Subordinated compound Poisson processes of order k

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Abstract   In this article, the compound Poisson process of order k (CPPoK) is introduced and its properties are discussed. Further, using mixture of tempered stable subordinators (MTSS) and its right continuous inverse, the two subordinated CPPoK with various distributional properties are studied. It is also shown that the space and tempered space fractional versions of CPPoK and PPoK can be obtained, which generalize the process defined in [Statist. Probab. Lett. 82 (2012), 852–858].

1   Introduction

The Poisson process has been the conventional model for count data analysis, and due to its popularity and applicability various researchers have generalized it in several directions; e.g., compound Poisson processes, weighted Poisson distributions, fractional (time-changed) versions of Poisson processes (see [6, 21, 13, 2, 1], and references therein). A handful of researchers have also studied the distributions and processes of order k in [11, 23, 18]. In particular, the discrete distribution of order k, introduced by Philippou et al. (see [24]), includes binomial, geometric and negative binomial distributions of order k. These distributions play an important role in

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several areas, such as reliability, statistics, finance, and actuarial risk analysis (see [19, 29, 4]).

In risk theory, the total claim amount is usually modeled by using a compound Poisson process, say $Z_t = \sum_{i=1}^{N(t)} Y_i$, where the compounding random variables Y_i are iid and the number of claims $N(t)$, which are independent of $\{Y_i\}_{i\geq 1}$, follows the Poisson distribution. Due to the restriction of single arrival in each inter-arrival time, the model is not suitable to use. Kostadinova and Minkova [10] introduced a Poisson process of order k (PPoK), which allows us to model the arrival in a group of size k. Recently, a time-changed version of Poisson processes of order k is studied by [29], which allows group arrivals and also accommodates the case of extreme events, which was not covered by [10]. In spite of its applicability, we observe that this model does not cover the phenomenon of underdispersion, where the variance is less than the mean (see [25, 32] and references therein). Therefore, a generalization of this model is essentially required and is proposed in this article.

To the best of our knowledge, such a generalization is not yet studied. Therefore, we introduce the compound Poisson process of order k (CPPoK) with the help of the Poisson process of order k (PPoK) and study its distributional properties. Recently, tempered stable processes and their inverses have been widely used for time-change (subordination) as their moments are finite and hence various real-life situations can be modeled easily (see [12, 26, 20]). Various versions of Poisson processes, using subordination like space and time-fractional Poisson processes have been studied in the literature (see [22, 17]). Hence, it is worth exploring the time-change of CPPoK with a special type of Lévy subordinator known as mixture of tempered stable subordinators, its right continuous inverse, and analyze some properties of these time-changed processes. These processes also generalize the process discussed in [22].

The article is organized as follows. In Section 2, we introduce CPPoK and derive some of its general properties along with martingale characterization property. In Section 3, we introduce two types of CPPoK with the help of MTSS and its right continuous inverse. Further we derive some important distributional properties.

Acronym

PoK	Poisson distribution of order k
PPoK	Poisson process of order k
PP	Poisson process
CPPoK	Compound Poisson process of order k
FDD	Finite-dimensional distribution
IID	Independent and identically distributed
MTSS	Mixture of tempered stable subordinators
TCPPoK	Time changed compound Poisson process of order k
pmf	Probability mass function
pgf	Probability generating function
LRD	Long-range dependence

2 Compound Poisson process of order k and its properties

2.1 Compound Poisson process of order k

In this section, we introduce CPPoK and derive its distributional properties. First, we define the Poisson distribution of order k.

Definition 1 ([23]). Let $N^{(k)} \sim PoK(\lambda)$, the Poisson distribution of order k (PoK) with rate parameter $\lambda > 0$, then the probability mass function (pmf) of $N^{(k)}$ is given by

$$\mathbb{P}[N^{(k)} = n] = \sum_{\substack{x_1, x_2, \ldots, x_k \geq 0 \\ \sum_{j=1}^{k} j x_j = n}} e^{-k\lambda} \frac{\lambda^{x_1 + x_2 + \ldots + x_k}}{x_1! x_2! \ldots x_k!}, \ n = 0, 1, \ldots,$$

where the summation is taken over all non-negative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = n$.

Philippou [23] showed the existence of PoK as a limiting distribution of negative binomial distribution of order k. Kostadinova and Minkova [10] later generalized PoK to evolve over time, in terms of a process which can be defined as follows.

Definition 2 ([10]). Let $\{N(t)\}_{t \geq 0}$ denote $PP(k\lambda)$, the Poisson process with rate parameter $k\lambda$, and $\{X_i\}_{i \geq 1}$ be a sequence of independent and identically distributed (IID) discrete uniform random variables with support on $\{1, 2, \ldots, k\}$. Also, assume that $\{X_i\}_{i \geq 1}$ and $\{N(t)\}_{t \geq 0}$ are independent. Then $\{N^{(k)}(t)\}_{t \geq 0}$, defined by $N^{(k)}(t) = \sum_{i=1}^{N(t)} X_i$ is called the Poisson process of order k (PPoK) and is denoted by $PPoK(\lambda)$.

However, the clumping behavior associated with the random phenomenon in [8] cannot be accommodated by PPoK [10]. Hence, there is a need to generalize this notion as well. Now we propose the following generalization of PPoK.

Definition 3. Let $\{N^{(k)}(t)\}_{t \geq 0}$ be the $PPoK(\lambda)$ and $\{Y_i\}_{i \geq 1}$ be a sequence of IID random variables, independent of $N^{(k)}(t)$, with cumulative distribution function (CDF) H. Then the process $\{Z(t)\}_{t \geq 0}$ defined by $Z(t) = \sum_{i=1}^{N^{(k)}(t)} Y_i$ is called the compound Poisson process of order k (CPPoK) and is denoted by $CPPoK(\lambda, H)$.

From the definition, it is clear that:

(i) for $k = 1$, $\{Z(t)\}_{t \geq 0}$ is $CPP(\lambda, H)$ the usual compound Poisson process.

(ii) for $H = \delta_1$, the Dirac measure at 1, $\{Z(t)\}_{t \geq 0}$ is $PPoK(\lambda)$.

(iii) for $k = 1$ and $H = \delta_1$, $\{Z(t)\}_{t \geq 0}$ is $PP(\lambda)$.

Next, we present a characterization of $CPPoK(\lambda, H)$, in terms of the finite-dimensional distribution (FDD).

Theorem 1. *Let* $\{Z(t)\}_{t \geq 0}$ *be as defined in Definition 3. Then the FDD, denoted as* $F_{Z(t_1), \ldots, Z(t_n)}(y_1, \ldots, y_n) = \mathbb{P}[Z(t_1) \leq y_1, \ldots, Z(t_n) \leq y_n]$ *has the following form*

$$F_{Z(t_1), \ldots, Z(t_n)}(y_1, \ldots, y_n) = \sum_{j_1, \ldots j_n} \prod_{l=1}^{n} p_{j_l}(\Delta t_l) \int_{-\infty}^{v_1} \ldots \int_{-\infty}^{v_n} \prod_{m=1}^{n} h_{Y_1}^{*j_m}(x_m) dx_m,$$

(1)

where the summation is taken over all non-negative integers $j_i \geq 0$, $i = 1, 2, \ldots, n$, $v_k = y_k - \sum_{l=1}^{k-1} x_l$, $k = 1, \ldots, n$, $\Delta t_l = t_l - t_{l-1}$, h is the density/pmf of H, p_{j_l} is the pmf of $PPoK(\lambda)$, and $(*j_n)$ represents the j_n-fold convolution.

Proof. Let $0 = t_0 \leq t_1 \leq \cdots \leq t_n = t$ be the partition of $[0, t]$. Since, the increments of $\{N^{(k)}(t)\}$ are independent and stationary, we can write $N^{(k)}(t_i) = \sum_{l=1}^{i} N^{(k)}(\Delta t_l)$, $i = 1, \ldots, n$, and $\mathbb{P}[N^{(k)}(t) = j] = p_j(t)$, $j = 0, 1, \ldots$.

$$F_{Z(t_1),\ldots,Z(t_n)}(y_1, \ldots, y_n) = \mathbb{P}\left[\sum_{i=1}^{N^{(k)}(t_1)} Y_i \leq y_1, \ldots, \sum_{i=1}^{N^{(k)}(t_n)} Y_i \leq y_n\right]$$

$$= \sum_{j_1,\ldots,j_n} \mathbb{P}\left[\sum_{i=1}^{j_1} Y_i \leq y_1, \ldots, \sum_{i=1}^{\sum_{l=1}^{n} j_l} Y_i \leq y_n\right] \prod_{l=1}^{n} p_{j_l}(\Delta t_l)$$

Let us denote $\sum_{i=1}^{j_1} Y_i = Y(j_1), \ldots, \sum_{i=j_1+\cdots+j_{n-1}+1}^{j_1+\cdots+j_n} Y_i = Y(j_n)$, then it becomes

$$F_{Z(t_1),\ldots,Z(t_n)}(y_1, \ldots, y_n)$$

$$= \sum_{j_1,\ldots,j_n} \prod_{l=1}^{n} p_{j_l}(\Delta t_l) \mathbb{P}\left[Y(j_1) \leq y_1, \ldots, \sum_{l=1}^{n} Y(j_l) \leq y_n\right]$$

$$= \sum_{j_1,\ldots,j_n} \prod_{l=1}^{n} p_{j_l}(\Delta t_l) \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n - \sum_{l=1}^{n-1} x_l} \prod_{m=1}^{n} h_{Y_1}^{*j_m}(x_m) dx_m$$

$$= \sum_{j_1,\ldots,j_n} \prod_{l=1}^{n} p_{j_l}(\Delta t_l) \int_{-\infty}^{v_1} \cdots \int_{-\infty}^{v_n} \prod_{m=1}^{n} h_{Y_1}^{*j_m}(x_m) dx_m.$$

$\square$

Remark 1. *For $n = 1$, (1) reduces to $\mathbb{P}[Z(t) \leq y] = \sum_{j=0}^{\infty} p_j(t) \int_{-\infty}^{y} h_{Y_1}^{*j}(x) dx$, which is the marginal distribution of CPPoK(λ, H).*

Remark 2. *For $n = 1$ and H has discrete distribution, the probability mass function (pmf) of CPPoK(λ, H), denoted as $\mathbb{P}[Z(t) = n] = P_n(t)$ is given as*

$$P_n(t) = \mathbb{P}[Z(t) = n] = \sum_{m=0}^{\infty} p_m(t) h_{Y_1}^{*m}(n),$$

where $p_m(t) = \mathbb{P}[N^{(k)}(t) = m]$, the pmf of PPoK($\lambda$) and, $h_{Y_1}^{*m}$ is the m-fold convolution of pmf of Y_1. Difference-differential equation satisfied by the pmf of CPPoK(λ, H) is given as

$$\frac{d}{dt} P_n(t) = \sum_{m=0}^{\infty} h_{Y_1}^{*m}(n) \frac{d}{dt} p_m(t) = \sum_{m=0}^{\infty} h_{Y_1}^{*m}(n)\left[-k\lambda p_m(t) + \lambda \sum_{j=1}^{m \wedge k} p_{m-j}(t)\right]$$

with the initial condition $P_n(0) = h_{Y_1}^{*m}(n)$, $n = 1, 2, \ldots$ and $P_0(0) = h_{Y_1}^{*m}(0)$ and $m \wedge k = \min(m, k)$.

Next, we present some examples of *CPPoK*(λ, H) by taking different distribution of H.

Example 1. Let H has geometric distribution with parameter $p \in (0, 1]$. Then the marginal distribution of *CPPoK*(λ, H) is given as

$$F_{Z(t)}(y) = \mathbb{P}[Z(t) \le y] = \sum_{j=0}^{\infty} p_j(t) \int_{-\infty}^{y} h_{Y_1}^{*j}(x)dx$$

$$= \sum_{j=0}^{\infty} I_{1-p}(j, y+1) p_j(t),$$

where $I_x = \frac{B(x;a,b)}{B(a,b)}$, $0 < x < 1$ is the regularized incomplete beta function, $B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}$ is the incomplete beta function.

Example 2. Let H has exponential distribution with parameter $\mu > 0$. Then the marginal distribution of *CPPoK*(λ, H) is given as

$$F_{Z(t)}(y) = \mathbb{P}[Z(t) \le y] = \sum_{j=0}^{\infty} p_j(t) \int_{-\infty}^{y} h_{Y_1}^{*j}(x)dx$$

$$= \sum_{j=0}^{\infty} \frac{\gamma(j, \mu y)}{(j-1)!} p_j(t),$$

where $\gamma(s, x) = \int_0^x e^{-t} t^{s-1} dt$, $x \ge 0$ is the incomplete gamma function.

Here we plot the pmf of the considered process when $Y_i \sim Geo(p)$, $p \in (0, 1]$. Fix $t = 1$, $\lambda = 1$, $p = 0.5$, and $k = 1$.
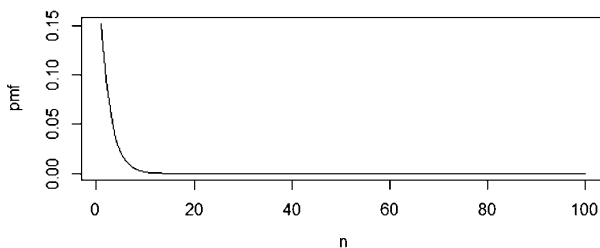

Fig. 1. Plot for $\mathbb{P}[Z(t) = n]$

Theorem 2. *The mean, variance and covariance of the process* $\{Z(t)\}_{t \ge 0}$ *can be expressed as*

(*i*) $\mathbb{E}[Z(t)] = \mathbb{E}[Y]\mathbb{E}[N^{(k)}(t)]$,

(*ii*) $Var[Z(t)] = Var(Y)\mathbb{E}[N^{(k)}(t)] + \mathbb{E}[Y]^2 Var(N^{(k)}(t))$.

(*iii*) $Cov[Z(s), Z(t)] = Var(Y)\mathbb{E}[N^{(k)}(s \wedge t)] + \mathbb{E}[Y]^2 Var[N^{(k)}(s \wedge t)]$, *where* $s \wedge t = \min(s, t)$.

Proof. $\mathbb{E}[Z(t)] = \mathbb{E}[\sum_{i=1}^{N^{(k)}(t)} Y_i] = \mathbb{E}[Y]\mathbb{E}[N^{(k)}(t)] = \frac{k(k+1)}{2}\lambda t \mathbb{E}[Y]$. Hence, part (i) is proved.

Next, we derive the expression for variance of $Z(t)$. Let $p_n(t)$ be the *pmf* of $PPoK(\lambda)$. Then

$$
\begin{aligned}
\mathrm{Var}[Z(t)] &= \mathbb{E}[Z(t) - \mathbb{E}[Z(t)]]^2 \\
&= \sum_{n=0}^{\infty} \mathbb{E}[[Z(t) - \mathbb{E}[Z(t)]]^2 | N_k(t) = n] p_n(t) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[S_n - \mathbb{E}[Y]\mathbb{E}[N^{(k)}(t)] \right]^2 p_n(t), \text{ where } S_n = \sum_{i=1}^{n} Y_i \\
&= \sum_{n=0}^{\infty} \mathbb{E}[(S_n - n\mathbb{E}[Y]) + (n\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[N^{(k)}(t)])]^2 p_n(t) \\
&= \sum_{n=0}^{\infty} \left[\mathrm{Var}(S_n) + \mathbb{E}[Y]^2 \mathbb{E}[n - \mathbb{E}[N^{(k)}(t)]]^2 \right] p_n(t) \\
&= \mathrm{Var}(Y)\mathbb{E}[N^{(k)}(t)] + \mathbb{E}[Y]^2 \mathrm{Var}(N^{(k)}(t)),
\end{aligned}
$$

which proves part (ii).

Now, in order to derive the covariance term, first we evaluate $\mathbb{E}[Z(s)Z(t)]$. So

$$
\begin{aligned}
\mathbb{E}[Z(s)Z(t)] &= \mathbb{E}[Z(s)]\mathbb{E}[Z(t) - Z(s)] + \mathbb{E}[Z(s)^2] \\
&= \mathbb{E}[Z(s)]\mathbb{E}[Z(t) - Z(s)] + \mathrm{Var}[Z(s)] + \mathbb{E}[Z(s)]^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Cov(Z(s), Z(t)) &= \mathbb{E}[Z(s)Z(t)] - \mathbb{E}[Z(s)]\mathbb{E}[Z(t)] \\
&= \mathrm{Var}(Y)\mathbb{E}[N^{(k)}(s \wedge t)] + \mathbb{E}[Y]^2 \mathrm{Var}[N^{(k)}(s \wedge t)],
\end{aligned}
$$

hence, part (iii) is proved. $\square$

Remark 3. *Now we present the mean and covariance formula for some specific cases of CPPoK(λ, H) that we discussed in Definition 3.*

S. No.	CPPoK(λ, H)	*Mean*	*Covariance*	*Variance*
1.	For $H = \delta_1$	$\mathbb{E}[N^{(k)}(t)]$	$Var[N^{(k)}(s \wedge t)]$	$Var[N^{(k)}(t)]$
2.	For $k = 1$	$\mathbb{E}[Y]\mathbb{E}[N(t)]$	$\mathbb{E}[Y^2]\mathbb{E}[N(s \wedge t)]$	$\mathbb{E}[Y^2]\mathbb{E}[N(t)]$
3.	For $H = \delta_1$ and $k = 1$	$\mathbb{E}[N(t)]$	$Var[N(s \wedge t)]$	$Var[N(t)]$

From this table, we make the following observations.

1. For $H = \delta_1$, it reduces to mean and covariance formula for *PPoK*(λ) (see [29]).

2. For $k = 1$, it reduces to mean and covariance formula for *CPP*(λ, H).

3. For $H = \delta_1$ and $k = 1$, it reduces to mean and covariance formula for *PP*(λ).

Further we present the plots of mean and variance when $Y_i \sim exp(\mu)$, $\mu > 0$ for different setting of parameters. Fix $t = 10$.
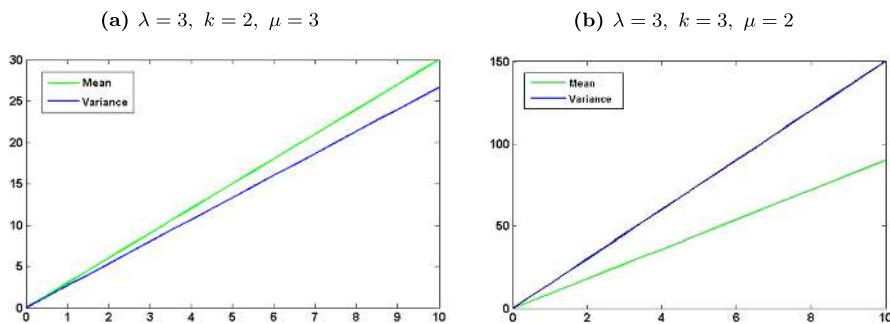
(a) $\lambda = 3, \; k = 2, \; \mu = 3$

(b) $\lambda = 3, \; k = 3, \; \mu = 2$

Fig. 2. Plots for mean and variance

2.2 Index of dispersion

In this subsection, we are going to discuss the index of dispersion of $CPPoK(\lambda, H)$.

Definition 4 ([16]). The index of dispersion for a counting process $\{Z(t)\}_{t \geq 0}$ is defined by

$$I(t) = \frac{\text{Var}[Z(t)]}{\mathbb{E}[Z(t)]}.$$

Then the stochastic process $\{Z(t)\}_{t \geq 0}$ is said to be overdispersed if $I(t) > 1$, underdispersed if $I(t) < 1$, and equidispersed if $I(t) = 1$.

Alternatively, Definition 4 can be interpreted as follows. A stochastic process $\{Z(t)\}_{t \geq 0}$ is over(under)-dispersed if $\text{Var}[Z(t)] - \mathbb{E}[Z(t)] > 0 (< 0)$. Therefore, we first calculate

$$
\begin{aligned}
\text{Var}[Z(t)] - \mathbb{E}[Z(t)] &= \frac{k(k+1)}{2} \lambda t \left[\text{Var}[Y] - \mathbb{E}[Y] + \frac{(2k+1)}{3} \mathbb{E}[Y]^2 \right] \\
&= \frac{k(k+1)}{2} \lambda t \left[\mathbb{E}[Y^2] - \mathbb{E}[Y] + \frac{2k-2}{3} \mathbb{E}[Y]^2 \right]. \quad (2)
\end{aligned}
$$

From the above definition, the following cases arise:

(i) If $Y_i's$ are over and equidispersed, then $CPPoK(\lambda, H)$ exhibits overdispersion.

(ii) If $Y_i's$ are underdispersed with non-negative integer valued random variable i.e., $[\mathbb{E}(Y^2) - \mathbb{E}(Y)] \geq 0$, then $CPPoK(\lambda, H)$ shows overdispersion.

(iii) If $Y_i's$ are underdispersed, then $CPPoK(\lambda, H)$ may show over, equi and underdispersion.

Further we present some examples to discuss the index of dispersion.

Example 3. If $Y_i \sim exp(\mu)$, where $\mu > 0$, then (2) becomes

$$\text{Var}[Z(t)] - \mathbb{E}[Z(t)] = \frac{k(k+1)}{2\mu} \lambda t \left[\frac{1}{\mu} \left(\frac{2k+4}{3} \right) - 1 \right].$$

(a) If $0 < \mu \leq 1$, then $CPPoK(\lambda, H)$ exhibits overdispersion.

(b) If $1 < \mu < \frac{2k+4}{3}$, then $CPPoK(\lambda, H)$ shows overdispersion.

(c) If $\mu = \frac{2k+4}{3}$, then $CPPoK(\lambda, H)$ shows equidispersion.

(d) If $\mu > \frac{2k+4}{3}$, then $CPPoK(\lambda, H)$ shows underdispersion.

Example 4. If $Y_i \sim Geo(p)$, where $p \in (0, 1]$, then (2) becomes

$$\text{Var}[Z(t)] - \mathbb{E}[Z(t)] = \frac{k(k+1)\lambda t}{2} \left(\frac{1-p}{p}\right)^2 \left[1 + \frac{2k+1}{3}\right].$$

If $0 < p < 1$, then $CPPoK(\lambda, H)$ exhibits overdispersion.

2.3 Long range dependence

In this subsection, we prove the long-range dependence (LRD) property for the $CPPoK(\lambda, H)$. There are several definitions available in literature. We used the definition given in [15].

Definition 5 ([15]). Let $0 \leq s < t$ and s be fixed. Assume a stochastic process $\{X(t)\}_{t \geq 0}$ has the correlation function $Corr[X(s), X(t)]$ that satisfies

$$c_1(s)t^{-d} \leq Corr[X(s), X(t)] \leq c_2(s)t^{-d},$$

for large t, $d > 0$, $c_1(s) > 0$ and $c_2(s) > 0$. i.e.,

$$\lim_{t \to \infty} \frac{Corr[X(s), X(t)]}{t^{-d}} = c(s)$$

for some $c(s) > 0$ and $d > 0$. We say that, $X(t)$ has the long-range dependence (LRD) property if $d \in (0, 1)$ and short-range dependence (SRD) property if $d \in (1, 2)$.

Proposition 2.1. *The $CPPoK(\lambda, H)$ has the LRD property.*

Proof. Let $0 \leq s < t$. Consider

$$Corr[Z(s), Z(t)] = \frac{Cov[Z(s), Z(t)]}{\sqrt{\text{Var}[Z(s)]\text{Var}[Z(t)]}},$$

$$= \sqrt{\frac{\frac{k(k+1)}{2}\lambda s \text{Var}(Y) + \frac{k(k+1)(2k+1)}{6}\lambda s \mathbb{E}[Y]^2}{\frac{k(k+1)}{2}\lambda t \text{Var}(Y) + \frac{k(k+1)(2k+1)}{6}\lambda t \mathbb{E}[Y]^2}}, \text{ from Theorem 2}$$

$$= c(s)t^{-1/2}, \tag{3}$$

where, $0 < c(s) = \sqrt{\frac{\frac{k(k+1)}{2}\lambda s \text{Var}(Y) + \frac{k(k+1)(2k+1)}{6}\lambda s \mathbb{E}[Y]^2}{\frac{k(k+1)}{2}\lambda \text{Var}(Y) + \frac{k(k+1)(2k+1)}{6}\lambda \mathbb{E}[Y]^2}}$, which decays like the power law $t^{-1/2}$. Hence $CPPoK(\lambda, H)$ has LRD property. □

Corollary 1. *Let $Y_i \equiv 1$, then $CPPoK(\lambda, H)$ reduces to $PPoK(\lambda)$ and $Corr[Z(s), Z(t)]$ becomes $s^{1/2}t^{-1/2}$. Hence LRD property also holds for PPoK (see [29, Lemma 3.1]).*

2.4 Martingale characterization for CPPoK

It is well known that the martingale characterization for homogeneous Poisson process is called Watanabe theorem (see [5]). Now, we extend this theorem for *CPPoK*(λ, H), where H is discrete distribution with support on $\mathbb{Z}^+$ and for this, we need following two lemmas.

Lemma 1. *Let* $D(t) = \sum_{j=1}^{N(t)} X_j$, $t \geq 0$ *be the compound Poisson process, where* $\{N(t)\}_{t \geq 0}$ *is PP*$(k\lambda)$ *and* $\{X_j\}_{j \geq 1}$ *are non-negative IID random variable, independent from* $\{N(t)\}_{t \geq 0}$, *with pmf* $\mathbb{P}(X_j = i) = \alpha_i$, $(i = 0, 1, 2, \ldots, j = 1, 2, \ldots)$. *Then* $\{D(t)\}_{t \geq 0}$ *can be represented as*

$$D(t) \stackrel{d}{=} \sum_{i=1}^{\infty} i Z_i(t), \quad t \geq 0,$$

where, $Z_i(t)$, $i = 1, 2, \ldots$ *are independent PP*$(k\lambda\alpha_i)$, *and the symbol* $\stackrel{d}{=}$ *denotes the equality in distribution.*

Proof. We prove this lemma by showing that the probability generating function (*pgf*) of L.H.S. and R.H.S. coincides. Let $G_{D(t)}(u)$ is the *pgf* of $\{D(t)\}_{t \geq 0}$, then

$$
\begin{aligned}
G_{D(t)}(u) = \mathbb{E}[u^{D(t)}] &= \sum_{n=0}^{\infty} \mathbb{E}[u^{X_1}]^n \mathbb{P}[N(t) = n] \\
&= \exp[-\lambda k t (1 - \mathbb{E}[u^X])] \\
&= \exp\left[\lambda k t \sum_{j=1}^{\infty} \alpha_j (u^j - 1)\right].
\end{aligned}
$$

Now we compute the *pgf* of $\sum_{i=1}^{\infty} i Z_i(t)$, i.e.,

$$
\begin{aligned}
\mathbb{E}[u^{\sum_{i=1}^{\infty} i Z_i(t)}] &= \prod_{i=1}^{\infty} \mathbb{E}[u^{i Z_i(t)}] \\
&= \prod_{i=1}^{\infty} \sum_{n_i=0}^{\infty} u^{i n_i} \mathbb{P}[Z_i(t) = n_i] \\
&= \prod_{i=1}^{\infty} \sum_{n_i=0}^{\infty} u^{i n_i} e^{-k\lambda\alpha_i t} \frac{(k\lambda\alpha_i t)^{n_i}}{n_i!} \\
&= \exp\left[\lambda k t \sum_{i=1}^{\infty} \alpha_i (u^i - 1)\right].
\end{aligned}
$$

Hence, this lemma is proved. $\qquad\qquad\square$

Lemma 2. *The pgf of* $Z(t) = \sum_{i=1}^{N^{(k)}(t)} Y_i$, $t \geq 0$ *has the following form*

$$G_{Z(t)}(u) = \exp\left[\lambda k t \sum_{j_1=1}^{\infty} \frac{q_{j_1}^{(1)} + q_{j_1}^{(2)} + \cdots + q_{j_1}^{(k)}}{k} (u^{j_1} - 1)\right],$$

where Y_i, $i = 1, 2, \ldots$ are non-negative integer valued IID random variables with $\mathbb{P}[Y_i = n] = q_n$, $n = 0, 1, \ldots$.

Proof. Let $G_{N^{(k)}(t)}(u)$ is the *pgf* of $\{N^{(k)}(t)\}_{t \geq 0}$, which is given by (see [29, Remark 2.2])

$$G_{N^{(k)}(t)}(u) = \exp[-k\lambda t + \lambda t\{u + u^2 + \cdots + u^k\}].$$

The *pgf* of $\{Z(t)\}_{t \geq 0}$, denoted as $G_{Z(t)}(u) = \mathbb{E}[u^{Z(t)}]$, is then given by

$$
\begin{aligned}
G_{Z(t)}(u) &= \exp[\lambda t\{G_Y(u) + \cdots + G_Y^{*k}(u)\} - k\lambda t], \text{ where } G_Y(u) = \mathbb{E}[u^Y] \\
&= \exp\left[\lambda t\left\{\sum_{j_1=0}^{\infty} q_{j_1} u^{j_1} + \sum_{j_1=0}^{\infty} q_{j_1}^{*2} u^{j_1} + \cdots + \sum_{j_1=0}^{\infty} q_{j_1}^{*k} u^{j_1}\right\} - k\lambda t\right] \\
&= \exp\left[\lambda t\left\{\sum_{j_1=0}^{\infty} q_{j_1} u^{j_1} + \cdots + \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{j_{k-1}} q_{j_k} \cdots q_{j_1-j_2} u^{j_1}\right\} - k\lambda t\right]
\end{aligned}
$$

Let us denote $q_{j_1}^{(n)} = \prod_{m=2}^{n} \sum_{j_m=0}^{j_{m-1}} q_{j_n} q_{j_{n-1}-j_n} \cdots q_{j_1-j_2}$, $n = 1, \ldots, k$.

$$
\begin{aligned}
&= \exp\left[\lambda t \sum_{j_1=0}^{\infty} q_{j_1}^{(1)}(u^{j_1} - 1) + \cdots + \lambda t \sum_{j_1=0}^{\infty} q_{j_1}^{(k)}(u^{j_1} - 1)\right] \\
&= \exp\left[\lambda k t \sum_{j_1=1}^{\infty} \frac{q_{j_1}^{(1)} + q_{j_1}^{(2)} + \cdots + q_{j_1}^{(k)}}{k}(u^{j_1} - 1)\right].
\end{aligned}
$$

$\square$

Remark 4. *Set $\alpha_{j_1} = \frac{q_{j_1}^{(1)} + \cdots + q_{j_1}^{(k)}}{k}$, $j_1 = 0, 1, 2, \ldots$. Substituting α_{j_1} in Lemma 1, we get the following relation*

$$D(t) \stackrel{d}{=} \sum_{i=1}^{\infty} i Z_i(t) \stackrel{d}{=} \sum_{i=1}^{N^{(k)}(t)} Y_i, \ t \geq 0, \tag{4}$$

where $Y_i's$ are non-negative integer valued random variables and $N^{(k)}(t)$ is PPoK(λ).

Theorem 3. *Let $\{Z(t)\}_{t \geq 0}$ be the $\mathcal{F}_t$-adapted stochastic process, where $\{\mathcal{F}_t\}$ is non-decreasing family of sub-sigma algebras. Then $\{Z(t)\}_{t \geq 0}$ is a CPPoK(λ, H), where H is discrete distribution with support on $\mathbb{Z}^+$ iff process $M(t) = Z(t) - \frac{k(k+1)}{2}\lambda t \mathbb{E}[Y]$, $t \geq 0$ is an $\mathcal{F}_t$ martingale.*

Proof. Let $\{Z(t)\}_{t \geq 0}$ be the $\mathcal{F}_t$ adapted stochastic process. If $\{Z(t)\}_{t \geq 0}$ is a compound Poisson process of order k, then

$$\mathbb{E}[M(t)|\mathcal{F}_s] = \mathbb{E}[Z(t) - \frac{k(k+1)}{2}\lambda t \mathbb{E}[Y]|\mathcal{F}_s], \ 0 \leq s \leq t.$$

$$= \quad \mathbb{E}[Z(t)|\mathcal{F}_s] - \frac{k(k+1)}{2}\lambda t \mathbb{E}[Y]$$

$$= \quad \mathbb{E}[Z(t) - Z(s)|\mathcal{F}_s] + \mathbb{E}[Z(s)|\mathcal{F}_s] - \frac{k(k+1)}{2}\lambda t \mathbb{E}[Y]$$

$$= \quad \mathbb{E}[Z(t) - Z(s)] + Z(s) - \frac{k(k+1)}{2}\lambda t \mathbb{E}[Y]$$

$$= \quad Z(s) - \frac{k(k+1)}{2}\lambda s \mathbb{E}[Y] = M(s).$$

Hence, the process $\{M(t)\}_{t \geq 0}$ is $\mathcal{F}_t$ martingale.

Since in Remark 4 it is shown that $\sum_{i=1}^{\infty} i Z_i(t) \stackrel{d}{=} \sum_{i=1}^{N^{(k)}(t)} Y_i$, then the other part easily follows using [31, Theorem 5.2]. □

Remark 5. *We know that CPP is a Lévy process and in* (4) *it is proved that CPPoK is equal in distribution to* $\{D(t)\}_{t \geq 0}$. *Hence, CPPoK is also a Lévy process and hence infinitely divisible.*

Remark 6. *The characteristic function of CPPoK*(λ, H) *can be written as*

$$\mathbb{E}[e^{iwZ(t)}] = \exp\left[t \sum_{j=1}^{\infty} (e^{iwj} - 1)k\lambda\alpha_j\right], \tag{5}$$

where, α_j, $j = 1, 2, \ldots$ *are as defined in Remark 4, and* $k\lambda\alpha_j = v_j$ *is called the Lévy measure of CPPoK*(λ, H).

1. *For* $K = 1$, (5) *reduces to* $\mathbb{E}[e^{iwZ(t)}] = \exp[t \sum_{j=1}^{\infty}(e^{iwj} - 1)\lambda\alpha_j]$, *which is the characteristic function of CPP*(λ, H).

2. *For* $H = \delta_1$, (5) *reduces to* $\mathbb{E}[e^{iwZ(t)}] = \exp[\lambda k t \sum_{j=1}^{k}(\frac{e^{iwj}}{k} - 1)]$, *which is the characteristic function of PPoK*(λ).

3. *For* $H = \delta_1$ *and* $k = 1$, (5) *reduces to* $\mathbb{E}[e^{iwZ(t)}] = \exp[\lambda t (e^{iw} - 1)]$, *which is the characteristic function of PP*(λ).

3   Main results

In this section, we recall the definitions of Lévy subordinator and its first exit time. Further, we define the subordinated versions of *CPPoK*(λ, H) and discuss their properties.

3.1   Lévy subordinator

A Lévy subordinator $\{D_f(t)\}_{t \geq 0}$ is a one-dimensional non-decreasing Lévy process whose Laplace transform (LT) can be expressed in the form (see [3])

$$\mathbb{E}[e^{-\lambda D_f(t)}] = e^{-tf(\lambda)}, \ \lambda > 0,$$

where the function $f : [0, \infty) \to [0, \infty)$ is called the Laplace exponent and

$$f(\lambda) = b\lambda + \int_0^{\infty} (1 - e^{-\lambda x})v(dx), \ b \geq 0.$$

Here b is the drift coefficient and ν is a non-negative Lévy measure on positive half-line satisfying

$$\int_0^\infty (x \wedge 1)\nu(dx) < \infty \text{ and } \nu([0, \infty)) = \infty,$$

which ensures that the sample paths of $D_f(t)$ are almost surely (a.s.) strictly increasing. Also, the inverse subordinator $\{E_f(t)\}_{t \geq 0}$ is the first exit time of the Lévy subordinator $\{D_f(t)\}_{t \geq 0}$, and it is defined as

$$E_f(t) = \inf\{r \geq 0 : D_f(r) > t\}, \ t \geq 0.$$

Next, we study $CPPoK(\lambda, H)$ by taking subordinator as mixture of tempered stable subordinators (MTSS).

3.2   CPPoK time changed by mixtures of tempered stable subordinators

The mixtures of tempered stable subordinators (MTSS) $\{S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)\}_{t \geq 0}$ is a Lévy process with LT (see [7])

$$\mathbb{E}[e^{-sS_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)}] = \exp\{-t(c_1((s + \mu_1)^{\alpha_1} - \mu_1^{\alpha_1}) + c_2((s + \mu_2)^{\alpha_2} - \mu_2^{\alpha_2}))\}, \ s > 0,$$

where $c_1 + c_2 = 1, c_1, c_2 \geq 0, \mu_1, \mu_2 > 0$ are tempering parameters and $\alpha_1, \alpha_2 \in (0, 1)$ are stability indices. The function $f(s) = c_1((s + \mu_1)^{\alpha_1} - \mu_1^{\alpha_1}) + c_2((s + \mu_2)^{\alpha_2} - \mu_2^{\alpha_2})$ is the Laplace exponent of MTSS. It can also be represented as sum of two independent tempered stable subordinators $S_{\alpha_1}^{\mu_1}(t)$ and $S_{\alpha_2}^{\mu_2}(t)$ as

$$S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t) = S_{\alpha_1}^{\mu_1}(c_1 t) + S_{\alpha_2}^{\mu_2}(c_2 t), \ c_1, c_2 \geq 0.$$

The mean and variance of MTSS are given as

$$\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] = t(c_1\alpha_1\mu_1^{\alpha_1-1} + c_2\alpha_2\mu_2^{\alpha_2-1}), \tag{6}$$

$$\text{Var}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] = t(c_1\alpha_1(1 - \alpha_1)\mu_1^{\alpha_1-2} + c_2\alpha_2(1 - \alpha_2)\mu_2^{\alpha_2-2}). \tag{7}$$

Definition 6. Let $\{S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)\}_{t \geq 0}$ be the Lévy subordinator satisfying $\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)^\rho] < \infty$ for all $\rho > 0$. Then the time-changed $CPPoK(\lambda, H)$, denoted by $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ is defined as

$$Z_1(t) = Z(S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)) = \sum_{i=1}^{N^{(k)}(S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t))} Y_i, \ t \geq 0,$$

where $\{Z(t)\}_{t \geq 0}$ is $CPPoK(\lambda, H)$, independent from $\{S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)\}_{t \geq 0}$.

Remark 7. *If $\alpha_1 = \alpha_2 = \alpha$, and $\mu_1 = \mu_2 = 0$, then MTSS becomes α-stable subordinator, reducing $Z_1(t)$ to $TCPPoK(\lambda, H, S_\alpha)$, which we call as space fractional CPPoK and written as*

$$Z(S_\alpha(t)) = \sum_{i=1}^{N^{(k)}(S_\alpha(t))} Y_i, \ t \geq 0.$$

Let $Y_i's \equiv 1$, then $Z(S_\alpha(t))$ reduces to space fractional PPoK, denoted as PPoK(λ, S_α). Its pgf is given as

$$G_{Z_1(t)}(u) = e^{[-t\lambda^\alpha(k-\sum_{j=1}^k u^j)^\alpha]}, \ t \geq 0.$$

It can be seen as extension of space fractional Poisson process (see [22]).

Remark 8. *If $\alpha_1 = \alpha_2 = \alpha$, and $\mu_1 = \mu$, $\mu_2 = 0$, then MTSS reduces to tempered α-stable subordinator and $Z_1(t)$ becomes tempered space fractional CPPoK, denoted as TCPPoK$(\lambda, H, S_\alpha^\mu)$, can be written as*

$$Z(S_\alpha^\mu(t)) = \sum_{i=1}^{N^{(k)}(S_\alpha^\mu(t))} Y_i, \ t \geq 0,$$

where $\mu > 0$ is tempering parameter. Substituting $Y_i's \equiv 1$, it becomes tempered space fractional PPoK, denoted as PPoK(λ, S_α^μ).

Theorem 4. *The finite dimensional distribution of TCPPoK$(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ has the following form*

$$F_{Z_1(t_1),...,Z_1(t_n)}(y_1,\ldots,y_n) = \sum_{j_1,\ldots j_n} \prod_{l=1}^n q_{j_l}(\Delta t_l) \int_{-\infty}^{v_1} \cdots \int_{-\infty}^{v_n} \prod_{m=1}^n h_{Y_1}^{*j_m}(x_m)dx_m,$$

(8)

where the summation is taken over all non-negative integers $j_i \geq 0$, $i = 1, 2, \ldots, n$, $\Delta t_l = t_l - t_{l-1}$, $v_k = y_k - \sum_{l=1}^{k-1} x_l, k = 1, \ldots, n$, h is the density of H, and $q_j(t) = \mathbb{P}[N^{(k)}(S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)) = j]$, where $\{S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)\}_{t\geq 0}$ is the MTSS and $\{N^{(k)}(t)\}_{t\geq 0}$ is PPoK(λ).

Proof. The result easily follows from the proof of Theorem 1. □

Now, we present some distributional properties of *TCPPoK$(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$*.

Theorem 5. *Let $0 < s \leq t < \infty$. Then the mean and covariance function of TCPPoK$(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ are given as*

(i) $\mathbb{E}[Z_1(t)] = \mathbb{E}[Z(1)]\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$,

(ii) $Cov[Z_1(s), Z_1(t)] = \mathbb{E}[Z(1)]^2 Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)] + \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)]Var[Z(1)]$.

On putting $s = t$ in part (ii), we can get the expression for variance of TCPPoK$(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$.

Proof. The proof follows from (see [14, Theorem 2.1]). □

Remark 9. *Here are some specific cases of mean and covariance for CPPoK$(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ that we discussed in Definition 3.*

(a) *For $H = \delta_1$, (i) reduces to $\mathbb{E}[Z_1(t)] = \mathbb{E}[N^{(k)}(1)]\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$ and (ii) reduces to $Cov[Z_1(s), Z_1(t)] = \mathbb{E}[N^{(k)}(1)]^2 Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)] + \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)]Var[N^{(k)}(1)]$ which is mean and covariance of TCPPoK-I as discussed in (see [29, Theorem 3.2]).*

(b) *For $k = 1$, (i) reduces to $\mathbb{E}[Z_1(t)] = \lambda \mathbb{E}[Y]\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$ and (ii) reduces to $Cov[Z_1(s), Z_1(t)] = (\lambda \mathbb{E}[Y])^2 Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)] + \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)]\lambda \mathbb{E}[Y^2]$ which is the mean and covariance of time-changed CPP(λ, H).*

(c) *For $H = \delta_1$ and $k = 1$, (i) reduces to $\mathbb{E}[Z_1(t)] = \mathbb{E}[N(1)]\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$ and (ii) reduces to $Cov[Z_1(s), Z_1(t)] = \mathbb{E}[N(1)]^2 Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)] + \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)]Var[N(1)]$ which is the mean and covariance of time-changed PP(λ).*

Now, we discuss the index of dispersion for $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$. For this, we evaluate

$$\begin{aligned} Var[Z_1(t)] - \mathbb{E}[Z_1(t)] = {} & \mathbb{E}[Z(1)]^2 Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] \\ & + \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]\{Var[Z(1)] - \mathbb{E}[Z(1)]\}. \end{aligned} \quad (9)$$

Since $\{S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)\}_{t\geq 0}$ is a Lévy subordinator, therefore $\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] > 0$. Thus the following cases arises:

(i) If $Z(1)$ is over/equidispersed, then $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ exhibits overdispersion.

(ii) If $Z(1)$ is underdispersed, then $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ may show over, under and equidispersion.

Moreover, we discuss the index of dispersion by taking some example of $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$.

Example 5. When H has exponential distribution with parameter $\mu > 0$, (9) becomes

$$\begin{aligned} Var[Z_1(t)] - \mathbb{E}[Z_1(t)] = {} & Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]\left[\frac{k(k+1)\lambda}{2\mu}\right]^2 + \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] \times \quad (10) \\ & \frac{k(k+1)\lambda}{2\mu}\left[\frac{2k+4}{3\mu} - 1\right]. \end{aligned}$$

1. If $0 < \mu \leq 1$, then $Z(1)$ is overdispersed. Since the sample paths of Lévy subordinator $S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)$ are strictly increasing (see [28, Theorem 21.3]), so $\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$ is positive, and both the terms of (10) is positive. Therefore $Z_1(t)$ shows overdispersion.

2. If $\mu = \frac{2k+4}{3}$, then $Z(1)$ is equidispersed. Second term becomes zero but the first terms of (10) is positive. Therefore $Z_1(t)$ shows overdispersion.

3. If $\mu > \frac{2k+4}{3}$, then $Z(1)$ is underdispersed. So the second term in (10) becomes negative and the following cases arises

   (a) if $\left[\frac{k(k+1)\lambda}{2\mu}\right]^2 Var[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] > \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]\frac{k(k+1)\lambda}{2\mu}\left[1 - \frac{2k+4}{3\mu}\right]$, then $Z_1(t)$ shows overdispersion.

(c) if $\left[\frac{k(k+1)\lambda}{2\mu}\right]^2 \text{Var}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] = \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]\frac{k(k+1)\lambda}{2\mu}\left[1 - \frac{2k+4}{3\mu}\right]$, then $Z_1(t)$ shows equidispersion.

(b) if $\left[\frac{k(k+1)\lambda}{2\mu}\right]^2 \text{Var}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] < \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]\frac{k(k+1)\lambda}{2\mu}\left[1 - \frac{2k+4}{3\mu}\right]$, then $Z_1(t)$ shows underdispersion.

3.3 Long-range dependence

Now we analyze the LRD property for $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$.

Theorem 6. *Let $\{Z_1(t)\}_{t\geq 0}$ be the time-changed CPPoK(λ, H). It has the LRD property.*

Proof. We have $\mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)^n] \sim (c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1})^n t^n$, as $t \to \infty$, from [7]. Therefore

$$\text{Var}[Z_1(t)] \sim \mathbb{E}[S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]\text{Var}[Z(1)] \sim Kt, \qquad (11)$$

where $K = (c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1})\text{Var}[Z(1)]$.

Let $0 < s < t < \infty$, then

$$
\begin{aligned}
Corr[Z_1(s), Z_1(t)] &= \frac{Cov[Z_1(s), Z_1(t)]}{\sqrt{\text{Var}[Z_1(s)]\text{Var}[Z_1(t)]}}, \\
&\sim \frac{\sqrt{Cov[Z_1(s), Z_1(t)]}}{t^{1/2}\sqrt{K}}, \text{ from (11)} \\
&= c(s)t^{-1/2},
\end{aligned}
$$

where $c(s) = \frac{\sqrt{Cov[Z_1(s),Z_1(t)]}}{\sqrt{K}} > 0$. Hence from the Definition 5, $TCPPoK(\lambda, H, S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ captures LRD property. □

3.4 CPPoK time changed by the first exit time of mixtures of tempered stable subordinators

In this subsection, we consider $CPPoK(\lambda, H)$ subordinated with first exit time of MTSS and discuss the asymptotic behavior of its moments.

The first exit time of Lévy subordinator $S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)$ also known as inverse subordinator is defined as

$$E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t) = \inf\{r \geq 0 : S_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(r) > t\}, \ t \geq 0.$$

Definition 7. Let $\{Z(t)\}_{t\geq 0}$ be the $CPPoK(\lambda, H)$ as discussed in Definition 3, then the subordinated CPPoK with first exit time of MTSS, denoted as $TCPPoK(\lambda, H, E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2})$ is defined as

$$Z_2(t) = Z(E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)) = \sum_{i=1}^{N^{(k)}(E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t))} Y_i, \ t \geq 0$$

where the process $\{Z(t)\}_{t\geq 0}$ is independent from $\{E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)\}_{t\geq 0}$.

Remark 10. *If $\mu_1 = \mu_2 = 0$, then $E_{\alpha_1,\alpha_2}^{\lambda_1,\lambda_2}(t)$ becomes inverse mixed stable subordinator as discussed in [9], reducing $Z_2(t)$ to TCPPoK(λ, H, E_{α_1,α_2}), which we call as mixed fractional CPPoK and written as*

$$Z_2(t) = \sum_{i=1}^{N^{(k)}(E_{\alpha_1,\alpha_2}(t))} Y_i, \ t \geq 0.$$

Proposition 3.1. *The marginal distribution of TCPPoK(λ, H, $E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}$) is given as*

$$F_{Z_2(t)}(y) = \int_0^\infty F_{Z(x)}(y)g_E(x,t)dx,$$

where $F_{Z(x)}(y) = \sum_{j=0}^\infty p_j(x) \int_{-\infty}^y h_{Y_1}^{*j}(z)dz$ is the marginal distribution of CPPoK(λ, H), and $g_E(x,t)$ is the density function of $E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)$.

Proof. We prove this proposition by using the conditioning argument on $E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)$. Then the rest follows from Theorem 1. □

Theorem 7. *The mean and covariance function of TCPPoK(λ, H, $E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}$) are given as*

1. $\mathbb{E}[Z_2(t)] = \mathbb{E}[Z(1)]\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$,

2. $Cov[Z_2(s), Z_2(t)] = Var[Z(1)]\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s)] + \mathbb{E}[Z(1)]^2 Cov[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(s), E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$.

Proof. The proof follows as given in [14, Theorem 2.1]. □

Now, we discuss the asymptotic behavior of moments of TCPPoK(λ, H, $E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}$). First we need the following Theorem (see [3, 30]).

Theorem 8 (Tauberian Theorem). *Let $l : (0,\infty) \to (0,\infty)$ be a slowly varying function at 0 (respectively ∞) and let $\rho \geq 0$. Then for a function $U : (0,\infty) \to (0,\infty)$, the following are equivalent.*

1. *$U(x) \sim x^\rho l(x)/\Gamma(1+\rho), x \to 0$ (respectively $x \to \infty$).*

2. *$\tilde{U}(s) \sim s^{-\rho-1}l(1/s), s \to \infty$ (respectively $s \to 0$), where $\tilde{U}(s)$ is the LT of $U(x)$.*

We know that LT of pth order moment of inverse subordinator $\{E_f(t)\}_{t \geq 0}$ is given by (see [16])

$$\mathscr{L}[\mathbb{E}(E_f(t))^p] = \frac{\Gamma(1+p)}{s(f(s))^p}, \ p > 0,$$

where $f(s)$ is the corresponding Bernstein function associated with Lévy subordinator.

Proposition 3.2. *The asymptotic form of mean and variance of $\{Z_2(t)\}_{t \geq 0}$ is given as*

$$\mathbb{E}[Z_2(t)] \sim \mathbb{E}[Z(1)]\frac{t}{c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}}, \ as \ t \to \infty.$$

$$Var[Z_2(t)] \sim Var[Z(1)]\frac{t}{c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}}, \ as \ t \to \infty.$$

Proof. Let $\tilde{M}(s)$ be the LT of $\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]$. Then

$$
\begin{aligned}
\tilde{M}(s) = \mathscr{L}[\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]] &= \frac{1}{s(c_1((s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}) + c_2((s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}))} \\
&\sim \frac{1}{s^2(c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1})}, \quad \text{as } s \to 0, \text{ (see [7]).}
\end{aligned}
$$

Then, by using Theorem 8, we have that

$$
\begin{aligned}
\mathbb{E}[Z_2(t)] &= \mathbb{E}[Z(1)]\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] \\
&\sim \mathbb{E}[Z(1)]\frac{t}{c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}}, \quad as\ t \to \infty.
\end{aligned}
$$

Now we compute the asymptotic behavior of variance of $\{Z_2(t)\}_{t\geq0}$. So

$$
\begin{aligned}
\mathscr{L}[\mathbb{E}(E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t))^2] &= \frac{2}{s(c_1((s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}) + c_2((s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}))^2} \\
&\sim \frac{2}{s^3(c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1})^2}, \quad \text{as } s \to 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\text{Var}Z_2(t) &= \text{Var}[Z(1)]\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] + \mathbb{E}[Z(1)]^2\text{Var}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] \\
&= \text{Var}[Z(1)]\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)] + \mathbb{E}[Z(1)]^2\{\mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)^2] - \mathbb{E}[E_{\alpha_1,\alpha_2}^{\mu_1,\mu_2}(t)]^2\} \\
&\sim \text{Var}[Z(1)]\frac{t}{c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}}, \quad as\ t \to \infty.
\end{aligned}
$$

$\square$

There are few studies available in the literature which consider underdispersion phenomenon such as [27] uses hyper-Poisson regression model, [25] uses weighted Poisson distribution, and [21] uses compound weighted Poisson distributions. In this work, we have proposed a model accomodating more number of parameters, which makes this model more flexible to use.

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