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SUBSEMIGROUPS OF SIMPLE SEMIGROUPS

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The purpose of this paper is to study the structure of subsemigroups of a simple semigroup S , especially in the case when S is completely simple. Also coset decompositions of S modulo some subsemigroups are studied.

Let S be a semigroup. A left ideal of S is a subset $L \subset S$ with $SL \subset L$. A right ideal of S is a subset R with $RS \subset R$. A subset which is both a left and right ideal of S is called a two-sided ideal of S .

The semigroup S itself and the zero element 0 (if S contains a zero element) are always two-sided ideals.

A minimal left ideal of S is a left ideal of S which does not contain any proper left ideal of S with the eventually exception of (0) (if S contains a zero element). Minimal right and two-sided ideals are defined analogously.

A semigroup S without zero and containing at least two elements is called *simple* if it does not contain any two-sided ideal $\neq S$.

A semigroup S with a zero element 0 containing at least two elements is said to be simple if it does not contain any two-sided ideal different from (0) and S itself.

Also a semigroup consisting of a single element is called simple. Of course, in this case this element can be considered as a zero element of S . Hence to avoid confusion by a semigroup with zero we shall usually mean a semigroup containing at least two elements (one of them being a zero element).

The purpose of this paper is to study subsemigroups of a simple semigroup. The question arises whether any subsemigroup of a simple semigroup is simple. In general the answer to this question is certainly negative since it is known (see R. H. BRUCK [1], p. 48 and G. B. PRESTON [6]): Any semigroup T can be embedded in a simple semigroup (with or without zero) containing a unit element. On the other hand every finite simple semigroup without zero possesses the mentioned property.

In section 1 of this paper we show that the answer to this question is positive if S is a compact simple semigroup without zero and we restrict ourselves to closed subsemigroups.

In section 2 we treat an analogous problem for simple semigroups with a zero.

Some corollaries and special cases of these results are needed in a forthcoming paper on convolution semigroups of measures on non-commutative semigroups (see [9]). Also the results of section 3 dealing with some coset decompositions of completely simple semigroups are proved for this purpose. All these results seem to be also of an independent interest. For this reason I have found it convenient to publish them separately.

1

A simple semigroup S is called *completely simple* if it contains at least one minimal left and at least one minimal right ideal of S .

Recall that if S is a completely simple semigroup without zero it can be written in the form $S = \bigcup_{\alpha \in A_1} R_\alpha = \bigcup_{\beta \in A_2} L_\beta$, where R_α and L_β run through all minimal right and left ideals of S respectively. Moreover, every minimal left ideal L_α is generated by an idempotent, i.e. there is an idempotent $e \in L_\alpha$ such that $L_\alpha = Se = L_\alpha \cdot e$. Clearly e (and every idempotent $\in L_\alpha$) is a right unit of L_α . Analogously for minimal right ideals. Also $R_\alpha L_\beta = R_\alpha \cap L_\beta$ is a group. Denoting $R_\alpha L_\beta = G_{\alpha\beta}$ we can write $S = \bigcup_{\alpha} \bigcup_{\beta} G_{\alpha\beta}$ as a union of pairwise disjoint maximal isomorphic groups $\subset S$. The $G_{\alpha\beta}$'s will be called the group-components of S .

Recall further that if S is completely simple with zero 0 (and S contains at least two elements) we have either $S^2 = 0$ or $S^2 = S$. In the first case S is of the form $S = \{0, a\}$ with $a^2 = a \cdot 0 = 0 \cdot a = 0^2 = 0$. In the second case S contains at least one idempotent $e \neq 0$ and it can be written in the form $S = \bigcup_{\beta} L_\beta$, where L_β runs through all minimal left ideals of S . For any two minimal left ideals $L_\alpha \neq L_\beta$ we have $L_\alpha \cap L_\beta = (0)$. Moreover, every minimal left ideal L_β contains at least one idempotent $\neq 0$ and it is generated by any such idempotent. Any minimal left ideal L_β of S can be written as a union of disjoint sets $L_\beta = \left\{ \bigcup_{\gamma} G_{\gamma\beta} \right\} \cup P_\beta$, where $G_{\gamma\beta}$ are (isomorphic) groups and P_β a semigroup with $P_\beta^2 = 0$. Analogous statements hold for minimal right ideals. (The proofs of all these statements can be found f.i. in the recent book of E. C. Пляпин [4].)

Remark. The restriction to completely simple semigroups though an essential one is itself not sufficient to obtain results of the kind mentioned in the introduction. If S is completely simple and T a subsemigroup of S then T need not be simple even in the case when T has an idempotent. Let f.i. S be the multiplicative group of real numbers >0 . S is then a (trivial) completely simple semigroup. Let T be the subsemigroup of real numbers ≥ 1 . Then T contains the idempotent 1 and is not simple, since f.i. the subsemigroup of all real numbers ≥ 2 is a proper ideal of T .

We first prove

Lemma 1.1. *Let S be a completely simple semigroup without zero and T a simple subsemigroup of S containing an idempotent. Then*

1. *T is completely simple. If T contains more than one element, T is completely simple without zero.*

2. *If L_α is a minimal left ideal of S , and $L'_\alpha = T \cap L_\alpha \neq \emptyset$, then L'_α is a minimal left ideal of T .*

3. *Conversely, if L'_α is a minimal left ideal of T , then there exists a uniquely determined minimal left ideal L_α of S such that $L'_\alpha = T \cap L_\alpha$.*

Proof. 1. An idempotent $e \neq 0$ of any semigroup is called primitive if there does not exist an idempotent $x \neq e$ and $x \neq 0$ such that $xe = e'z = x$.

It is known that all non-zero idempotents of a completely simple semigroup (with zero or without zero) are primitive. Further it is known (see D. REES [7]): If a simple semigroup T contains a non-zero primitive idempotent, then T is completely simple. In our case: Since T is simple and it contains an idempotent $e \in S$, T is completely simple if e is not a zero element of T or if T reduces to e .

Suppose that T contains more than one element and $e = z$ is a zero of T . We show that this case is impossible. Let be $a \in T$, $a \neq z$. By definition of a zero element we have $az = za = z$. The idempotent $z \in T$ is contained in a group-component $G_{\alpha\beta}$ of S . The element a cannot be contained in $G_{\alpha\beta}$ since otherwise $az = z^2$ (in $G_{\alpha\beta}$) would imply $a = z$. Hence $a \in G_{\gamma\delta}$ for some γ, δ and $G_{\alpha\beta} \cap G_{\gamma\delta} = \emptyset$. Denote by e' the unit element of $G_{\gamma\delta}$, hence $e' \neq z$. The relation $az = z$ implies $e'az = e'z$, $az = e'z$, $z = e'z$; analogously $za = z$ implies $zae' = ze'$, $za = ze'$, $z = ze'$. Hence $z = e'z = ze'$. But this contradicts to the fact that e' is a primitive idempotent of S .

2. Let L_α be a minimal left ideal of S and suppose $L_\alpha \cap T = L'_\alpha \neq \emptyset$. Then for $a \in T$ and $x \in L'_\alpha$ we have $ax \in aL'_\alpha \subset aL_\alpha \subset L_\alpha$, further $ax \in T$, $T \subset L_\alpha$, hence $L_\alpha \subset T \cap L_\alpha = L'_\alpha$. Therefore L'_α is a left ideal of T .

We next prove that L'_α is a minimal left ideal of T . Since T is completely simple, it is the union of its minimal left ideals. Hence L'_α is either a minimal left ideal of T or there is a minimal left ideal L of T such that $L \subsetneq L'_\alpha$. Suppose this second case. Let e be the idempotent contained in L . We have $L'_\alpha e \subset L'_\alpha L \subset L$. On the other hand every idempotent $e \in L_\alpha$ is a right unit of the semigroup L_α , hence $L'_\alpha e = L'_\alpha$. Therefore $L'_\alpha \subset L$, i.e. $L'_\alpha = L$, contrary to the supposition.

3. Write $S = \bigcup_{\beta \in A_2} L_\beta$ as the decomposition of S into the union of its minimal left ideals. Let L'_α be a minimal left ideal of T . We have

$$(1) \quad L'_\alpha = L'_\alpha \cap S = L'_\alpha \cap \left\{ \bigcup_{\beta \in A_2} L_\beta \right\} = \bigcup_{\beta \in A_2} \{L'_\alpha \cap L_\beta\}.$$

If $L_\alpha \cap L_\beta \neq \emptyset$, then $T(L_\alpha \cap L_\beta) \subset L_\alpha \cap L_\beta$, hence $L_\alpha \cap L_\beta$ is a left ideal of T . Since L'_α is minimal, there is exactly one non-empty summand on the right hand side of (1). Hence there is a unique minimal left ideal of S — say L_α , $\alpha \in \Lambda_2$, — such that $L'_\alpha = L_\alpha \cap L_\beta$. We have $L'_\alpha \subset T$, $L'_\alpha \subset L_\alpha$, hence $L'_\alpha \subset T \cap L_\alpha$. On the other hand it has been proved above that $T \cap L_\alpha$ is a minimal left ideal of T , therefore $L'_\alpha = T \cap L_\alpha$, q.e.d.

Remark. The following problem arises. Does there exist a completely simple semigroup S containing a simple subsemigroup T without idempotents (hence T simple but not completely simple)?

Write $S = \bigcup_{\gamma \in \Lambda_1} \bigcup_{\delta \in \Lambda_2} G_{\gamma\delta}$ and suppose that $T \subset S$ is simple. Then there is at least one couple $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$ such that $T \cap G_{\alpha\beta} = P_{\alpha\beta} \neq \emptyset$. We show that the semigroup $P_{\alpha\beta}$ itself must be a simple semigroup. Choose $a, b \in P_{\alpha\beta}$. We then also have $a^3 \in P_{\alpha\beta} \subset T$. Since T is simple, there exist $x, y \in T$ such that $xa^3y = b$. If $x \in R_\gamma$ for some γ , we have $b = xa^3y \in R_\gamma a^3 y \subset R_\gamma$, therefore $\gamma = \alpha$; hence $x \in R_\alpha$ and $x \in T \cap R_\alpha$. Analogously $y \in T \cap L_\beta$. Further we have $\xi = xa \in R_\alpha G_{\alpha\beta} = R_\alpha R_\alpha L_\beta = R_\alpha L_\beta = G_{\alpha\beta}$, and $\xi = xa \in T$. $T = T$, hence $\xi \in T \cap G_{\alpha\beta} = P_{\alpha\beta}$. Analogously $\eta = ay \in P_{\alpha\beta}$. Since $(xa)a(ay) = \xi a \eta = b$, this implies: To every couple $a, b \in P_{\alpha\beta}$ there exist elements $\xi, \eta \in P_{\alpha\beta}$ such that $\xi a \eta = b$. Hence $P_{\alpha\beta}$ is simple. (Note also that $P_{\alpha\beta}$ being a subset of $G_{\alpha\beta}$ satisfies the left and right cancellation law.)

We have proved: *If there is a completely simple semigroup containing a simple subsemigroup without idempotents, there must exist a group G such that G contains a simple subsemigroup H without idempotents.* Conversely, the existence of such a group G would give a positive answer to our question, since G is itself a (trivial) completely simple semigroup.

Now such groups really exist. Consider f.i. the set G of all couples (a, b) of real numbers with $a \neq 0$ and introduce in G a multiplication by $(a, b) \odot (c, d) = (ac, bc + d)$. Then G is a group with $(1, 0)$ as unit element. Next let S be the subset of all couples (a, b) with $a > 0$, $b > 0$. Then S is easily seen to be a simple semigroup (without idempotents). This shows that the assumption in Lemma 1.1 that T contains an idempotent is an essential one. {Analogous examples due to O. ANDERSEN can be found in the recent book [3], p. 51.}

If $G_{\alpha\beta}$ is commutative (and hence all $G_{\gamma\delta}$ commutative), then $P_{\alpha\beta}$ is commutative and since a simple commutative semigroup without zero is a group, $P_{\alpha\beta}$ is a group. This implies:

Corollary 1.1. *Let S be completely simple without zero and let the group-components of S be commutative. Then every simple subsemigroup of S is completely simple.*

The following lemma is well known. We prove it only for the sake of completeness (since we shall use it several times).

Lemma 1.2. *Let S be a semigroup without zero containing a minimal left ideal and suppose that S is the union of its minimal left ideals. Then S is a simple semigroup.*

Proof. Write $S = \bigcup_{\alpha \in A_2} L_\alpha$, where L_α runs through all minimal left ideals of S . Suppose that N is a two-sided ideal of S , $N \subset S$. Then $NS \subset N$. On the other hand $NS = N\{\bigcup_{\alpha} L_\alpha\} = \bigcup_{\alpha} NL_\alpha$, and since (with respect to the minimality of L_α) $NL_\alpha = L_\alpha$, we have $NS = \bigcup_{\alpha} L_\alpha = S$, i.e. $S \subset N$, therefore $S = N$. This proves our lemma.

Now we introduce a topological restriction. If S is a semigroup and at the same time a topological space, and the multiplication is continuous, S is called a topological semigroup. Supposing that S is a Hausdorff compact space the corresponding semigroup will be called a compact semigroup. It is known that any compact semigroup contains always at least one idempotent. Moreover if S has not a zero, there exists at least one (non-zero) minimal left ideal and at least one (non-zero) minimal right ideal, and the minimal ideals are closed. This implies (see K. NUMAKURA [5], p. 103): A compact simple semigroup without zero is completely simple.

We now prove:

Theorem 1.1. *Let S be a compact simple semigroup without zero. Then every closed subsemigroup T of S is a completely simple semigroup.*

Remark. In the finite case this was proved by A. K. СУШКЕВИЧ ([10], p. 59).

Proof. Let $S = \bigcup_{\alpha \in A_2} L_\alpha$ be the decomposition of S into the union of its minimal left ideals. Consider the set of all left ideals L_α for which $L_\alpha \cap T \neq \emptyset$. Let this set be $\{L_\gamma, \gamma \in A'_2\}$. For every $\gamma \in A'_2$ denote $L_\gamma \cap T = L'_\gamma$.

The closed subset $L'_\gamma \neq \emptyset$ is a left ideal of T . In fact, for every $a \in T$ and every $x \in L'_\gamma$ we have $ax \in aL'_\gamma \subset SL_\gamma = L_\gamma$, further $ax \in T$. $T \subset T$, hence $ax \in T \cap L_\gamma = L'_\gamma$.

We next show that L'_γ is a minimal left ideal of T . Let L' be a left ideal of T and suppose that $L' \subset L'_\gamma$. Since T is closed and hence compact (in the relative topology), T is a compact semigroup and there is necessarily a minimal left ideal L^* of T with $L^* \subset L'$. Again L^* is closed (hence compact), therefore it contains an idempotent e . We have $e \in L^* \subset L' \subset L'_\gamma \subset L_\gamma$. In particular $L'_\gamma e \subset L'_\gamma L^* \subset TL^* \subset L^*$. On the other hand every idempotent $e \in L_\gamma$ is a right unit of the semigroup L_γ , hence $L'_\gamma e = L'_\gamma$. Therefore $L'_\gamma \subset L^*$ and $L^* = L'_\gamma$. This proves that L'_γ is a minimal left ideal of T .

It follows from $T = T \cap S = \bigcup_{\gamma \in A'_2} L'_\gamma$ that T is the union of its minimal left ideals. By Lemma 1.2 we conclude that T is a simple semigroup. Since we can prove by the same argument that T contains also a minimal right ideal of T , T is completely simple.

Remark. The following (trivial) example shows that the supposition that T is closed cannot be – in general – dropped. Let G be the group of complex numbers

$\{z \mid |z| = 1\}$ in the obvious topology, $z_0 = e^{2\pi i \vartheta}$ with an irrational ϑ and $T = \{z_0^n \mid n = 1, 2, \dots\}$. The sub-semigroup T is clearly not simple, since f.i. $T^2 \neq T$. Also the subsemigroup $T_0 = T \cup \{1\}$ (containing an idempotent) is not simple since it contains an infinity of ideals of T_0 .

Lemma 1,3. *Let S be a completely simple semigroup without zero and T a simple subsemigroup of S containing an idempotent. Then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having the same idempotents as T . The semigroup T_1 can be written in the form $T_1 = \left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} \cap \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\}$ with suitably chosen minimal right and left ideals R_α, L_β of S respectively.*

Proof. By Lemma 1,1 T is completely simple. Therefore we may write $T = \bigcup_{\alpha \in A'_1} R'_\alpha = \bigcup_{\beta \in A'_2} L'_\beta$, where R'_α, L'_β run through all minimal left and right ideals of T respectively.

Choose — in the sense of Lemma 1,1 — R_α, L_β such that $R'_\alpha = R_\alpha \cap T, L'_\beta = L_\beta \cap T$. We have

$$\begin{aligned} T = T^2 &= \left\{ \bigcup_{\alpha \in A'_1} R'_\alpha \right\} \cdot \left\{ \bigcup_{\beta \in A'_2} L'_\beta \right\} = \bigcup_{\alpha} \bigcup_{\beta} \{R'_\alpha L'_\beta\} \subset \bigcup_{\alpha \in A'_1} \bigcup_{\beta \in A'_2} \{R_\alpha L_\beta\} = \\ &= \left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} \cap \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\} = \bigcup_{\alpha \in A'_1} \bigcup_{\beta \in A'_2} G_{\alpha\beta} = T_1, \end{aligned}$$

and T_1 is a semigroup which contains exactly the same idempotents as T .

For $\beta \in A'_2$ denote $L''_\beta = L_\beta \cap T_1$. We prove that L''_β is a minimal left ideal of T_1 . If $x \in L''_\beta, a \in T_1$, we have $ax \in aL''_\beta \subset aL_\beta \subset L_\beta$, further $ax \in T_1 \cdot T_1 \subset T_1$, hence $ax \in T_1 \cap L_\beta = L''_\beta$. This proves that L''_β is a left ideal of T_1 . Now

$$\begin{aligned} L''_\beta &= L_\beta \cap T_1 = L_\beta \cap \left\{ \left(\bigcup_{\alpha \in A'_1} R_\alpha \right) \cap \left(\bigcup_{\beta \in A'_2} L_\beta \right) \right\} = L_\beta \cap \left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} = \\ &= \left\{ \bigcup_{\alpha \in A'_1} \{L_\beta \cap R_\alpha\} \right\} = \bigcup_{\alpha \in A'_1} G_{\alpha\beta}. \end{aligned}$$

Suppose that L is a left ideal of T_1 and $L \subset L''_\beta$. Then there is at least one summand $G_{\alpha_0\beta}, \alpha_0 \in A'_1$ with $G_{\alpha_0\beta} \cap L \neq \emptyset$. Now a left ideal of any semigroup which has a non-empty intersection with a subgroup contains the whole subgroup. Therefore $G_{\alpha_0\beta} \subset L$. In particular L contains the idempotent $e_{\alpha_0\beta}$. We have $e_{\alpha_0\beta} \in L \subset L''_\beta \subset L_\beta$ and $L''_\beta e_{\alpha_0\beta} \subset L''_\beta L \subset L$. On the other hand every idempotent $\in L_\beta$ is a right unit of L_β , in particular $L''_\beta e_{\alpha_0\beta} = L''_\beta$. Therefore $L''_\beta \subset L$, i.e. $L = L''_\beta$. This proves that L''_β is a minimal left ideal of T_1 . Analogously we prove that for $\alpha \in A'_1$ $R_\alpha \cap T_1 = R''_\alpha$ is a minimal right ideal of T_1 .

Since $T_1 = \left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} \cap \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\} = T_1 \cap \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\} = \bigcup_{\beta \in A'_2} \{L_\beta \cap T_1\} = \bigcup_{\beta \in A'_2} L''_\beta$ and T_1 contains also a minimal right ideal, it follows by Lemma 1,2 that T_1 is a completely simple semigroup.

Let now finally T_2 be a simple subsemigroup of S with $T_2 \supset T$. T_1 is a union of disjoint maximal groups of S . Since also T_2 is completely simple, it is also a union

of groups. The maximal groups belonging to different idempotents are disjoint. If there were $T_2 - T_1 \neq \emptyset$, T_2 would contain at least one group-component, and hence at least one idempotent, not contained in T_1 . This contradiction proves the maximality of T_1 and completes the proof of our theorem.

The suppositions of Lemma 1,3 hold if, in particular, S is compact and simple, and T is any closed subsemigroup of S . In this case T_1 is closed. To prove this denote $A = \bigcup_{\alpha \in A'_1} R_\alpha$, $B = \bigcup_{\beta \in A'_2} L_\beta$. Then $T_1 = A \cap B$ and it is sufficient to prove that A and B are closed. Since S is the union of all minimal right ideals of S , $S - A$ is clearly the largest right ideal of S that does not meet T , hence the largest right ideal contained in the open set $S - T$. Therefore it is sufficient to prove that the largest right ideal R^* of S contained in $S - T$ is open. Let $x \in R^*$. Since R^* is a right ideal, we have $x \cup xS \subset R^* \subset S - T$. Since $S - T$ is open, we may apply a lemma of A. D. WALLACE ([11], Lemma 1) which says that there is an open set V about x such that $V \cup VS \subset S - T$. Now since $V \cup VS$ is a right ideal, we have $V \cup VS \subset R^*$, hence $V \subset R^*$. This proves the following

Corollary 1,3. *If S is compact simple, $T \subset S$ closed, then the maximal subsemigroup T_1 of S having the same idempotents as T is closed.*

If in Lemma 1,3 the subsemigroup T contains a maximal group of S , then all group-components of T are maximal groups of S and $T = T_1$.

This combined with Theorem 1,1, Lemma 1,3 and Corollary 1,3 gives

Theorem 1,2. *If S is a compact simple semigroup without zero and T a closed subsemigroup of S , then there exists a unique greatest subsemigroup $T_1 \supset T$ having the same idempotents as T . The semigroup T_1 is closed and completely simple and it can be written in the form $T_1 = \{\bigcup_{\alpha} R_\alpha\} \cap \{\bigcup_{\beta} L_\beta\}$ with suitably chosen minimal right and left ideals R_α and L_β of S respectively. If, moreover, T contains a maximal group of S , then $T = T_1$.*

2

In this section we shall try to find analogous results for completely simple semigroups with zero.

If S is compact simple with zero, and T a closed subsemigroup of S , T need not be simple even in the case when T contains an idempotent $\neq 0$. This can be shown on the example $S = \{0, a_1, a_2, a_3, a_4\}$ with the multiplication table

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	a_2	0	0
a_2	0	0	0	a_1	a_2
a_3	0	a_3	a_4	0	0
a_4	0	0	0	a_3	a_4

The subsemigroup $T = \{0, a_1, a_2\}$ contains the idempotent $a_1 \neq 0$ but T is not a simple semigroup since $\{0, a_2\}$ is clearly a proper two-sided ideal of T different from (0) and T .

Lemma 2,1. *Let S be a completely simple semigroup with zero 0 for which $S^2 \neq 0$. Let T be a simple subsemigroup of S containing an idempotent but not containing the zero element 0 . Then*

1. *T is a completely simple semigroup. If $\text{card } T > 1$, T is a completely simple semigroup without zero.*

2. *If L_α is a minimal left ideal of S and $T \cap L_\alpha = L'_\alpha \neq \emptyset$, then L'_α is a minimal left ideal of T .*

3. *Conversely, if L'_α is a minimal left ideal of T , then there exists a uniquely determined minimal left ideal L_α of S such that $L'_\alpha = T \cap L_\alpha$.*

Proof. 1. If T reduces to a single element, T is completely simple. If T has more than one element, then since T is simple and contains an idempotent e , T is completely simple if e is not a zero element of T . For in this case e (being a primitive idempotent of S) is also a primitive idempotent of T and we may use the known result mentioned at the beginning of the proof of Lemma 1,1.

We show that the case that T contains at least two elements and $e = z$ is the zero element of T is impossible. Let be $a \in T$, $a \neq z$. Write S as the union of minimal left ideals $S = \bigcup_{\beta} L_{\beta}$ and suppose $a \in L_{\beta}$. As remarked above, L_{β} can be written in the form $L_{\beta} = \left\{ \bigcup_{\alpha} G_{\alpha\beta} \right\} \cup P_{\beta}$, where $G_{\alpha\beta}$ are maximal groups of S and P_{β} is a semigroup with $P_{\beta}^2 = 0$. The element a is not contained in P_{β} since otherwise we would have $a^2 = 0$, $z = az = a^2z = 0$, which contradicts to $z \neq 0$. Therefore there is a group $G_{\alpha_0\beta}$ with $a \in G_{\alpha_0\beta}$. Analogously z is contained in a minimal left ideal $L_{\delta} = \left\{ \bigcup_{\gamma} G_{\gamma\delta} \right\} \cup P_{\delta}$ and z cannot be contained in P_{δ} since otherwise we would have $z = z^2 = 0$, contrary to the assumption. Hence there is a group $G_{\gamma_0\delta}$ with $z \in G_{\gamma_0\delta}$. We have $G_{\gamma_0\delta} \cap G_{\alpha_0\beta} = \emptyset$ since $G_{\gamma_0\delta} = G_{\alpha_0\beta}$ and $z^2 = az (=z)$ would imply $a = z$. If e' is the unit element of the group $G_{\alpha_0\beta}$, $az = z$ implies $e'az = e'z$, $az = e'z$, $z = e'z$ and $za = z$ implies $zae' = ze'$, $za = ze'$, $z = ze'$; hence $e'z = ze' = z$. Since $z \neq 0$ and $e' \neq z$, this contradicts to the fact that e' is a primitive idempotent of S .

2. The proofs of the second and third assertions follow in the same lines as in Lemma 1,1.

Theorem 2,1. *Let S be a compact simple semigroup with zero 0 satisfying $S^2 \neq 0$. Let T be a closed subsemigroup of S which contains more than one element and which does not contain 0 . Then T is a completely simple semigroup without zero.*

Remark. The case T has one element is trivial.

Proof. Write $S = \bigcup_{\alpha \in A_2} L_\alpha$, L_α running through all minimal left ideals of S .¹⁾ Let $\{L_\gamma \mid \gamma \in A'_2 \subset A_2\}$ be the set of all minimal left ideals of S for which $L'_\gamma = L_\gamma \cap T \neq \emptyset$. Since $0 \text{ non} \in T$, $L_\gamma^2 \neq (0)$ for every $\gamma \in A'_2$ and $T = S \cap T = \{\bigcup_{\alpha \in A_2} L_\alpha\} \cap T = \bigcup_{\gamma \in A'_2} L'_\gamma$. Clearly the closed subset L'_γ is again a left ideal of T .

We prove that L'_γ is a minimal left ideal of T . Suppose that there is a left ideal L of T with $L \subset L'_\gamma$. Since T is closed (hence compact), there is a minimal left ideal L^* of T with $L^* \subset L \subset L'_\gamma$ and L^* is closed. L^* contains an idempotent $e \neq 0$. The relation $e \in L^* \subset L \subset L'_\gamma \subset L_\gamma$ implies $L_\gamma e \subset L_\gamma L_\gamma = L_\gamma^2$. Since $L_\gamma^2 \neq (0)$ and $L'_\gamma \subset L_\gamma^2$, we have $L_\gamma^2 \neq (0)$, and with respect to the minimality of L_γ we have $L_\gamma^2 = L_\gamma$ and $L_\gamma e = L_\gamma$. Hence e is a right unit of L_γ , therefore $L_\gamma e = L'_\gamma$. Since L^* is a left ideal of T , we have $L'_\gamma = L'_\gamma e \subset L'_\gamma L^* \subset L^*$, i.e. $L'_\gamma \subset L^*$, whence $L'_\gamma = L^*$. This proves that L'_γ is a minimal left ideal of T .

Now Lemma 1,2 implies that $T = \bigcup_{\gamma \in A'_2} L'_\gamma$ is a simple semigroup and hence – by Lemma 2,1 – T is a completely simple semigroup without zero.

Lemma 2,2. *Let S be a completely simple semigroup with zero 0 satisfying $S^2 \neq (0)$ and T a simple subsemigroup of S containing an idempotent but not containing the zero element 0 . Then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having (exactly) the same idempotents as T . The semigroup T_1 can be written in the form*

$$T_1 = \left[\left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} \cap \left\{ \bigcap_{\beta \in A'_2} L_\beta \right\} \right] - \{0\}$$

with suitably chosen minimal right and left ideals R_α, L_β of S respectively.

Proof. By Lemma 2,1 T is completely simple (without zero if $\text{card } T > 1$) and we may write $T = \bigcup_{\alpha \in A'_1} R'_\alpha = \bigcup_{\beta \in A'_2} L'_\beta$, where R'_α, L'_β run through all minimal left and right ideals of T respectively. Choose – in the sense of Lemma 2,1 – R_α and L_β such that $R'_\alpha = R_\alpha \cap T$, $L'_\beta = L_\beta \cap T$. We have

$$T = T^2 = \left\{ \bigcup_{\alpha \in A'_1} R'_\alpha \right\} \cdot \left\{ \bigcup_{\beta \in A'_2} L'_\beta \right\} = \bigcup_{\alpha \in A'_1} \bigcup_{\beta \in A'_2} \{R'_\alpha L'_\beta\}.$$

Each of the R'_α, L'_β 's is a group and $L'_\beta R'_\alpha = T$. Denote $R'_\alpha L'_\beta = G'_{\alpha\beta}$ and let be $e_{\alpha\beta}$ the unit element of $G'_{\alpha\beta}$.

We have $0 \text{ non} \in L'_\beta R'_\alpha$ and $L'_\beta R'_\alpha \subset L_\beta R_\alpha$, and since $L_\beta R_\alpha$ is a two-sided ideal of S , we have $L_\beta R_\alpha = S$. Further $0 \text{ non} \in R'_\alpha L'_\beta$ and $R'_\alpha L'_\beta \subset R_\alpha L_\beta$, hence $R_\alpha L_\beta \neq \{0\}$. It is known (A. H. CLIFFORD [2], R. P. RICH [8]) that if R, L are two minimal right and left ideals of S respectively and $LR = S$, $RL \neq \{0\}$, then $RL = R \cap L$ is a group

¹⁾ We use the following result: A compact simple semigroup with zero is completely simple (see f.i. [12]).

with zero. Hence for $\alpha \in A'_1, \beta \in A'_2, R_\alpha L_\beta = R_\alpha \cap L_\beta$ is a group with zero (having $e_{\alpha\beta}$ as unity element). Write $R_\alpha L_\beta = G_{\alpha\beta} \cup \{0\}$. We then have

$$T = \bigcup_{\alpha \in A'_1} \bigcup_{\beta \in A'_2} \{R'_\alpha L'_\beta\} \subset \bigcup_{\alpha \in A'_1} \bigcup_{\beta \in A'_2} R_\alpha L_\beta = \bigcup_{\alpha \in A'_1} \bigcup_{\beta \in A'_2} \{R_\alpha \cap L_\beta\} = \left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} \cap \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\}.$$

Since T does not contain 0 we may write

$$T \subset T_1 = \left[\left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} \cap \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\} \right] - \{0\},$$

and T_1 contains clearly the same idempotents as T . Moreover, T_1 is a union of maximal groups of S .

To prove that T_1 is a semigroup it is sufficient to show that $a, b \in T_1$ imply $ab \neq 0$. Let be $a \in R_\alpha \cap L_\beta, b \in R_\gamma \cap L_\delta$. Denote by \bar{a} the element of $[R_\alpha \cap L_\beta] - \{0\} = G_{\alpha\beta}$ for which $\bar{a}a = e_{\alpha\beta}$, by \bar{b} the element of $[R_\gamma \cap L_\delta] - \{0\} = G_{\gamma\delta}$ for which $b\bar{b} = e_{\gamma\delta}$. If there were $ab = 0$, we would have $\bar{a}ab\bar{b} = 0$, i.e. $e_{\alpha\beta}e_{\gamma\delta} = 0$. But $e_{\alpha\beta}, e_{\gamma\delta}$ are contained in the semigroup T which does not contain 0. Hence $ab \neq 0$.

The proof that T_1 is simple follows by the same argument as in Lemma 1,3.

It rests to prove the maximality property of T_1 . Suppose that T_2 is a simple subsemigroup of S with $T_2 \supseteq T_1$. If T_2 contains the zero element, T_2 has an idempotent not contained in T_1 . We may suppose therefore that T_2 does not contain 0. The proof then follows in the same way as in the proof of Lemma 1,3.

Lemma 2,2 together with Theorem 2,1 implies

Theorem 2,2. *If S is a compact simple semigroup with zero satisfying $S^2 \neq 0$ and T a closed subsemigroup of S which does not contain 0, then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having exactly the same idempotents as T . The semigroup T_1 can be written in the form $T_1 = \left[\left\{ \bigcup_{\alpha} R_\alpha \right\} \cap \left\{ \bigcup_{\beta} L_\beta \right\} \right] - \{0\}$ with suitably chosen minimal right and left ideals R_α, L_β of S . If, moreover, T contains a maximal group of S , then $T = T_1$.*

Remark. The largest right and left ideals contained in $S - T$ are open and $A = \left\{ \bigcup_{\alpha \in A'_1} R_\alpha \right\} - \{0\}, B = \left\{ \bigcup_{\beta \in A'_2} L_\beta \right\} - \{0\}$ respectively are their complements in S . Hence $T_1 = A \cap B$ is again closed.

3

Let S be a completely simple semigroup without zero and H a simple subsemigroup of S containing all idempotents $\in S$. By Lemma 1,1 H is then completely simple.

We shall study coset decompositions of S modulo H . The possibility of such decompositions is a priori not evident.

We shall write $S = \bigcup_{\alpha \in A_2} L_\alpha$, where L_α runs through all minimal left ideals of S and $H = \bigcup_{\alpha \in A_2} L'_\alpha$, where L'_α runs through all minimal left ideals of H . By Lemma 1,1 there

is a one-to-one correspondence between L_α and L'_α such that $L'_\alpha = L_\alpha \cap H$. The decompositions $S = \bigcup_{\beta \in A_1} R_\beta$, $H = \bigcup_{\beta \in A_1} R'_\beta$ have an analogous meaning.

Denote further $R_\alpha L_\beta = R_\alpha \cap L_\beta = G_{\alpha\beta}$, $R'_\alpha L'_\beta = R'_\alpha \cap L'_\beta = G'_{\alpha\beta}$. We then have $S = \bigcup_{\alpha \in A_1} \bigcup_{\beta \in A_2} G_{\alpha\beta}$, $H = \bigcup_{\alpha \in A_1} \bigcup_{\beta \in A_2} G'_{\alpha\beta}$; the groups $G_{\alpha\beta}$ are pairwise disjoint; further $G'_{\alpha\beta} \subset G_{\alpha\beta}$. The unit element of $G_{\alpha\beta}$ will be denoted by $e_{\alpha\beta}$.

Lemma 3,1. *If $a \in S$, then $a \in Ha$.*

Proof. Since $a \in S$, there is a group $G_{\mu\nu}$ with $a \in G_{\mu\nu}$. Hence $a = e_{\mu\nu}a \in Ha$.

Lemma 3,2. *If $L_\alpha \neq L_\beta$ and $a \in L_\alpha$, $b \in L_\beta$, then $Ha \cap Hb = \emptyset$.*

Proof. $Ha \subset SL_\alpha \subset L_\alpha$, $Hb \subset SL_\beta \subset L_\beta$ and since $L_\alpha \cap L_\beta = \emptyset$, we have $Ha \cap Hb = \emptyset$.

Lemma 3,3. *If $a \in L_\alpha$, then for any $\lambda \in A_2$ we have $L'_\lambda a = L'_\alpha a$ and $L'_\lambda a = Ha$.*

Proof. The element a is contained in a minimal right ideal of S , say R_β . Then $a \in R_\beta \cap L_\alpha = G_{\beta\alpha}$ and $a = e_{\beta\alpha}a$. We have $L'_\lambda a = L'_\lambda e_{\beta\alpha}a$. The set $L'_\lambda e_{\beta\alpha}$ is contained in H and by a known theorem it is also a minimal left ideal of H . But $L'_\lambda e_{\beta\alpha} \subset L'_\lambda L_\alpha \subset L_\alpha$. Hence $L'_\lambda e_{\beta\alpha} \subset L_\alpha \cap H = L'_\alpha$, and with respect to the minimality of L'_λ we have $L'_\lambda e_{\beta\alpha} = L'_\alpha$. Hence $L'_\lambda a = (L'_\alpha e_{\beta\alpha})a = L'_\alpha a$. Moreover, $Ha = (\bigcup_{\lambda \in A_2} L'_\lambda)a = \bigcup_{\lambda \in A_2} (L'_\lambda a) = L'_\alpha a$. This proves our assertion.

Lemma 3,4. *If $a, b \in L_\alpha$, then either $Ha \cap Hb = \emptyset$ or $Ha = Hb$.*

Proof. Since $Ha = L'_\alpha a$, $Hb = L'_\alpha b$, it is sufficient to show that $L'_\alpha a \cap L'_\alpha b \neq \emptyset$ implies $L'_\alpha a = L'_\alpha b$.

Suppose $L'_\alpha a \cap L'_\alpha b \neq \emptyset$. Then there exist $u, v \in L'_\alpha$ such that $ua = vb$. The relation $u \in L'_\alpha = L_\alpha \cap H = L_\alpha \cap \{\bigcup_{\beta \in A_1} R'_\beta\} = \bigcup_{\beta \in A_1} (L_\alpha \cap R'_\beta) = \bigcup_{\beta \in A_1} G'_{\beta\alpha}$ implies that there is a $\gamma \in A_1$ with $u \in G'_{\gamma\alpha}$. Find the element $u' \in G'_{\gamma\alpha}$ with $u'u = e_{\gamma\alpha} \in L'_\alpha$. We then have $u'ua = u'vb$, i.e. $e_{\gamma\alpha}a = u'vb$. Hence $L'_\alpha a = (L'_\alpha e_{\gamma\alpha})a = (L'_\alpha u'v)b \subset L'_\alpha L'_\alpha L'_\alpha b = L'_\alpha b$, i.e. $L'_\alpha a \subset L'_\alpha b$. By the same argument we prove $L'_\alpha b \subset L'_\alpha a$, hence $L'_\alpha a = L'_\alpha b$, q.e.d.

Since (by Lemma 3,1) $a \in Ha$, we have $S = \bigcup_{\eta \in S} H\eta$ and omitting equal summands we get a decomposition (2) $S = \bigcup_{\xi \in A} H\xi$ with pairwise disjoint summands, where ξ runs through a subset $A \subset S$.

If L_α is fixed chosen, then for any $\xi \in L_\alpha$ we have $L'_\alpha \xi \subset L'_\alpha L_\alpha \subset L_\alpha$, and since $L'_\alpha \xi = H\xi$, we conclude that L_α can be covered by cosets of the form $L'_\alpha \xi$, $\xi \in L_\alpha$. Again omitting equal terms we may write $L_\alpha = \bigcup_{\xi \in A_\alpha} L'_\alpha \xi$, where A_α is a suitably chosen subset of L_α and the summands are disjoint.

It is easy to find a complete system of the ξ 's which are sufficient for the construction of such a decomposition. Write $L_\alpha = \bigcup_{\beta \in A_1} G_{\beta\alpha}$ and choose a fixed summand, say $G_{\delta\alpha}$,

$\delta \in A_1$. Construct the right coset decomposition of the group $G_{\delta\alpha}$ modulo the subgroup $G'_{\delta\alpha}$ of the form $G_{\delta\alpha} = \bigcup_{v \in I} G'_{\delta\alpha} \cdot \eta_{\delta\alpha}^{(v)}$. We then have

$$\begin{aligned} L_\alpha &= \bigcup_{\beta \in A_1} G_{\beta\alpha} = \bigcup_{\beta \in A_1} e_{\beta\alpha} G_{\delta\alpha} = \bigcup_{\beta \in A_1} e_{\beta\alpha} \left\{ \bigcup_{v \in I} G'_{\delta\alpha} \eta_{\delta\alpha}^{(v)} \right\} = \bigcup_{v \in I} \left\{ \bigcup_{\beta \in A_1} e_{\beta\alpha} G'_{\delta\alpha} \eta_{\delta\alpha}^{(v)} \right\} = \\ &= \bigcup_{v \in I} \left\{ \bigcup_{\beta \in A_1} G'_{\beta\alpha} \eta_{\delta\alpha}^{(v)} \right\} = \bigcup_{v \in I} L'_\alpha \eta_{\delta\alpha}^{(v)}. \end{aligned}$$

Note further that since $G'_{\delta\alpha} \eta_{\delta\alpha}^{(v)} \neq G'_{\delta\alpha} \eta_{\delta\alpha}^{(\mu)}$ for $\mu \neq v$, we have $L'_\alpha \eta_{\delta\alpha}^{(v)} \cap L'_\alpha \eta_{\delta\alpha}^{(\mu)} = \emptyset$. Finally, since $S = \bigcup_{\alpha \in A_2} L_\alpha = \bigcup_{\alpha \in A_2} \bigcup_{v \in I} L'_\alpha \eta_{\delta\alpha}^{(v)} = \bigcup_{\alpha \in A_2} \bigcup_{v \in I} H \eta_{\delta\alpha}^{(v)}$, we see that the ξ 's in (2) can be chosen in an arbitrary – but fixed – minimal right ideal R_δ of S .

Summarily we proved:

Theorem 3.1. *If $S = \bigcup_{\alpha \in A_1} \bigcup_{\beta \in A_2} G_{\alpha\beta}$ is a completely simple semigroup without zero and $H = \bigcup_{\alpha \in A_1} \bigcup_{\beta \in A_2} G'_{\alpha\beta}$ a simple subsemigroup of S containing all idempotents $\in S$, then:*

1. *There exists a coset decomposition $S = \bigcup_{\xi^{(v)} \in A} H \xi^{(v)}$ into pairwise disjoint classes, A being a suitably chosen subset of S .*
2. *A can be chosen as a subset of an arbitrary fixed chosen minimal right ideal of S .*
3. *If L_α is a minimal left ideal of S and $L'_\alpha = L_\alpha \cap H$, then there exists a coset decomposition of L_α of the form $L_\alpha = \bigcup_{\xi^{(\mu)} \subset A_\alpha} L'_\alpha \xi^{(\mu)}$ with pairwise disjoint summands.*
4. *If $G_{\delta\alpha} = \bigcup_{v \in I} G'_{\delta\alpha} \eta_{\delta\alpha}^{(v)}$ ($\delta \in A_1$ fixed) is the coset decomposition of the group $G_{\delta\alpha}$ with respect to the subgroup $G'_{\delta\alpha}$, we may choose $A_\alpha = \{\eta_{\delta\alpha}^{(v)}, v \in I\}$.*

In some applications (see the forthcoming paper [9]) double coset decompositions mod (H, K) are needed. We end therefore our investigations with the proof of the following theorem:

Theorem 3.2. *Let $S = \bigcup_{\alpha \in A_1} \bigcup_{\beta \in A_2} G_{\alpha\beta}$ be a completely simple semigroup without zero and H, K two simple subsemigroups of S both containing all idempotents $\in S$. Then there exists a double coset decomposition of S into pairwise disjoint classes of the form $S = \bigcup_{\xi^{(v)} \in B} H \xi^{(v)} K$, $B \subset S$. Denoting $G'_{\alpha\beta} = H \cap G_{\alpha\beta}$, $G''_{\alpha\beta} = K \cap G_{\alpha\beta}$, we have $HaK \cap G_{\alpha\beta} = G'_{\alpha\beta} a G''_{\alpha\beta}$ for any $a \in S$. Further $G'_{\alpha\beta} a G''_{\alpha\beta}$ is exactly one class of the double coset decomposition of the group $G_{\alpha\beta}$ modulo $(G'_{\alpha\beta}, G''_{\alpha\beta})$.*

Proof. 1. If $a \in S$, there is a group $G_{\alpha\beta}$ with $a \in G_{\alpha\beta}$. If $e_{\alpha\beta}$ is the unit element of $G_{\alpha\beta}$, we have $a = e_{\alpha\beta} a e_{\alpha\beta} \in HaK$. Hence $S = \bigcup_{x \in S} HxK$.

We show that if $a, b \in S$, then either $HaK = HbK$ or $HaK \cap HbK = \emptyset$. Suppose $HaK \cap HbK \neq \emptyset$. Then there exist elements $u, w \in H, v, z \in K$ such that

$$(3) \quad uav = wbz.$$

Suppose $a \in R_\alpha, u \in L'_\beta$ for some α and β (L'_β being a minimal left ideal of H). $e_{\alpha\beta} \in R_\alpha \cap L'_\beta$ is a right unit for every element $\in L'_\beta$ and a left unit for every element $\in R_\alpha$. Since L'_β is a minimal left ideal of H , we have $L'_\beta u = L'_\beta$, hence there exists an $u^* \in L'_\beta \subset H$ with $u^*u = e_{\alpha\beta}$. Let analogously be $a \in L_\gamma, v \in R'_\delta$ for some γ and δ (R'_δ being a minimal right ideal of K). The idempotent $e_{\delta\gamma} \in R'_\delta \cap L_\gamma$ is a left unit for every element $\in R'_\delta$ and a right unit for every element $\in L_\gamma$. Since R'_δ is a minimal right ideal of K , we have $vR'_\delta = R'_\delta$, hence there is an element $v^* \in R'_\delta \subset K$ such that $vv^* = e_{\delta\gamma}$. The relation (3) implies

$$\begin{aligned} u^*uavv^* &= u^*wbzv^*, \\ e_{\alpha\beta}ae_{\delta\gamma} &= (u^*w)b(zv^*), \\ a &= (u^*w)b(zv^*). \end{aligned}$$

We have therefore $HaK = H(u^*w)b(zv^*)K \subset HbK$. Analogously we prove $HbK \subset HaK$, hence $HaK = HbK$.

Omitting in $S = \bigcup_{x \in S} HxK$ the classes which occur more times, we get a decomposition of the required form. This proves the first statement of our theorem.

2. Consider next the set $G'_{\rho\sigma}aG''_{\tau\omega}$. We prove that this product is independent of σ and τ (i.e. we have the same set for every couple σ, τ). Let f.i. be $a \in G_{\mu\nu}$; then $e_{\mu\nu}ae_{\mu\nu} = a$ and

$$G'_{\rho\sigma}aG''_{\tau\omega} = (G'_{\rho\sigma}e_{\mu\nu})a(e_{\mu\nu}G''_{\tau\omega}) = G'_{\rho\nu}aG''_{\mu\omega}$$

and this is clearly independent of σ and τ . Further

$$(4) \quad HaK = \left[\bigcup_{\rho \in A_1} \bigcup_{\sigma \in A_2} G'_{\rho\sigma} \right] a \left[\bigcup_{\tau \in A_1} \bigcup_{\omega \in A_2} G''_{\tau\omega} \right] = \bigcup_{\rho} \bigcup_{\sigma} \bigcup_{\tau} \bigcup_{\omega} G'_{\rho\sigma}aG''_{\tau\omega}$$

and

$$G'_{\rho\sigma}aG''_{\tau\omega} \subset G'_{\rho\sigma}aG_{\tau\omega} = R_\rho L'_\sigma a R_\tau L_\omega \subset R_\rho \cap L_\omega = G_{\rho\omega}.$$

Now α, β being fixed $G_{\alpha\beta}$ contains those and only those summands of (4) which are of the form $G'_{\alpha\sigma}aG''_{\tau\beta}$. Since all these summands are identical we may choose $\sigma = \beta, \tau = \alpha$. Thus we have $HaK \cap G_{\alpha\beta} = G'_{\alpha\beta}aG''_{\alpha\beta}$. This proves the second assertion.

3. It is possible to characterise the set $T_{\alpha\beta} = G'_{\alpha\beta}aG''_{\alpha\beta}$ in terms of the elements of the group $G_{\alpha\beta}$. Clearly $T_{\alpha\beta} = (G'_{\alpha\beta}e_{\alpha\beta})a(e_{\alpha\beta}G''_{\alpha\beta})$. Further $e_{\alpha\beta}ae_{\alpha\beta} \subset R_\alpha L_\beta a R_\alpha L_\beta \subset R_\alpha L_\beta = G_{\alpha\beta}$. Hence, denoting $a_{\alpha\beta} = e_{\alpha\beta}ae_{\alpha\beta} \in G_{\alpha\beta}$, we have $T_{\alpha\beta} = G'_{\alpha\beta}a_{\alpha\beta}G''_{\alpha\beta}$ and $a_{\alpha\beta} \in T_{\alpha\beta}$ since $e_{\alpha\beta}a_{\alpha\beta}e_{\alpha\beta} = a_{\alpha\beta}$. This says that $T_{\alpha\beta}$ is a class of the double coset decomposition of $G_{\alpha\beta}$ modulo the subgroups $(G'_{\alpha\beta}, G''_{\alpha\beta})$. Our theorem is completely proved.

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Резюме

ПОДПОЛУГРУППЫ ПРОСТЫХ ПОЛУГРУПП

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава

Целью настоящей работы является исследование строения подполугрупп простых (в особенности вполне простых) полугрупп. Приведем некоторые результаты:

Если S — вполне простая полугруппа без нуля и H — простая подполугруппа, обладающая идемпотентом, то H вполне проста. Если S — простая быкомпактная полугруппа, то всякая замкнутая подполугруппа вполне проста.

Аналогичные результаты имеют место и для вполне простых полугрупп с нулем, если в качестве H рассматриваются подполугруппы, не содержащие нуля полугруппы S .

Существует тесная связь между односторонними идеалами полугрупп S и H .

Исследуется строение максимальных подполугрупп из S , имеющие в точности все идемпотенты данной подполугруппы H .

Пусть S — вполне проста и H, K — две подполугруппы из S , содержащие все идемпотенты $\epsilon \in S$. В отделе 3 исследуется разложение полугруппы S по двойному модулю (H, K) .