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SUBORDINATION FOR BMOA

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1. Introduction.

Let p be arbitrarily fixed with p>0 throughout this paper. Let f(z) be a function analytic in the unit disk $U=\{|z|<1\}$. The H_p norm $||f||_p$ of f is defined by

(1.1)
$$||f||_{p} = \lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{p} d\theta \right\}^{1/p}.$$

The family of all f for which $||f||_p$ is finite is denoted by $H_p(U)$ and called Hardy class. For every $a \in U$, $t_a(z) = (z+a)/(1+\bar{a}z)$ be the conformal map of U onto itself with $t_a(0) = a$, and we set $f_a(z) = f(t_a(z)) - f(a)$. The **BMO** norm $B_p(f)$ of f is defined by

(1.2)
$$B_p(f) = \sup\{\|f_a\|_p : a \in U\}.$$

The family of all f for which $B_p(f)$ is finite is denoted by **BMO**A(U), where **BMO**A stands for "Analytic functions of Bounded Mean Oscillation". It is known that the family **BMO**A(U) does not depend on the value of p ([2]). In the following, abbreviating the index p, we simply write as

$$(1.3) B(f) = B_p(f)^p.$$

Let ϕ denote an analytic function in U with $|\phi(z)| < 1$ for $z \in U$. Applying the subordination principle, we easily see that

$$(1.4) B(f \circ \phi) \leq B(f)$$

holds for every $f \in BMOA(U)$. Here $f \circ \phi$ denotes the composite function of f by ϕ , *i.e.* $(f \circ \phi)(z) \equiv f(\phi(z))$. In this paper, we deal with the equality problem for the inequality (1.4).

A bounded analytic function ϕ in U is called an *inner function*, if its nontangential boundary values are of modulus 1 almost everywhere on the unit circle T. Ryff [5] characterized inner functions by a property of preserving H_p norms of analytic functions:

Ryff's Theorem ([6, Theorem 3, p. 351]). Let ϕ be an analytic function in Received February 13, 1989; revised April 20, 1989 U with $|\phi(z)| < 1$ for $z \in U$ and $\phi(0)=0$. If ϕ is an inner function, then the equality

(1.5)
$$\|f \circ \phi\|_p = \|f\|_p$$

holds for every $f \in H_p(U)$. Conversely, if (1.5) holds for some nonconstant $f \in H_p(U)$, then ϕ is an inner function.

It follows from the Ryff theorem that if ϕ is an inner function, then equality holds in (1.4) for every $f \in BMOA(U)$ (see Theorem 2.1 in the next section). As for the converse, however, we did not know whether equality occurs or not in (1.4) for some noninner function ϕ . Recently, the author [3] showed that there is a noninner function ϕ for which equality holds in (1.4) for every function f analytic in U and continuous on the closure \overline{U} of U. In the present paper, we generalize this to the desired case for every $f \in BMOA(U)$, that is, we show that there are fairly many noninner functions ϕ such that equality holds in (1.4) for every $f \in BMOA(U)$.

2. Subordination by inner functions.

In this section, to make this paper self-contained, we show that equality holds in (1.4) for every inner function ϕ and for every $f \in BMOA(U)$:

THEOREM 2.1. If ϕ is an inner function, then the equality

 $(2.1) B(f \circ \phi) = B(f)$

holds for every $f \in BMOA(U)$.

Before stating the proof of the theorem, we introduce notation concerning the least harmonic majorants of subharmonic functions, which we use throughout this paper. Let $S^+(U)$ denote the family of all nonnegative subharmonic functions in U. For $s \in S^+(U)$ we denote by \hat{s} the least harmonic majorant of s in U. Here, in case that s admits no harmonic majorants in U, we set $\hat{s}(z)$ $\equiv +\infty$.

The author and Suita [5] generalized the *Ryff theorem* cited above to the case of nonnegative subharmonic functions on Riemann surfaces. We restate it in the case for $S^+(U)$:

LEMMA 2.2. If ϕ is an inner function, then the equality

(2.2)
$$(s \circ \phi)^{\hat{}}(z) \equiv (\hat{s} \circ \phi)(z)$$
.

holds for every $s \in S^+(U)$, and conversely if (2.2) holds for some $s \in S^+(U)$ which is not harmonic in whole U, then ϕ is inner.

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For the proof, see [5, pp. 316-318].

Returning to the case of an analytic function f, we set $s_a(z) = |f(z) - f(a)|^p$ for every $a \in U$. On noting that the equality

$$||f_a||_p^p = (s_a \circ t_a)^{(0)} = (\hat{s}_a \circ t_a)(0) = \hat{s}_a(a)$$

holds for any $f \in BMOA(U)$, where we used (2.2) with $\phi = t_a$, we obtain from (1.2) the following lemma:

LEMMA 2.3. For every
$$f \in BMOA(U)$$

(2.3) $B_p(f) = \sup\{\hat{s}_a(a) : a \in U\}.$

Proof of the theorem. By a theorem of Frostman [1], the range $\phi(U)$ of any inner function ϕ covers U except possibly for a set of (*logarithmic*) capacity zero. In particular, we see that $\phi(U)$ is dense in U. Therefore, by using (2.3) and (2.2), we obtain

$$B_p(f) = \sup\{\hat{s}_a(a) : a \in U\}$$
$$= \sup\{\hat{s}_a(a) : a \in \phi(U)\}$$
$$= \sup\{\hat{s}_a(\phi(b)) : b \in U\}$$
$$= \sup\{(s_a \circ \phi)^{\hat{}}(b) : b \in U\}$$
$$= B(f \circ \phi).$$

Here, we set $a = \phi(b)$.

3. Subordination by noninner functions.

In this section we show that equality holds in (1.4) for every $f \in BMOA(U)$ even for some noninner functions ϕ :

THEOREM 3.1. There are some noninner functions ϕ for which the equality (2.1) holds for every $f \in BMOA(U)$.

Proof. We construct functions ϕ which satisfy the condition of the theorem. Let ϕ be any inner function which has a singularity (a point at which ϕ can not be analytically continued) at a point ζ on the unit circle T. We easily see that there are a lot of such inner functions. In fact, we can take as ϕ any inner function which has as a factor a *Blaschke product* whose zeros converge to ζ or a singular inner function whose associated singular measure has a positive mass at ζ . Let D be a subdomain of U which contains $\{z \in U : |z - \zeta| < \delta\}$ for a sufficiently small $\delta > 0$. Let λ be a universal covering map of D. It is easily seen that λ is inner if and only if U-D is of capacity zero. Now we further assume that U-D is of capacity positive, so that λ is noninner. Finally

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we set $\phi = \phi \circ \lambda$. Applying the Ryff theorem, we easily see that ϕ is also non-inner.

For $s \in S^+(U)$ we denote by $\hat{s}_D(z)$ the least harmonic majorant of s in D. The following lemma is merely a restatement of a wellknown fact that a universal covering map preserves the least harmonic majorant of any subharmonic function. As for the proof, see for example [5, Lemma 1, p. 316]:

LEMMA 3.2. The equality

 $(3.1) \qquad (s \circ \lambda)^{\hat{}}(z) \equiv (\hat{s} \circ \lambda)(z)$

holds for every $s \in S^+(U)$.

We set $s(z) = |g(z)|^p$ for $g \in H_p(U)$. Let D_1 be a simply-connected subdomain of D whose boundary contains an open interval on T containing ζ . In fact, for example, we can take as $D_1 = \{z \in U : |z - \zeta| < \delta\}$ for a sufficiently small $\delta > 0$. It is known that $\hat{s}(z)$ and $\hat{s}_D(z)$ are expressed as integrals by the harmonic measure for the point z in D_1 of their boundary values on the boundary of D_1 , respectively. Here note that these integrals are expressed via the Poisson integral formula by conformally mapping the domain D_1 onto the unit disk U. Since their boundary values coincide almost everywhere on the open interval on Tcontaining ζ , we see that $\hat{s}(z) - \hat{s}_D(z)$ approaches to 0 as z approaches to ζ . Therefore, by noting Lemma 3.2, we obtain the following lemma:

LEMMA 3.3 Set $s(z) = |g(z)|^p$ for any $g \in H_p(U)$. If $b_n \in U$ is a sequence such that $\lambda(b_n) \rightarrow \zeta$ as $n \rightarrow \infty$ and that the limit

$$\lim_{n\to\infty} (\hat{s}_D \lambda)(b_n)$$

exists, then the limit

$$\lim_{n\to\infty}(s\circ\lambda)^{\hat{}}(b_n)$$

exists and the two limits coincide:

(3.2)
$$\lim_{n \to \infty} (\hat{s} \circ \lambda)(b_n) = \lim_{n \to \infty} (s \circ \lambda)^{\hat{s}}(b_n) \,.$$

Now we continue the proof of the theorem. Since we assumed ψ to be an inner function with a singularity at ζ , we see by noting a local version of the Frostman theorem [1] cited in the previous section that the range set $R(\phi, \zeta)$ at ζ of ψ convers U except possibly for a set of capacity zero. Here, the range set $R(\phi, \zeta)$ is defined by

(3.3)
$$R(\phi, \zeta) = \bigcap_{\delta > 0} \phi(\{z \in U : |z - \zeta| < \delta\}).$$

In particular, $R(\phi, \zeta)$ is dense in U. Let ε be an arbitrarily fixed positive number. On noting Lemma 2.2, we can take a point $a \in R(\phi, \zeta)$ such that

$$(3.4) B(f) - \varepsilon < \hat{s}_a(a) \,.$$

The definition (3.3) means that we can take a sequence $a_n \in D$ such that $\phi(a_n) = a$ for every *n* and that $a_n \rightarrow \zeta$ as $n \rightarrow \infty$. Finally we take a sequence $b_n \in U$ such that $\lambda(b_n) = a_n$ for every *n*. By using (2.2) we see

$$\begin{split} \hat{s}_a(a) &= \hat{s}_a(\phi(a_n)) \\ &= (s_a \circ \phi)^{\hat{}}(a_n) \\ &= (s_a \circ \phi)^{\hat{}}(\lambda(b_n)) \,. \end{split}$$

Therefore by letting $n \to \infty$ in this equality and applying Lemma 3.3 as $g(z) = f(\phi(z)) - f(a)$, we see

(3.5)
$$\hat{s}_{a}(a) = \lim_{n \to \infty} (s_{a} \circ \psi)^{\hat{}}(\lambda(b_{n}))$$
$$= \lim_{n \to \infty} (s_{a} \circ \psi \circ \lambda)^{\hat{}}(b_{n}).$$

Since we have set $\phi = \phi \circ \lambda$, writing $h(z) = f \circ \phi$, we easily see $|h(z) - h(b_n)|^p = (s_a \circ \phi \circ \lambda)(z)$ for every *n*. Therefore, on noting Lemma 2.3, we see that for every *n*

(3.6)
$$(s_a \circ \phi \circ \lambda)^{\hat{}}(b_n) \leq B(f \circ \phi).$$

Combining (1.4), (3.4), (3.5) and (3.6), we obtain (2.1), as asserted, since ε in (3.4) can be chosen arbitrarily small.

4. Concluding remarks.

We do not know as yet whether there are noninner functions ϕ of different type from ones constructed in the proof of Theorem 3.1 for which the equality (2.1) holds for every $f \in BMOA(U)$. The functions ϕ constructed in the proof of Theorem 3.1 are easily seen to satisfy the condition that for some point $\zeta \in T$ the range set $R(\phi, \zeta)$ covers U except possibly for a set of capacity zero. This condition, however, is not sufficient in order that the equality (2.1) holds for every $f \in BMOA(U)$. In fact, for example, let $\mu(z)$ be the conformal map of U onto $U \cap \{z : Re \ z > -k\}$ with $\mu(0)=0$, where 0 < k < 1, and ψ to be an inner function with a singularity at a point $\zeta \in T$. If we set $\phi(z) = \{\mu(\phi(z))\}^2$, we can prove that the strict inequality $B(f \circ \phi) < B(f)$ holds for every function f analytic in U and continuous on \overline{U} (cf. [4, Example 6.3, p. 169]).

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