Subordinators which are infinitely divisible w.r.t. time: Construction, properties, and simulation of max-stable sequences and infinitely divisible laws

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Abstract. The concept of a Lévy subordinator is generalized to a family of non-decreasing stochastic processes, which are parameterized in terms of two Bernstein functions. Whereas the independent increments property is only maintained in the Lévy subordinator special case, the considered family is always strongly infinitely divisible with respect to time, meaning that a path can be represented in distribution as a finite sum with arbitrarily many summands of independent and identically distributed paths of another process. Besides distributional properties of the process, we present two applications to the design of accurate and efficient simulation algorithms. First, each member of the considered family corresponds uniquely to an exchangeable max-stable sequence of random variables, and we demonstrate how the associated extreme-value copula can be simulated exactly and efficiently from its Pickands dependence measure. Second, we show how one obtains different series and integral representations for infinitely divisible probability laws by varying the parameterizing pair of Bernstein functions, without changing the law of one-dimensional margins of the process. As a particular example, we present an exact simulation algorithm for compound Poisson distributions from the Bondesson class, for which the generalized inverse of the distribution function of the associated Stieltjes measure can be evaluated accurately.

1. Introduction

We recall that a *Lévy subordinator* $L = \{L_t\}_{t \geq 0}$ is a non-decreasing stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with independent and stationary increments, whose paths are almost surely right-continuous and start at $L_0 = 0$, see Bertoin (1999) for a textbook treatment. Intuitively, Lévy subordinators are

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the continuous-time analog of discrete-time random walks with non-negative increments. The law of L, that is its finite-dimensional distributions, is fully determined by the law of any random variable L_t with $t > 0$, whose Laplace transform is given by

$$\mathbb{E}\left[e^{-x\,L_t}\right] = e^{-t\,\Psi_L(x)}, \quad \Psi_L(x) = \mu_L\,x + \int_{(0,\infty]} \left(1 - e^{-u\,x}\right) \nu_L(\mathrm{d}u), \quad x \geq 0, \ (1.1)$$

where the so-called *Lévy measure* ν_L satisfies $\int_{(0,\infty]} \min\{u,1\}\,\nu_L(\mathrm{d}u) < \infty$ and $\mu_L \geq 0$ is a drift constant. The function Ψ_L is a so-called *Bernstein function*, see Schilling et al. (2010) for a textbook treatment, and the number $\nu(\{\infty\})$ is called the *killing rate* of L, because it corresponds to an exponential rate at which L jumps to the absorbing graveyard state $\{\infty\}$, i.e. is "killed." The so-called *Lévy–Khinchin formula* (1.1) establishes a one-to-one correspondence between Bernstein functions and Lévy subordinators, so that Ψ_L (or equivalently the pair (μ_L, ν_L)) provides a convenient analytical description of the law of L.

The purpose of the present article is to embed the concept of a Lévy subordinator into a larger family of non-decreasing processes that can be parameterized in terms of a pair (Ψ_L, Ψ_F) of two Bernstein functions. On the one hand, the enlarged family of processes still satisfies the concept of being *strongly infinitely divisible with respect to time*, as explained below, which renders it a natural generalization from an algebraic viewpoint. On the other hand, our generalization is inspired by two practical applications: Firstly, the processes can be used to construct and simulate multivariate extreme-value distributions. Second, they provide a reasonable framework to derive series representations for infinitely divisible laws on the positive half-axis, which can be used for simulation.

For a pair (F, L) of a distribution function F of some non-negative random variable with finite, positive mean $\int_0^\infty 1 - F(x)\,\mathrm{d}x \in (0,\infty)$ and a Lévy subordinator L without drift, the present article studies distributional properties of the stochastic process

$$H_t := H_t^{(F,L)} = \int_0^\infty -\log\left\{F\left(\frac{s}{t}-\right)\right\}\mathrm{d}L_s, \quad t \geq 0, \tag{1.2}$$

the integral being defined pathwise in the usual Riemann–Stieltjes sense, and with $F(x-) := \lim_{u \nearrow x} F(u)$. Since F is right-continuous, so is $t \mapsto F(s/t-)$ for each fixed s. This implies that $t \mapsto H_t$ is almost surely right-continuous as integral over right-continuous functions. By definition, $H_0 = 0$ (using the notations $1/0 := \infty$ and $F(\infty-) := 1$), and $H_t \in [0,\infty]$ for $t \geq 0$, i.e. in particular $H_t = \infty$ is possible. An alternative representation for H is given by the formula

$$H_t = \int_0^\infty L_{s\,t}\,\mathrm{d}\left(\log(F(s))\right), \quad t \geq 0, \tag{1.3}$$

which is a consequence of Tonelli's Theorem:

$$\int_{(0,\infty)} L_{s\,t}\,\mathrm{d}\left(\log(F(s))\right) = \int_{(0,\infty)} \int_{(0,s\,t]} \mathrm{d}L_u\,\mathrm{d}\left(\log(F(s))\right)$$

$$= \int_{(0,\infty)} \int_{[u/t,\infty)} \mathrm{d}\left(\log(F(s))\right)\mathrm{d}L_u$$

$$= \int_{(0,\infty)} -\log\left\{F\left(\frac{s}{t}-\right)\right\}\mathrm{d}L_u = H_t.$$

According to Mansuy (2005, Paragraph (2.2)) this shows that $H = \{H_t\}_{t \geq 0}$ is *strongly infinitely divisible with respect to time (strong IDT)*[1], meaning that for arbitrary $n \in \mathbb{N}$ we have the distributional equality

$$\{H_t\}_{t \geq 0} \overset{d}{=} \Big\{ \sum_{i=1}^{n} H_{\frac{t}{n}}^{(i)} \Big\}_{t \geq 0},$$

where $H^{(i)}$, $i = 1, \ldots, n$, denote independent copies of H. In particular, H_t is an infinitely divisible random variable and there exists a Bernstein function Ψ_H such that $\exp(-t \Psi_H)$ equals the Laplace transform of H_t for arbitrary $t \geq 0$.

We call a pair (F, L) *admissible*, whenever H_t is not almost surely equal to the trivial process $H_t = \infty \cdot \mathbf{1}_{\{t > 0\}}$, which can happen in general. Our interest in this semi-parametric family of stochastic processes is fueled both by theoretical and practical aspects. In the following Section 2 we study distributional properties, whereas Sections 3 and 4 give applications of the presented class of processes to the design of simulation algorithms. More precisely, the following list outlines the organization of the remaining article and the contributions made.

Section 2: From the viewpoint of the theory on infinite divisibility, the considered family of stochastic processes is a natural extension of the concept of a Lévy subordinator. Lévy subordinators arise in the special case when F corresponds to a Bernoulli distribution with success probability $\exp(-1)$, see Example 2.8. Strong IDT processes have first been introduced in Mansuy (2005) and further examples have been studied in Es-sebaiy and Ouknine (2008); Hakassou and Ouknine (2013). These references give some examples of strong IDT processes with an emphasis on Gaussian processes. A LePage series representation for strong IDT processes without Gaussian component, in particular for non-negative strong IDT processes, is derived in Kopp and Molchanov (2018), and has been refined in the non-decreasing case by Mai (2019). The present investigation demonstrates that the considered processes form a large, and particularly tractable, subfamily of non-decreasing strong IDT processes. In particular, its two parameter-"dimensions" F and L are explained to be dual in a certain sense, see Lemma 2.3 and Section 4. Regarding applications, in the particular case when F is continuous and satisfies $F(1) = 1$ the process H is applied as a time-change in a model for stock prices in James and Zhang (2011), building upon earlier work of Bender and Marquardt (2009) who study a more general family of convoluted subordinators for the same purpose.

We demonstrate how several distributional properties of H can be inferred conveniently from the parameterizing pair (F, L), thus pave the way to an analytical treatment of H via its parameters. In particular, in addition to the defining integral representation we present a canonical LePage series representation for H in the spirit of Kopp and Molchanov (2018). We also present necessary and sufficient analytical conditions for the probability law of H_1 to be of compound Poisson type or to have an atom at ∞. Section 2.3 further studies the natural filtration $\{\mathcal{F}_t^H\}_{t \geq 0}$ of the process H. While Lévy subordinators have independent increments, we demonstrate how the support of the probability measure dF controls the ability of H to "*see into the future.*" In particular, for bounded support of dF, the increment $H_{t+h} - H_t$ can be decomposed into a sum of one part that is measurable with respect to $\mathcal{F}_t^H$, and another part that is independent thereof.

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[1]This property is called *time-stable* in Kopp and Molchanov (2018).

Section 3: For each realization of H, the function $x \mapsto 1 - \exp(-H_x)$, $x \geq 0$, is the distribution function of some non-negative random variable. In other words, $1 - \exp(-H)$ is a random variable taking values in the set of distribution functions of non-negative random variables. Due to Mai and Scherer (2014, Theorem 5.3), an iid sequence $\{Y_k\}_{k \in \mathbb{N}}$ of random samples drawn from this a priori simulated distribution function is an *exchangeable min-stable exponential sequence*, meaning that $\min\{Y_1/t_1, \ldots, Y_d/t_d\}$ has an exponential distribution, whose exponential rate we denote by

$$\ell(t_1, \ldots, t_d, 0, 0, \ldots) \in [0, \infty), \quad d \in \mathbb{N}, \ t_1, \ldots, t_d \geq 0 \text{ (not all zero).}$$

Equivalently, $\{1/Y_k\}_{k \geq 1}$ is an *exchangeable max-stable sequence*. If Y_1 has unit mean, i.e. if $-\log\left(\mathbb{E}[\exp\{-H_1\}]\right) = 1$, the associated function $\ell : [0, \infty)_{00}^{\mathbb{N}} \to [0, \infty)$ is called a *stable tail dependence function*, defined on the set $[0, \infty)_{00}^{\mathbb{N}}$ of non-negative sequences that are eventually zero. The stable tail dependence function uniquely characterizes the law of $\{Y_k\}_{k \in \mathbb{N}}$ and, equivalently, the law of H. Since min- (resp. max-) stability is closely related to multivariate extreme-value theory, an understanding of the law of H is thus tantamount with the understanding of an associated family of multivariate extreme-value copulas, see Gudendorf and Segers (2010) for background on the subject. In particular, a simulation algorithm for the random vector $(Y_1, \ldots, Y_d)$ is equivalent to one for the associated extreme-value copula.

Section 3 shows how the random vector $(Y_1, \ldots, Y_d)$ can be simulated exactly. To this end, we make use of a simulation algorithm presented in Dombry et al. (2016), which requires to simulate from the so-called *Pickands dependence measure* associated with $(Y_1, \ldots, Y_d)$, a finite measure on the d-dimensional unit simplex. In the present situation, we demonstrate how this simulation can be achieved efficiently and accurately.

Section 4: Fixing $t = 1$, the random variable H_1 has an infinitely divisible law on $[0, \infty]$, which is invariant with respect to many changes in the parameterizing pair (F, L). This fact can be used to derive different series representations for the same infinitely divisible law from Definition (1.2), when either L is of compound Poisson type or the support of $\mathrm{d}F$ is bounded. In spirit, this methodology is quite similar to seminal ideas in Bondesson (1982), who proposes alternative series representations for infinitely divisible laws on $\mathbb{R}$. Section 4 demonstrates how the (F, L)-parameterization of H_1 provides a very convenient setting to derive a simulation algorithm for distributions from the so-called *Bondesson class*, whenever the associated *Stieltjes measure* is given in a more convenient form than the Lévy measure. In fact, if L is chosen as a compound Poisson subordinator with unit exponential jumps, the definition of H_1 defines a bijection between the Bondesson class and distribution functions F having finite mean and left-end point of support equal to zero. This is to some extent analogous to traditional results for the case when L is a standard Gamma process, in which case it is well known that the definition of H_1 defines a bijection between the so-called *Thorin class* and distribution functions F satisfying a certain integrability condition, see James et al. (2008, Propositions 1.1 and 1.6).

As a final remark regarding our exposition in Section 4 we like to point out that our investigation of the law of H_1 is in general different from a series of seemingly similar articles studying so-called *exponential functionals of Lévy processes*. Only

in the special case $F(x) = \exp(-\exp(-x))$ the random variable H_1 defines an exponential functional of a Lévy process. In this case, Behme (2015, Proposition 3.5) states that if the law of L_1 lies in the Bondesson family and satisfies $\mathbb{E}[\log(L_1)] < \infty$, then the law of H_1 lies in the Thorin class. Whereas exponential functionals of Lévy processes are defined in terms of integrands of the form $t \mapsto \exp(-\xi_t)$ for a Lévy process ξ, which might in general be different from the trivial process $\xi_t = t$, one crucial difference in the present investigation is that our integrand – albeit being always deterministic – needs not be of exponential form (i.e. we consider arbitrary F and not only $F(x) = \exp(-\exp(-x))$), so H_1 in general is not the exponential functional of a Lévy process.

Finally, Section 5 concludes.

2. Anatomy of the process H

2.1. *Technical preliminaries and admissibility.* Throughout, we denote by $L = \{L_t\}_{t \geq 0}$ a (possibly killed) Lévy subordinator without drift and with Lévy measure ν_L on $(0, \infty]$, i.e. with killing rate $\nu(\{\infty\})$. We assume that ν_L is non-zero, i.e. L is not identically zero. Its associated Bernstein function is denoted by

$$\Psi_L(x) := \int_{(0,\infty]} \left(1 - e^{-u\,x}\right) \nu_L(\mathrm{d}u), \quad x \geq 0,$$

implicitly using the short-hand notations $\exp(-\infty) := 0$ and $0 \cdot \infty := 0$ in order to enforce $\Psi_L(0) = 0$.

Remark 2.1 (Why only driftless L are considered?). A positive drift μ_L of the Lévy subordinator L would imply a drift of the process H, namely

$$\mu_H = \mu_L \int_0^\infty -\log\{F(s)\}\,\mathrm{d}s. \tag{2.1}$$

On the one hand, this is inconvenient, because it requires an additional integrability condition on F, so that (2.1) exists at all. We will always postulate that F satisfies $\int_0^\infty 1 - F(s)\,\mathrm{d}s < \infty$, which is a weaker condition. For instance, the distribution function $F(x) = \exp(-x^{-2})$ may be admissible in the following but (2.1) is not finite. On the other hand, the assumption of a driftless Lévy subordinator is without loss of generality. To explain this, recall that H falls into the family of non-decreasing strong IDT processes. It follows from a structural result in Mai (2019) that, just like for the subfamily of Lévy subordinators, such processes can be decomposed uniquely into $H_t = \mu_H\,t + \tilde{H}_t$, with a drift $\mu_H \geq 0$ and a non-decreasing strong IDT process $\tilde{H}$ without drift. This allows us to concentrate our study on the driftless case, because the more general case is simply obtained by adding a drift a posteriori.

For later reference, we introduce the following sets of distribution functions (cdfs)

$$\mathfrak{F}_c := \left\{F \text{ cdf of some } X \in [0, \infty) \,:\, c = \mathbb{E}[X] = \int_0^\infty 1 - F(s)\,\mathrm{d}s\right\},$$

$$\mathfrak{F} := \bigcup_{c \in (0,\infty)} \mathfrak{F}_c.$$

The following result is derived in Mai (2018) and occupies a commanding role with regards to an analytic treatment of the stochastic properties of H, thus is repeated

here for the sake of an educational exposition. In particular, it shows that the parameterizing pair (F, L) may conveniently be characterized analytically in terms of two Bernstein functions Ψ_L (characterizing the law of L) and Ψ_F (characterizing F).

Lemma 2.2 (Bernstein functions associated with $\mathfrak{F}$). *For $F \in \mathfrak{F}$ the function*

$$\Psi_F(x) := \int_0^\infty 1 - F(s)^x \, \mathrm{d}s, \quad x \geq 0, \tag{2.2}$$

is a Bernstein function and the mapping $F \mapsto \Psi_F$ is a bijection between $\mathfrak{F}$ and the set of Bernstein functions without drift. The Lévy measure ν_F associated with Ψ_F is determined in terms of F by the equation $\nu_F((t, \infty]) = F^{-1}(\exp(-t))$, $t > 0$, where F^{-1} denotes the generalized inverse of F. The inverse mapping $\nu \mapsto F_\nu$ from the set of Lévy measures on $(0, \infty]$ to $\mathfrak{F}$ is given by

$$F_\nu(t) = \begin{cases} 0 & , \text{ if } t < \nu(\{\infty\}) \\ e^{-S_\nu^{-1}(t)} & , \text{ if } \nu(\{\infty\}) \leq t < \nu((0, \infty]) \, , \\ 1 & , \text{ else} \end{cases}$$

where S_ν^{-1} denotes the generalized inverse of $S_\nu(t) := \nu((t, \infty])$.

Proof: This is precisely Mai (2018, Lemma 3), where a detailed proof can be found. The basic idea is that under the mapping $u \mapsto -\log\{F(u)\}$ the Lebesgue measure on $(0, \infty)$ is mapped to a measure ν_F and the integral (2.2) defining Ψ_F becomes

$$\Psi_F(x) = \int_{(0, \infty]} 1 - e^{-u\,x} \, \nu_F(\mathrm{d}u).$$

The assumption $F \in \mathfrak{F}$ is equivalent to integrability of $1 - F$, which is equivalent to $\int_{(0, \infty]} \min\{x, 1\} \, \nu_F(\mathrm{d}x) < \infty$, showing that ν_F is a Lévy measure and hence Ψ_F a Bernstein function. $\qquad\square$

We always assume that F in the definition of H lies in $\mathfrak{F}$, and it is convenient to study the law of H in terms of the function

$$\ell(\vec{t}) := -\log\left(\mathbb{E}\left[e^{-\sum_{k \geq 1} H_{t_k}}\right]\right), \quad \vec{t} \in [0, \infty)_{00}^{\mathbb{N}}.$$

Furthermore, the Laplace exponent of the infinitely divisible random variable H_1 is denoted by Ψ_H in the sequel, i.e.

$$\Psi_H(x) := -\log\left(\mathbb{E}\left[e^{-x\,H_1}\right]\right), \quad x \geq 0.$$

Lemma 2.3 (Stable tail dependence function associated with H). *Let $F \in \mathfrak{F}$ and L be a driftless, non-zero Lévy subordinator. Then*

$$\ell(\vec{t}) = \int_{(0, \infty]} \int_0^\infty \left(1 - \prod_{k \geq 1} F\left(\frac{s}{t_k}\right)^y\right) \mathrm{d}s \, \nu_L(\mathrm{d}y), \quad \vec{t} \in [0, \infty)_{00}^{\mathbb{N}}.$$

Furthermore, we have the following expressions for the Bernstein function Ψ_H, which indicate a duality between the parameters L and F:

$$\Psi_H(x) = \int_{(0, \infty]} \Psi_F(x\,y) \, \nu_L(\mathrm{d}y) = \int_{(0, \infty]} \Psi_L(x\,y) \, \nu_F(\mathrm{d}y)$$

$$= \int_{(0, \infty]} \Psi_L\left(-\log\{F(u)^x\}\right) \mathrm{d}u.$$

Proof: Writing out the Riemann–Stieltjes definition of H, introducing for $N \in \mathbb{N}$ and $R \gg 1$ the notation

$$s_{n,N}^{(R)} := \frac{1}{R} + \frac{n\left(R - \frac{1}{R}\right)}{N}, \quad n = 0, \dots, N,$$

we have

$$H_t = \lim_{R \to \infty} \int_{\frac{1}{R}}^{R} -\log\left\{F\left(\frac{s}{t}-\right)\right\} \mathrm{d}L_s$$

$$= \lim_{R \to \infty} \lim_{N \to \infty} \sum_{n=1}^{N} -\log\left\{F\left(\frac{s_{n-1,N}^{(R)}}{t}-\right)\right\} \left(L_{s_{n,N}^{(R)}} - L_{s_{n-1,N}^{(R)}}\right).$$

Using the bounded convergence theorem in $(*)$ and the stationary and independent increment property of L in $(**)$ implies

$$\mathbb{E}\left[e^{-\sum_{k=1}^{d} H_{t_k}}\right]$$

$$= \mathbb{E}\left[\exp\left\{-\lim_{R \to \infty} \lim_{N \to \infty} \sum_{n=1}^{N}\left(L_{s_{n,N}^{(R)}} - L_{s_{n-1,N}^{(R)}}\right) \sum_{k=1}^{d} -\log\left[F\left(\frac{s_{n-1,N}^{(R)}}{t_k}-\right)\right]\right\}\right]$$

$$\overset{(*)}{=} \lim_{R \to \infty} \lim_{N \to \infty} \mathbb{E}\left[\exp\left\{-\sum_{n=1}^{N}\left(L_{s_{n,N}^{(R)}} - L_{s_{n-1,N}^{(R)}}\right) \sum_{k=1}^{d} -\log\left[F\left(\frac{s_{n-1,N}^{(R)}}{t_k}-\right)\right]\right\}\right]$$

$$\overset{(**)}{=} \lim_{R \to \infty} \lim_{N \to \infty} \prod_{n=1}^{N} \mathbb{E}\left[\exp\left\{-L_{\frac{R-\frac{1}{R}}{N}} \sum_{k=1}^{d} -\log\left[F\left(\frac{s_{n-1,N}^{(R)}}{t_k}-\right)\right]\right\}\right]$$

$$= \lim_{R \to \infty} \lim_{N \to \infty} \prod_{n=1}^{N} \exp\left\{-\frac{R-\frac{1}{R}}{N} \int_{(0,\infty]}\left(1 - \prod_{k=1}^{d} F\left(\frac{s_{n-1,N}^{(R)}}{t_k}-\right)^u\right) \nu_L(\mathrm{d}u)\right\}$$

$$= \lim_{R \to \infty} \exp\left\{-\int_{\frac{1}{R}}^{R} \int_{(0,\infty]}\left(1 - \prod_{k=1}^{d} F\left(\frac{s}{t_k}-\right)^u\right) \nu_L(\mathrm{d}u)\,\mathrm{d}s\right\}$$

$$= \exp\left\{-\int_{0}^{\infty} \int_{(0,\infty]}\left(1 - \prod_{k=1}^{d} F\left(\frac{s}{t_k}-\right)^u\right) \nu_L(\mathrm{d}u)\,\mathrm{d}s\right\}.$$

The order of the two remaining integrations can be switched by Tonelli's Theorem. When integrating with respect to $\mathrm{d}s$, it is further possible to change from $F(s-)$ to $F(s)$, since the at most countably many jump times of F play no role in the integration.

By a completely analogous computation we obtain the equality

$$\Psi_H(x) = \int_{(0,\infty]} \Psi_F(x\,y)\,\nu_L(\mathrm{d}y). \tag{2.3}$$

Alternatively, (2.3) also follows for $x \in \mathbb{N}$ from the formula for ℓ in the special case $t_1 = \dots = t_x = 1$ (and all other entries of $\vec{t}$ equal to zero) and for general $x \geq 0$ by the facts that (i) the right-hand side of (2.3) defines a Bernstein function by Schilling et al. (2010, Corollary 3.7) and (ii) a Bernstein function is uniquely determined by its values on $\mathbb{N}$, see Gnedin and Pitman (2008, p. 36). That the roles of L and F can be switched in (2.3) is an obvious consequence of Tonelli's Theorem. The last claimed representation is a consequence of the change of measure in the proof of Lemma 2.2. □

We view ℓ as a mapping from the set $[0,\infty)_{00}^{\mathbb{N}}$ of sequences which are eventually zero to $[0,\infty]$. The substitution $u = s/r$ shows that ℓ is homogeneous of order one, that is $\ell(r \cdot \vec{t}) = r\,\ell(\vec{t})$ for $r \geq 0$ and $\vec{t} \in [0,\infty)_{00}^{\mathbb{N}}$. The mapping ℓ specifies the law of H uniquely, i.e. its finite-dimensional distributions. This is due to the fact that the law of the random vector $(H_{t_1}, \ldots, H_{t_d})$ is uniquely determined by the values of its multivariate Laplace transform on $\mathbb{N}^d$, which follows from the Stone–Weierstrass Theorem (polynomials are dense in the space of continuous functions on $[0,1]^d$). Consequently, the mapping ℓ, which in turn by Lemma 2.3 is specified by F and ν_L, is a convenient analytical description for the law of H.

The condition $\mathbb{E}[X] > 0$ in the definition of $\mathfrak{F}$ implies $F(0) < 1$, which results in $\lim_{t\to\infty} H_t = \infty$ a.s.. The condition $\mathbb{E}[X] < \infty$ in the definition of $\mathfrak{F}$ is necessary (but needs not be sufficient) to have $\mathbb{P}(H_t < \infty) > 0$ for $t > 0$, i.e. necessary for admissibility. In order to explain this, Lemma 2.2 implies in particular that the integral $\int_0^\infty 1 - F(s)^x \, \mathrm{d}s$ is finite for all $x > 0$ if and only if it is finite for a single fixed $x > 0$. Now $\mathbb{P}(H_t < \infty) > 0$ is equivalent to $\ell(t, 0, 0, \ldots) < \infty$ and from Lemma 2.3 we see that this necessarily requires $\int_0^\infty 1 - F(s)^y \, \mathrm{d}s$ to be finite for those y on which $\nu_L(\mathrm{d}y)$ puts mass, which must be somewhere by the assumption $\nu_L \neq 0$. Hence, in particular $\mathbb{E}[X] = \int_0^\infty 1 - F(s) \, \mathrm{d}s < \infty$.

While the condition $\mathbb{E}[X] < \infty$ in the definition of $\mathfrak{F}$ is necessary to prevent the non-interesting case $H_t = \infty \cdot \mathbf{1}_{\{t>0\}}$, we have claimed that this needs not be sufficient but depends on the specific choice of L and F. To this end, we introduce the set

$$\mathfrak{F}_L := \left\{ F \in \mathfrak{F} \, : \, \Psi_H(1) = \int_{(0,\infty]} \Psi_F(y) \, \nu_L(\mathrm{d}y) < \infty \right\}$$

of L-*admissible* distribution functions.

Lemma 2.4 (Admissibility). *The following statements are equivalent.*
  (a) (F, L) *is admissible, i.e.* $\mathbb{P}(H_t < \infty) > 0$ *for some (hence all)* $t > 0$.
  (b) $F \in \mathfrak{F}_L$, *i.e.* $\Psi_H(1) < \infty$.
  (c) $\ell(\vec{t}) < \infty$ *for arbitrary* $\vec{t} \in [0,\infty)_{00}^{\mathbb{N}}$.

Proof: The equivalence of (a) and (b) is obvious, as well as the fact that (c) implies (a) and (b). The only non-obvious statement is that admissibility in (a) implies (c). Denote the biggest argument of $\vec{t}$ by $t_{[d]} := \max\{t_1, t_2, \ldots\}$ and $d := \max\{n \in \mathbb{N} : t_n > 0\}$. We compute

$$\ell(\vec{t}) \leq \int_{(0,\infty]} \int_0^\infty \left(1 - F\big(\tfrac{s}{t_{[d]}}\big)^{d\,y} \right) \mathrm{d}s \, \nu_L(\mathrm{d}y)$$

$$= t_{[d]} \int_{(0,\infty]} \Psi_F(d\,y) \, \nu_L(\mathrm{d}y) \leq t_{[d]}\, d \int_{(0,\infty]} \Psi_F(y) \, \nu_L(\mathrm{d}y) \overset{(*)}{<} \infty,$$

where $(*)$ follows from $(L$-$)$admissibility and the above inequality from the estimate $g(d\,y) \leq d\,g(y)$, which holds for any concave function g on $[0,\infty)$ with $g(0) = 0$ and $d \in \mathbb{N}$, such as $g = \Psi_F$. To see this, for $d \in \mathbb{N}$, concavity and $g(0) = 0$ imply that

$$d\,g(y) = d\,g\Big(\frac{d-1}{d} \cdot 0 + \frac{1}{d}\,d\,y\Big) \geq d\,\Big(\frac{d-1}{d}\,g(0) + \frac{1}{d}\,g(d\,y)\Big) = g(d\,y).$$

$\square$

Remark 2.5 (ν_L non-finite $\Rightarrow \nu_F(\{\infty\}) = 0$). If (F, L) is admissible and ν_L is non-finite, then necessarily $\nu_F(\{\infty\}) = 0$, which is briefly explained. Since ν_L is

non-finite, we have $\Psi_L(\infty) := \lim_{x \to \infty} \Psi_L(x) = \infty$. Admissibility implies with the help of Lemma 2.3 that

$$\infty > \Psi_H(1) = \int_{(0,\infty]} \Psi_L(y)\,\nu_F(\mathrm{d}y) \geq \Psi_L(\infty)\,\nu_F(\{\infty\}) = \infty\,\nu_F(\{\infty\}),$$

thus necessarily $\nu_F(\{\infty\}) = 0$.

Lemma 2.6 (Simple admissibility criteria).

    (a) *Let $F \in \mathfrak{F}$ and L a non-zero Lévy subordinator without drift. If Ψ_F is continuous at zero, bounded, and satisfies $\lim_{x\downarrow 0} \Psi'_F(x) < \infty$, the pair (F, L) is admissible, no matter how L is chosen.*

    (b) *Let L be a (driftless) compound Poisson subordinator with intensity $\beta > 0$ and jump size distribution $\mathbb{P}(J \in \mathrm{d}y)$, where J denotes a generic jump size random variable on $(0, \infty]$. Then*

$$\mathfrak{F}_L = \Big\{ F \in \mathfrak{F} \, : \, \mathbb{E}\big[\Psi_F(J)\big] < \infty \Big\}.$$

    In particular, if $\mathbb{E}[J] < \infty$ we have $\mathfrak{F}_L = \mathfrak{F}$.

Proof:

    (a) By the assumptions on Ψ_F there exists $c > 0$ such that $\Psi_F(x) \leq c\,\min\{x, 1\}$ for all $x \geq 0$. We thus observe

$$\Psi_H(1) = \int_{(0,\infty]} \Psi_F(x)\,\nu_L(\mathrm{d}x) \leq c \int_{(0,\infty]} \min\{x, 1\}\,\nu_L(\mathrm{d}x) < \infty.$$

    (b) The claim follows from $\nu_L(\mathrm{d}y) = \beta\,\mathbb{P}(J \in \mathrm{d}y)$ and the fact that $\Psi_F(x) \leq x\,\Psi_F(1)$ for $x \geq 1$ by concavity of Ψ_F.

$$\square$$

2.2. *Distributional properties.* Lemma 2.7 derives an alternative stochastic representation of the process $\{H_t\}_{t\geq0}$. This representation constitutes a particular special case of Kopp and Molchanov (2018, Theorem 4.2). To this end, recall from Lemma 2.4 that for an admissible pair (F, L) the measure $\Psi_F(z)\,\nu_L(\mathrm{d}z)/\Psi_H(1)$ defines a probability measure on $(0, \infty]$.

Lemma 2.7 (LePage series representation). *Let (F, L) be an admissible pair, i.e. $F \in \mathfrak{F}_L$. Let $\{Z_k\}_{k\geq1}$ be an iid sequence drawn from the probability measure $\frac{\Psi_F(z)}{\Psi_H(1)}\,\nu_L(\mathrm{d}z)$. Independently, let $\{\epsilon_k\}_{k\geq1}$ be an iid sequence of unit exponential random variables. We then have the following equality in distribution:*

$$\{H_t\}_{t\geq0} \overset{d}{=} \Big\{ -\sum_{k\geq1} \log\Big[F\Big(\frac{(\epsilon_1 + \ldots + \epsilon_k)\,\Psi_F(Z_k)}{t\,\Psi_H(1)} - \Big)^{Z_k}\Big]\Big\}_{t\geq0}.$$

Proof: Denoting by $\delta_{(x,z)}$ the Dirac measure at a point (x, z) in the plane, we note that $P := \sum_{k\geq1} \delta_{(\epsilon_1 + \ldots + \epsilon_k, Z_k)}$ is a Poisson random measure on $[0, \infty) \times (0, \infty]$ with mean measure $\mathrm{d}x \times \big(\frac{\Psi_F(z)}{\Psi_H(1)}\,\nu_L(\mathrm{d}z)\big)$ by Resnick (1987, Proposition 3.8). Consequently, denoting the stochastic process on the right-hand side of the claim by

$\{\tilde{H}_t\}_{t\geq 0}$, the Laplace functional formula for Poisson random measure Resnick (1987, Proposition 3.6) yields

$$\mathbb{E}\Big[e^{-\sum_{j\geq 1}\tilde{H}_{t_j}}\Big] = \exp\Big(-\int_{(0,\infty]}\int_0^\infty\Big(1-\prod_{j\geq 1}F\Big(\frac{x}{t_j}-\Big)^z\Big)\,\mathrm{d}x\,\nu_L(\mathrm{d}z)\Big)$$

$$= e^{-\ell(\vec{t})} = \mathbb{E}\Big[e^{-\sum_{j\geq 1}H_{t_j}}\Big]$$

for arbitrary $\vec{t}\in[0,\infty)_{00}^{\mathbb{N}}$, establishing the claim. □

Example 2.8 (The special case of a Lévy subordinator). Suppose that $F(x) = \exp(-1)+(1-\exp(-1))\,\mathbf{1}_{[1,\infty)}(x)$ is the cdf of a Bernoulli distribution with associated Bernstein function

$$\Psi_F(x) = \int_0^\infty 1 - F(s)^x\,\mathrm{d}s = 1-e^{-x},\quad x\geq 0.$$

An arbitrary (driftless) Lévy subordinator L leads to an admissible pair (F,L) by Lemma 2.6 and obviously we have $H = L$. So Lemma 2.7 provides an infinite series representation for an arbitrary Lévy subordinator L, namely

$$\{L_t\}_{t\geq 0} = \Big\{\sum_{k\geq 1} Z_k\,\mathbf{1}_{\big\{(\epsilon_1+\ldots+\epsilon_k)\frac{1-e^{-Z_k}}{\Psi_L(1)}\leq t\big\}}\Big\}_{t\geq 0}. \tag{2.4}$$

In the special case when $L = N$ is a standard (unit intensity) Poisson process, this formula boils down to the well-known counting process representation

$$\{N_t\}_{t\geq 0} = \Big\{\sum_{k\geq 1}\mathbf{1}_{\{\epsilon_1+\ldots+\epsilon_k\leq t\}}\Big\}_{t\geq 0}. \tag{2.5}$$

Representation (2.4) is a quite natural generalization of (2.5), and by Lemma 2.7 it is general enough to comprise all Lévy subordinators. In particular, it is worth mentioning that the probability law of the Z_k is $(1-\exp(-z))\,\nu_L(\mathrm{d}z)/\Psi_L(1)$, which can be any probability law on $(0,\infty]$. This means that, conversely, if ρ is an arbitrary probability law on $(0,\infty]$ and $\{Z_k\}_{k\geq 1}$ is an iid sequence drawn from ρ, (2.4) defines a Lévy subordinator L without drift and with associated Lévy measure $\nu_L(\mathrm{d}z) = (1-\exp(-z))^{-1}\,\rho(\mathrm{d}z)$.

Remark 2.9 (Series representation for ID$[0,\infty]$). Example 2.8 shows that Lemma 2.7 provides a series representation for infinitely divisible laws on $[0,\infty]$ (abbreviated ID$[0,\infty]$ in the sequel), namely

$$L_1 \stackrel{d}{=} \sum_{k\geq 1} Z_k\,\mathbf{1}_{\big\{(\epsilon_1+\ldots+\epsilon_k)\big(1-e^{-Z_k}\big)\leq\Psi_L(1)\big\}},$$

where $\{Z_k\}_{k\geq 1}$ is an iid sequence distributed according to the probability measure $(1-e^{-z})\,\nu_L(\mathrm{d}z)/\Psi_L(1)$, independent of $\{\epsilon_k\}_{k\geq 1}$. This representation is always an infinite series, even when L_1 has a compound Poisson distribution.

Recall that an element of ID$[0,\infty]$ is of compound Poisson type if its associated Bernstein function Ψ is bounded, i.e. $\lim_{x\to\infty}\Psi(x) < \infty$.

Lemma 2.10 (When does H_t have a compound Poisson distribution?). *Let (F,L) be admissible. The Bernstein function Ψ_H is bounded if and only if the following two conditions are satisfied:*

(a) Ψ_L *is bounded, i.e. L is a compound Poisson subordinator.*

(b) Ψ_F is bounded, i.e. the random variable $X \sim F$ has bounded support.

Proof: Recall from Lemma 2.3 that Ψ_H may be represented as

$$\Psi_H(x) = \int_0^\infty \Psi_L\Big(-\log\big(F(s)^x\big)\Big)\,\mathrm{d}s. \tag{2.6}$$

The sufficiency of the conditions (a) and (b) is clear from this representation, since Ψ_L is bounded and the improper integral $\int_0^\infty [\ldots]\,\mathrm{d}s$ is actually a finite integral $\int_0^{u_F}[\ldots]\,\mathrm{d}s$, where u_F is the right-end point of the support of $\mathrm{d}F$. Necessity is more difficult to observe. To this end, assume that Ψ_L is unbounded. There is an $\epsilon > 0$ and a $\delta > 0$ such that $f(z) := -\log(F(z)) > \delta$ for $z \in (0,\epsilon)$ by the condition $\mathbb{E}[X] > 0$ in the definition of $\mathfrak{F}$. Consequently,

$$\Psi_H(x) \geq \int_0^\epsilon \Psi_L(x\,\delta)\,\mathrm{d}s = \Psi_L(x\,\delta)\,\epsilon$$

is obviously unbounded. Hence, we already see that Ψ_L needs to be bounded. In this case, L is of compound Poisson type, i.e. there is a Laplace transform φ of a positive random variable on $(0,\infty]$ and a number $\beta > 0$ such that $\Psi_L = \beta\,(1-\varphi)$. Consequently, we observe from (2.6) that

$$\Psi_H(x) = \beta \int_0^\infty \Big(1 - \varphi\big(x\,f(s)\big)\Big)\,\mathrm{d}s.$$

By assumption, we know that Ψ_H is bounded, so we see with the help of Fatou's Lemma that

$$\infty > \lim_{x\to\infty} \Psi_H(x) = \lim_{x\to\infty} \beta \int_0^\infty \Big(1 - \varphi\big(x\,f(s)\big)\Big)\,\mathrm{d}s$$

$$\geq \beta \int_0^\infty \Big(1 - \liminf_{x\to\infty} \varphi\big(x\,f(s)\big)\Big)\,\mathrm{d}s.$$

But φ tends to zero as $u \to \infty$, since it is the Laplace transform of a positive random variable. Consequently, the assumption $f(z) > 0$ for arbitrarily large $z > 0$ leads to the contradiction $\infty > \infty$. Rather, we must have that $f(z) = 0$ for $z \geq T$ with some finite T, which turns the last inequality into

$$\infty > \beta \int_0^T 1\,\mathrm{d}s = \beta\,T,$$

and is not a contradiction. $\qquad\square$

Recall that an element of $\mathrm{ID}[0,\infty]$ is said to *have killing*, if it assigns positive mass to $\{\infty\}$, which is the case if and only if its associated Lévy measure ν satisfies $\nu(\{\infty\}) > 0$. In terms of the associated Bernstein function Ψ, this means $\Psi(x) > \epsilon > 0$ for all positive $x > 0$. In this case, we also say that the Bernstein function Ψ *has killing*.

Lemma 2.11 (When does H_t have a positive killing rate?). *Let (F,L) be admissible. The Bernstein function Ψ_H has killing if and only if at least one of the following two conditions is satisfied:*

(a) Ψ_F *has killing, i.e. the left end point of the support of $X \sim F$ is strictly positive.*

(b) Ψ_L *has killing, i.e. $\nu_L(\{\infty\}) > 0$.*

In this case, if Ψ_F (resp. Ψ_L) has killing, then necessarily ν_L (resp. ν_F) is finite.

Proof: Sufficiency of the conditions (a) and (b): By Lemma 2.2, Ψ_F has killing if and only if the left-end point of the support of $X \sim F$ is strictly positive. In this case, $\Psi_F(x) > \epsilon > 0$ for all $x > 0$ and one immediately observes from (2.3) that H has killing in this case (since L, hence ν_L, is non-zero by assumption) and ν_L is finite. Also, if Ψ_L has killing, $\Psi_H(x) > \epsilon > 0$ from Lemma 2.3 for all $x > 0$, hence H has killing and ν_F is finite.

Necessity of the conditions (a) and (b): Assume that neither (a) nor (b) holds. Equation (2.3) can be re-written as

$$\Psi_H(x) = \int_{(0,\infty]} \int_{(0,\infty]} \left(1 - e^{-x\,u\,s}\right) \nu_F(\mathrm{d}s)\, \nu_L(\mathrm{d}u)$$

$$= \int_{(0,\infty)} \int_{(0,\infty)} \left(1 - e^{-x\,u\,s}\right) \nu_F(\mathrm{d}s)\, \nu_L(\mathrm{d}u),$$

where the last equality follows from the assumptions that $\nu_L(\{\infty\}) = \nu_F(\{\infty\}) = 0$. Taking the limit as $x \searrow 0$ on both sides of this equation, and using the bounded convergence theorem, it follows that $\lim_{x \searrow 0} \Psi_H(x) = 0$, so H has no killing.   □

Since we are only interested in a description of the probability law of $H = H^{(F,L)}$, it is helpful to briefly ponder on potential redundancies, i.e. to investigate the question: *When do two different pairs $(F, L) \neq (\tilde{F}, \tilde{L})$ lead to exactly the same probability law of the associated processes H?* To address this issue in a mathematically rigorous manner, we introduce the equivalence relation

$$(F, L) \sim (\tilde{F}, \tilde{L}) :\Leftrightarrow \text{ the law of } H^{(F,L)} \text{ equals that of } H^{(\tilde{F}, \tilde{L})}.$$

The equivalence class of an admissible pair (F, L) is denoted by $[F, L]$, or also by $[F, \Psi_L]$, in the following. Unfortunately, we did not accomplish to find a convenient analytical description of the equivalence class $[F, L]$ in general, so this remains an interesting open problem. It is not difficult, however, to see that for an admissible pair (F, L) the equivalence class $[F(c_1\,.)^{c_2}, c_1\,\Psi_L(./c_2)]$ is invariant with respect to $c_1, c_2 > 0$. Depending on the admissible pair, however, there can be even more redundancies, and the following example demonstrates this quite extremely.

Example 2.12 (The curious case $F = $ Fréchet distribution). With a parameter $\theta \in (0, 1)$ and $c_\theta := \Gamma(1 - \theta)^{-1/\theta}$ consider the Fréchet distribution function $F(x) := \exp(-c_\theta\, x^{-1/\theta}) \mathbf{1}_{\{x > 0\}}$ and observe that $\Psi_F(x) = x^\theta$. For arbitrary $y > 0$ it is not difficult to compute

$$\int_0^\infty \left(1 - \prod_{k \geq 1} F\left(\frac{s}{t_k}\right)^y\right) \mathrm{d}s = y^\theta \left(\sum_{k \geq 1} t_k^{\frac{1}{\theta}}\right)^\theta, \quad \vec{t} \in [0, \infty)_{00}^{\mathbb{N}}.$$

Let L be a Lévy subordinator (without drift) and such that $F \in \mathfrak{F}_L$, i.e. such that $\Psi_H(1) = \int_{(0,\infty]} y^\theta\, \nu_L(\mathrm{d}y) < \infty$. It follows that

$$\ell(\vec{t}) = \int_{(0,\infty]} y^\theta\, \nu_L(\mathrm{d}y) \left(\sum_{k \geq 1} t_k^{\frac{1}{\theta}}\right)^\theta = \Psi_H(1) \left(\sum_{k \geq 1} t_k^{\frac{1}{\theta}}\right)^\theta.$$

Consequently, the function ℓ, hence the law of H, depends on the choice of L only via the scalar $\Psi_H(1)$. In particular, we have

$$[F, \Psi_L] = \left\{(F, \tilde{L}) \,:\, c = \int_{(0,\infty]} y^\theta\, \nu_{\tilde{L}}(\mathrm{d}y) < \infty\right\}$$

for arbitrary L satisfying $c = \int_{(0,\infty]} y^\theta \, \nu_L(\mathrm{d}y) < \infty$, so in order to study the probability law of H (up to a scaling constant c) it is sufficient to choose one particular such L. Choosing $L = N$, i.e. a standard Poisson process, we know from Mai (2018) that $\{H_t\}_{t\geq 0} \stackrel{d}{=} \{M_\theta \, t^{1/\theta}\}_{t\geq 0}$ with a θ-stable random variable M_θ. In contrast to Lévy subordinators, which have independent increments, this stochastic process looks peculiar at first glimpse. The whole path of the process is already known if one just observes H_t for one $t > 0$. This phenomenon is studied in more detail in Section 2.3.

2.3. *The natural filtration of H.*

We investigate the amount of information one can obtain by observing the process H up to some time $t > 0$. We define by $\mathcal{F}_t^H = \sigma(H_s : 0 \leq s \leq t)$ the information from observing H up to time $t > 0$, similarly we define and interpret $\mathcal{F}_t^L$. We have already seen in Example 2.12 that $\{\mathcal{F}_t^H\}_{t\geq 0}$ can be a quite trivial filtration. Here is a second motivating example with the same phenomenon, although significantly more difficult to see.

Example 2.13 (The case $L = N$ and $F(x) = 1 - \exp(-x)$). Let $L = N$ be a Poisson process with unit intensity, whose sequence of consecutive jump times we denote by $\{\tau_k\}_{k\geq 1}$, and F the distribution function of the unit exponential law. In particular, $\mathrm{d}F$ has unbounded support. One can show that $\mathcal{F}_t^H = \mathcal{F}_\infty^H$ for all $t > 0$. In words, this means that the whole path of H is determined completely by the path on $[0, t]$ for arbitrarily small $t > 0$. To see this, we show that the function

$$z \mapsto H_z = \sum_{k\geq 1} - \log\left(1 - e^{-\frac{\tau_k}{z}}\right)$$

is almost surely holomorphic on $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathcal{R}(z) > 0\}$. Since holomorphic functions on $\mathbb{C}_+$ are determined everywhere, once they are determined on a small real interval, such as $(0, t) \subset \mathbb{C}_+$ for $t > 0$, the claim follows. Why is $z \mapsto H_z$ holomorphic? Using Gilman et al. (2007, Theorem 7.2, p. 124), it is sufficient to prove that the defining series of H_z converges uniformly on all compact subsets of $\mathbb{C}_+$. Each compact subset of $\mathbb{C}_+$ is contained within a set of the form $\{z \in \mathbb{C}_+ : \epsilon \leq |z| \leq c, \mathcal{R}(z) \geq \epsilon\}$ for some constants $\infty > c > \epsilon > 0$. On this set, we have for arbitrary real $x > 0$ that

$$\left| e^{-\frac{x}{z}} \right| = e^{-\frac{x\,\mathcal{R}(z)}{|z|^2}} \leq e^{-\frac{x\,\epsilon}{c^2}}. \tag{2.7}$$

Furthermore, the law of the iterated logarithm implies that (a.s.)

$$\liminf_{k\to\infty} \frac{\tau_k - k}{\sqrt{2\,k\,\log(\log(k))}} = -1.$$

Consequently, we obtain for almost all $k \in \mathbb{N}$ the estimate

$$\tau_k = k + \sqrt{2\,k\,\log(\log(k))} \, \frac{\tau_k - k}{\sqrt{2\,k\,\log(\log(k))}} \geq k - \sqrt{2\,k\,\log(\log(k))}\,(1 + \epsilon) \geq k/2. \tag{2.8}$$

These estimates, together with the identity $-\log(1-x) = \sum_{m\geq 1} x^m/m$ for real $x \in [0,1)$, imply for n large enough that

$$\Big| \sum_{k\geq n} -\log\Big(1 - e^{-\frac{\tau_k}{z}}\Big) \Big| \leq \sum_{k\geq n}\sum_{m\geq 1} \frac{1}{m} \Big| e^{-\frac{\tau_k\, m}{z}} \Big| \overset{(2.7)}{\leq} \sum_{k\geq n}\sum_{m\geq 1} \frac{1}{m} e^{-\frac{\tau_k\, m\, \epsilon}{c^2}}$$

$$\overset{(2.8)}{\leq} \sum_{k\geq n}\sum_{m\geq 1} \frac{1}{m} e^{-\frac{k\, m\, \epsilon}{2\, c^2}} = \sum_{m\geq 1} \frac{1}{m} \frac{e^{-\frac{n\, m\, \epsilon}{2\, c^2}}}{1 - e^{-\frac{m\, \epsilon}{2\, c^2}}}$$

$$\leq \frac{1}{1 - e^{-\frac{\epsilon}{2\, c^2}}} \frac{e^{-\frac{n\, \epsilon}{2\, c^2}}}{1 - e^{-\frac{n\, \epsilon}{2\, c^2}}}.$$

Since the last expression tends to zero as $n \to \infty$, uniform convergence of the defining series on all compact subsets of $\mathbb{C}_+$ is shown.

The observation in Examples 2.12 and 2.13 is a general rule, as the following lemma shows.

Lemma 2.14 (Filtering out L from H). *Let (F, L) be admissible. We denote by $u_F = F^{-1}(1)$ the right-end point of the support of dF. Then $\mathcal{F}_t^H = \mathcal{F}_{u_F\, t}^L$ for arbitrary $t \geq 0$. In particular, if $u_F = \infty$ this implies for arbitrary $t > 0$ that*

$$\mathcal{F}_t^H = \mathcal{F}_\infty^L = \mathcal{F}_\infty^H.$$

Proof: We first assume that $u_F < \infty$, which implies that the defining integral representation (1.2) of H becomes the series representation

$$H_t = \sum_{k:\tau_k \leq u_F\, t} -\log\Big(F\big(\tfrac{\tau_k}{t}-\big)\Big) J_k, \tag{2.9}$$

where $\tau_1 < \tau_2 < \ldots$ denote the at most countably many jump times of L in $(0, u_F\, t)$ with respective jump sizes $J_1, J_2, \ldots$.

The inclusion $\mathcal{F}_t^H \subset \mathcal{F}_{u_F\, t}^L$ is immediate from (2.9).

The inclusion $\mathcal{F}_t^H \supset \mathcal{F}_{u_F\, t}^L$ requires us to recover the jump times $\{\tau_k\}$ and jump sizes $\{J_k\}$ of L up to time $u_F\, t$ from observing the process H up to time $t > 0$. This can be done inductively.

If $H_t = 0$, then $\tau_1 > u_F\, t$ and we are done. Else, we recover the first jump time by $\tau_1 := \inf\{s \in [0, t] : H_s > 0\}$ as well as $J_1 = \lim_{\epsilon \searrow 0} H_{\tau_1 + \epsilon}/ - \log\big(F\big(\tfrac{\tau_1}{\tau_1 + \epsilon}\big)\big)$.

If $H_t = -\log\big(F\big(\tfrac{\tau_1}{t}-\big)\big) J_1$, then $\tau_2 > u_F\, t$ and we are done. Else, we recover the second jump time by $\tau_2 := \inf\{s \in (\tau_1, t] : H_s + \log\big(F\big(\tfrac{\tau_1}{s}-\big)\big) J_1 > 0\}$ as well as

$$J_2 = \lim_{\epsilon \searrow 0} \frac{H_{\tau_2 + \epsilon} + \log\big(F\big(\tfrac{\tau_1}{\tau_2 + \epsilon}\big)\big) J_1}{-\log\big(F\big(\tfrac{\tau_2}{\tau_2 + \epsilon}\big)\big)}.$$

This procedure is repeated until either $\tau_n > u_F\, t$ for some $n \in \mathbb{N}$ if L is of compound Poisson type, or ad infinitum otherwise.

Before we prove the statement in the case $u_F = \infty$, we collect some further useful facts in the case $u_F < \infty$. We decompose the increments of H into two parts. To

this end, we observe for $h > 0$ using (1.3) that

$$
H_{t+h} - H_t = \int_0^{u_F} \left(L_{s\,(t+h)} - L_{u_F\,t} + L_{u_F\,t} - L_{s\,t} \right) \mathrm{d}\left(\log F(s) \right)
$$

$$
= \int_{\frac{u_F\,t}{t+h}}^{u_F} \left(L_{s\,(t+h)} - L_{u_F\,t} \right) \mathrm{d}\left(\log F(s) \right) + \int_0^{\frac{u_F\,t}{t+h}} \left(L_{s\,(t+h)} - L_{u_F\,t} \right) \mathrm{d}\left(\log F(s) \right)
$$

$$
+ \int_0^{u_F} \left(L_{u_F\,t} - L_{s\,t} \right) \mathrm{d}\left(\log F(s) \right)
$$

$$
= \underbrace{\int_{u_F\,t}^{u_F\,(t+h)} \left(L_v - L_{u_F\,t} \right) \mathrm{d}\left(\log F(\frac{v}{t+h}) \right)}_{\text{positive random variable, independent of } \mathcal{F}^L_{u_F\,t}} + \underbrace{\int_0^{u_F\,t} \left(L_{u_F\,t} - L_v \right) \mathrm{d}\left(\log \frac{F(v/t)}{F(v/(t+h))} \right)}_{\text{positive random variable, } \mathcal{F}^L_{u_F\,t}\text{-measurable}}
$$

$$
=: X_1 + X_2.
$$

We can continue to investigate X_1 and X_2 and find:

$$
X_1 \stackrel{d}{=} \int_0^{u_F} L_{x\,y}\, \mathrm{d}\left(\log F(\frac{h\,y + u_F\,t}{t+h}) \right) = - \int_0^{u_F} \log F(\frac{h\,y + u_F\,t}{h+t}-)\, \mathrm{d}L_{h\,y} \geq 0,
$$

$$
X_2 = \int_0^{u_F\,t} - \log F(\frac{v}{t+h}-)\, \mathrm{d}L_v - H_t \geq 0. \tag{2.10}
$$

Now we prove the claim assuming $u_F = \infty$. We fix $t, h > 0$ and approximate $F \in \mathfrak{F}$ by the sequence of distribution functions $F_n(x) := \mathbf{1}_{\{x \geq n\}} + \mathbf{1}_{\{x < n\}}\, F(x)$, for $n \in \mathbb{N}$, and observe that

$$
H_t^{(n)} := H_t^{(F_n, L)} = \int_0^n - \log F(s/t-)\, \mathrm{d}L_s \longrightarrow H_t = H_t^{(F,L)} \text{ for } n \to \infty.
$$

This, in turn, implies that $H_{t+h}^{(n)} - H_t^{(n)} \longrightarrow H_{t+h} - H_t$ for $n \to \infty$ (almost surely). As shown above, the random variable $H_{t+h}^{(n)} - H_t^{(n)}$ equals the sum of $X_2^{(n)}$ that is measurable with respect to $\mathcal{F}_t^{H^{(n)}}$, and another part $X_1^{(n)}$ which is independent thereof. We observe further from (2.10) that

$$
X_2^{(n)} = \int_0^\infty - \log F_n\left(\frac{v}{t+h}- \right) \mathbf{1}_{\{v < u_{F_n}\,t\}}\, \mathrm{d}L_v - H_t.
$$

Since F_n converges in a monotonically decreasing manner to F, the integrand in the last expression converges in a monotonically increasing fashion to $- \log F(v/(t+h)-)$. Thus, the monotone convergence theorem, applied pathwise, shows that $X_2^{(n)}$ converges almost surely to $H_{t+h} - H_t$. Consequently, $X_1^{(n)}$ tends to zero almost surely as $n \to \infty$. Since $X_2^{(n)}$ is measurable with respect to $\mathcal{F}_t^{H^{(n)}}$ for all n, we conclude that $H_{t+h} - H_t$ is measurable with respect to $\mathcal{F}_t^H$, finishing the argument. $\qquad\square$

Example 2.15 (The case $- \log F(x) = (1 - x)_+$). The resulting process H is called *Holmgren–Liouville convoluted subordinator* in Bender and Marquardt (2009) and has also been studied in Bernhart et al. (2015). Investing it in the present context (note that $u_F = 1$), we find for X_1, X_2 defined as in the proof of Lemma 2.14 above

that

$$\Psi_1(\alpha) := -\log\left(\mathbb{E}\left[e^{-\alpha X_1}\right]\right) = h\int_0^1 \Psi_L\left(\alpha(1 - \frac{h\,y + t}{h + t})\right)\mathrm{d}y$$

$$= h\int_0^1 \Psi_L\left(\alpha(1 - y)\,\frac{h}{h + t}\right)\mathrm{d}y = h\,\Psi_H\left(\alpha\,\frac{h}{h + t}\right),$$

$$X_2 = \frac{h}{h + t}\left(L_t - H_t\right).$$

The function Ψ_1 tends to zero as t increases to infinity, provided Ψ_L has no killing. This is intuitive in the sense that the larger t, the more we know about the increment $H_{t+h} - H_t$ for fixed $h > 0$.

Example 2.16 (The case $F(x) = \exp(-c_\theta\,x^{-1/\theta})$). Let M_θ be a positive random variable with Laplace transform $x \mapsto \exp(-x^\theta)$, $\theta \in (0,1)$, then the associated strong IDT process is given as $H_t = M_\theta\,t^{1/\theta}$, see Example 2.12. This is a peculiar stochastic process, as observing it at only one fixed $t > 0$ corresponds to knowing it everywhere. We have furthermore seen in Example 2.12 that the choice of the Lévy subordinator L is arbitrary, provided admissibility. In particular, we are free to choose a compound Poisson process with unit intensity and unit exponentially distributed jumps, i.e. $\Psi_L(x) = x/(x + 1)$, which is a convenient choice for the following considerations. We truncate F via $F_n(x) := \mathbf{1}_{\{x \geq n\}} + \mathbf{1}_{\{x < n\}}\,F(x)$ like in the proof of Lemma 2.14. Clearly, we then have $u_{F_n} = n < \infty$ and F_n converges to F pointwise. We find for $X_1^{(n)}, X_2^{(n)}$ as defined in the proof of Lemma 2.14 that

$$-\log\left(\mathbb{E}\left[e^{-\alpha X_1^{(n)}}\right]\right) = h\int_0^n \frac{c_\theta\,(h + t)^{1/\theta}}{c_\theta\,(h + t)^{1/\theta} + (h\,y + n\,t)^{1/\theta}}\,\mathrm{d}y \longrightarrow 0, \qquad (n \to \infty),$$

$$X_2^{(n)} = c_\theta\left((t + h)^{1/\theta} - t^{1/\theta}\right)\int_0^{n\,t} v^{-1/\theta}\,\mathrm{d}L_v \longrightarrow H_{t+h} - H_t, \qquad (n \to \infty),$$

both observations confirming our knowledge about H_t from Lemma 2.14.

3. Application 1: Simulation of extreme-value copulas

Throughout this section, we fix one admissible pair (F, L). The fact that ℓ in Lemma 2.3 is homogeneous of order 1 implies by virtue of Mai and Scherer (2014, Theorem 5.3) that the infinite exchangeable sequence of random variables

$$Y_k := \inf\{t > 0 : H_t > \epsilon_k\}, \quad k \in \mathbb{N}, \tag{3.1}$$

with $\{\epsilon_k\}_{k\in\mathbb{N}}$ independent unit exponentials, independent of H, is *min-stable multivariate exponential* with survival function given by

$$\mathbb{P}(Y_1 > t_1, Y_2 > t_2, \ldots) = e^{-\ell(\vec{t})}, \quad \vec{t} \in [0, \infty)_{00}^{\mathbb{N}}.$$

This means that $\min\{Y_1/t_1, Y_2/t_2, \ldots\}$ has a univariate exponential distribution with rate $\ell(\vec{t})$, not all t_k equal to zero. The exponential rate of each component Y_k equals $\Psi_H(1)$ and it is convenient to normalize it to $\Psi_H(1) = 1$, which then implies that the d-variate function $\ell_d(t_1, \ldots, t_d) := \ell(t_1, \ldots, t_d, 0, 0, \ldots)$ defines a so-called *stable tail dependence function* in dimension d. If for (F, L) the law of $H = H^{(F,L)}$ does not satisfy $\Psi_H(1) = 1$, we can always change from F to $F(c\,.)$ for some $c > 0$ such that $H^{(F(c\,.),L)}$ has this property. This c is even unique, i.e. for any given (F, L) there is a unique $c > 0$ such that $H^{(F(c\,.),L)}$ satisfies $\Psi_H(1) = 1$. Given this

normalization, the function $C_d(u_1, \ldots, u_d) := \exp\{-\ell_d(-\log(u_1), \ldots, -\log(u_d))\}$, $u_1, \ldots, u_d \in (0,1)$, defines a so-called *extreme-value copula*. For background on the latter, the reader is referred to Joe (1997); Nelsen (2006); Gudendorf and Segers (2010). Loosely speaking, extreme-value copulas are the dependence structures behind the limit of appropriately normalized componentwise maxima of independent and identically distributed random vectors, which is of paramount interest in multivariate extreme-value theory. The relationship between strong IDT processes and multivariate extreme-value theory has already been investigated in the present authors' references Mai and Scherer (2014); Bernhart et al. (2015); Mai (2018, 2019). On one hand, the stochastic model (3.1) can directly be used to simulate the random vector $\mathbf{U} := \exp(-\mathbf{Y}) \sim C_d$, where $\mathbf{Y} = (Y_1, \ldots, Y_d)$. However, due to the in general inconvenient integral definition of H in (1.2), the infinite series representation of H in Lemma 2.7, and the fact that the increments of H are typically not independent, this is a non-trivial task in general, although feasible in particular cases, an example with L being a Poisson process and support of $\mathrm{d}F$ bounded is provided in Mai (2018, Section 3.1). However, there is an alternative approach to accomplish the simulation in the general case, as described in the remainder of this section.

It is well-known from de Haan and Resnick (1977); Ressel (2013) that the stable tail-dependence function ℓ_d is uniquely associated with a probability measure on the unit simplex $S_d := \{(q_1, \ldots, q_d) \in [0,1]^d : q_1 + \ldots + q_d = 1\}$ subject to the constraint that each component has expected value $1/d$. To wit, there exists a random vector $\mathbf{Q} = (Q_1, \ldots, Q_d)$, uniquely determined in law, taking values in S_d and satisfying $\mathbb{E}[Q_k] = 1/d$ for each $k = 1, \ldots, d$, such that

$$\ell_d(t_1, \ldots, t_d) = d\,\mathbb{E}[\max\{t_1\, Q_1, \ldots, t_d\, Q_d\}], \quad t_1, \ldots, t_d \geq 0, \qquad (3.2)$$

which is called the *Pickands representation* of ℓ_d, named after Pickands III (1981). It is important to notice that $\mathbf{Q} = \mathbf{Q}^{[d]}$ depends on the dimension d. In particular, in our situation where d is arbitrary the first d components of $\mathbf{Q}^{[d+1]}$ are not equal in distribution to $\mathbf{Q}^{[d]}$, not even when re-scaled. In order to simplify notation, however, we omit to highlight this dependence on d for the rest of this section.

The simulation algorithm in Dombry et al. (2016, Algorithm 1), based on a seminal idea by Schlather (2002), shows how to simulate a random vector $\mathbf{U} = \exp(-\mathbf{Y}) \sim C_d$ exactly and efficiently, if one has at hand a simulation algorithm for the vector $\mathbf{Q}$. More precisely, it is shown that

$$\mathbf{Y} \overset{d}{=} \left(\frac{1}{Z_1^{(K)}}, \ldots, \frac{1}{Z_d^{(K)}}\right), \qquad (3.3)$$

$$\left(Z_1^{(n)}, \ldots, Z_d^{(n)}\right) \overset{d}{=} \left(\max_{k=1,\ldots,n}\left\{\frac{d\,Q_1^{(k)}}{\epsilon_1 + \ldots + \epsilon_k}\right\}, \ldots, \max_{k=1,\ldots,n}\left\{\frac{d\,Q_d^{(k)}}{\epsilon_1 + \ldots + \epsilon_k}\right\}\right), \; n \geq 1,$$

where $\{\mathbf{Q}^{(k)}\}_{k \geq 1}$ denote independent copies of $\mathbf{Q}$, independently of $\{\epsilon_k\}_{k \geq 1}$ iid unit exponentials, and K equals the smallest $n \in \mathbb{N}$ for which $d/(\epsilon_1 + \ldots + \epsilon_{n+1})$ is smaller than the minimal component of $\left(Z_1^{(n)}, \ldots, Z_d^{(n)}\right)$. Thus, deriving an efficient and exact simulation algorithm for $\mathbf{Q}$ is essentially the key to deriving an efficient and exact simulation algorithm for the extreme-value copula C_d along the model (3.3). The purpose of the present section is to demonstrate how this is possible. Concluding, one concrete application of the stochastic processes considered in the

present article is to enlarge the repertoire of extreme-value copulas for which exact
and efficient simulation strategies are available.

3.1. *The Pickands representation of ℓ_d.* We assume that $\Psi_H(1) = 1$ so that ℓ_d is a
proper stable tail dependence function. This condition implies that

$$\int_{(0,\infty]} \Psi_F(z)\,\nu_L(\mathrm{d}z) = 1,$$

so that $\Psi_F(z)\,\nu_L(\mathrm{d}z)$ is a probability measure on $(0, \infty]$. Notice that this probability
measure has already been important in Lemma 2.7. In the sequel, we show that
it also occupies a commanding role when determining the Pickands representation
of ℓ_d. Denoting by Z a generic random variable drawn from this probability law,
observe that

$$\ell_d(t_1, \ldots, t_d) = \int_{(0,\infty]} \iint_{[0,\infty)^d} \max\{t_1\,x_1, \ldots, t_d\,x_d\} \prod_{i=1}^{d} \mathrm{d}F^y(x_i)\,\nu_L(\mathrm{d}y)$$

$$= \mathbb{E}\left[\frac{1}{\Psi_F(Z)} \iint_{[0,\infty)^d} \max\{t_1\,x_1, \ldots, t_d\,x_d\} \prod_{i=1}^{d} \mathrm{d}F^Z(x_i) \right]$$

$$= d\,\mathbb{E}\left[\iint_{[0,\infty)^d} \max_{i=1,\ldots,d} \left\{ t_i\,\frac{x_i}{\sum_{k=1}^{d} x_k} \right\} \frac{1}{d} \sum_{k=1}^{d} \frac{x_k\,\mathrm{d}F^Z(x_k)}{\Psi_F(Z)} \prod_{i \neq k}^{d} \mathrm{d}F^Z(x_k) \right].$$

The important observation from this computation is that the vector

$$(q_1, \ldots, q_d) := \left(\frac{x_1}{\sum_{k=1}^{d} x_k}, \ldots, \frac{x_d}{\sum_{k=1}^{d} x_k} \right)$$

takes values in S_d and, conditioned on Z, the measure

$$\frac{1}{d} \sum_{k=1}^{d} \frac{x_k\,\mathrm{d}F^Z(x_k)}{\Psi_F(Z)} \prod_{i \neq k}^{d} \mathrm{d}F^Z(x_k)$$

is a probability measure on $[0, \infty)^d$. To see this, notice that conditioned on Z, the
measure $\frac{x\,\mathrm{d}F^Z(x)}{\Psi_F(Z)}$ is a probability measure on $[0, \infty)$, since

$$\int_{[0,\infty)} \frac{x\,\mathrm{d}F^Z(x)}{\Psi_F(Z)} = \frac{\int_0^\infty 1 - F(s)^Z\,\mathrm{d}s}{\Psi_F(Z)} = 1.$$

Consequently, we have found the unique Pickands dependence measure, as summa-
rized in the following lemma.

Lemma 3.1 (Pickands representation of ℓ_d). *A random sample of $\mathbf{Q}$, which is
uniquely determined in law by (3.2), can be drawn according to the following algo-
rithm:*

- *Draw D uniformly distributed on $\{1, \ldots, d\}$.*
- *Draw a sample of the random variable $Z \sim \Psi_F(z)\,\nu_L(\mathrm{d}z)$.*
- *Draw d independent and identically distributed random variables $X_1, \ldots, X_d$
  from the distribution function F^Z.*
- *Draw a random variable M from the probability law $x\,\mathrm{d}F^Z(x)/\Psi_F(Z)$.*

- *Compute the random vector $(W_1, \ldots, W_d)$, defined by*

$$W_k := \begin{cases} X_k & , \; if \; k = D \\ M & , \; else \end{cases}.$$

- *Return*

$$\boldsymbol{Q} := \Big(\frac{W_1}{\sum_{k=1}^d W_k}, \ldots, \frac{W_d}{\sum_{k=1}^d W_k} \Big).$$

Remark 3.2 (Expected runtime of the algorithm in Lemma 3.1). When using the stochastic representation (3.3) together with Lemma 3.1 to simulate the extreme-value copula C_d, the runtime of the simulation algorithm is random itself. However, Dombry et al. (2016, Proposition 4) shows that the expected value of K in (3.3) equals $d \, \mathbb{E}[\max\{Y_1, \ldots, Y_d\}]$, which in the present situation can be computed in closed form by

$$\mathbb{E}[K] = d \, \mathbb{E}[\max\{Y_1, \ldots, Y_d\}] = d \, \mathbb{E}\Big[\int_0^\infty \Big(1 - \big(1 - e^{-H_t}\big)^d \Big) \, \mathrm{d}t \Big]$$

$$= -d \sum_{k=1}^d \binom{d}{k} (-1)^k \int_0^\infty \mathbb{E}\Big[e^{-k\,H_t}\Big] \, \mathrm{d}t = -d \sum_{k=1}^d \binom{d}{k} (-1)^k \int_0^\infty e^{-t\,\Psi_H(k)} \, \mathrm{d}t$$

$$= -d \sum_{k=1}^d \binom{d}{k} \frac{(-1)^k}{\Psi_H(k)} \leq d^2.$$

The last estimate follows from the estimate

$$\mathbb{E}\Big[\int_0^\infty \Big(1 - \big(1 - e^{-H_t}\big)^d \Big) \, \mathrm{d}t \Big] = \mathbb{E}\big[\Psi_{1-e^{-H}}(d)\big] \leq \underbrace{\mathbb{E}[\Psi_{1-e^{-H}}(1)]}_{=1 \; by \; normalization} \; d,$$

where the inequality follows from the argument in the proof of Lemma 2.4. Since the simulation of $\boldsymbol{Q}$ in Lemma 3.1 itself is apparently of linear order in the dimension d, the total expected runtime for the exact simulation of the extreme-value copula C_d according to the representation (3.3) with the help of Lemma 3.1 has expected order between d^2 and d^3 and can be computed explicitly in terms of the Bernstein function Ψ_H.

In the sequel, we work out some concrete examples, demonstrating the versatility of Lemma 3.1.

3.2. *Examples.*

Example 3.3 (The Lévy subordinator case). Consider the distribution function $F(x) = \exp(-1) + (1 - \exp(-1)) \, \mathbf{1}_{[1,\infty)}(x)$ with associated Bernstein function

$$\Psi_F(x) = 1 - e^{-x}, \quad x \geq 0.$$

An arbitrary Lévy subordinator L leads to an admissible pair (F, L) and obviously $H = L$. Furthermore, $\Psi_H(1) = 1$ is satisfied whenever $\Psi_L(1) = 1$. It is well-known that C_d equals the survival copula of a so-called *Marshall–Olkin distribution*, named after Marshall and Olkin (1967), for details see Mai and Scherer (2009, 2011). Furthermore, it is observed for $z > 0$ that a random variable $X \sim F^z$ is Bernoulli-distributed with success probability $1 - \exp(-z)$. In order to simulate from the Pickands measure, additionally required is also a simulation algorithm for

$M \sim x \, \mathrm{d}F^z(x)/\Psi_F(z)$ with given z. With X_z a Bernoulli random variable with success probability $1 - \exp(-z)$, it is observed that

$$\mathbb{P}(M \le x) = \int_0^x \frac{s}{\Psi_F(z)} \, \mathrm{d}F^z(s) = \frac{1}{1 - e^{-z}} \, \mathbb{E}[\mathbf{1}_{\{X_z \le x\}} \, X_z] = \mathbf{1}_{\{x \ge 1\}},$$

hence $M \equiv 1$. Summarizing, the random vector $\mathbf{Q} = (Q_1, \ldots, Q_d)$ can be simulated as follows:

- Draw a random variable D which is uniformly distributed on $\{1, \ldots, d\}$.
- Draw the random variable $Z \sim (1 - \exp(-z)) \, \nu_L(\mathrm{d}z)$.
- Simulate $X_1, \ldots, X_d$ iid Bernoulli variables with success probability $1 - \exp(-Z)$.
- Compute the random vector $(W_1, \ldots, W_d)$ as

$$W_k := \begin{cases} X_k & , \text{ if } k \ne D \\ 1 & , \text{ if } k = D \end{cases}.$$

- Return $\mathbf{Q}$, where $Q_k := W_k / \left(\sum_{i=1}^d W_i \right)$, for $k = 1, \ldots, d$.

To illustrate, we consider one particular Lévy subordinator, which is of interest in the context of regenerative composition structures, see Gnedin and Pitman (2005, Section 8). Define the Lévy measure by

$$\nu_L(\mathrm{d}x) = \frac{e^{-\theta x}}{\left(1 - e^{-x}\right)^{\alpha+1}} \left\{ \alpha \, e^{-x} + \theta \left(1 - e^{-x}\right) \right\} \frac{\Gamma(2 + \theta - \alpha)}{\Gamma(\theta + 1) \, \Gamma(1 - \alpha)},$$

with $\theta \ge 0$ and $0 \le \alpha < 1$. Then

$$\Psi_L(x) = x \frac{\Gamma(x + \theta) \, \Gamma(2 + \theta - \alpha)}{\Gamma(1 + \theta) \, \Gamma(x + \theta + 1 - \alpha)}$$

and $\Psi_L(1) = 1$. The required probability distribution of Z has density

$$z \mapsto e^{-z \theta} \left(1 - e^{-z}\right)^{-\alpha} \left\{ \alpha \, e^{-z} + \theta \left(1 - e^{-z}\right) \right\} \frac{\Gamma(2 - \alpha + \theta)}{\Gamma(\theta + 1) \, \Gamma(1 - \alpha)}$$

$$= (1 - \alpha) \, g_{\theta, 2 - \alpha}(z) + \alpha \, g_{\theta + 1, 1 - \alpha}(z),$$

where $g_{p,q}$ denotes the density of $-\log(B)$ for B having a Beta distribution with parameters (p, q). Apparently, this mixture of transformed Beta distributions can be simulated exactly and efficiently, making the simulation algorithm for $(Y_1, \ldots, Y_d)$ feasible. Notice that this simulation of $(Y_1, \ldots, Y_d)$ implies an efficient alternative simulation algorithm for the random variable $\mathcal{C}_d$ in Gnedin and Pitman (2005, Section 8), as $\mathcal{C}_d$ is defined as the composition of the integer d obtained as the vector of cardinalities of the subsets of $\{Y_1, \ldots, Y_d\}$ with identical elements.

Example 3.4 (The case $L = N$). In the special case when $L = N$ is a standard Poisson process, one observes that $Z \equiv 1$ and Lemma 3.1 boils down to a result in Mai (2018, Lemma 4). It can thus be viewed as a generalization thereof.

Example 3.5 (The case $F(x) = \min\{x, 1\}$). The stable tail dependence function ℓ_d is studied in Bernhart et al. (2015, Theorem 2). However, no simulation algorithm for $\mathbf{Q}$ has been found in that reference, a gap which we now fill. It is observed that

$$\Psi_F(y) = \int_0^1 \left(1 - s^y\right) \mathrm{d}s = \frac{y}{y + 1}, \quad y \ge 0,$$

is the Bernstein function associated with a compound Poisson subordinator with unit exponential jumps and unit intensity. Hence, any L leads to an admissible pair (F, L) by Lemma 2.6. In order to ensure $\Psi_H(1) = 1$, ν_L must be normalized such that $\int_{(0,\infty]} z/(z+1)\,\nu_L(\mathrm{d}z) = 1$. For any fixed $z > 0$ the distribution function $F(x)^z = \min\{x^z, 1\}$ is trivial to simulate from via the inversion method, see Mai and Scherer (2017, p. 234). Furthermore, the distribution function of the random variable $M \sim x\,\mathrm{d}F^z(x)/\Psi_F(z)$ is given by $x \mapsto \min\{x^{z+1}, 1\}$, which is also easy to simulate from by the inversion method. Consequently, the simulation algorithm in Lemma 3.1 is straightforward to implement, whenever the Lévy subordinator L is chosen such that the probability law of Z, that is

$$\mathbb{P}(Z \in \mathrm{d}z) = \frac{z}{z+1}\,\nu_L(\mathrm{d}z),$$

can be simulated from.

Example 3.6 (The case $F(x) = \min\{\exp(x-1), 1\}$). Recall from Example 2.15 that the resulting process H is the Holmgren–Liouville convoluted subordinator considered in Bender and Marquardt (2009). The stable tail dependence function ℓ_d is computed in closed form in Bernhart et al. (2015, Theorem 1). It is given by

$$\ell_d(x_1, \ldots, x_d)$$
$$= \frac{d\,\Psi_H(d)}{\sum_{j=1}^d x_{[j]}^{-1}} - \sum_{i=1}^{d-1} \left(\frac{d-i+1}{\sum_{j=i}^d x_{[j]}^{-1}} - \frac{d-i}{\sum_{j=i+1}^d x_{[j]}^{-1}} \right) \Psi_H\Big(d-i-\sum_{j=i+1}^d \frac{x_{[i]}}{x_{[j]}}\Big),$$

where $x_{[1]} \leq \ldots \leq x_{[d]}$ is the ordered list of $x_1, \ldots, x_d$. However, no simulation algorithm for $\mathbf{Q}$ has been found in that reference, a gap which we now fill. It is observed that

$$\Psi_F(y) = \int_0^1 \left(1 - e^{y\,s - y}\right)\mathrm{d}s = 1 - \frac{1 - e^{-y}}{y}, \quad y \geq 0,$$

is the Bernstein function associated with a compound Poisson subordinator with unit intensity and jumps that are uniformly distributed on $[0, 1]$. Hence, any L leads to an admissible pair (F, L) by Lemma 2.6 and Bernhart et al. (2015, Lemma 3) shows that the Bernstein function Ψ_H runs through all Bernstein functions with non-increasing Lévy density when L is varied. In order to ensure $\Psi_H(1) = 1$, ν_L must be normalized such that $\int_{(0,\infty]} \left(1 - (1 - e^{-z})/z\right)\nu_L(\mathrm{d}z) = 1$. For any fixed $z > 0$ the distribution function $F(x)^z = \min\{\exp(z\,x - z), 1\}$ is easy to simulate from via the inversion method. Furthermore, the density f_M of the random variable $M \sim x\,\mathrm{d}F^z(x)/\Psi_F(z)$ is computed to be

$$f_M(x) = \frac{z^2\,e^{-z}}{z + e^{-z} - 1}\,x\,e^{x\,z}\,\mathbf{1}_{(0,1)}(x).$$

This density is bounded and rejection-acceptance sampling with comparison density $g(x) = 2x\mathbf{1}_{(0,1)}(x)$, which itself is easy to simulate via the inversion method, can be implemented to achieve an exact simulation scheme of M, see Mai and Scherer (2017, p. 235). Consequently, the simulation algorithm in Lemma 3.1 is straightforward to implement, whenever the Lévy subordinator L is chosen such that the probability law of Z, that is

$$\mathbb{P}(Z \in \mathrm{d}z) = \left(1 - \frac{1 - e^{-z}}{z}\right)\nu_L(\mathrm{d}z),$$

can be simulated from.

Figure 3.1 schematically visualizes those admissible pairs (F, L) for which either former literature or the present article provides knowledge about the associated extreme-value copula C_d, or about the probability distribution of H_1.
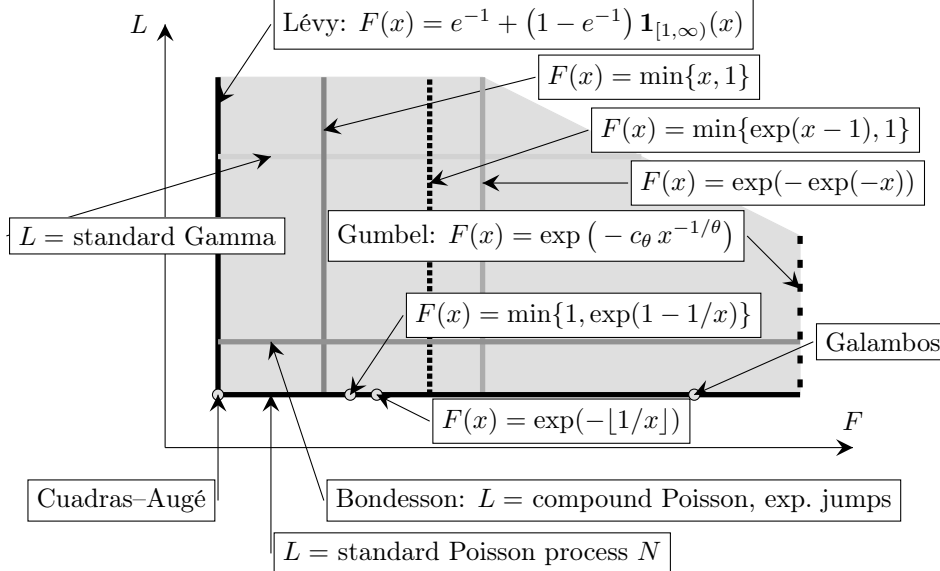

FIGURE 3.1. Illustration of pairs (F, L) occurring in previous literature or the present article. Vertical and horizontal lines of the same greyscale and line-type combination are complementary in the sense of the distributional equality (4.2). The Lévy case is covered in Example 3.3 and the link to Marshall–Olkin copulas originates from Mai and Scherer (2009, 2011). The case $F(x) = \min\{x, 1\}$ relates to Example 3.5, see Bernhart et al. (2015) for details. The law of H_1 in case $F(x) = \exp(-\exp(-x))$ is studied in Behme (2015, Proposition 3.5). The Holmgren–Liouville convoluted subordinator in the case $F(x) = \min\{\exp(x - 1), 1\}$ relates to Example 3.6 and has been studied in Bender and Marquardt (2009); James and Zhang (2011); Bernhart et al. (2015). The Gumbel case is covered in Examples 2.12 and 2.16, see also Mai (2018), the name stems from Gumbel (1960, 1961). The Galambos and the Cuadras–Augé copulas are named after Galambos (1975) and Cuadras and Augé (1981), respectively. The case of a standard Poisson process is discussed in Example 3.4 and Mai (2018). The cases $F(x) = \exp(-\lfloor 1/x \rfloor)$ or $F(x) = \min\{1, \exp(1 - 1/x)\}$ for L a standard Poisson process appear in Kopp and Molchanov (2018, Example 5.1) or Kopp and Molchanov (2018, Example 4.6), respectively. The Bondesson family relates to Lemma 4.2. The probability law of H_1 when L is a standard Gamma process is related to generalized Gamma convolutions and Dirichlet means, see James et al. (2008).

4. Application 2: Series representations for infinitely divisible laws

In this section we fix $t = 1$ and investigate the probability law of the random variable H_1, which is an element in $\mathrm{ID}[0, \infty]$. In particular, we are interested in series representations for H_1 that are suitable for efficient random number generation. We recall from Lemma 2.3 that the Laplace exponent of H_1 equals

$$\Psi_H(x) = \int_{(0,\infty]} \int_{(0,\infty]} \left(1 - e^{-x s u}\right) \nu_F(\mathrm{d}s)\, \nu_L(\mathrm{d}u), \quad x \geq 0. \qquad (4.1)$$

This double integral representation shows that the roles of F and L can be switched without changing the one-dimensional marginal distribution of H_1. More precisely,

$$\int_0^\infty -\log\left(F(s-)\right) \mathrm{d}L_s \stackrel{d}{=} \int_0^\infty -\log\left(F^{(L)}(s-)\right) \mathrm{d}L_s^{(F)}, \qquad (4.2)$$

where for a given Lévy subordinator L the function $F^{(L)} \in \mathfrak{F}$ is uniquely determined by the equality $\Psi_{F^{(L)}} = \Psi_L$, and for a given $F \in \mathfrak{F}$ the Lévy subordinator $L^{(F)}$ is uniquely determined in law by the equality $\Psi_{L^{(F)}} = \Psi_F$. Recall that the bijection between the set of Lévy measures on $(0, \infty]$ and $\mathfrak{F}$ is explicitly stated in Lemma 2.2.

The integral definition of H_1 becomes an infinite series whenever the integrator L is a (driftless) compound Poisson process, i.e. if Ψ_L is bounded. Switching the roles of L and F according to the duality (4.2), we also obtain a series representation for H_1 if $L^{(F)}$ is a (driftless) compound Poisson subordinator, i.e. if Ψ_F is bounded.

In the present section we are particularly interested in the case when L is a compound Poisson subordinator with exponentially distributed jumps. However, before we consider this case, we find it educational to briefly point out two choices for L that imply a connection of the current article to previous literature. On the one hand, if L is a standard Gamma process, i.e. $\Psi_L(x) = \log(1+x)$, the law of H_1 runs through the Thorin class when F is varied, see James et al. (2008, Propositions 1.1 and 1.6). On the other hand, the upcoming example is due to Bondesson (1982) and relates to the choice of a standard Poisson process for L.

Example 4.1 (Series representations for $\mathrm{ID}[0, \infty]$ from duality). On the left-hand side of (4.2) let $F(x) = \exp(-1) + (1 - \exp(-1))\, \mathbf{1}_{[1,\infty)}(x)$, such that $\Psi_F(x) = 1 - \exp(-x)$ and $L^{(F)} = N$ is a Poisson process with unit intensity, whose unit exponential inter-arrival times we denote by $\{\epsilon_k\}_{k \geq 1}$ and the associated jump times are $\{\tau_k\}_{k \geq 1}$. Then (4.2) becomes

$$L_1 = \int_0^\infty -\log\left(F(s-)\right) \mathrm{d}L_s \stackrel{d}{=} \sum_{k \geq 1} S_{\nu_L}^{-1}(\underbrace{\epsilon_1 + \ldots + \epsilon_k}_{\tau_k})\, \mathbf{1}_{\{\underbrace{\epsilon_1 + \ldots + \epsilon_k}_{\tau_k} \leq \nu_L((0,\infty])\}},$$

where $S_{\nu_L}(t) := \nu_L\left((t, \infty]\right)$ is the survival function of the Lévy measure ν_L of L, and $S_{\nu_L}^{-1}$ its generalized inverse. In particular, this series is almost surely finite in case of a compound Poisson distribution, i.e. if Ψ_L is bounded (resp. ν_L is finite). This is more or less the representation of an infinitely divisible law in terms of a series representation involving only independent exponentials that Bondesson (1982) proposes as a basis for his simulation ansatz (restricted to laws on $[0, \infty]$ in the present context). It is of particular use in those cases where the survival function of the Lévy measure has an inverse $S_{\nu_L}^{-1}$ in closed form. If, in addition, the Lévy measure is finite, i.e. one has a compound Poisson distribution, one obtains an exact simulation algorithm. When simulating this compound Poisson law, the

representation is useful if we have no simulation algorithm for the jump size distribution at hand, but we are able to compute the inverse of the Laplace transform of the jump size distribution in closed form.

The series representation in Example 4.1 was based on the choice $L = N$ of a Poisson process as integrator. In the sequel, we choose as integrator a compound Poisson process with unit exponential jumps. In this case, Lemma 4.2 shows that the law of H_1 lies in the so-called *Bondesson family* $\mathrm{BO}(0, \infty)$, see Bondesson (1981). This means that the survival function of the Lévy measure equals the Laplace transform of a measure ρ on $(0, \infty)$, called the *Stieltjes measure*. From an analytical viewpoint, the Bernstein function Ψ_H, which is said to be *complete* in this case, can be represented as

$$\Psi_H(x) = \int_0^\infty \frac{x}{x + u}\, \rho(\mathrm{d}u), \quad x \geq 0,$$

with the Stieltjes measure ρ on $(0, \infty)$ satisfying $\int_0^\infty (1 + u)^{-1}\, \rho(\mathrm{d}u) < \infty$. There are examples for laws in $\mathrm{BO}(0, \infty)$ for which the Stieltjes measure ρ is way more convenient to handle than the associated Lévy measure[2], we provide some examples below. For these cases, Lemma 4.2 provides a series representation in a similar spirit as that for $\mathrm{ID}[0, \infty]$ in Example 4.1.

We fix L as a compound Poisson subordinator with unit exponential jumps and unit intensity, i.e. $\Psi_L(x) = x/(x + 1)$. Furthermore, we let $F \in \mathfrak{F}$ be arbitrary, and only assume that the left-end point of the support of $X \sim F$ equals zero, which by Lemma 2.2 is equivalent to postulating $\nu_F(\{\infty\}) = 0$, i.e. Ψ_F has no killing. In this situation, the measure

$$\rho_F(A) := \nu_F\left(\left\{\frac{1}{u} \,:\, u \in A\right\}\right) \tag{4.3}$$

is well-defined for all Borel sets $A \subset (0, \infty)$, and we observe that

$$\int_0^\infty \frac{1}{1 + t}\, \rho_F(\mathrm{d}t) = \int_0^\infty \frac{u}{1 + u}\, \nu_F(\mathrm{d}u) \leq \int_0^\infty \min\{u, 1\}\, \nu_F(\mathrm{d}u) < \infty,$$

so ρ_F is a proper Stieltjes measure. To simplify notation, we denote

$$\hat{\mathfrak{F}} := \left\{F \in \mathfrak{F} \,:\, \nu_F(\{\infty\}) = 0\right\}$$

and obtain the following result. The series representation in part (c) is a special case of the series representation discussed in Bondesson (1982, p. 862).

Lemma 4.2 (Series representation in terms of the Stieltjes measure). *Let $F \in \hat{\mathfrak{F}}$ and L a compound Poisson subordinator with unit intensity and unit exponentially distributed jumps, i.e. $\Psi_L(x) = x/(x + 1)$.*

(a) *The law of H_1 is in $\mathrm{BO}(0, \infty) \subset \mathrm{ID}[0, \infty]$ with associated Stieltjes measure ρ_F.*

(b) *The mapping $F \mapsto \rho_F$, defined by (4.3) and Lemma 2.2, defines a bijection between $\hat{\mathfrak{F}}$ and $\mathrm{BO}(0, \infty)$. The inverse mapping $\rho \mapsto F_\rho$ is*

$$F_\rho(x) = \begin{cases} e^{-g_\rho^{-1}(x)} & , \text{ if } x < \rho\big((0, \infty)\big) \\ 1 & , \text{ else} \end{cases}, \quad g_\rho(x) := \rho\big((0, 1/x)\big),$$

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[2]See, e.g., Families 3, 5, 27, 28, 29, 33, 35, 45, 46, 63, 64, 88, and 89 in the list of complete Bernstein functions of Schilling et al. (2010, Chapter 15).

where g_ρ^{-1} *denotes the generalized inverse of the non-increasing function* g_ρ.

(c) *We have the following equality in law, with* $\{\epsilon_k, J_k\}_{k\geq 1}$ *iid unit exponentials:*

$$H_1 \overset{d}{=} \sum_{k\geq 1} J_k \, g_\rho^{-1}(\epsilon_1 + \ldots + \epsilon_k) \, \mathbf{1}_{\{\epsilon_1 + \ldots + \epsilon_k \leq \rho((0,\infty))\}}.$$

Proof: (a) We compute, using (4.1), $\Psi_L(x) = x/(x+1)$, and (4.3) that

$$\Psi_H(x) = \int_0^\infty \Psi_L(x\,u)\,\nu_F(\mathrm{d}u) = \int_0^\infty \frac{x\,u}{x\,u+1}\,\nu_F(\mathrm{d}u)$$
$$= \int_0^\infty \frac{x}{x+\frac{1}{u}}\,\nu_F(\mathrm{d}u) = \int_0^\infty \frac{x}{x+u}\,\rho_F(\mathrm{d}u).$$

(b) We recall from Lemma 2.2 that $F \mapsto \nu_F$ is a bijection between $\hat{\mathfrak{F}}$ and Lévy measures on $(0,\infty)$. Furthermore, it is clear that the mapping $\nu_F \mapsto \rho_F$ is injective. Left to show is only that every Stieltjes measure is attainable, i.e. that $\nu_F \mapsto \rho_F$ is surjective. To this end, it is sufficient to show for an arbitrary Stieltjes measure ρ that the measure

$$\nu(A) := \rho\Big(\Big\{\frac{1}{u} \,:\, u \in A\Big\}\Big), \quad A \subset (0,\infty) \text{ a Borel set,}$$

is a Lévy measure on $(0,\infty)$. To this end, we observe that

$$\nu\big((1,\infty)\big) = \rho\big((0,1)\big) \leq \int_0^1 \frac{2}{1+t}\,\rho(\mathrm{d}t) \leq 2\int_0^\infty \frac{1}{1+t}\,\rho(\mathrm{d}t) < \infty,$$
$$\int_0^1 u\,\nu(\mathrm{d}u) = \int_1^\infty \frac{1}{t}\,\rho(\mathrm{d}t) \leq \int_1^\infty \frac{2}{t+1}\,\rho(\mathrm{d}t) \leq 2\int_0^\infty \frac{1}{t+1}\,\rho(\mathrm{d}t) < \infty.$$

The claimed expression for F_ρ can be retrieved directly from Lemma 2.2, while noticing that $\rho_F\big((0,1/x)\big) = \nu_F\big((x,\infty)\big)$.

(c) This is a direct consequence of part (b) and the definition of the process H.

$\square$

Like the series representation in Example 4.1 for $\mathrm{ID}[0,\infty]$ is useful if the Lévy measure is nice, the representation in Lemma 4.2(c) can be used to construct simulation algorithms for infinitely divisible distributions from the Bondesson family, when the Stieltjes measure ρ is nice. In case of a compound Poisson distribution the series is even finite, hence the simulation algorithm is exact. We provide some examples to demonstrate this procedure.

Example 4.3 (Exact simulation of compound Poisson laws). Let ρ be some finite measure on $(0,\infty)$, hence it automatically is a Stieltjes measure. We denote by X a generic random variable in $\mathrm{BO}(0,\infty)$ associated with this Stieltjes measure. There exists a distribution function G of some non-negative random variable with $G(0) = 0$ and a constant $\beta := \rho\big((0,\infty)\big) > 0$ such that $\rho\big((0,x]\big) = \beta\,G(x)$. The Lévy measure ν associated with this Stieltjes measure ρ is determined by its survival function, which satisfies $\nu\big((x,\infty)\big) = \beta\,\varphi_G(x)$, where $\varphi_G(x) := \beta \int_0^\infty \exp(-x\,u)\,\mathrm{d}G(u)$ denotes the Laplace transform of $\mathrm{d}G$. We observe

$$g_\rho^{-1}(y) = \frac{1}{G^{-1}(y/\beta)}, \quad 0 < y < \beta,$$

where G^{-1} is the generalized inverse of the distribution function G. Consequently, according to Lemma 4.2(c) the compound Poisson distribution X with intensity β and jump size density $-\varphi_G'$ has the finite series representation

$$X \stackrel{d}{=} \sum_{k \geq 1} \frac{J_k}{G^{-1}\big((\epsilon_1 + \ldots + \epsilon_k)/\beta\big)} \mathbf{1}_{\{\epsilon_1 + \ldots + \epsilon_k \leq \beta\}}, \tag{4.4}$$

with $\{\epsilon_k, J_k\}_{k \geq 1}$ iid unit exponentials. It is not difficult to come up with examples for G such that G^{-1} is in closed form, but neither is a simulation algorithm for the density $-\varphi_G'$ at hand, nor is the inverse of φ_G in closed form. In such a situation, (4.4) provides a convenient basis for an exact simulation algorithm. One particular example is given by Family 45 in Schilling et al. (2010, Chapter 15): we have $\beta = 0.5$ and $G(x) = \min\{x, 1\}(2 - \min\{x, 1\})$ and obtain $G^{-1}(x) = 1 - \sqrt{1 - x}$ for $x \in (0, 1)$, leading to

$$X \stackrel{d}{=} \sum_{k \geq 1} \frac{J_k \, \mathbf{1}_{\{\epsilon_1 + \ldots + \epsilon_k \leq 0.5\}}}{1 - \sqrt{1 - 2(\epsilon_1 + \ldots + \epsilon_k)}}.$$

The associated Bernstein function is given by $\Psi(x) = x(1 + x)\log(1 + 1/x) - x$.

Example 4.4 (A few non-compound Poisson examples). With a parameter $\theta > 0$, we consider the complete, unbounded Bernstein functions

$$\Psi_1(x) = \frac{x}{\sqrt{x + \theta}}, \quad \Psi_2(x) = \frac{x}{\theta - x} \log\left(\frac{\theta}{x}\right), \quad \Psi_3(x) = \sqrt{x} \arctan\left(\sqrt{\frac{x}{\theta}}\right),$$

which correspond to Families 5, 33, and 64 in Schilling et al. (2010, Chapter 15). For each of them, the associated Stieltjes measure has a convenient form, namely

$$\rho_1(\mathrm{d}x) = \frac{\mathbf{1}_{\{x > \theta\}} \, \mathrm{d}x}{\pi \sqrt{x - \theta}}, \quad \rho_2(\mathrm{d}x) = \frac{\mathrm{d}x}{\theta + x}, \quad \rho_3(\mathrm{d}x) = \frac{\mathbf{1}_{\{x > \theta\}} \, \mathrm{d}x}{2\sqrt{x}},$$

and the associated function g_ρ^{-1} from Lemma 4.2 can be computed in closed form, to wit

$$g_{\rho_1}^{-1}(x) = \frac{1}{\theta + (\pi x/2)^2}, \quad g_{\rho_2}^{-1}(x) = \frac{1}{\theta(e^x - 1)}, \quad g_{\rho_3}^{-1}(x) = \frac{1}{(x + \sqrt{\theta})^2}.$$

5. Conclusion

We have studied a semi-parametric family of non-decreasing stochastic processes $H = \{H_t\}_{t \geq 0}$, which comprises (possibly killed) Lévy subordinators without drift as a special case. Whereas a Lévy subordinator L is conveniently specified by a Bernstein function Ψ_L, the process H is specified by a pair (Ψ_F, Ψ_L) of two Bernstein functions. From a theoretical point of view, we have demonstrated that the parameterization in terms of a distribution function F and a Lévy subordinator L provides a convenient apparatus to study distributional properties of H. In particular, we have established a canonical series representation and have studied the natural filtration of H, highlighting that the independent increment property is exclusive to the Lévy subordinator subfamily. From a practical point of view, we have explained how to simulate exactly and accurately the d-variate extreme-value copula associated with each process H. Furthermore, we have used the derived setting to establish a series representation for infinitely divisible laws from the Bondesson family that is given in terms of its Stieltjes measure only.

Acknowledgments

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