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SUBORDINATION CONDITIONS FOR A CLASS OF NON-BAZILEVIČ TYPE DEFINED BY USING FRACTIONAL q-CALCULUS OPERATORS

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Abstract. In this article, we introduce and investigate a new class of non-Bazilevič functions with respect to k-symmetric points defined by using fractional q-calculus operators and q-differentiation. Several interesting subordination results are derived for the functions belonging to this class in the open unit disc. Furthermore, we point out some new and known consequences of our main result.
Keywords: Fractional calculus; q-calculus; q-transform analysis.

1. Introduction and preliminaries

Fractional calculus appears more and more frequently for the modelling of relevant systems in several fields of applied sciences. These equations play important roles, not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create a mathematical model of many physical phenomena [5, 6, 7, 14, 19, 20, 35].

Fractional q-calculus is the q-extension of ordinary fractional calculus. The theory of q-calculus operators has been applied lately in areas such as ordinary fractional calculus, optimal control problems, solutions of the q-difference, q-differential and q-integral equations, q-transform analysis and many more. Recently, there has been a significant increase of activity in the area of q-calculus due to its applications in mathematics, statistics and physics. For more details, one may refer to the books [9, 10, 13] and the recent papers [1, 2, 8, 15, 26] on the subject.

Purohit and Raina [21] used fractional q-calculus operators in investigating certain classes of functions which are analytic in the open disk. Recently, many authors have introduced new classes of analytic functions using q-calculus operators. For

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some recent investigations on the classes of analytic functions defined by using q-calculus operators and related topics, we refer the reader to [3, 17], [22]-[25], [29, 30] and the references cited therein. In the present paper, we aim at introducing a new class of non-Bazilevič type involving fractional q-calculus operators. Certain interesting subordination results are also derived for the functions belonging to this class.

We first give various definitions and notations in q-calculus which are useful to understand the subject of this paper.

For any complex number α, q-shifted factorials are defined as

$$(\alpha; q)_0 = 1, \qquad (\alpha; q)_n = \prod_{k=0}^{n-1}(1-\alpha q^k), \qquad n \in \mathbb{N}, \tag{1.1}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \qquad (n > 0),$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q,q)_\infty (1-q)^{1-x}}{(q^x; q)_\infty}, \qquad (0 < q < 1).$$

If $|q| < 1$, the definition (1.1) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j).$$

In view of the relation

$$\lim_{q\to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n,$$

we observe that the q-shifted factorial (1.1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

Also, the q-derivative and q-integral of a function on a subset of $\mathbb{C}$ are, respectively, given by (see [10] pp. 19 –22)

$$D_q f(z) = \frac{f(z) - f(zq)}{(1-q)z}, \qquad (z \neq 0, q \neq 0) \tag{1.2}$$

and

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{1.3}$$

Therefore, the q-derivative of $f(z) = z^n$, where n is a positive integer is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q\, z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q} = q^{n-1} + \cdots + 1$$

and is called the q-*analogue* of n. As $q \to 1$, we have $[n]_q = q^{n-1} + \cdots + 1 \to 1 + \cdots + 1 = n$.

We now define the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [21].

Definition 1.1. (**Fractional q-integral operator**) The fractional q-integral operator $I^{\delta}_{q,z}$ of a function $f(z)$ of order δ is defined by

$$I^{\delta}_{q,z} f(z) \equiv D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z-tq)_{\delta-1} f(t)\, d_q t, \qquad (\delta > 0), \tag{1.4}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the q-binomial function $(z-tq)_{\delta-1}$ is given by

$$(z-tq)_{\delta-1} = z^{\delta-1}\, {}_1\Phi_0[q^{-\delta+1}; -; q, tq^{\delta}/z].$$

The series ${}_1\Phi_0[\delta; -; q, z]$ is single valued when $|\arg(z)| < \pi$ and $|z| < 1$ (see for details [10], pp. 104-106). Therefore, the function $(z-tq)_{\delta-1}$ in (1.4) is single valued when $|\arg(-tq^{\delta}/z)| < \pi$, $|tq^{\delta}/z| < 1$ and $|\arg(z)| < \pi$.

Definition 1.2. (**Fractional q-derivative operator**) The fractional q-derivative operator $D^{\delta}_{q,z}$ of a function $f(z)$ of order δ is defined by

$$D^{\delta}_{q,z} f(z) \equiv D_{q,z}\, I^{1-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\delta)}\, D_{q,z} \int_0^z (z-tq)_{-\delta}\, f(t)\, d_q t, \qquad (0 \le \delta < 1), \tag{1.5}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1.1.

Definition 1.3. (**Extended fractional q-derivative operator**) Under the hypotheses of Definition 1.2, the fractional q-derivative for a function $f(z)$ of order δ is defined by

$$D^{\delta}_{q,z} f(z) = D^{m}_{q,z}\, I^{m-\delta}_{q,z} f(z), \tag{1.6}$$

where $m-1 \le \delta < 1$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}$ denotes the set of natural numbers.

Remark 1.1. It follows from Definition 1.2 that

$$D^{\delta}_{q,z}\, z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\delta)}\, z^{n-\delta} \quad (\delta \ge 0,\ \ n > -1).$$

2. The Class $\mathcal{N}_{q,k}^{m}(\lambda,\delta,\gamma;\phi)$

Let $\mathcal{H}(a,n)$ denote the class of functions $f(z)$ of the form

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots,\quad (z\in\mathcal{U}),\tag{2.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z\in\mathbb{C}:\,|z|<1\}$. In particular, let $\mathcal{A}$ be the subclass of $\mathcal{H}(0,1)$ containing functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n.\tag{2.2}$$

We denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$, the classes of all functions in $\mathcal{A}$ which are, respectively, univalent, starlike, convex and close-to-convex in $\mathcal{U}$. Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ with

$$w(0)=0,\quad |w(z)|<1\quad (z\in\mathcal{U}),$$

such that

$$f(z)=g(w(z))\qquad (z\in\mathcal{U}).$$

We denote this subordination by $f(z)\prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathcal{U}$, then $f(z)\prec g(z)\quad (z\in\mathcal{U})\Longleftrightarrow f(0)=g(0)$ and $f(\mathcal{U})\subset g(\mathcal{U})$.
Let k be a positive integer and let $\varepsilon_k=\exp(\frac{2\pi i}{k})$. For $f\in\mathcal{A}$ let

$$f_k(z)=\frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{-j}f(\varepsilon_k^{j}z).\tag{2.3}$$

The function f is said to be starlike with respect to k-symmetric points if it satisfies

$$Re\left(\frac{zf'(z)}{f_k(z)}\right)>0,\quad z\in\mathcal{U}.\tag{2.4}$$

We denote by $S_s^{(k)}$ the subclass of $\mathcal{A}$ consisting of all functions starlike with respect to k-symmetric points in $\mathcal{U}$. The class $S_s^{(2)}$ was introduced and studied by Sakaguchi [27]. We also note that different subclasses of $S_s^{(k)}$ can be obtained by replacing condition (2.4) by

$$Re\left(\frac{zf'(z)}{f_k(z)}\right)\prec h(z),$$

where $h(z)$ is a given convex function, with $h(0)=1$ and $Re\,h(z)>0$.

Using $D_{q,z}^{\delta}$, we define a q-differintegral operator $\Omega_{q,z}^{\delta}:\mathcal{A}\longrightarrow\mathcal{A}$, as follows:

$$\begin{aligned}\Omega_{q,z}^{\delta}f(z) &= \frac{\Gamma_q(2-\delta)}{\Gamma_q(2)}z^{\delta}\,D_{q,z}^{\delta}\,f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)}\,a_nz^n,\\ &\quad (\delta<2;\ n\in\mathbb{N};\ 0<q<1;\ z\in\mathcal{U}),\end{aligned}\tag{2.5}$$

where $D^{\delta}_{q,z} f(z)$ in (2.5) represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 \leq \delta < 2$. Here we note that $\Omega^{0}_{q,z} f(z) = f(z)$.
We now define a linear multiplier fractional q-differintegral operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ as follows:

$$\begin{aligned} \mathscr{D}^{\delta,0}_{q,\lambda} f(z) &= f(z), \\ \mathscr{D}^{\delta,1}_{q,\lambda} f(z) &= (1-\lambda)\Omega^{\delta}_{q,z} f(z) + \lambda\, z\, D_q(\Omega^{\delta}_{q,z} f(z)), \qquad (\lambda \geq 0), \\ \mathscr{D}^{\delta,2}_{q,\lambda} f(z) &= \mathscr{D}^{\delta,1}_{q,\lambda}(\mathscr{D}^{\delta,1}_{q,\lambda} f(z)), \\ &\vdots \\ \mathscr{D}^{\delta,m}_{q,\lambda} f(z) &= \mathscr{D}^{\delta,1}_{q,\lambda}(\mathscr{D}^{\delta,m-1}_{q,\lambda} f(z)), \qquad m \in \mathbb{N}. \end{aligned} \tag{2.6}$$

If $f(z)$ is given by (2.2), then by (2.6), we have

$$\mathscr{D}^{\delta,m}_{q,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)} \big[1 - \lambda + [n]_q \lambda\big] \right)^{m} a_n z^n.$$

It can be seen that, by specializing the parameters the operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ reduces to many known and new integral and differential operators. In particular, when $\delta = 0$ the operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ reduces to the operator introduced by AL-Oboudi [4] and for $\delta = 0$, $\lambda = 1$ it reduces to the operator introduced by Sălăgean [28].

Throughout this paper we assume that

$$f^{m}_{q,k}(\lambda,\delta;z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-j} (\mathscr{D}^{\delta,m}_{q,\lambda} f(\varepsilon_k^{j} z)) = z + \cdots, \qquad (f \in \mathcal{A}). \tag{2.7}$$

Clearly, for $k = 1$, we have

$$f^{m}_{q,1}(\lambda,\delta;z) = \mathscr{D}^{\delta,m}_{q,\lambda} f(z).$$

Let $\mathcal{P}$ denote the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in $\mathcal{U}$ and for which $Re\{h(z)\} > 0, \quad (z \in \mathcal{U})$.

We now introduce a new subclass $\mathcal{N}^{m}_{q,k}(\lambda,\delta,\gamma;\phi)$ of analytic functions using the linear multiplier q-fractional differintegral operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ defined by (2.6). It is interesting to note that the class $\mathcal{N}^{m}_{q,k}(\lambda,\delta,\gamma;\phi)$ is a generalization of the class of non-Bazilevič functions. The notion of the class of non-Bazilevič functions $\mathcal{N}(\alpha)$ was first introduced by Obradović [18] in 1998. Until now, the class of non-Bazilevič functions was studied in a direction of determining necessary conditions over α that embed this class into the class of univalent functions or its subclass. In recent years, many papers have appeared in the literature concerned with extending the results contained in Obradović's paper [18]. Tuneski and Darus [33] obtained Fekete-Szegő inequality for the class of non-Bazilevič functions. Using the concept of non-Bazilevič class, Wang et al [34] studied many subordination results for the generalized class of non-Bazilevič functions.

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}_{q,k}^{m}(\lambda, \delta, \gamma; \phi)$ if and only if

$$D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)\left(\frac{z}{f_{q,k}^{m}(\lambda, \delta; z)}\right)^{1+\gamma} \prec \phi(z), \qquad (z \in \mathcal{U}),$$

where $0 \leq \gamma \leq 1$, $\phi \in \mathcal{P}$ and $f_{q,k}^{m}(\lambda, \delta; z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$.

It is easy to verify the following relations: If $q \to 1^-$, then

1. $\mathcal{N}_{q,1}^{0}(\lambda, \delta, 0; \frac{1+z}{1-z}) = \mathcal{S}^*$
2. $\mathcal{N}_{q,k}^{0}(\lambda, \delta, 0; \frac{1+z}{1-z}) = \mathcal{S}_s^{(k)}$
3. $\mathcal{N}_{q,1}^{0}(\lambda, \delta, \gamma; \frac{1+z}{1-z}) = \mathcal{N}(\gamma)$, the class of non-Bazilevič functions of type γ.

In this article, we derive some sufficient conditions for functions belonging to the class $\mathcal{N}_{q,k}^{m}(\lambda, \delta, \gamma; \phi)$. In order to prove our results we need the following results.

Lemma 2.1. (*[16], see also [11]*) *Let h be convex in $\mathcal{U}$, with $h(0) = a, \beta \neq 0$ and $Re\, \beta \geq 0$. If $g \in \mathcal{H}(a, n)$ and*

$$g(z) + \frac{zg'(z)}{\beta} \prec h(z),$$

then

$$g(z) \prec \phi(z) \prec h(z),$$

where

$$\phi(z) = \frac{\beta}{n\, z^{\beta/n}} \int_0^z h(t)\, t^{(\beta/n)-1}\, dt.$$

The function ϕ is convex and is the best (a, n)-dominant.

Lemma 2.2. (*[16], see also [32]*) *Let h be starlike in $\mathcal{U}$, with $h(0) = 0$. If $g \in \mathcal{H}(a, n)$ satisfies*

$$zg'(z) \prec h(z),$$

then

$$g(z) \prec \phi(z) = a + n^{-1} \int_0^z h(t)\, t^{-1}\, dt.$$

The function ϕ is convex and is the best (a, n)-dominant.

In the following, we give the q-analogues of the above lemmas.

Proposition 2.1. *Let h be convex in $\mathcal{U}$, with $h(0) = a, \beta \neq 0$ and $Re\,\beta \geq 0$. If $g \in \mathcal{H}(a,n)$ and*

$$g(z) + \frac{z\,D_q(g(z))}{\beta} \prec h(z),$$

then

$$g(z) \prec \phi(z) \prec h(z),$$

where

$$\phi(z) = \frac{\beta}{n\,z^{\beta/n}} \int_0^z h(t)\,t^{(\beta/n)-1}\,d_qt.$$

The function ϕ is convex and is the best (a, n)-dominant.

Proposition 2.2. *Let h be starlike in $\mathcal{U}$, with $h(0) = 0$. If $g \in \mathcal{H}(a,n)$ satisfies*

$$z\,D_q(g(z)) \prec h(z),$$

then

$$g(z) \prec \phi(z) = a + n^{-1} \int_0^z h(t)\,t^{-1}\,d_qt.$$

The function ϕ is convex and is the best (a, n)-dominant.

3. Subordination Results for the Class $\mathcal{N}^m_{q,k}(\lambda, \delta, \gamma; \phi)$

Theorem 3.1. *Let $f \in \mathcal{A}$ with $f(z)$ and $f^m_{q,k}(\lambda, \delta; z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and let h be convex in $\mathcal{U}$, with $h(0) = 1$ and $Re\,h(z) > 0$. If*

$$\left\{ D_q\big(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\big) \left(\frac{z}{f^m_{q,k}(\lambda,\delta;z)}\right)^{1+\gamma} \right\}^2 \left[3 + 2\gamma + \frac{2z\,D^2_q\big(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\big)}{D_q\big(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\big)} - 2(1+\gamma)\frac{z\,D_q\big(f^m_{q,k}(\lambda,\delta;z)\big)}{f^m_{q,k}(\lambda,\delta;z)} \right] \prec h(z), \tag{3.1}$$

then

$$D_q\big(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\big)\left(\frac{z}{f^m_{q,k}(\lambda,\delta;z)}\right)^{1+\gamma} \prec \phi(z) = \sqrt{Q(z)}, \tag{3.2}$$

where

$$Q(z) = \frac{1}{z}\int_0^z h(t)d_qt,$$

the function $\phi(z)$ is convex and is the best dominant.

Proof. Let $p(z) = D_q\big(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\big)\left(\frac{z}{f^m_{q,k}(\lambda,\delta;z)}\right)^{1+\gamma} \qquad (z \in \mathcal{U}; z \neq 0; f \in \mathcal{A}).$

Then $p(z) \in \mathcal{H}(1,1)$ with $p(z) \neq 0$ in $\mathcal{U}$. Since h is convex, it can be easily verified that Q is convex and univalent. We now set $P(z) = p^2(z)$. Then $P(z) \in \mathcal{H}(1,1)$ with $P(z) \neq 0$ in $\mathcal{U}$. By logarithmic q-differentiation (see [12]) we have,

$$\frac{\ln q}{q-1}\frac{z\,D_q\big(P(z)\big)}{P(z)} = \frac{2\ln q}{q-1}\left[\frac{z\,D_q^2\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)} + (1+\gamma)\left(1 - \frac{z\,D_q\big(f_{q,k}^m(\lambda,\delta;z)\big)}{f_{q,k}^m(\lambda,\delta;z)}\right)\right].$$

Therefore, by (3.1) we have

$$P(z) + z\,D_q\big(P(z)\big) \prec h(z). \tag{3.3}$$

Now, by Proposition 2.1 with $\beta = 1$, we deduce that

$$P(z) \prec Q(z) \prec h(z),$$

and Q is the best dominant of (3.3). Since $Re\,h(z) > 0$ and $Q(z) \prec h(z)$ we also have $Re\,Q(z) > 0$. Hence, the univalence of Q implies the univalence of $\phi(z) = \sqrt{Q(z)}$, and

$$p^2(z) = P(z) \prec Q(z) = \phi^2(z),$$

which implies that $p(z) \prec \phi(z)$. Since Q is the best dominant of (3.3), we deduce that $\phi(z)$ is the best dominant of (3.2). This completes the proof. $\square$

Corollary 3.1. *Let $f \in \mathcal{A}$ with $f_{q,k}^m(\lambda,\delta;z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$. If $Re\big(\Psi(z)\big) > \alpha$, $(0 \le \alpha < 1)$, where*

$$\Psi(z) = \left\{D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)\left(\frac{z}{f_{q,k}^m(\lambda,\delta;z)}\right)^{1+\gamma}\right\}^2 \tag{3.4}$$

$$\times\left[3 + 2\gamma + \frac{2z\,D_q^2\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)}{D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)} - 2(1+\gamma)\frac{z\,D_q\big(f_{q,k}^m(\lambda,\delta;z)\big)}{f_{q,k}^m(\lambda,\delta;z)}\right], \tag{3.5}$$

then

$$Re\left\{D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m}f(z)\big)\left(\frac{z}{f_{q,k}^m(\lambda,\delta;z)}\right)^{1+\gamma}\right\} > \mu(\alpha),$$

where $\mu_q(\alpha) = \big[2(1-\alpha)(q-1)\frac{\ln 2}{\ln q} + (2\alpha-1)\big]^{\frac{1}{2}}$, and this result is sharp.

Proof. Let $h(z) = \dfrac{1+(2\alpha-1)z}{1+z}$ with $0 \le \alpha < 1$. Then from Theorem 3.1, it follows that $Q(z)$ is convex and $Re\,Q(z) > 0$. Also we have,

$$\min_{|z|\le 1} Re\,\phi(z) = \min_{|z|\le 1} Re\,\sqrt{Q(z)} = \sqrt{Q(1)} = \left[2(1-\alpha)(q-1)\frac{\ln 2}{\ln q} + (2\alpha-1)\right]^{\frac{1}{2}}.$$

This completes the proof the corollary. $\square$

By setting $m = 0$ and letting $q \to 1^-$ in Corollary 3.1, we have the following corollary.

Corollary 3.2. *[31] Let $f \in \mathcal{A}$ with $f'(z)$ and $f_k(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$. If*

$$Re\left\{\left(f'(z)\left(\frac{z}{f_k(z)}\right)^{1+\gamma}\right)^2\left[3+2\gamma+\frac{2zf''(z)}{f'(z)}-2(1+\gamma)\frac{zf_k'(z)}{f_k(z)}\right]\right\} > \alpha, \quad (0 \leq \alpha < 1),$$

then

$$Re\left\{f'(z)\left(\frac{z}{f_k(z)}\right)^{1+\gamma}\right\} > \mu(\alpha),$$

where $\mu(\alpha) = \left[2(1-\alpha)\ln 2 + (2\alpha - 1)\right]^{\frac{1}{2}}$, *and this result is sharp.*

Further, setting $\gamma = 0$ in Corollary 3.2 we obtain

Corollary 3.3. *[31] Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$. If*

$$Re\left\{\left(\frac{zf'(z)}{f_k(z)}\right)^2\left[3+2\frac{zf''(z)}{f'(z)}-2\frac{zf_k'(z)}{f_k(z)}\right]\right\} > \alpha, \quad (0 \leq \alpha < 1),$$

then

$$Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > \mu(\alpha),$$

where $\mu(\alpha) = \left[2(1-\alpha)\ln 2 + (2\alpha - 1)\right]^{\frac{1}{2}}$, *and this result is sharp.*

Theorem 3.2. *Let $f \in \mathcal{A}$ with $f(z)$ and $f_{q,k}^m(\lambda, \delta; z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and h be starlike in $\mathcal{U}$, with $h(0) = 0$. If*

$$\frac{\ln q}{q-1}\left[\frac{z\,D_q^2\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)}{D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)}+(1+\gamma)\left(1-\frac{z\,D_q\big(f_{q,k}^m(\lambda,\delta;z)\big)}{f_{q,k}^m(\lambda,\delta;z)}\right)\right] \prec h(z), \ (z \in \mathcal{U}), \tag{3.6}$$

then

$$D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)\left(\frac{z}{f_{q,k}^m(\lambda,\delta;z)}\right)^{1+\gamma} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\, d_qt\right), \tag{3.7}$$

the function $\phi(z)$ is convex and is the best dominant.

Proof. Let $p(z) = D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)\left(\dfrac{z}{f_{q,k}^m(\lambda,\delta;z)}\right)^{1+\gamma} \qquad (z \in \mathcal{U}; z \neq 0; f \in \mathcal{A}).$
Then $p(z) \in \mathcal{H}(1,1)$ with $p(z) \neq 0$ in $\mathcal{U}$. Thus we can define an analytic function $P(z) = \log p(z)$. Clearly $P \in \mathcal{H}(0,1)$. Now by logarithmic q-differentiation we have,

$$\frac{\ln q}{q-1}\frac{z\,D_q(p(z))}{p(z)} = \frac{\ln q}{q-1}\left[\frac{z\,D_q^2\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)}{D_q\big(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\big)}+(1+\gamma)\left(1-\frac{z\,D_q\big(f_{q,k}^m(\lambda,\delta;z)\big)}{f_{q,k}^m(\lambda,\delta;z)}\right)\right].$$

Therefore, by (3.6) we obtain

$$z\,D_q(P(z)) \prec h(z). \tag{3.8}$$

Now by using Proposition 2.2, we deduce that $\quad P(z) \prec Q(z) = \int_0^z \frac{h(t)}{t}\, d_q t,$

and Q is the best dominant of (3.8). Converting back we obtain

$$p(z) = \exp P(z) \prec \exp Q(z) = \phi(z),$$

and since Q is the best dominant of (3.8), we deduce that ϕ is the best dominant of (3.7). This completes the proof. $\square$

By setting $m = 0$ and letting $q \to 1^-$ in Theorem 3.2, we have the following corollary.

Corollary 3.4. *[31] Let $f \in \mathcal{A}$ with $f'(z)$ and $f_k(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and h be starlike in $\mathcal{U}$, with $h(0) = 0$. If*

$$\frac{zf''(z)}{f'(z)} + (1+\gamma)\left(1 - \frac{zf_k'(z)}{f_k(z)}\right) \prec h(z), \qquad (z \in \mathcal{U}),$$

then

$$f'(z)\left(\frac{z}{f_k(z)}\right)^{1+\gamma} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\, dt\right),$$

and q is the best dominant.

Also, putting $\gamma = 0$ in Corollary 3.4 we obtain

Corollary 3.5. *[31] Let $f \in \mathcal{A}$ with $f'(z)$ and $f_k(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and h be starlike in $\mathcal{U}$, with $h(0) = 0$. If*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf_k'(z)}{f_k(z)} \prec h(z) \qquad (z \in \mathcal{U}),$$

then

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\, dt\right),$$

and q is the best dominant.

Finally, we conclude this article by remarking that, by suitably choosing the parameters, one can easily obtain a number of inequalities from the main results (Theorem 3.1 & Theorem 3.2). Moreover, the class $\mathcal{N}_{q,k}^m(\lambda, \delta, \gamma; \phi)$, defined in this paper, can be used in the investigation of various geometric properties like, the coefficient estimates, distortion bounds, radii of starlikeness, convexity and close to convexity etc. in the unit disk.

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