x_{m}, A^{m} x_{n}\right) \geqq 0$ for every finite set $x_{0}, x_{1}, \cdots, x_{r}$ in $H$, and
(2) there exists a positive constant $c$ such that

$$
\sum_{m, n=0}^{r}\left(A^{n+1} x_{m}, A^{m+1} x_{n}\right) \leqq c \cdot \sum_{m, n=0}^{r}\left(A^{n} x_{m}, A^{m} x_{n}\right)
$$

for every finite set $x_{0}, \cdots, x_{r}$ in $H$.
Theorem 2. (Bram[1]) Let $A$ be an operator on $H$, and suppose that $\sum_{m, n=0}^{r}\left(A^{n} x_{m}, A^{m} x_{n}\right) \geqq 0$ for every finite set $x_{0}, x_{1}, \cdots, x_{r}$ in $H$. Then

$$
\sum_{m, n=0}^{r}\left(A^{n+1} x_{m}, A^{m+1} x_{n}\right) \leqq\|A\|^{2} \cdot \sum_{m, n=0}^{r}\left(A^{n} x_{m}, A^{m} x_{n}\right)
$$

for every finite set $x_{0}, x_{1}, \cdots, x_{r}$ in $H$.

Lemma 1. Let $H$ be cyclic with respect to a subnormal operator $A$ on $H$, and let $B$, acting on $K$, be the minimal normal extension of $A$. Then $K$ is cyclic with respect to $B$.

Proof. Let $x$ be a cyclic vector for $H$ with respect to $A$, i.e., $H=$ $\vee\left\{A^{n} x ; n \geqq 0\right\}$. Let $M=\bigvee\left\{B^{* m} B^{n} x ; m, n \geqq 0\right\}$. Then, since $B^{n} x=A^{n} x$ for all $n \geqq 0$, we have $H \subset M$; moreover $M$ reduces $B$. But $B$ is the minimal normal extension of $A$ so that $M=K, K$ is cyclic with respect to $B$.

Let $x$ be a cyclic vector for a subnormal operator $A$ on $H$, let $\mu=(E(\lambda) x, x)$ where $E(\lambda)$ denotes the resolution of the identity for the minimal normal extension $B$, acting on $K$, of $A$ and let $D_{1}$ be the linear manifold in $K$ consisting of all vectors of the form $f(B) x$ where $f$ is a bounded Borel function on the spectrum $\sigma(B)$ of $B$.

By Lemma 1, $D_{1}$ is dense in $K$ and we see easily that the operator $V_{1}$ from $D_{1}$ to $L_{2}(d \mu ; \sigma(B))$ defined by $V_{1} f(B) x=f$ has a unique extension $V$ from the closure $\widetilde{D_{1}}=K$ of $D_{1}$ to the $L^{2}$-closure of the set of all bounded Borel functions, i.e., to $L^{2}(d \mu ; \sigma(B))$ and that $V$ is an isometric isomorphism between $K$ and $L^{2}(d \mu ; \sigma(B))$.

Lemma 2. If $A$ is a subnormal operator on $H$ with a cyclic vector $x$ in $H$ and if $H^{2}(d \mu ; \sigma(B))$ be the $L^{2}$-closure of the set $P$ of all complex polynomials in $\lambda$, defined on the spectrum $\sigma(B)$ of the minimal normal extension B, acting on $K$, of $A$ with respect to the Lebesgue-Stieltjes measure $d \mu=d(E(\lambda) x, x)$, then $H$ and $H^{2}(d \mu ; \sigma(B))$ are isomorphic by the mapping $V$ defined as above.

Proof. Since $H=\bigvee\left\{A^{n} x ; n \geqq 0\right\}$, for any $y$ in $H$, there exists some sequence $p_{n}$ in $P$ such that $y=\lim _{n \rightarrow \infty} p_{n}(A) x$, and since $p_{n}(A) x=p_{n}(B) x$ for all $n \geqq 0$, we have

$$
\int\left|p_{n}(\lambda)-p_{m}(\lambda)\right|^{2} d(E(\lambda) x, x) \rightarrow 0 \text { as } m, n \rightarrow \infty,
$$

so that there exists a function $p_{y}$ in $H^{2}(d \mu ; \sigma(B))$ such that

$$
\int\left|p_{y}(\lambda)-p_{n}(\lambda)\right|^{2} d(E(\lambda) x, x) \rightarrow 0 \text { as } n \rightarrow \infty
$$

Since the existence of $p_{y}$ is independent of the choice of the sequence $p_{n}$ in $P$, the operator $V$ defined by $V y=p_{y}$ is well-defined and clearly $V$ is an isometry from $H$ into $H^{2}(d \mu ; \sigma(B))$.

Conversely, by the definition of $H^{2}(d \mu ; \sigma(B))$, for any $p$ in $H^{2}(d \mu ; \sigma(B))$, there exists a sequence $p_{n}$ in $P$ such that

$$
\int\left|p(\lambda)-p_{n}(\lambda)\right|^{2} d \mu \rightarrow 0 \text { as } n \rightarrow \infty .
$$

Hence, $\left\{p_{n}(B) x\right\}$ is a Cauchy sequence in $H$, and hence there exists a vector $y$ in $H$ such that $\left\|p_{n}(A) x-y\right\| \rightarrow 0$ as $n \rightarrow \infty$. This means that the operator $V$ is an isometry from $H$ onto $H^{2}(d \mu ; \sigma(B))$, and the proof is completed.

ThEOREM 3. If $A$ is a subnormal operator on $H$ with a cyclic vector $x$ in $H$ and if $T$ is a bounded linear operator on $H$ which commutes with $A$, then $T$ is subnormal and there exists a Borel measurable function $p_{T}(\lambda)$ in

$$
H^{\infty}(d \mu ; \sigma(B))=H^{2}(d \mu ; \sigma(B)) \cap L^{\infty}(d \mu ; \sigma(B))
$$

such that $T y=p_{T}(B) y$ for all $y$ in $H$, where $B$ denotes the minimal normal extension of $A$ and $d \mu=d(E(\lambda) x, x), E(\lambda)$ denotes the spectral measure of $B$.

Proof. Since $H=\bigvee\left\{A^{n} x ; n \geqq 0\right\}, T x=\lim _{n \rightarrow \infty} p_{n}(A) x$ for some sequence $\left\{p_{n}(A)\right\}, p_{n} \in P$ and since $p_{n}(A) x=p_{n}(B) x$, we have

$$
\int\left|p_{n}(\lambda)-p_{m}(\lambda)\right|^{2} d(E(\lambda) x, x) \rightarrow 0 \text { as } m, n \rightarrow \infty,
$$

so that there exists a Borel measurable function $p_{T}$ in $H^{2}(d \mu ; \sigma(B))$ such that

$$
\int\left|p_{T}(\lambda)\right|^{2} d(E(\lambda) x, x)<\infty \text { and } \int\left|p_{T}(\lambda)-p_{n}(\lambda)\right|^{2} d(E(\lambda) x, x) \rightarrow 0
$$

as $n \rightarrow \infty$ (see [5; page 348]) and hence
$x \in D\left(p_{T}(B)\right)=\left\{y \in K ; \int\left|p_{T}(\lambda)\right|^{2} d(E(\lambda) y, y)<\infty\right\}$ and $p_{n}(A) x=p_{n}(B) x \rightarrow p_{T}(B) x$ from which we have $T x=p_{T}(B) x$.

Since $T$ commutes with $A$, for any $p$ in $P$, we have $T p(A) x=p(A) T x$ $=p(B) p_{T}(B) x=p_{T}(B) p(B) x=p_{T}(B) p(A) x$. Hence, if $y \in H$, and $q_{n}(A) x \rightarrow y$ with $q_{n}$ in $P$, then $p_{T}(B) q_{n}(A) x=T q_{n}(A) x \rightarrow T y$ because $T$ is bounded, and since $p_{T}(B)$ is closed, it follows that $H \subset D\left(p_{T}(B)\right)$ and $p_{T}(B) y=T y$ for all $y$ in $H$. Hence, also, since $T H \subset H$, we have $H \subset D\left(p_{T}(B)^{n}\right)=D\left(p_{T}(B)^{* n}\right)$ for all non-negative integers $n$, and $p_{T}(B)^{n} y=T^{n} y$ for all $y$ in $H$.

Let $N=\vee\left\{p_{T}(B)^{*} y ; y \in H, n \geqq 0\right\}$, then clearly we have $H \subset N \subset K$. If $y_{0}, y_{1}, \cdots, y_{r}$ in $H$, then we have

$$
\begin{aligned}
\sum_{m, n=0}^{r}\left(T^{m} y_{n}, T^{n} y_{m}\right) & =\sum_{m, n=0}^{r}\left(p_{T}(B)^{m} y_{n}, p_{T}(B)^{n} y_{m}\right) \\
& =\left\|\sum_{n=0}^{r} p_{T}(B)^{* n} y_{n}\right\|^{2} \geqq 0
\end{aligned}
$$

Hence, by Theorem 1 and 2, $T$ is subnormal. By Theorem 2, it follows that for any finite set $y_{0}, y_{1}, \cdots, y_{r}$ in $H$, we have

$$
\sum_{m, n=0}^{r}\left(T^{m+1} y_{n}, T^{n+1} y_{m}\right) \leqq\|T\|^{2} \cdot \sum_{m, n=0}^{r}\left(T^{m} y_{n}, T^{n} y_{m}\right)
$$

i.e., $\quad \sum_{m, n=0}^{r}\left(p_{T}(B)^{m+1} y_{n}, p_{T}(B)^{n+1} y_{m}\right) \leqq\|T\|^{2} . \sum_{m, n=0}^{r}\left(p_{T}(B)^{m} y_{n}, p_{T}(B)^{n} y_{m}\right)$,

$$
\text { or }\left\|p_{T}(B) \cdot \sum_{n=0}^{r} p_{T}(B)^{* n} y_{n}\right\|^{2} \leqq\|T\|^{2} \cdot\left\|\sum_{n=0}^{r} p_{T}(B)^{* n} y_{n}\right\|^{2},
$$

which shows that $p_{T}(B)$ is bounded on a dense linear subset of $N$.
Since $p_{T}(B)$ is closed, $N \subset D\left(p_{T}(B)\right)$ and $\left\|p_{T}(B) y\right\| \leqq\|T\| \cdot\|y\|$ for all $y$ in $N$. We observe that $N$ reduces $p_{r}(\mathrm{~B})$ and also $B$. Hence, by the minimality of $B, N=K$ and $K=D\left(p_{T}(B)\right)$. This implies that $p_{T}(B)$ is bounded on $K$ and hence $p_{T}(\lambda) \in L^{\infty}(d \mu ; \sigma(B))$ which completes the proof.

COROLlary 1. If $A$ is a subnormal operator on $H$ with a cyclic vector and if $T$ is a bounded linear operator on $H$ which commutes with $A$ and $A^{*}$, then $T$ is normal.

Proof. By Theorem 3, $T$ and $T^{*}$ are subnormal. Since every subnormal operator $S$ on $H$ is hyponormal (i.e., $\|S x\| \geqq\left\|S^{*} x\right\|$ for all $x \in H$ ), $T$ is normal.

As the consequence, we see easily that if $A$ is a subnormal operator with a cyclic vector and if $R(A)^{\prime}$ is the commutant of the von Neumann algebra $R(A)$ generated by a single operator $A$, then $R(A)^{\prime}$ is abelian, in particular, if $A$ is normal, then $R(A)$ is maximal abelian.

Let $A$ be a subnormal operator on $H$ with a cyclic vector $x, B$ its minimal normal extension and let $\alpha(A)$ be the set of all bounded linear operators on $H$ which commutes with $A$, then, by Theorem 3, for any $T$ in $\alpha(A)$, there exists a Borel measurable function $p_{T}(\lambda)$ in $H^{\infty}(d \mu ; \sigma(B))$ such that $T y=p_{T}(B) y$ for all $y$ in $H$. Let $L_{\alpha(A)}$ be the set of all functions $p_{T}(\lambda), T \in \alpha(A)$, then we can show

Lemma 3. $L_{\alpha(A)}=H^{\infty}(d \mu ; \sigma(B))$.
Proof. If $p \in H^{\infty}(d \mu ; \sigma(B))$, then, by the definition of $H^{\infty}(d \mu ; \sigma(B))$, there exists a sequence $p_{n}$ in $P$ such that

$$
\int\left|p_{n}(\lambda)-p(\lambda)\right|^{2} d(E(\lambda) x, x) \rightarrow 0 \text { as } n \rightarrow \infty
$$

Since $p \in L^{\infty}(d \mu ; \sigma(B))$, we can define the bounded linear operator $p(B)$ on $K$ such that $\left\|p_{n}(B) x-p(B) x\right\| \rightarrow 0$ as $n \rightarrow \infty$ and $p(B) x \in H$. (because $\left.p_{n}(B) x=p_{n}(A) x \in H\right)$. Since $H=\bigvee\left\{A^{n} x ; n \geqq 0\right\}$ and for any $q$ in $P$, $p(B) q(A) x=p(B) q(B) x=q(B) p(B) x=q(A) p(B) x \in H$, by the boundedness of $p(B)$, we have $p(B) y \in H$ for all $y \in H$. Therefore the restriction $p(B) \mid H$ of $p(B)$ on its invariant subspace $H$ clearly commutes with $A$, i.e., $p(B) \mid H \in \alpha(A)$, and, by the definition of $L_{\alpha(A)}, p \in L_{\alpha(A)}$. The converse inclusion is clear by Theorem 3.

Let $\widetilde{r}_{0}(\sigma(A))$ be the uniform closure of the set $r_{0}(\sigma(A))$ of all rational functions with no pole in the spectrum $\sigma(A)$ of an operator $A$. If $r(\lambda) \in \widetilde{r_{0}}(\sigma(A))$, then $r(A) \in \alpha(A)$ and hence, if $A$ is a subnormal operator on $H$ with a cyclic vector, then, by Theorem 3 and Lemma $3, r(\lambda) \in H^{\infty}(d \mu ; \sigma(B))$. However, it might not be true that $\widetilde{r_{0}}(\sigma(B)) \subset H^{\infty}(d \mu ; \sigma(B))$ where $B$ is the minimal normal extension of $A$.

ThEOREM 4. If $A$ is a subnormal operator on $H$ with a cyclic vector and if $B$ is its minimal normal extension on $K$, then $\widetilde{r_{0}}(\sigma(B)) \subset H^{\infty}(d \mu ; \sigma(B))$ if and only if $\sigma(A)=\sigma(B)$.

Proof. Since $\widetilde{r_{0}}(\sigma(A)) \subset H^{\infty}(d \mu ; \sigma(B))$, we have only to show that $\widetilde{r_{0}}(\sigma(B))$ $\subset H^{\infty}(d \mu ; \sigma(B))$ implies $\sigma(A)=\sigma(B)$.

It is known that $\sigma(B) \subset \sigma(A)$ by Halmos [4] and that $\sigma(A) \subset \sigma(B) \cup h(B)$ by $\operatorname{Bram}[1]$, where $h(B)$ denotes the union of all holes of $\sigma(B)$. Hence we have only to show that $\widetilde{r_{0}}(\sigma(B)) \subset H^{\circ}(d \mu ; \sigma(B))$ implies $h(B) \subset \rho(A)$, where $\rho(A)$ denotes the resolvent set of $A$.

If $\lambda_{0}$ is an arbitrary point in $h(B)$ and if $r(\lambda)=\left(\lambda-\lambda_{0}\right)^{-1}$, then $r(\lambda) \in \widetilde{r_{0}}(\sigma(B))$ and hence $r(\lambda) \in H^{\infty}(d \mu ; \sigma(B))$. Therefore, by Lemma 3, there exists an operator $T$ in $\alpha(A)$ such that $T y=r(B) y$ for all $y \in H$. Hence, for any $y \in H$, we have

$$
\begin{aligned}
\left(A-\lambda_{0} I\right) T y & =T\left(A-\lambda_{0} I\right) y=r(B)\left(B-\lambda_{0} I\right) y \\
& =\int r(\lambda)\left(\lambda-\lambda_{0}\right) d E(\lambda) y=y .
\end{aligned}
$$

This means that $A-\lambda_{0} I$ has a bounded inverse $T$, i.e., $\lambda_{0} \in \rho(A)$. Therefore $h(B) \subset \rho(A)$.

Example. To show that a subnormal operator $A$ need not be normal even when $A$ has a cyclic vector and $\sigma(A)=\sigma(B)$, where $B$ is the minimal normal extension of $A$, let $D$ be the closed unit disk, $\mu$ the normalized Lebesgue measure in $D, K=L^{2}(d \mu ; D), B=L_{z}$, i.e., $L_{z} f(z)=z \cdot f(z)$ for all $f(z) \in L^{2}(d \mu ; D)$. Let $H$ be the $L^{2}$-closure of the set $P$ of all complex polynomials in $z$, defined on $D$, with respect to $\mu$, and set $A=L_{z} \mid H$, then clearly $A$ is a subnormal operator on $H$ with a cyclic vector $u(z)=1$ and its minimal normal extension is $B$ on $K$. Since $\sigma(B)=D, h(B)=\emptyset$. And hence, by the same reason as in the proof of Theorem 4, $\sigma(A)=\sigma(B)$.

If we set $z=r \exp (i \theta)$, then $d \mu(z)=\frac{1}{\pi} r d r d \theta$; and hence,

$$
\int_{D} z z^{n} d \mu(z)=\frac{1}{\pi} \int_{0}^{1} \int_{0}^{2 \pi} r^{n+1} e^{i(n+1) \theta} r d r d \theta
$$

$$
=\frac{1}{\pi} \int_{0}^{1} r^{n+2} d r \cdot \int_{0}^{2 \pi} e^{i(n+1) \theta} d \theta=0
$$

for all $n \geqq 0$. This implies that $\bar{z} \perp H$ and $\bar{z} \in K$. Therefore $H \subsetneq K$ and $A$ is non-normal by the minimality of $B$.

THEOREM 5. If $B$ is the minimal normal extension of a subnormal operator $A$ on $H$ and if $\widetilde{r}_{0}(\sigma(B)) \cap \overline{\tilde{r}_{0}(\sigma(B))} \neq\{c \cdot 1\}$ (the bar denotes the complex conjugate), then there exists a non-trivial closed invariant subspace of $H$ for $A$.

Proof. For any fixed non-zero vector $x$ in $H$, we may assume that $H=\bigvee\left\{A^{n} x ; n \geqq 0\right\}$, because if $\vee\left\{A^{n} x ; n \geqq 0\right\} \subsetneq H$, then $\vee\left\{A^{n} x ; n \geqq 0\right\}$ is clearly a non-trivial closed invariant subspace of $A$.

In the case where $\sigma(A)=\sigma(B)$, for any $r(\lambda)$ in $\widetilde{r}_{0}(\sigma(B)) \cap \overline{\widetilde{r}_{0}(\sigma(B)),} r(\lambda) \neq c \cdot 1$, we have $r(A), r(A)^{*} \in \alpha(A)$ and $r(A) \neq c \cdot I$; and hence $r(A)$ is normal by Corollary 1 . In this case, clearly, $A$ has reducing subspaces.

In the other case, we have $\sigma(A) \cap h(B) \neq \emptyset$ by the same reason as in the proof of Theorem 4. Since $A y_{n}-\gamma y_{n}=B y_{n}-\gamma y_{n}$ for all $y_{n}$ in $H, \sigma_{a p}(A) \subset \sigma_{a p}(B)$ and easily we have $\sigma_{p}(S) \cup \sigma_{c}(S) \subset \sigma_{a p}(S)$ for any bounded linear operator $S$, where $\sigma_{p}(S), \sigma_{c}(S)$ and $\sigma_{a p}(S)$ denote the point spectrum, the continuous spectrum and the approximate point spectrum of $S$, respectively. From this, we have $\gamma \in \sigma_{r}(A)$ if $\gamma \in \sigma(A) \cap h(B)$, where $\sigma_{r}(A)$ denotes the residual spectrum of $A$. Hence $\bar{\gamma} \in \sigma_{p}\left(A^{*}\right)$. Let $M=\left\{y \in H ; A^{*} y=\bar{\gamma} y\right\}$, then the subspace $H \ominus M$ is clearly a non-trivial closed invariant subspace of $A$.

It is known that if $\sigma(B)$ has two dimensional Lebesgue measure zero, then $\widetilde{r}_{0}(\sigma(B))=C(\sigma(B))$, where $C(\sigma(B))$ denotes the set of all continuous functions on $\sigma(B)$ (see[7]). Hence, we have

Corollary 2. (Wermer [6]) If $B$ is the minimal normal extension of a subnormal operator $A$ and if the spectrum $\sigma(B)$ has two dimensional Lebesgue measure zero, then there exists a non-trivial closed invariant subspace of $A$.

REMARK 1. It is clear that $H^{\infty}(d \mu ; \sigma(B))=L^{\infty}(d \mu ; \sigma(B))$ if and only if $A$ is normal and also that $H^{\infty}(d \mu ; \sigma(B)) \cap \overline{H^{\circ}(d \mu ; \sigma(B))}=\{c \cdot 1\}$ if and only if the von Neumann algebra $R(A)$ generated by a single operator $A$ is the full operator algebra on $H$.

As an application of the representation theorem of a subnormal operator $A$ on $H$ with a cyclic vector, we can give the description of the existence of a non-trivial closed invariant subspace of $A$ in terms of $L^{2}(d \mu ; \sigma(B)), H^{2}(d \mu ; \sigma(B))$ and $H^{\circ}(d \mu ; \sigma(B))$ as follows:

A subnormal operator $A$ on $H$ with a cyclic vector $x$ in $H$ has a nontrivial closed invariant subspace if and only if
(*) there exists a function $q$, not identically zero, in $H^{2}(d \mu ; \sigma(B))$ such that

$$
\left[L^{2}(d \mu ; \sigma(B)) \ominus \overline{H^{\infty}(d \mu ; \sigma(B))} \cdot q\right] \cap H^{2}(d \mu ; \sigma(B)) \neq\{0\}
$$

where the bar denotes the complex conjugate and $B$ is the minimal normal extension on $K$ of $A$.

As spacial cases of the condition (*), we have
(case 1) there exists a function $q$ in $H^{2}(d \mu ; \sigma(B))$ such that

$$
H^{\circ}(d \mu ; \sigma(B)) \cdot q \cap\left[L^{2}(d \mu ; \sigma(B)) \ominus H^{2}(d \mu ; \sigma(B))\right] \neq\{0\} \text { and }
$$

(case 2) there exists a function $q$ in $H^{2}(d \mu ; \sigma(B))$ such that

$$
H^{\overline{0}}(d \bar{\mu} ; \sigma(B)) \cdot q \cap H^{2}(d \mu ; \sigma(B)) \neq\{c \cdot q\} .
$$

In the case 1 , the subnormal operator $A$ has a non-trivial closed invariant subspace $M$ such that $M$ is the closure of the range of some operator $T$ in $\alpha(A)$. In fact, by the assumption, there exists a non-constant function $p$ in $H^{\infty}(d \mu ; \sigma(B))$ such that $\bar{p} \cdot q \in\left[L^{2}(d \mu ; \sigma(B)) \ominus H^{2}(d \mu ; \sigma(B))\right]$; and hence, for a vector $y$ in $H$ corresponding to $q$ by Lemma 2 and an operator $p(B) \mid H$ in $\alpha(A)$ corresponding to $p$ by Lemma 3, $(p(B) \mid H)^{*} y=0$. Let $H \ominus M=\left\{y \in H ;(p(B) \mid H)^{*} y=0\right\}$, then $M$ is the desirous subspace.

In the case 2 , the subnormal operator $A$ on $H$ has a non-trivial closed invariant subspace $M$ such that $M=\left\{y \in H ;\|T y\|=\left\|T^{*} y\right\|\right\}$ for some $T$ in $\alpha(A)$. Hence, we have

Theorem 6. If $A$ is a subnormal operator on $H$ and if there exists a non-zero vector $y$ in $H$ such that $\|T y\|=\left\|T^{*} y\right\|$ for some $T$ in $\alpha(A)$, $0 \neq T \neq c \cdot I$, then $A$ has a non-trivial closed invariant subspace.

Proof. We may assume that $A$ has a cyclic vector and that $T$ is nonnormal. Let $M=\left\{y \in H ;\|T y\|=\left\|T^{*} y\right\|\right\}$, then $M$ is non-trivial by the assumption. Since, by Theorem 3, $T$ is subnormal and hence $T$ is hyponormal, i.e., $(S=) T * T-T T^{*} \geqq 0$. Therefore $M$ is the null space of the non-negative selfadjoint operator $S$ and is a closed subspace of $H$. Let $B$ be the minimal normal extension on $K$ of $A$, then, by Theorem 3, there exists a function $p_{T}(\lambda)$ in $H^{\infty}(d \mu ; \sigma(B))$ such that $T y=p_{T}(B) y$ for all $y$ in $H$. The invariantness of $M$, under $A$, follows from $p_{T}(B)^{*} A y=p_{T}(B)^{*} B y=B p_{T}(B)^{*} y=A p_{T}(B)^{*} y \in H$ for all $y$ in $M$ because, for any $y$ in $H, p_{T}(B)^{*} y \in H$ if and only if $\|T y\|=\left\|T^{*} y\right\|$.

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