Subspace Fitting with Diversely Polarized Antenna Arrays

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***Abstract*—Diversely polarized antenna arrays are widely used in RF applications. The diversity of response provided by such arrays can greatly improve direction finding performance over arrays sensitive to only one polarization component. For d emitters, directly implementing a multidimensional estimation algorithm (e.g., maximum likelihood) would require a search for $3d$ parameters: d directions of arrival (DOA's), and $2d$ polarization parameters. In this paper, we present a more efficient solution based on the so-called noise subspace fitting (NSF) algorithm. In particular, we show how to decouple the NSF search into a two-step procedure, where the DOA's are estimated separately. The polarization parameters are then obtained by solving a linear system of equations. The advantage of this approach is that the search dimension is reduced by a factor of three, and no initial polarization estimate is required. In addition, the algorithm can be shown to yield asymptotically minimum variance estimates provided no perfectly coherent signals are present. Simulation examples are included to compare the NSF approach with a similar generalization of the MUSIC algorithm.**

I. Introduction

MOST antenna arrays used in radar and communications systems are sensitive to polarization differences between one received signal and another. Some antennas (e.g., crossed dipoles) are specially designed for this purpose and measure each polarization component separately. Other antennas superimpose the polarization components and yield some combined response. For example, a vertically polarized monopole mounted on the fuselage of an aircraft will typically have a significant nonzero horizontal polarization response due to coupling between the antenna and aircraft body.

Provided the array response for each polarization component can be separately calibrated, diversely polarized arrays are advantageous because they possess an extra degree of signal discrimination that can be used to improve detection and estimation performance. Despite this fact, relatively few algorithms for direction of arrival (DOA) estimation have been developed to take advantage of such arrays. Recent work in this area has focused on implementations for adaptive arrays [1], algorithms for exploiting multiple polarization dimensions (including both electrical and magnetic field components) [2], and performance analyses [3], [4]. Extensions of MUSIC, root-MUSIC, and other algorithms to diversely polarized arrays have been considered in [5], [6], [7], but similar ideas for multidimensional algorithms such as maximum likelihood have not been extensively studied. Notable exceptions include a recent algorithm that involves application of ESPRIT to a uniform linear array of crossed dipoles [8], and a simulated annealing approach presented in [9].

Conceptually, there is no real difficulty in extending multidimensional DOA estimation algorithms to the diversely polarized case. For example, one obvious method would be to simply augment the dimension of the parameter search to include the polarization parameters as well as the DOA's. However, this approach does not take advantage of the fact that the polarization parameters lie in a "linear parameter subspace" of the full parameter space [10]. Therefore, such an approach triples the dimension of the parameter search, and requires one to somehow obtain initial estimates of the signal polarization before the DOA's can be consistently estimated.

The goal of this paper is to show how a more computationally efficient solution can be obtained without sacrificing estimation accuracy. Our approach is based on the noise subspace fitting (NSF) algorithm described in [11], [12]. The proposed method exploits the special structure of the manifold for diversely polarized arrays to decouple the polarization estimation from that of the DOA's. In particular, the DOA's are obtained by a search procedure of standard order, while the polarization parameters are determined separately (without the need for initial estimates) by solving a linear system of equations. If consistent initial estimates of the DOA's are available (e.g., from an initial application of this or some other algorithm, or from some prior estimates), our decoupled NSF approach can be shown to yield asymptotically efficient (minimum variance) parameter estimates.

In the next section, we briefly present the narrowband data model assumed in this work, and review the MUSIC-based DOA/polarization estimator of [5]. Section III provides a derivation of our NSF approach, and addresses the issue of parameter identifiability. An asymptotic performance analysis of the algorithm is undertaken in Section IV, and the paper concludes with some representative numerical examples in Section V.

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II. Background and Modeling Assumptions

We will use the standard linear data model assumed in most narrowband direction finding (DF) applications. Although our discussion is confined to the case where only a single location parameter is to be estimated per signal, the algorithm presented in the next section is easily extended to the multiple parameter case (e.g., estimation of azimuth and elevation angles, etc.).

A. Standard Data Model

For an m-element array of sensors, we will define $\mathbf{a}_k \in \mathbb{C}^m$ to be the complex array response for the k-th narrowband emitter. The l-th element of $\mathbf{a}_k$ represents the gain and phase response of the l-th sensor to emitter signal k, with respect to some reference point. The array response vector is a function of the *signal parameters* of interest. Herein, we assume that $\mathbf{a}_k$ only depends on the azimuthal angle and the polarization parameters of the k-th signal. Other parameters (such as elevation, range, etc.) are assumed to be fixed.

When sampled at some time instant t, the outputs of the m array elements are stacked in a vector $\mathbf{x}(t) \in \mathbb{C}^m$. Assuming d narrowband (co-channel) emitters are present, and using linear superposition, the array output $\mathbf{x}(t)$ may be written as

$$\mathbf{x}(t) = (\mathbf{A}\mathbf{s}(t)) + \mathbf{n}(t),$$

where the signal vector $\mathbf{s}(t) \in \mathbb{C}^d$ is composed of the received signal waveforms sampled at time t, $\mathbf{n}(t)$ is additive noise, and where

$$\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_d]. \tag{1}$$

The number of signals d is assumed to be known (i.e., correctly estimated).

If no noise were present, the observations $\mathbf{x}(t)$ would be confined entirely to the d-dimensional subspace of $\mathbb{C}^m$ defined by the column space of $\mathbf{A}$. Determining the DOA's for the no-noise case is therefore simply a matter of finding the signal parameters that make the range space of $\mathbf{A}$ equal to that spanned by the observations. A modification is of course necessary in the presence of noise, since the observations are then "full-rank" with probability one. The approach taken by subspace-based methods is to first estimate the dominant subspace of the observations, and then find the d distinct vectors $\mathbf{a}_k$ (parametrized by the signal parameters) that are in some sense closest to this subspace.

The subspace estimation step is usually achieved by performing an eigendecomposition on the covariance matrix $\mathbf{R}$ of the received data. Assuming that the noise and signals are uncorrelated, and that the noise is spatially white, we have

$$\mathbf{R} = \mathscr{E}\{\mathbf{x}(t)\mathbf{x}^*(t)\} = \mathbf{A}\mathbf{S}\mathbf{A}^* + \sigma^2\mathbf{I}, \tag{2}$$

where $\mathscr{E}\{\}$ denotes expectation, $\{\}^*$ denotes a complex conjugate transpose, σ^2 is the noise power at each antenna, and $\mathbf{S}$ is the covariance matrix of the emitter signals. For deterministic signals, $\mathbf{S}$ can be thought of as the following limit:

$$\mathbf{S} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \mathbf{s}(t)\mathbf{s}^*(t).$$

In this paper, we will require that $\mathbf{S}$ be full rank d (e.g., no perfect multipath is present). Note also that the assumption of spatially white noise can be relaxed provided the spatial covariance of the noise is known.

Using the model of (2), it is easily shown that the eigendecomposition of $\mathbf{R}$ has the following form:

$$\mathbf{R} = \sum_{i=1}^{m} \lambda_i \mathbf{e}_i \mathbf{e}_i^* = \mathbf{E}_s \mathbf{\Lambda}_s \mathbf{E}_s^* + \sigma^2 \mathbf{E}_n \mathbf{E}_n^*, \tag{3}$$

where $\mathbf{E}_s = [\mathbf{e}_1 \cdots \mathbf{e}_d]$, $\mathbf{E}_n = [\mathbf{e}_{d+1} \cdots \mathbf{e}_m]$, and $\lambda_1 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_m = \sigma^2$. The key observations to be made here are that the column space of $\mathbf{A}$ and $\mathbf{E}_s$ coincide, and that $\mathbf{A}^*\mathbf{E}_n = 0$. This fact suggests that the signal parameters can be determined as the values that make $\mathbf{a}_k$ orthogonal to $\mathbf{E}_n$, for $k = 1, \cdots, d$.

The above result is independent of the presence of noise and the signal-to-noise ratio, provided a perfect measurement of $\mathbf{R}$ is available. However, since only a finite amount of data can be collected, the best we can do in practice is to form the sample average

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=1}^{N} \mathbf{x}(t)\mathbf{x}^*(t),$$

and use the eigendecomposition of $\hat{\mathbf{R}}$ to form estimates $\hat{\mathbf{E}}_s$, $\hat{\mathbf{E}}_n$, $\hat{\mathbf{\Lambda}}_s$, and $\hat{\sigma}^2$.

B. Diversely Polarized Array Data

The response of a diversely polarized array can be decomposed into two parts, each due to a separate polarization component.[1] These responses are both a function of the direction-of-arrival (DOA), denoted by θ, and both are assumed to be available (e.g., via some calibration procedure). Let $\mathbf{a}_h(\theta)$ and $\mathbf{a}_v(\theta)$ represent the array responses for the two polarization components (the subscripts h and v could, for example, stand for *horizontal* and *vertical*). The net response of the array $\mathbf{a}(\theta, \phi_h, \phi_v)$ in a given direction θ is assumed to be given by a linear combination of the responses due to each polarization:

$$\mathbf{a}(\theta, \phi_h, \phi_v) = \mathbf{a}_h(\theta)\phi_h + \mathbf{a}_v(\theta)\phi_v, \tag{4}$$

where ϕ_h and ϕ_v are complex scalars representing the relative contribution of each polarization component in the response. For d emitter signals, define

$$\mathbf{A}_h(\boldsymbol{\theta}) = [\mathbf{a}_h(\theta_1) \cdots \mathbf{a}_h(\theta_d)],$$
$$\mathbf{A}_v(\boldsymbol{\theta}) = [\mathbf{a}_v(\theta_1) \cdots \mathbf{a}_v(\theta_d)],$$

[1] As described in [2], [4], polarization (both electric and magnetic) can be measured in more than two dimensions. However, most RF antennas in use today measure only two separate electric polarizations, and this is the case that we restrict ourselves to.

so that the "steering" matrix for the array is given by

$$\mathbf{A} = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathbf{A}_h(\boldsymbol{\theta})\boldsymbol{\Phi}_h + \mathbf{A}_v(\boldsymbol{\theta})\boldsymbol{\Phi}_v \in \mathbb{C}^{m \times d}, \quad (5)$$

where

$$\boldsymbol{\theta} = [\theta_1, \cdots, \theta_d],$$
$$\boldsymbol{\phi} = [\phi_{h1}, \cdots, \phi_{hd}, \phi_{v1}, \cdots, \phi_{vd}]^T,$$
$$\boldsymbol{\Phi}_h = \operatorname{diag}\{\phi_{h1}, \cdots, \phi_{hd}\},$$
$$\boldsymbol{\Phi}_v = \operatorname{diag}\{\phi_{v1}, \cdots, \phi_{vd}\}.$$

Note that any given diagonal element of $\boldsymbol{\Phi}_h$ could be zero, as long as the corresponding element in $\boldsymbol{\Phi}_v$ is nonzero, and vice versa. The diagonal elements of these matrices are also unique only to within a scale factor (since the signal waveforms $\mathbf{s}(t)$ are unknown).

Some arrays are composed of antennas (e.g., crossed dipole pairs) that measure each polarization component separately, hence their response might more appropriately be written as

$$\mathbf{a}(\theta, \phi_h, \phi_v) = \begin{bmatrix} \mathbf{a}_h(\theta)\phi_h \\ \mathbf{a}_v(\theta)\phi_v \end{bmatrix}. \quad (6)$$

However, such arrays may be cast in the framework of (5) by defining

$$\mathbf{A}_h(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{a}_h(\theta_1) & \cdots & \mathbf{a}_h(\theta_d) \\ 0 & \cdots & 0 \end{bmatrix},$$
$$\mathbf{A}_v(\boldsymbol{\theta}) = \begin{bmatrix} 0 & \cdots & 0 \\ \mathbf{a}_v(\theta_1) & \cdots & \mathbf{a}_v(\theta_d) \end{bmatrix}.$$

In other words, an m-element array of the form (6) may be thought of as a $2m$-element array like (4) whose first m elements are sensitive to horizontal polarization only, and whose last m elements are sensitive only to vertical polarization. Thus, there is no loss of generality in restricting our attention to the case described by (4).

The fact that the response of a diversely polarized array is a linear function of the polarization parameters ϕ_{hi} and ϕ_{vi} is the reason that it is possible to find a more efficient solution than a brute force search for all parameters simultaneously. This linear dependence on polarization means that the array manifold is not a one-dimensional "rope" in $\mathbb{C}^m$ (a vector continuum), but rather a continuum of planes (a *bivector* continuum) [10]. As we will see below and in the next section, this property allows the estimation of the DOA's to be decoupled from that of the polarization parameters.

C. A *MUSIC-Based Approach*

In the standard MUSIC algorithm [5], the DOA's are estimated by searching one by one for those values of θ that make $\mathbf{a}(\theta)$ nearly orthogonal to $\hat{\mathbf{E}}_n$, according to the following measure:

$$V_{\mathrm{MU}}(\theta) = \frac{\mathbf{a}^*(\theta)\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{a}(\theta)}{\mathbf{a}^*(\theta)\mathbf{a}(\theta)}. \quad (7)$$

The d smallest (separated) minima of $V_{\mathrm{MU}}(\theta)$ are then taken to be the estimates of the DOA's. For a diversely polarized array, if we define

$$\overline{\mathbf{A}}(\theta) = \begin{bmatrix} \mathbf{a}_h(\theta) & \mathbf{a}_v(\theta) \end{bmatrix},$$
$$\overline{\boldsymbol{\phi}} = \begin{bmatrix} \phi_h \\ \phi_v \end{bmatrix},$$

the MUSIC cost function becomes

$$V_{\mathrm{MU}}(\theta, \overline{\boldsymbol{\phi}}) = \frac{\overline{\boldsymbol{\phi}}^*\overline{\mathbf{A}}^*(\theta)\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\overline{\mathbf{A}}(\theta)\overline{\boldsymbol{\phi}}}{\overline{\boldsymbol{\phi}}^*\overline{\mathbf{A}}^*(\theta)\overline{\mathbf{A}}(\theta)\overline{\boldsymbol{\phi}}}. \quad (8)$$

Since $\overline{\boldsymbol{\phi}}$ is unconstrained (except for $\overline{\boldsymbol{\phi}} \neq 0$), minimizing $V_{\mathrm{MU}}(\theta, \overline{\boldsymbol{\phi}})$ with respect to $\overline{\boldsymbol{\phi}}$ is equivalent to finding, as a function of θ, the following minimum generalized eigenvalue and eigenvector [5]:

$$\overline{\mathbf{A}}^*(\theta)\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\overline{\mathbf{A}}(\theta)\mathbf{z}_{\min} = \lambda_{\min}\overline{\mathbf{A}}^*(\theta)\overline{\mathbf{A}}(\theta)\mathbf{z}_{\min}. \quad (9)$$

As proposed in [5], the DOA estimates can then be found by viewing $\lambda_{\min}$ as a function of θ, and searching for its minima. The polarization of the signal with DOA estimate $\hat{\theta}_i$ is then taken to be the eigenvector associated with $\lambda_{\min}(\hat{\theta}_i)$; i.e., $\overline{\boldsymbol{\phi}}_i = \mathbf{z}_{\min}(\hat{\theta}_i)$.

In the next section, we present a multidimensional algorithm based on subspace fitting concepts that, like the MUSIC-based solution above, decouples the estimation of DOA and polarization. As will be demonstrated in Section V, the multidimensional nature of the algorithm allows it to achieve performance superior to MUSIC, particularly in situations involving highly correlated signals.

III. A Noise Subspace Fitting Approach

In the noise subspace fitting (NSF) approach [11], [12], the DOA's are estimated as those that minimize the following cost function:

$$V_{\mathrm{NSF}}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \operatorname{Tr}\left(\mathbf{A}^*(\boldsymbol{\theta}, \boldsymbol{\phi})\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\phi})\mathbf{W}\right),$$

where $\mathbf{W} = \mathbf{W}^* > 0$ is a $d \times d$ weighting matrix. MUSIC is a special case of NSF for the choice[2] $\mathbf{W} = \mathbf{I}$. In general (e.g., when $\mathbf{W}$ is not diagonal), the NSF algorithm must resort to a multidimensional search for the DOA's, hence our referring to it earlier as a *multidimensional* algorithm.

The choice of the weighting $\mathbf{W}$ is critical to algorithm performance. While *any* symmetric, positive definite $\mathbf{W}$ will yield consistent parameter estimates, only a consistent estimate of the following weighting will, under certain conditions (see Section IV), yield statistically efficient (minimum variance) estimates:

$$\mathbf{W}_{\mathrm{NSF}} = \left(\mathbf{A}^*(\boldsymbol{\theta}, \boldsymbol{\phi})\mathbf{E}_s(\boldsymbol{\Lambda}_s - \sigma^2\mathbf{I})^{-2}\boldsymbol{\Lambda}_s\mathbf{E}_s^*\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\phi})\right)^{-1}. \quad (10)$$

[2]This is true provided the constraint $\hat{\theta}_i \neq \hat{\theta}_k$, $i \neq k$, can be maintained during the minimization process. This is often not possible when the emitters are closely spaced, and hence in such cases MUSIC may lose its capability to resolve all sources.

For this reason, NSF is usually implemented in two stages. In the first, a consistent estimate of the signal parameters is obtained either by application of NSF with an arbitrary weighting or by application of some other consistent method. The second step then consists of recomputing the estimates using NSF and the optimal weighting in (10). The performance advantage of optimally weighted NSF over MUSIC is especially evident in situations involving highly correlated signals.

In the case of a diversely polarized array, one might expect that NSF would require a search in a $5d$-dimensional space (d DOA's and $2d$ complex parameters in $\boldsymbol{\phi}$). This search can be reduced to $3d$ by removing the redundancy in the scaling of the polarization parameters (see below). The goal of this section is to show that, as with MUSIC [5], a simpler solution is possible that entirely decouples the DOA and polarization estimation. In particular, the method we present involves only a standard d-dimensional search for the DOA's, followed by a step in which the polarization parameters are solved for directly.

For a diversely polarized array, the NSF cost function becomes

$$V_{\mathrm{NSF}} = \mathrm{Tr}\left(\boldsymbol{\Phi}_h^*\mathbf{A}_h^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_h\boldsymbol{\Phi}_h\mathbf{W}\right) + \mathrm{Tr}\left(\boldsymbol{\Phi}_h^*\mathbf{A}_h^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_v\boldsymbol{\Phi}_v\mathbf{W}\right) + \mathrm{Tr}\left(\boldsymbol{\Phi}_v^*\mathbf{A}_v^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_h\boldsymbol{\Phi}_h\mathbf{W}\right) + \mathrm{Tr}\left(\boldsymbol{\Phi}_v^*\mathbf{A}_v^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_v\boldsymbol{\Phi}_v\mathbf{W}\right),$$

where the arguments of $\mathbf{A}_h$ and $\mathbf{A}_v$ have been dropped for simplicity. It is easily shown that, for a diagonal matrix $\boldsymbol{\Phi}$ and arbitrary square matrices $\mathbf{X}$ and $\mathbf{Y}$, the relation

$$\mathrm{Tr}\,(\boldsymbol{\Phi}^*\mathbf{X}\boldsymbol{\Phi}\mathbf{Y}) = \boldsymbol{\phi}^*(\mathbf{X}\odot\mathbf{Y}^T)\boldsymbol{\phi} \tag{11}$$

holds, where $\boldsymbol{\phi}$ is the vector formed from the diagonal elements of $\boldsymbol{\Phi}$, and $\odot$ represents a Hadamard/Schur (element by element) product. Using (11), the NSF criterion may be rewritten as

$$\begin{aligned} V_{\mathrm{NSF}} = {} & \boldsymbol{\phi}_h^*\left[\left(\mathbf{A}_h^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_h\right)\odot\mathbf{W}^T\right]\boldsymbol{\phi}_h \\ & + \boldsymbol{\phi}_h^*\left[\left(\mathbf{A}_h^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_v\right)\odot\mathbf{W}^T\right]\boldsymbol{\phi}_v \\ & + \boldsymbol{\phi}_v^*\left[\left(\mathbf{A}_v^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_h\right)\odot\mathbf{W}^T\right]\boldsymbol{\phi}_h \\ & + \boldsymbol{\phi}_v^*\left[\left(\mathbf{A}_v^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_v\right)\odot\mathbf{W}^T\right]\boldsymbol{\phi}_v, \end{aligned}$$

where, as above, $\boldsymbol{\phi}_h$ and $\boldsymbol{\phi}_v$ represent the diagonal elements of $\boldsymbol{\Phi}_h$ and $\boldsymbol{\Phi}_v$, respectively. A more compact matrix representation of V_{NSF} is possible, as follows:

$$V_{\mathrm{NSF}}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \boldsymbol{\phi}^*\mathbf{M}_s(\boldsymbol{\theta})\boldsymbol{\phi}, \tag{12}$$

where $\boldsymbol{\phi} = [\boldsymbol{\phi}_h^T \quad \boldsymbol{\phi}_v^T]^T$ and

$$\mathbf{M}_s(\boldsymbol{\theta}) = \begin{bmatrix} \left(\mathbf{A}_h^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_h\right)\odot\mathbf{W}^T & \left(\mathbf{A}_h^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_v\right)\odot\mathbf{W}^T \\ \left(\mathbf{A}_v^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_h\right)\odot\mathbf{W}^T & \left(\mathbf{A}_v^*\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*\mathbf{A}_v\right)\odot\mathbf{W}^T \end{bmatrix}.$$

Note that in the formulation of NSF in (12), $\boldsymbol{\phi}$ contains only the $2d$ complex polarization parameters and not the DOA's, while $\mathbf{M}_s(\boldsymbol{\theta})$ is a function of the DOA's only.[3] Because of the form of the cost function in (12), it is natural to suppose that minimizing V_{NSF} is equivalent to finding the $\boldsymbol{\theta}$ that minimizes the smallest eigenvalue of $\mathbf{M}_s(\boldsymbol{\theta})$, and then estimating the polarization parameters from the eigenvector associated with the minimum eigenvalue. However, this is not the case since constraints on $\boldsymbol{\phi}$ (other than $\boldsymbol{\phi}^*\boldsymbol{\phi} = 1$) are necessary to guarantee identifiability of the DOA's.

To see why this is so, suppose that two signals are present with DOA's θ_1 and θ_2, and polarization parameters ϕ_{h1}, ϕ_{v1}, ϕ_{h2}, and ϕ_{v2}. With no noise or infinite data (and no perfectly correlated signals), the noise eigenvectors are orthogonal to the steering vectors:

$$\mathbf{E}_n^*\left(\begin{aligned} &\left[\mathbf{a}_h(\theta_1) \quad \mathbf{a}_h(\theta_2)\right]\begin{bmatrix}\phi_{h1} & 0 \\ 0 & \phi_{h2}\end{bmatrix} \\ &+\left[\mathbf{a}_v(\theta_1) \quad \mathbf{a}_v(\theta_2)\right]\begin{bmatrix}\phi_{v1} & 0 \\ 0 & \phi_{v2}\end{bmatrix}\end{aligned}\right) = 0. \tag{13}$$

In an identifiable parameterization, this would be the only set of parameters that results in a steering matrix $\mathbf{A}$ orthogonal to $\mathbf{E}_n$. However, if we let both $\phi_{h2} \to 0$ and $\phi_{v2} \to 0$, then (13) can be satisfied for *arbitrary* $\theta_2 = \theta$:

$$\mathbf{E}_n^*\left(\begin{aligned} &\left[\mathbf{a}_h(\theta_1) \quad \mathbf{a}_h(\theta)\right]\begin{bmatrix}\phi_{h1} & 0 \\ 0 & 0\end{bmatrix} \\ &+\left[\mathbf{a}_v(\theta_1) \quad \mathbf{a}_v(\theta)\right]\begin{bmatrix}\phi_{v1} & 0 \\ 0 & 0\end{bmatrix}\end{aligned}\right) = 0.$$

Thus, to guarantee unique DOA estimates, a constraint on $\boldsymbol{\phi}$ is required that prevents $\phi_{hi} \to 0$ and $\phi_{vi} \to 0$ simultaneously for $1 \le i \le d$.

Since the polarization parameters can be scaled arbitrarily, one approach might be simply to force $\boldsymbol{\Phi}_v = \mathbf{I}$ and solve only for $\boldsymbol{\phi}_h$:

$$\left\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}_h\right\} = \arg\min_{\boldsymbol{\theta}, \boldsymbol{\phi}_h}\left[\boldsymbol{\phi}_h^* \quad \mathbf{e}^T\right]\begin{bmatrix}\mathbf{M}_{s,hh} & \mathbf{M}_{s,hv} \\ \mathbf{M}_{s,vh} & \mathbf{M}_{s,vv}\end{bmatrix}\begin{bmatrix}\boldsymbol{\phi}_h \\ \mathbf{e}\end{bmatrix}, \tag{14}$$

where $\mathbf{e}$ is a vector composed of d ones, and $\mathbf{M}_s(\boldsymbol{\theta})$ has been partitioned into $d \times d$ blocks. Since the cost function is quadratic in $\boldsymbol{\phi}_h$, the estimation of $\boldsymbol{\phi}_h$ is separable from that of the DOA's. Setting $\partial V_{\mathrm{NSF}}/\partial\boldsymbol{\phi}_h = 0$ yields

$$\hat{\boldsymbol{\phi}}_h = -\mathbf{M}_{s,hh}^{-1}\mathbf{M}_{s,hv}\mathbf{e}, \tag{15}$$

[3]If the optimal NSF weighting is used, $\mathbf{W}$ and hence $\mathbf{M}_s(\boldsymbol{\theta})$ depend implicitly on the polarization. However, as with the DOA's, using any initial consistent polarization estimate to fix $\mathbf{W}$ will yield estimates that are asymptotically as accurate as those that would be obtained if $\mathbf{W}$ were allowed to vary with polarization.

and substituting this expression into (14) leads to a concentrated cost function that depends only on $\boldsymbol{\theta}$:

$$V_{\mathrm{NSF}}(\boldsymbol{\theta}) = \mathbf{e}^T(\mathbf{M}_{s,vv} - \mathbf{M}_{s,vh}\mathbf{M}_{s,hh}^{-1}\mathbf{M}_{s,hv})\mathbf{e}. \quad (16)$$

In this case, $V_{\mathrm{NSF}}(\boldsymbol{\theta})$ becomes the sum of the elements of the Schur complement of $\mathbf{M}_s(\boldsymbol{\theta})$. An algorithm employing this approach would consist of the following two steps:

1. Find the $\boldsymbol{\theta}$ that minimizes the sum of the elements of

$$\mathbf{M}_{s,vv} - \mathbf{M}_{s,vh}\mathbf{M}_{s,hh}^{-1}\mathbf{M}_{s,hv}.$$

2. Solve for $\hat{\boldsymbol{\phi}}_h$ using $\hat{\boldsymbol{\theta}}$ in (15).

In some situations, it may be desirable to remove the implicit assumption that all of the signals have a nonzero (vertical) polarization ϕ_v. While the problem can easily be reformulated to force $\boldsymbol{\Phi}_h = \mathbf{I}$, the following more general problem formulation could be considered:

$$\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\} = \arg\min_{\theta,\phi} \boldsymbol{\phi}^*\mathbf{M}_s(\boldsymbol{\theta})\boldsymbol{\phi} \quad \text{subject to } \mathbf{B}^*\boldsymbol{\phi} = \mathbf{e}, \quad (17)$$

where $\mathbf{B}$ is a $2d \times d$ complex matrix used to specify the constraint. Our initial constraint $\boldsymbol{\Phi}_v = \mathbf{I}$ is achieved by setting

$$\mathbf{B} = \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix}.$$

On the other hand, to allow either polarization component to be zero, $\mathbf{B}$ could be chosen as

$$\mathbf{B} = \frac{1}{1-j}\begin{bmatrix} \mathbf{I} \\ \mathbf{I} \end{bmatrix},$$

which implements the constraint

$$\mathrm{Re}\{\boldsymbol{\phi}_h + \boldsymbol{\phi}_v\} = \mathrm{Im}\{\boldsymbol{\phi}_h + \boldsymbol{\phi}_v\} = \mathbf{e}, \quad (18)$$

where $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote real and imaginary parts, respectively. However, this particular constraint is unable to exactly represent polarization ratios $\phi_h/\phi_v = -1$ (although it can come arbitrarily close). It is clear that, for any linearly constrained problem of the form (17), certain polarization combinations can not be exactly represented. Although there exist nonlinear constraints that circumvent this problem,[4] an efficient separable solution like that presented here may be difficult to obtain. Because the set of polarization pairs not representable by a given linear constraint is a set of measure zero, the use of (14) or (17) should not pose a problem in practice.

To solve (17), we incorporate the constraint into the cost function by noting that all $\boldsymbol{\phi}$ that satisfy $\mathbf{B}^*\boldsymbol{\phi} = \mathbf{e}$ can be parameterized as

$$\boldsymbol{\phi} = \mathbf{B}\mathbf{e} + \mathbf{B}^\perp\boldsymbol{\psi},$$

where $\boldsymbol{\psi}$ is an arbitrary complex d-vector and $\mathbf{B}^\perp$ is a $2d \times d$ matrix that satisfies $\mathbf{B}^*\mathbf{B}^\perp = 0$. Writing the cost function in terms of $\mathbf{e}$ and $\boldsymbol{\psi}$, and setting $\partial V_{\mathrm{NSF}}/\partial\boldsymbol{\psi} = 0$ allows us to separate out the estimation of $\boldsymbol{\psi}$:

$$\hat{\boldsymbol{\psi}} = -\left[\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}^\perp\right]^{-1}\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}\mathbf{e}, \quad (19)$$

from which we obtain an estimate of $\boldsymbol{\phi}$:

$$\hat{\boldsymbol{\phi}} = \left(\mathbf{I} - \mathbf{B}^\perp\left[\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}^\perp\right]^{-1}\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\right)\mathbf{B}\mathbf{e}. \quad (20)$$

Substituting this solution into (17) gives an unconstrained minimization problem in only $\boldsymbol{\theta}$:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\theta} \frac{1}{2}\mathbf{e}^T(\mathbf{M}_s(\boldsymbol{\theta}) - \mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}^\perp \cdot\left[\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}^\perp\right]^{-1}\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\Big)\mathbf{e}, \quad (21)$$

where $\mathbf{e}$ is now of length $2d$. This constrained algorithm is implemented exactly as the previous one:

1. Find the $\boldsymbol{\theta}$ that minimizes the sum of the elements of

$$\mathbf{M}_s(\boldsymbol{\theta}) - \mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}^\perp\left[\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta})\mathbf{B}^\perp\right]^{-1}\mathbf{B}^{\perp *}\mathbf{M}_s(\boldsymbol{\theta}).$$

2. Solve for $\hat{\boldsymbol{\phi}}$ using $\hat{\boldsymbol{\theta}}$ in (20).

The discussion above indicates that a properly constrained $\boldsymbol{\phi}$ is necessary to maintain an identifiable array parametrization. In Section IV, it will be shown that the specific choice of the constraint does not affect the asymptotic performance of the algorithm (except possibly on a set of measure zero, as described above), as long as $\boldsymbol{\phi}$ remains unique. Further conditions on the array parametrization are required to guarantee identifiability of both the DOA's and the polarization parameters, but the derivation of necessary and sufficient conditions for identifiability is quite complicated in general. This issue is addressed in more detail in the following section.

IV. Performance Analysis

In this section we examine the statistical properties of the proposed estimator, under the assumption that the number of snapshots N is large. For purposes of this analysis, we assume that the observed noise vectors constitute a zero-mean, stationary complex Gaussian random process, with second-order moments given by

$$\mathscr{E}\{\mathbf{n}(t)\mathbf{n}^*(s)\} = \sigma^2\mathbf{I}\delta_{t,s}, \qquad \mathscr{E}\{\mathbf{n}(t)\mathbf{n}^T(s)\} = 0, \quad (22)$$

where $\delta_{t,s}$ is the Kronecker delta. We also assume that the signal waveforms $\mathbf{s}(t)$ are arbitrary (second-order ergodic) random processes, but we make no assumption on the probability distribution or the temporal correlation properties of the signals.

A. Identifiability and Consistency

We begin by first investigating under what conditions the NSF estimates are consistent. As $N \to \infty$, $\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^*$ converges with probability 1 (w.p.1) to $\mathbf{E}_n\mathbf{E}_n^*$. If the horizontal and vertical response vectors are differentiable, the NSF criterion function converges w.p.1, uniformly in $\boldsymbol{\theta}$, to the limit function

$$\bar{V}_{\mathrm{NSF}}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathrm{Tr}(\mathbf{A}^*(\boldsymbol{\theta}, \boldsymbol{\phi})\mathbf{E}_n\mathbf{E}_n^*\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\phi})\mathbf{W}). \quad (23)$$

[4]One such constraint is the following: $|\phi_h|^2 + |\phi_v|^2 = 1$ and $\angle\phi_v = 0$.

Clearly, $\bar{V}_{\mathrm{NSF}}(\boldsymbol{\theta}, \boldsymbol{\phi}) \geq 0$ with equality if and only if

$$\mathbf{A}^*(\boldsymbol{\theta}, \boldsymbol{\phi})\mathbf{E}_n = 0, \tag{24}$$

or, in other words

$$\mathbf{E}_n^* \mathbf{a}(\theta_k, \phi_k) = 0 \quad \text{for } k = 1, \cdots, d. \tag{25}$$

We see that as in the standard DOA estimation problem, the parametrization of the combined array response vector must be assumed to be *unambiguous*. That is, to uniquely find the parameters of d signals, any collection of steering vectors $\mathbf{a}_k = \mathbf{a}(\theta_k, \phi_k)$ for $k = 1, \cdots, d + 1$ (with distinct signal parameters) must form a linearly independent set. If this is the case, the only possible solutions to (25) are the d true DOA's and their corresponding polarization parameters.

For diversely polarized arrays, we must check the rank of all $m \times (d + 1)$ matrices of the form $\mathbf{A}_h\mathbf{\Phi}_h + \mathbf{A}_v\mathbf{\Phi}_v$, where, as discussed in the previous section, the diagonal elements of $\mathbf{\Phi}_h$ and $\mathbf{\Phi}_v$ are subject to certain constraints. If $\mathbf{A}_h\mathbf{\Phi}_h + \mathbf{A}_v\mathbf{\Phi}_v$ is rank deficient, there exists a nonzero vector $\mathbf{z}$ satisfying

$$(\mathbf{A}_h\mathbf{\Phi}_h + \mathbf{A}_v\mathbf{\Phi}_v)\mathbf{z} = 0. \tag{26}$$

This condition may be rewritten as

$$\begin{bmatrix} \mathbf{A}_h & \mathbf{A}_v \end{bmatrix} \boldsymbol{\phi}' = 0, \tag{27}$$

where

$$\boldsymbol{\phi}' \equiv \begin{bmatrix} \boldsymbol{\phi}'_h \\ \boldsymbol{\phi}'_v \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\phi}_h \odot \mathbf{z} \\ \boldsymbol{\phi}_v \odot \mathbf{z} \end{bmatrix}.$$

When $2(d + 1) \leq m$, or equivalently when $d \leq m/2 - 1$, no such $\boldsymbol{\phi}'$ will exist provided $[\mathbf{A}_h \; \mathbf{A}_v]$ is full rank for all possible combinations of the DOA's. When $d > m/2 - 1$, "false" solutions to (27) do exist, and hence there are certain combinations of signal parameters that cause ambiguities. However, if $d \leq m - 2$, "most" combinations will still yield unique answers; i.e., the set of signal parameters that cause ambiguities has measure zero in the set of all possible signal parameters. To see this, note that the nullspace of $[\mathbf{A}_h \; \mathbf{A}_v]$ is of dimension $2(d + 1) - m$. For fixed $\boldsymbol{\phi}$ (with the linear constraint $\mathbf{B}^*\boldsymbol{\phi} = \mathbf{e}$ in force), the dimensionality of the $2(d + 1)$-vector $\boldsymbol{\phi}'$ is equal to that of $\mathbf{z}$, viz. $d + 1$. Thus, if $d + 1 \leq 2(d + 1) - m$, we can solve (27) with respect to $\mathbf{z}$ for any fixed value of $\boldsymbol{\phi}$. If on the other hand $d + 1 > 2(d + 1) - m$, or equivalently $d \leq m - 2$, the set of $d + 1$-dimensional constrained vectors $\boldsymbol{\phi}$ that permits a solution to (27) has dimensionality $2(d + 1) - m < d + 1$, and hence has measure zero in the set of all possible $\boldsymbol{\phi}$-vectors. This result is summarized in the following theorem:

Theorem 4.1: Assuming an m-element diversely polarized antenna array, let $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$ represent the DOA and polarization parameters of d signals impinging upon the array, and suppose $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ are the corresponding NSF estimates using N snapshots from the array. Then

$$\lim_{N \to \infty} \left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right) = (\boldsymbol{\theta}_0, \boldsymbol{\phi}_0) \qquad \text{w.p.1},$$

provided the following conditions are satisfied:

- $[\mathbf{A}_h(\boldsymbol{\theta}) \; \mathbf{A}_v(\boldsymbol{\theta})]$ full rank for all $\boldsymbol{\theta} \in \mathbb{R}^{d+1}$ (with distinct elements),
- $d \leq m - 2$,
- ϕ_{hi} and ϕ_{vi} not both zero for any $i = 1, \cdots, d$,
- $\mathbf{B}^*\boldsymbol{\phi}_0 = \mathbf{e}$.

Proof: See discussion above. ∎

The condition $d \leq m - 2$ was first presented in [6], but not rigorously derived. Since the DOA's of $m - 1$ signals are generally identifiable using identically polarized antennas, uniquely determining both the DOA's and polarization parameters with a diversely polarized array reduces the maximum number of resolvable signals by one.

B. Asymptotic Accuracy

With the assumption of consistent parameter estimates, the results of [12] can be used to conduct a standard statistical analysis of estimate variance. Although [12] only deals with the case of one signal parameter per source, the results immediately extend to the case considered herein. Defining the $3d$-vector of unknown, unconstrained, parameters as

$$\boldsymbol{\eta} = \left[\boldsymbol{\theta}^T, \mathrm{Re}\,(\boldsymbol{\psi})^T, \mathrm{Im}\,(\boldsymbol{\psi})^T\right]^T, \tag{28}$$

the following theorem can be shown to hold.

Theorem 4.2: Assume that the combined array response parametrization is unambiguous. Furthermore, let $\hat{\boldsymbol{\eta}}$ be obtained by inserting the NSF estimates $\hat{\boldsymbol{\psi}}$ and $\hat{\boldsymbol{\theta}}$ into (28). Then, the normalized estimation error, $\sqrt{N}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})$, has a limiting (for $N \to \infty$) zero-mean Gaussian distribution with covariance matrix

$$N\mathscr{E}\left\{(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})^T\right\} = \mathbf{C}, \tag{29}$$

where the ij-th element of the inverse of $\mathbf{C}$ is given by

$$(\mathbf{C}^{-1})_{ij} = \frac{2}{\sigma^2} \mathrm{Re}\left[\mathrm{Tr}\left\{\mathbf{A}_i^*(\mathbf{I} - \mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*)\mathbf{A}_j\mathbf{W}^T\right\}\right]. \tag{30}$$

Here, $\mathbf{A}_i$ denotes the partial derivative of $\mathbf{A}$ with respect to the i-th component of $\boldsymbol{\eta}$, and $\mathbf{W}$ is the weighting matrix given in (10). Note that $\mathbf{C}$ should be evaluated at the true DOA's and polarization parameters.

In addition, if the signals are zero-mean complex Gaussian distributed, the asymptotic variance of the NSF parameter estimates is equal to the Cramér-Rao lower bound, and hence is the minimum achievable variance of any estimator.

Proof: The proof requires only a slight modification of results in [12, 13], and thus will not be given here. ∎

The expression (30) depends on the particular constraint one chooses for the polarization parameters. However, it can be shown that the upper left block of $\mathbf{C}$, i.e., the part that gives the accuracy of the DOA estimates, does *not* depend on the constraint. It is also true that the accuracy of any uniquely defined function of the original polarization parameters, such as ϕ_{kh}/ϕ_{kv}, is also asymptotically independent of the constraint used for finding the NSF DOA estimates. The easiest way to prove the above statements is to invoke the invariance principle of maximum likelihood (ML) estimation (e.g., see [14]). This rule states that the ML estimates of uniquely defined functions of the parameters are independent (for finite data) of the specific parametrization used to calculate these quantities. Because the NSF method is asymptotically a large sample realization of the ML estimator, we conclude that the *asymptotic* accuracy of any uniquely defined quantity using the NSF method is independent of the parametrization used for estimating that quantity. However, for finite data sets the variance of the estimation errors may indeed be somewhat affected by the choice of constraint, as seen in the next section.

V. Simulation Examples

To compare the performance of the above NSF methods with the MUSIC-based approach of [5], two simple simulation studies were conducted. In both examples, a 10 element uniform linear array (ULA) was assumed for the $\mathscr{A}_h$ manifold, while $\mathscr{A}_v$ was generated by slightly perturbing the positions of a 10 element ULA (to guarantee diverse polarization). Spatially white additive noise of unit variance was assumed to be present at each antenna element. In the first example, two 10 dB emitters were simulated at 0° and 10° relative to broadside, with polarization defined by $\phi_{h1} = \phi_{h2} = 1$, and $\phi_{v1} = e^{j6\pi/5}$, $\phi_{v2} = e^{j\pi/10}$, respectively. All sensors were assumed to have unity gain, so the effective SNR for each source was between 12 and 13 dB. A total of 100 snapshots were taken from the array in 250 independent trials, and the RMS estimation error was calculated for the DOA's and polarization parameters for MUSIC and NSF (using two types of constraints).

The results are plotted in Figs. 1 and 2 for the emitter at 10°. In the figures, constraint 1 refers to the NSF algorithm using (18), while constraint 2 indicates that the NSF algorithm assumed $\phi_h = \mathbf{e}$. When the signals are 50% correlated or less, we see that all three methods give similar results. At higher correlation levels, the relative performance of NSF becomes increasingly better than that of MUSIC. Note that in this particular example, the use of NSF with constraint 2 leads to consistently better performance than constraint 1. Although our analysis indicates that both constraints must asymptotically give the same performance,[5] in this case constraint 2 provides a slight advantage for "small" N.

In the second example, the emitter at 10° was moved to 8°, the correlation between the two signals was fixed at 90%, and signal power was varied from −5 to 20 dB. The polarization of the signals in this case was assumed to be $\phi_{h1} = \phi_{h2} = 1$ and $\phi_{v1} = e^{j\pi/20}$, $\phi_{v2} = e^{-j\pi/20}$, respectively. To demonstrate the validity of the asymptotic error expressions of Section IV, N was chosen to be somewhat larger ($N = 500$) than in the previous example, and a total of 500 independent trials were conducted for each

[5]The magnitude of N required to ensure the validity of the asymptotic expressions derived in Section IV is very scenario-dependent. Difficult cases (i.e., those involving high correlation, low SNR, closely spaced emitters, etc.) may require N to be quite large indeed, while much smaller values for N are sufficient in "easier" cases.

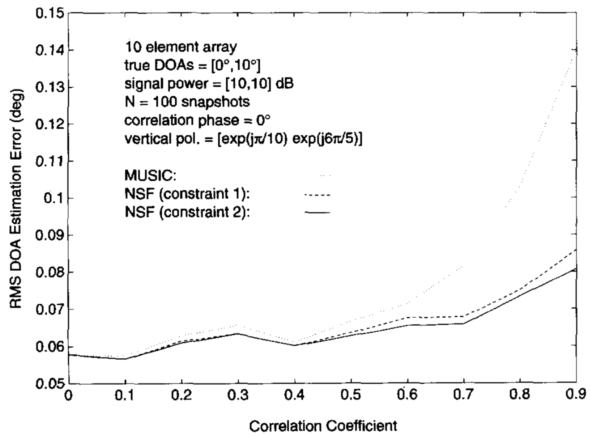

Fig. 1. Comparison of MUSIC and NSF DOA estimates, case 1.

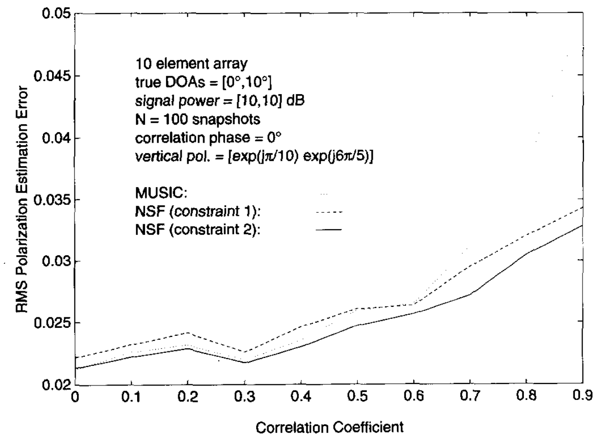

Fig. 1. Comparison of MUSIC and NSF polarization estimates, case 1.

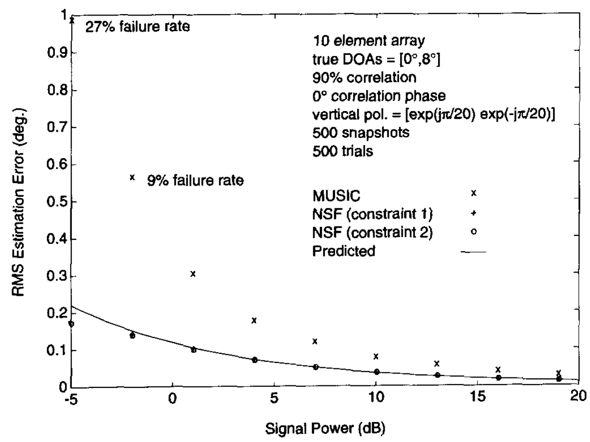

Fig. 3. Comparison of MUSIC and NSF DOA estimates, case 2.

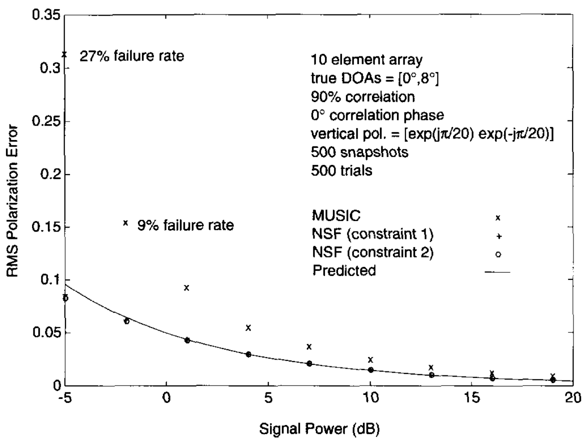

Fig. 4. Comparison of MUSIC and NSF polarization estimates, case 2.

value of signal power. The sample and predicted RMS DOA and polarization estimation errors are shown in Figs. 3 and 4. In this case, the advantage of NSF is increasingly evident at lower SNR. The failure rates shown next to the MUSIC results at -2 and -5 dB indicate the percentage of the trials where MUSIC was unable to resolve the two emitters. Note that the asymptotics have clearly "set in" for $N = 500$ in this example, as evidenced by the excellent agreement between actual and predicted NSF performance, and by the fact that both constraints yield essentially identical results.

VI. Conclusions

We have presented a statistically efficient algorithm for DOA estimation using diversely polarized arrays. The proposed technique separates the estimation of the DOA's from that of the polarization through use of the noise subspace fitting criterion function. In particular, while a multidimensional search is required to estimate the DOA parameters, the polarization parameters can be calculated explicitly. Numerical comparisons with a MUSIC-based approach revealed that our method may give significantly improved performance, particularly in scenarios involving highly correlated signals. We have also presented conditions on the array, polarization parameters, and number of emitters that guarantee an identifiable parameterization and consistent parameter estimates.

References

[1] R. T. Compton, Jr., "The tripole antenna: An adaptive array with full polarization flexibility," *IEEE Trans. Antennas Propagation*, vol. AP-29, pp. 944–952, Nov. 1981.

[2] A. Nehorai and E. Paldi, "Vector sensor processing for electromagnetic source localization," in *Proc. 25th Asilomar Conference on Signals, Systems, and Computers* (Asilomar, CA), pp. 566–572, Nov. 1991.

[3] A. J. Weiss and B. Friedlander, "Performance analysis of diversely polarized antenna arrays," *IEEE Trans. Sig. Proc.*, vol. SP-39, no. 7, pp. 1589–1603, July 1991.

[4] A. Nehorai and E. Paldi, "Vector sensor processing for electromagnetic source localization," Tech. Rep. No. 9209, Yale Univ., May 1992.

[5] R. Schmidt, A signal subspace approach to multiple emitter location and spectral estimation, Ph.D. dissertation, Stanford Univ., 1981.

[6] E. R. Ferrara and T. M. Parks, "Direction finding with an array of antennas having diverse polarizations," *IEEE Trans. Antennas Propagation*, vol. AP-31, no. 2, pp. 231–236, Mar. 1983.

[7] B. Friedlander and A. J. Weiss, "A direction finding algorithm for diversely polarized arrays," *Digital Signal Processing—A Review Journal*, vol. 2, no. 3, pp. 123–134, July 1992.

[8] J. Li and R. Compton, Jr., "Angle and polarization estimation using ESPRIT with a polarization sensitive array," *IEEE Trans. Antennas Propagation*, vol. AP-39, pp. 1376–1383, Sept. 1991.

[9] I. Ziskind and M. Wax, "Maximum likelihood localization of diversely polarized sources by simulated annealing," *IEEE Trans. Antennas Propagation*, vol. AP-38, pp. 1111–1114, July 1990.

[10] R. Schmidt, "Multilinear array manifold interpolation," *IEEE Trans. Sig. Proc.*, vol. 40, no. 4, pp. 857–866, April 1992.

[11] H. Clergeot, S. Tressens, and A. Ouamri, "Performance of high-resolution frequencies estimation methods compared to the Cramér-Rao bounds," *IEEE Trans. Acoust. Speech, Signal Processing* vol. 37, no. 11, pp. 1703–1720, Nov. 1989.

[12] P. Stoica and K. Sharman, "Maximum likelihood methods for direction-of-arrival estimation," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. 38, no. 7, pp. 1132–1143, July 1990.

[13] B. Ottersten, M. Viberg, P. Stoica, and A. Nehorai, "Exact and large sample ML techniques for parameter estimation and detection in array processing," in *Radar Array Processing*, Haykin, Litva, and Shepherd, eds. Berlin: Springer-Verlag, pp. 99–151, 1993.

[14] D. R. Cox and D. V. Hinkley, *Theoretical Statistics*. London: Chapman and Hall, Ltd., 1974.

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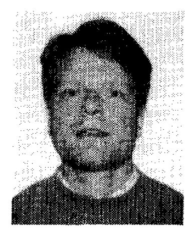

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