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Research Article

Subordination and Superordination on Schwarzian Derivatives

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Let the functions q_1 be analytic and let q_2 be analytic univalent in the unit disk. Using the methods of differential subordination and superordination, sufficient conditions involving the Schwarzian derivative of a normalized analytic function f are obtained so that either $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$ or $q_1(z) \prec 1 + zf''(z)/f'(z) \prec q_2(z)$. As applications, sufficient conditions are determined relating the Schwarzian derivative to the starlikeness or convexity of f.

1. Introduction

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. We will write $\mathcal{H} \equiv \mathcal{H}[1,1]$. Denote by $\mathcal{A}$ the subclass of $\mathcal{H}[0,1]$ consisting of normalized functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U). \tag{1.1}$$

Let $\mathcal{S}^*$ and $\mathcal{K}$, respectively, be the familiar subclasses of $\mathcal{A}$ consisting of starlike and convex functions in U.

The Schwarzian derivative $\{f, z\}$ of an analytic, locally univalent function f is defined by

$$\{f, z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1.2}$$

Owa and Obradović [1] proved that if $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{1}{2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f,z\}\right] > 0, \tag{1.3}$$

then $f \in \mathcal{K}$. Miller and Mocanu [2] proved that if $f \in \mathcal{A}$ satisfies one of the following conditions:

$$\begin{gathered}\Re\left[\left(1+\frac{zf''(z)}{f'(z)}\right) + \alpha z^2\{f,z\}\right] > 0 \quad (\Re\alpha \geq 0),\\ \Re\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f,z\}\right] > 0,\end{gathered} \tag{1.4}$$

or

$$\Re\left[\left(1+\frac{zf''(z)}{f'(z)}\right)e^{z^2\{f,z\}}\right] > 0, \tag{1.5}$$

then $f \in \mathcal{K}$. In fact, Miller and Mocanu [2] found conditions on $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ such that

$$\Re\left\{\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right)\right\} > 0 \tag{1.6}$$

implies $f \in \mathcal{K}$. Each of the conditions mentioned above readily followed by choosing an appropriate ϕ. Miller and Mocanu [2] also found conditions on $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that

$$\Re\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right)\right\} > 0 \tag{1.7}$$

implies $f \in \mathcal{S}^*$. As applications, if $f \in \mathcal{A}$ satisfies either

$$\Re\left[\alpha\left(\frac{zf'(z)}{f(z)}\right) + \beta\left(1+\frac{zf''(z)}{f'(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)z^2\{f,z\}\right] > 0 \quad (\alpha,\beta \in \mathbb{R}), \tag{1.8}$$

or

$$\Re\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)} + z^2\{f,z\}\right)\right] > -\frac{1}{2}, \tag{1.9}$$

then $f \in \mathcal{S}^*$.

Let f and F be members of $\mathcal{H}(U)$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, written $f(z) \prec F(z)$, if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$, such that $f(z) = F(w(z))$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

In this paper, sufficient conditions involving the Schwarzian derivatives are obtained for functions $f \in \mathcal{A}$ to satisfy either

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z) \quad \text{or} \quad q_1(z) \prec 1 + \frac{zf''(z)}{f'(z)} \prec q_2(z), \tag{1.10}$$

where the functions q_1 are analytic and q_2 is analytic univalent in U. In Section 2, a class of admissible functions is introduced. Sufficient conditions on functions $f \in \mathcal{A}$ are obtained so that $zf'(z)/f(z)$ is subordinated to a given analytic univalent function q in U. As a consequence, we obtained the result (1.7) of Miller and Mocanu [2] relating the Schwarzian derivatives to the starlikeness of functions $f \in \mathcal{A}$.

Recently, Miller and Mocanu [3] investigated certain first- and second-order differential superordinations, which is the dual problem to subordination. Several authors have continued the investigation on superordination to obtain sandwich-type results [4–20]. In Section 3, superordination is investigated on a class of admissible functions. Sufficient conditions involving the Schwarzian derivatives of functions $f \in \mathcal{A}$ are obtained so that $zf'(z)/f(z)$ is superordinated to a given analytic subordinant q in U. For q_1 analytic and q_2 analytic univalent in U, sandwich-type results of the form

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z) \tag{1.11}$$

are obtained. This result extends earlier works by several authors.

Section 4 is devoted to finding sufficient conditions for functions $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec 1 + \frac{zf''(z)}{f'(z)} \prec q_2(z). \tag{1.12}$$

As a consequence, we obtained the result (1.6) of Miller and Mocanu [2].

To state our results, we need the following preliminaries. Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\right\}, \tag{1.13}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ and $Q(1) \equiv Q_1$.

Definition 1.1 (see [2, Definition 2.3a, page 27]). Let Ω be a set in $\mathbb{C}, q \in Q$ and let n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; z) \notin \Omega \tag{1.14}$$

whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and

$$\Re\left\{\frac{t}{s}+1\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}, \tag{1.15}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

If $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (1.14) reduces to

$$\psi(q(\zeta), k\zeta q'(\zeta); z) \notin \Omega, \tag{1.16}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq n$.

Definition 1.2 (see [3, Definition 3, page 817]). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; \zeta) \in \Omega \tag{1.17}$$

whenever $r = q(z)$, $s = zq'(z)/m$, and

$$\Re\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\}, \tag{1.18}$$

$z \in U$, $\zeta \in \partial U$, and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

If $\psi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, then the admissibility condition (1.17) reduces to

$$\psi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega, \tag{1.19}$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n$.

Lemma 1.3 (see [2, Theorem 2.3b, page 28]). *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ satisfies*

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \in \Omega, \tag{1.20}$$

then $p(z) \prec q(z)$.

Lemma 1.4 (see [3, Theorem 1, page 818]). *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in Q(a)$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U, then*

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\right\} \tag{1.21}$$

implies $q(z) \prec p(z)$.

2. Subordination and starlikeness

We first define the following class of admissible functions that are required in our first result.

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_1$. The class of admissible functions $\Phi_S[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega \tag{2.1}$$

whenever

$$\begin{gathered} u = q(\zeta), \quad v = q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)} \quad (q(\zeta) \neq 0), \\ \Re\left\{\frac{2w + u^2 - 1 + 3(v-u)^2}{2(v-u)}\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \end{gathered} \tag{2.2}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq 1$.

Theorem 2.2. *Let $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$. If $\phi \in \Phi_S[\Omega, q]$ and*

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) : z \in U\right\} \subset \Omega, \tag{2.3}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z). \tag{2.4}$$

Proof. Define the function p by

$$p(z) := \frac{zf'(z)}{f(z)}. \tag{2.5}$$

A simple calculation yields

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}. \tag{2.6}$$

Further computations show that

$$z^2\{f, z\} = \frac{zp'(z) + z^2p''(z)}{p(z)} - \frac{3}{2}\left[\frac{zp'(z)}{p(z)}\right]^2 + \frac{1 - p^2(z)}{2}. \tag{2.7}$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}^3$ by

$$u = r, \qquad v = r + \frac{s}{r}, \qquad w = \frac{s+t}{r} - \frac{3}{2}\left[\frac{s}{r}\right]^2 + \frac{1-r^2}{2}. \tag{2.8}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, r + \frac{s}{r}, \frac{s+t}{r} - \frac{3}{2}\left[\frac{s}{r}\right]^2 + \frac{1-r^2}{2}; z\right). \tag{2.9}$$

The proof will make use of Lemma 1.3. Using (2.5), (2.6), and (2.7), from (2.9) we obtain

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) = \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right). \tag{2.10}$$

Hence (2.3) becomes

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega. \tag{2.11}$$

A computation using (2.8) yields

$$\frac{t}{s} + 1 = \frac{2w + u^2 - 1 + 3(v-u)^2}{2(v-u)}. \tag{2.12}$$

Thus the admissibility condition for $\phi \in \Phi_S[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for ψ as given in Definition 1.1. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.3, $p(z) \prec q(z)$ or

$$\frac{zf'(z)}{f(z)} \prec q(z). \tag{2.13}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. In this case, the class $\Phi_S[h(U), q]$ is written as $\Phi_S[h, q]$. The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3. *Let $\phi \in \Phi_S[h, q]$. If $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ satisfies*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \prec h(z), \tag{2.14}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z). \tag{2.15}$$

Following similar arguments as in [2, Theorem 2.3d, page 30], Theorem 2.3 can be extended to the following theorem where the behavior of q on ∂U is not known.

Theorem 2.4. *Let h and q be univalent in U with $q(0) = 1$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(i) *$\phi \in \Phi_S[h, q_\rho]$ for some $\rho \in (0,1)$, or*

(ii) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_S[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ satisfies (2.14), then

$$\frac{zf'(z)}{f(z)} \prec q(z). \tag{2.16}$$

The next theorem yields the best dominant of the differential subordination (2.14).

Theorem 2.5. *Let h be univalent in U, and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), q(z) + \frac{zq'(z)}{q(z)}, \frac{zq'(z) + z^2 q''(z)}{q(z)} - \frac{3}{2}\left(\frac{zq'(z)}{q(z)}\right)^2 + \frac{1 - q^2(z)}{2}; z\right) = h(z) \tag{2.17}$$

has a solution q with $q(0) = 1$ and one of the following conditions is satisfied:

(1) *$q \in Q_1$ and $\phi \in \Phi_S[h, q]$,*

(2) *q is univalent in U and $\phi \in \Phi_S[h, q_\rho]$ for some $\rho \in (0,1)$, or*

(3) *q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_S[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ satisfies (2.14), then

$$\frac{zf'(z)}{f(z)} \prec q(z), \tag{2.18}$$

and q is the best dominant.

Proof. Applying the same arguments as in [2, Theorem 2.3e, page 31], we first note that q is a dominant from Theorems 2.3 and 2.4. Since q satisfies (2.17), it is also a solution of (2.14), and therefore q will be dominated by all dominants. Hence q is the best dominant. □

We will apply Theorem 2.2 to two specific cases. First, let $q(z) = 1 + Mz$, $M > 0$.

Theorem 2.6. *Let Ω be a set in $\mathbb{C}$, and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy the admissibility condition*

$$\phi\left(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{kMe^{i\theta}}{1 + Me^{i\theta}}, L; z\right) \notin \Omega \tag{2.19}$$

whenever $z \in \mathcal{U}$, $\theta \in \mathbb{R}$, *with*

$$\Re\left\{ \left(2L + \left(1 + Me^{i\theta}\right)^2 - 1\right)\left(e^{-i\theta} + M\right) + \frac{3k^2M^2}{e^{-i\theta} + M} \right\} \geq 2k^2M \tag{2.20}$$

for all real θ *and* $k \geq 1$.

If $f \in \mathcal{A}$ *with* $f(z)f'(z)/z \neq 0$ *satisfies*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega, \tag{2.21}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M. \tag{2.22}$$

Proof. Let $q(z) = 1 + Mz$, $M > 0$. A computation shows that the conditions on ϕ implies that it belongs to the class of admissible functions $\Phi_S[\Omega, 1 + Mz]$. The result follows immediately from Theorem 2.2. □

In the special case $\Omega = q(\mathcal{U}) = \{\omega : |\omega - 1| < M\}$, the conclusion of Theorem 2.6 can be written as

$$\left|\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) - 1\right| < M \Longrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M. \tag{2.23}$$

Example 2.7. The functions $\phi_1(u, v, w; z) := (1 - \alpha)u + \alpha v$, $(\alpha \geq 2(M - 1) \geq 0)$ and $\phi_2(u, v, w; z) := v/u$, $(0 < M \leq 2)$ satisfy the admissibility condition (2.19) and hence Theorem 2.6 yields

$$\begin{gathered}\left|(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < M \Longrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (\alpha \geq 2(M - 1) \geq 0), \\ \left|\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right| < M \Longrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (0 < M \leq 2).\end{gathered} \tag{2.24}$$

By considering the function $\phi(u, v, w; z) := u(v-1) + \lambda(u-1)$ with $0 < M \leq 1$, $\lambda + 2 - M \geq 0$, it follows again from Theorem 2.6 that

$$\left|\frac{z^2f''(z)}{f(z)} + \lambda\left(\frac{zf'(z)}{f(z)} - 1\right)\right| \leq M(2 + \lambda - M) \Longrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < M. \tag{2.25}$$

This above implication was obtained in [21, Corollary 2, page 583].

A second application of Theorem 2.2 is to the case $q(\mathcal{U})$ being the half-plane $q(\mathcal{U}) = \{w : \Re w > 0\} =: \Delta$.

Theorem 2.8. *Let Ω be a set in $\mathbb{C}$ and let the function $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfy the admissibility condition*

$$\phi(i\rho, i\tau, \xi + i\eta; z) \notin \Omega \tag{2.26}$$

for all $z \in \mathcal{U}$ and for all real ρ, τ, ξ and η with

$$\rho\tau \geq \frac{1}{2}(1 + 3\rho^2), \quad \rho\eta \geq 0. \tag{2.27}$$

Let $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$. If

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega, \tag{2.28}$$

then $f \in \mathcal{S}^*$.

Proof. Let $q(z) := (1 + z)/(1 - z)$; then $q(0) = 1$, $E(q) = \{1\}$ and $q \in Q_1$. For $\zeta := e^{i\theta} \in \partial\mathcal{U} \setminus \{1\}$, we obtain

$$q(\zeta) = i\rho, \qquad \zeta q'(\zeta) = -\frac{(1 + \rho^2)}{2}, \qquad \zeta^2 q''(\zeta) = \frac{(1 + \rho^2)(1 - i\rho)}{2}, \tag{2.29}$$

where $\rho := \cot(\theta/2)$. Note that

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right) = 0 \quad (\zeta \neq 1). \tag{2.30}$$

We next describe the class of admissible functions $\Phi_S[\Omega, (1 + z)/(1 - z)]$ in Definition 2.1. For $\zeta \neq 1$,

$$u = q(\zeta) =: i\rho, \qquad v = q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)} = i\left[\rho + \frac{k(1 + \rho^2)}{2\rho}\right] =: i\tau, \qquad w = \xi + i\eta \tag{2.31}$$

with

$$\Re\left\{\frac{2w + u^2 - 1 + 3(v - u)^2}{2(v - u)}\right\} = \frac{2\rho\eta}{k(1 + \rho^2)}. \tag{2.32}$$

Thus the admissibility condition for functions in $\Phi_S[\Omega, (1 + z)/(1 - z)]$ is equivalent to (2.26), whence $\phi \in \Phi_S[\Omega, (1 + z)/(1 - z)]$. From Theorem 2.2, we deduce that $f \in S^*$. □

When $h(z) = (1 + z)/(1 - z)$, then $h(\mathcal{U}) = \Delta = q(\mathcal{U})$. Writing the class of admissible functions $\Phi_S[h(\mathcal{U}), \Delta]$ as $\Phi_S[\Delta]$, the following result is a restatement of (1.7), which is an immediate consequence of Theorem 2.8.

Corollary 2.9 (see [2, Theorem 4.6a, page 244]). *Let $\phi \in \Phi_S[\Delta]$. If $f \in \mathcal{A}$ with $f(z)f'(z)/z \neq 0$ satisfies*

$$\Re\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right)\right\} > 0, \tag{2.33}$$

then $f \in \mathcal{S}^*$.

3. Superordination and starlikeness

Now we will give the dual result of Theorem 2.2 for differential superordination.

Definition 3.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_S[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega \tag{3.1}$$

whenever

$$\begin{gathered} u = q(z), \quad v = q(z) + \frac{zq'(z)}{mq(z)} \quad (q(z) \neq 0,\ zq'(z) \neq 0), \\ \Re\left\{\frac{2w + u^2 - 1 + 3(v-u)^2}{2(v-u)}\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\}, \end{gathered} \tag{3.2}$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

Theorem 3.2. *Let $\phi \in \Phi'_S[\Omega, q]$, and $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$. If $zf'(z)/f(z) \in Q_1$ and $\phi(zf'(z)/f(z), 1 + zf''(z)/f'(z), z^2\{f,z\}; z)$ is univalent in U, then*

$$\Omega \subset \left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) : z \in U\right\} \tag{3.3}$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}. \tag{3.4}$$

Proof. With $p(z) = zf'(z)/f(z)$, and

$$\psi(r, s, t; z) = \phi\left(r, \frac{r+s}{r}, \frac{s+t}{r} + \frac{3}{2}\left(\frac{s}{r}\right)^2 + \frac{1-r^2}{2}; z\right) = \phi(u, v, w; z), \tag{3.5}$$

equations (2.10) and (3.3) yield

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\}. \tag{3.6}$$

Since

$$\frac{t}{s}+1=\frac{2w+u^2-1+3(v-u)^2}{2(v-u)}, \tag{3.7}$$

the admissibility condition for $\phi \in \Phi'_S[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.4, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{zf'(z)}{f(z)}. \tag{3.8}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. With $\Phi'_S[h(U), q]$ as $\Phi'_S[h, q]$, Theorem 3.2 can be written in the following form.

Theorem 3.3. *Let* $q \in \mathcal{H}$, h *be analytic in* U *and* $\phi \in \Phi'_S[h, q]$. *If* $f \in \mathcal{A}$, $f'(z)f(z)/z \neq 0$, $zf'(z)/f(z) \in Q_1$ *and* $\phi(zf'(z)/f(z), 1+zf''(z)/f'(z), z^2\{f,z\}; z)$ *is univalent in* U, *then*

$$h(z) \prec \phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \tag{3.9}$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}. \tag{3.10}$$

Theorems 3.2 and 3.3 can only be used to obtain subordinants of differential superordinations of the form (3.3) or (3.9). The following theorem proves the existence of the best subordinant of (3.9) for an appropriate ϕ.

Theorem 3.4. *Let* h *be analytic in* U *and* $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\phi\left(q(z), q(z)+\frac{zq'(z)}{q(z)}, \frac{zq'(z)+z^2q''(z)}{q(z)}-\frac{3}{2}\left(\frac{zq'(z)}{q(z)}\right)^2+\frac{1-q^2(z)}{2}; z\right)=h(z) \tag{3.11}$$

has a solution $q \in Q_1$. *Let* $\phi \in \Phi'_S[h, q]$, *and* $f \in \mathcal{A}$ *with* $f'(z)f(z)/z \neq 0$. *If* $zf'(z)/f(z) \in Q_1$ *and*

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \tag{3.12}$$

is univalent in U, *then*

$$h(z) \prec \phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \tag{3.13}$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}, \tag{3.14}$$

and q is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.5, and is therefore omitted. □

Combining Theorems 2.3 and 3.3, we obtain the following sandwich-type theorem.

Corollary 3.5. *Let h_1 and q_1 be analytic functions in U, let h_1 be an analytic univalent function in U, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_S[h_2, q_2] \cap \Phi'_S[h_1, q_1]$. Let $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$. If $zf'(z)/f(z) \in \mathcal{H} \cap Q_1$ and $\phi(zf'(z)/f(z), 1+zf''(z)/f'(z), z^2\{f,z\}; z)$ is univalent in U, then*

$$h_1(z) \prec \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \prec h_2(z) \tag{3.15}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z). \tag{3.16}$$

4. Schwarzian derivatives and convexity

We introduce the following class of admissible functions.

Definition 4.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{Sc}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi\left(q(\zeta), k\zeta q'(\zeta) + \frac{1 - q^2(\zeta)}{2}; z\right) \notin \Omega, \tag{4.1}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq 1$.

Theorem 4.2. *Let $\phi \in \Phi_{Sc}[\Omega, q]$, and $f \in \mathcal{A}$ with $f'(z) \neq 0$. If*

$$\left\{\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) : z \in U\right\} \subset \Omega, \tag{4.2}$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z). \tag{4.3}$$

Proof. Define the function p by

$$p(z) := 1 + \frac{zf''(z)}{f'(z)}. \tag{4.4}$$

Clearly $p \in \mathcal{A}$, and a simple calculation yields

$$z^2\{f, z\} = zp'(z) + \frac{1 - p^2(z)}{2}. \tag{4.5}$$

Define the transformation from $\mathbb{C}^2$ to $\mathbb{C}^2$ by

$$u = r, \qquad v = s + \frac{1 - r^2}{2}. \tag{4.6}$$

Let

$$\psi(r, s; z) = \phi(u, v; z) = \phi\left(r, s + \frac{1 - r^2}{2}; z\right). \tag{4.7}$$

The proof will make use of Lemma 1.3. Using (4.4) and (4.5), from (4.7), we obtain

$$\psi(p(z), zp'(z); z) = \phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right). \tag{4.8}$$

Hence (4.2) becomes

$$\psi(p(z), zp'(z); z) \in \Omega. \tag{4.9}$$

From (4.7), we see that the admissibility condition for $\phi \in \Phi_{Sc}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.3, $p(z) \prec q(z)$ or

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z). \tag{4.10}$$

□

We will denote by $\Phi_{Sc}[h, q]$ the class $\Phi_{Sc}[h(U), q]$, where h is the conformal mapping of U onto $\Omega \neq \mathbb{C}$. Proceeding similarly as in the previous section, the following results can be established, which we state without proof.

Theorem 4.3. *Let $\phi \in \Phi_{Sc}[h, q]$. If $f \in \mathcal{A}$ with $f'(z) \neq 0$ satisfies*

$$\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \prec h(z), \tag{4.11}$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec q(z). \tag{4.12}$$

We extend Theorem 4.3 to the case where the behavior of q on ∂U is not known.

Theorem 4.4. *Let $\Omega \subset \mathbb{C}$ and let q be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_{Sc}[h, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ with $f'(z) \neq 0$ satisfies* (4.2)*, then* (4.12) *holds.*

Theorem 4.5. *Let Ω be a set in $\mathbb{C}$, $q(z) = 1 + Mz$, $M > 0$, and $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfy*

$$\phi\left(1+Me^{i\theta}, \frac{2(k-1)-Me^{i\theta}}{2}Me^{i\theta}; z\right) \notin \Omega \tag{4.13}$$

whenever $z \in U$, $\theta \in \mathbb{R}$ and $k \geq 1$. Let $f \in \mathcal{A}$ with $f'(z) \neq 0$. If

$$\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \in \Omega, \tag{4.14}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < M. \tag{4.15}$$

In the special case $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, Theorem 4.5 gives the following: let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfy

$$\left|\phi\left(1+Me^{i\theta}, \frac{2(k-1)-Me^{i\theta}}{2}Me^{i\theta}; z\right) - 1\right| \geq M \tag{4.16}$$

whenever $z \in U$, $\theta \in \mathbb{R}$, and $k \geq 1$; if $f \in \mathcal{A}$ with $f'(z) \neq 0$ satisfies

$$\left|\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) - 1\right| < M, \tag{4.17}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < M. \tag{4.18}$$

With $\phi(u, v; z) = u + v$, we get the following:

Example 4.6. If $0 < M < 2$, and $f \in \mathcal{A}$ with $f'(z) \neq 0$ satisfies

$$\left|\frac{zf''(z)}{f'(z)} + z^2\{f,z\}\right| < M, \tag{4.19}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < M. \tag{4.20}$$

We next apply Theorem 4.2 to the particular case corresponding to $q(U)$ being a half-plane $q(U) = \Delta$.

Theorem 4.7. *Let Ω be a set in $\mathbb{C}$. Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ satisfy the admissibility condition*

$$\phi(i\rho, \eta; z) \notin \Omega \tag{4.21}$$

for all $z \in U$, and for all real ρ and η with $\eta \le 0$. Let $f \in \mathcal{A}$ with $f'(z) \neq 0$. If

$$\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega, \tag{4.22}$$

then $f \in \mathcal{K}$.

Let $h(z) = (1+z)/(1-z)$. Clearly, $h(U) = \Delta$. Writing the class of admissible functions $\Phi_{Sc}[h(U), \Delta]$ as $\Phi_{Sc}[\Delta]$, the following result is a restatement of (1.6), which is an immediate consequence of Theorem 4.7.

Corollary 4.8 (see [2, Theorem 4.6b, page 246]). *Let $\phi \in \Phi_{Sc}[\Delta]$. If $f \in \mathcal{A}$ with $f'(z) \neq 0$ satisfies*

$$\Re\left\{\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right)\right\} > 0, \tag{4.23}$$

then $f \in \mathcal{K}$.

Definition 4.9. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{Sc}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi\left(q(z), \frac{zq'(z)}{m} + \frac{1 - q^2(z)}{2}; \zeta\right) \in \Omega, \tag{4.24}$$

$z \in U$, $\zeta \in \partial U$, and $m \ge 1$.

Now we will give the dual result of Theorem 4.2 for differential superordination.

Theorem 4.10. *Let $\phi \in \Phi'_{Sc}[\Omega, q]$, and $f \in \mathcal{A}$ with $f'(z) \neq 0$. If $1 + zf''(z)/f'(z) \in Q_1$ and $\phi(1 + zf''(z)/f'(z), z^2\{f, z\}; z)$ is univalent in U, then*

$$\Omega \subset \left\{\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) : z \in U\right\} \tag{4.25}$$

implies

$$q(z) \prec 1 + \frac{zf''(z)}{f'(z)}. \tag{4.26}$$

Proof. With $p(z) = 1 + zf''(z)/f'(z)$ and

$$\psi(r, s; z) = \phi\left(r, s + \frac{1-r^2}{2}; z\right) = \phi(u, v; z), \tag{4.27}$$

from (4.8) and (4.25), we have

$$\Omega \subset \{\psi(p(z), zp'(z); z) : z \in U\}. \tag{4.28}$$

From (4.6), we see that the admissibility condition for $\phi \in \Phi'_{Sc}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.4, $q(z) \prec p(z)$ or

$$q(z) \prec 1 + \frac{zf''(z)}{f'(z)}. \tag{4.29}$$

□

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 4.10.

Theorem 4.11. *Let $q \in \mathcal{H}$, let h be analytic in U and $\phi \in \Phi'_{Sc}[h, q]$. Let $f \in \mathcal{A}$ with $f'(z) \neq 0$. If $1 + zf''(z)/f'(z) \in Q_1$ and $\phi(1 + zf''(z)/f'(z), z^2\{f, z\}; z)$ is univalent in U, then*

$$h(z) \prec \phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \tag{4.30}$$

implies

$$q(z) \prec 1 + \frac{zf''(z)}{f'(z)}. \tag{4.31}$$

Combining Theorems 4.3 and 4.11, we obtain the following sandwich-type theorem.

Corollary 4.12. *Let h_1 and q_1 be analytic functions in U, let h_1 be analytic univalent in U, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{Sc}[h_2, q_2] \cap \Phi'_{Sc}[h_1, q_1]$. Let $f \in \mathcal{A}$ with $f'(z) \neq 0$. If $1 + zf''(z)/f'(z) \in \mathcal{H} \cap Q_1$ and $\phi(1 + zf''(z)/f'(z), z^2\{f, z\}; z)$ is univalent in U, then*

$$h_1(z) \prec \phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \prec h_2(z) \tag{4.32}$$

implies

$$q_1(z) \prec 1 + \frac{zf''(z)}{f'(z)} \prec q_2(z). \tag{4.33}$$

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