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# SUBSPACES OF $L_\infty(G)$ WITH UNIQUE TOPOLOGICAL LEFT INVARIANT MEAN

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## 1. INTRODUCTION

In what follows we denote by $G$ always a locally compact Hausdorff group with left invariant Haar measure. Let $A$ be an $L_1(G)$-submodule of $L_\infty(G)$ which is left invariant and containing the constant functions. A mean on $A$ is a linear functional $m$ on $A$ such that $m(\bar{g}) = \overline{m(g)}$ for all $g \in A$ (the bar denoting complex conjugation), $m(1) = 1$, and $m(g) \geqq 0$ if $g \geqq 0$ locally almost everywhere. A mean $m$ on $A$ is called *left invariant* (LIM) if $m({}_a g) = m(g)$ for all $a$ in $G$ and all $g$ in $A$. A *topologically left invariant* mean (TLIM) on $A$ is a mean $m$ such that $m(\varphi * g) = m(g)$ for all $g \in A$ and all $\varphi \in P(G) = \{\varphi \in L_1(G) : \varphi \geqq 0, \|\varphi\|_1 = 1\}$.

It is well known (see e.g. [1], [6] and [8]) that on each of the spaces $AP(G)$ and $W(G)$, being respectively the sets of almost periodic and weakly almost periodic functions in $L_\infty(G)$ there exists a unique LIM; and it is also the unique TLIM. In section 2 we construct two new subspaces of $L_\infty(G)$, one of them containing properly $AP(G)$ and the other $W(G)$, such that on each of these new spaces there exists a unique TLIM. For Abelian $G$ with dual $\hat{G}$ the first space coincides precisely with the space of those functions which are almost periodic at every point of $\hat{G}$, as introduced by Loomis in [7].

All of these results are shown by use of the so-called $\tau_c$- and $\tau_w$-topologies, which have been introduced in [2] and [3]. For convenience we repeat here their definitions. The space $L_\infty(G)$ may be embedded into $B(L_1(G), L_\infty(G))$ by the operator $\Phi$ such that $\Phi(g)(f) = f * g$ ($f \in L_1(G)$, $g \in L_\infty(G)$, $*$ the convolution product). Since $B(L_1(G), L_\infty(G))$ carries naturally the strong and the weak operator topology, $\Phi$ allows us to consider their induced topologies on $L_\infty(G)$, which we denote by $\tau_c$ and $\tau_w$ respectively. These topologies may also be introduced in another manner; indeed, each $f \in L_1(G)$ induces by convolution an operator $C_f$ on $L_\infty(G)$ which is continuous when $L_\infty(G)$ carries its norm topology $\| \|_\infty$; the weak topology on $L_\infty(G)$ under the convolution operators $C_f : L_\infty(G) \to (L_\infty(G), \| \|_\infty)$ then coincides with $\tau_c$, while $\tau_w$ is the weak topology on $L_\infty(G)$ under the same set of operators $C_f : L_\infty(G) \to (L_\infty(G), w)$, where $w$ denotes the weak topology on $L_\infty(G)$. So we immediately obtain $w^* \leqq \tau_w \leqq \tau_c \leqq \| \ \|$, and $w^* \leqq \tau_w \leqq w \leqq \| \ \|_\infty$. Moreover, $\tau_c \equiv \| \ \|_\infty$ iff $G$ is discrete.

All other nonexplained notations and definitions are taken from [8].

## 2. SUBSPACES OF $L_\infty(G)$ WITH UNIQUE TLIM

We start with the following lemma.

**Lemma 2.1.** *Let $A$ be an $L_1(G)$-submodule of $L_\infty(G)$. A* LIM *$m$ on $A$ is a* TLIM *iff $m$ is continuous for the induced $\tau_c$-topology.*

Proof. Let $m$ be a TLIM on $A$. If $g$ is a fixed function in $A$ and $(g_\lambda)_{\lambda\in\Lambda}$ is a net in $A$ that $\tau_c$-converges to $g$, then the net $(\varphi * g_\lambda)_{\lambda\in\Lambda}$ is $\| \ \|_\infty$-convergent to $\varphi * g$, for each $\varphi \in P(G)$.

Since $m$ is always continuous for the $\| \ \|_\infty$-topology, the result follows from the fact that $m(\varphi * h) = m(h)$ for all $\varphi \in P(G)$ and all $h \in A$.

Conversely, let $m$ be a LIM on $A$ which is $\tau_c$-continuous. Using the left invariance of $m$ we obtain that $m({}_af * g) = m(f * g)$ for all $a \in G, f \in L_1(G)$, $g \in A$. In particular, the functional $f \to m(f * g)$ on $L_1(G)$ is linear, bounded, and left invariant, and so there exists a constant (depending on $g$), say $c(g)$, such that $m(f * g) = c(g) \int_G f(t) \, dt$ for all $f \in L_1(G)$; this leads to $m(\varphi * g) = c(g)$ for $\varphi \in P(G)$. Let then $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity in $L_1(G)$ such that each $e_\lambda$ belongs to $P(G)$. For $g$ in $A$, the net $(e_\lambda * g)_{\lambda\in\Lambda}$ is $\tau_c$-convergent to $g$. So, due to the $\tau_c$-continuity of $m$ we obtain $c(g) = m(e_\lambda * g) \to m(g)$, while due to the $\| \ \|_\infty$-continuity of $m$ we also have $c(g) = m((\varphi * e_\lambda) * g) \to m(\varphi * g)$, for all $\varphi \in P(G)$. Hence $m$ is a TLIM on $A$. ∎

We now construct Banach subspaces of $L_\infty(G)$ on which there exists a unique TLIM.

To this end, call a function $g$ in $L_\infty(G)$ *right almost periodic with respect to $\tau_c$* ($r - \tau_c -$ a.p.) iff the set $\{g_a : a \in G\}$ of right translates of $g$ is relatively compact with respect to $\tau_c$. We denote the set of these functions by $R - \tau_c -$ AP. Analogously, using the $\tau_w$-topology we may define the set $R - \tau_w -$ AP. Since the spaces $(L_\infty(G), \tau_c)$ and $(L_\infty(G), \tau_w)$ are Hausdorff topological vector spaces, it may be verified that both sets are right invariant linear subspaces of $L_\infty(G)$.

**Lemma 2.2.**

$$g \in R - \tau_c - \mathrm{AP} \Leftrightarrow f * g \in AP(G), \quad \forall f \in L_1(G),$$

$$g \in R - \tau_w - \mathrm{AP} \Leftrightarrow f * g \in W(G), \quad \forall f \in L_1(G).$$

Proof. We only give the proof of the first equivalence. Since for any $f \in L_1(G)$, each operator $C_f : (L_\infty(G), \tau_c) \to (L_\infty(G), \| \ \|_\infty)$ with $C_f(g) = f * g$ is continuous, one implication is quickly verified using the fact that $(f * g)_a = f * g_a$.

To prove the inverse implication, let $\Phi : L_\infty(G) \to B(L_1(G), L_\infty(G))$ be the operator defined in the introduction, and put $A = \{(\Phi(g))_a : a \in G\}$, where we define

$(\Phi(g))_a(f) = (f * g)_a = \Phi(g_a)(f)$. Then $A \subset B(L_1(G), L_\infty(G))$, and an adaptation of exercise VI.9.2 in [4] shows that $A$ is relatively compact in the strong operator topology. The result then follows from the definition of $\tau_c$.

The proof of the second equivalence is analogous. ∎

From lemma 2.2 we derive that both sets $R - \tau_c -$ AP and $R - \tau_w -$ AP are $\tau_c$-closed. Indeed, if $(g_\lambda)_{\lambda \in \Lambda}$ is a net in one of these sets such that $(g_\lambda)_{\lambda \in \Lambda}$ $\tau_c$-converges to $g$, then the net $(f * g_\lambda)_{\lambda \in \Lambda}$, which is in either $AP(G)$ or $W(G)$, is $\| \; \|_\infty$-convergent to $f * g$, for each $f$ in $L_1(G)$. Since both sets $AP(G)$ and $W(G)$ are $\| \; \|_\infty$-closed, the limit function $g$ also belongs to either $R - \tau_c -$ AP or $R - \tau_w -$ AP.

Of course $R - \tau_c -$ AP and $R - \tau_w -$ AP are also $\| \; \|_\infty$-closed (hence they are Banach subspaces) since $\tau_c \leqq \| \; \|_\infty$; being convex sets, they are also $\tau_w$-closed.

In order to obtain our next result, we state [2, coroll. 3 and 4] in the form of the following lemma; $\mathrm{cl}_\tau B$ denotes the closure of a set $A$ in the topology $\tau$.

**Lemma 2.3.** *Let $S$ be a $\tau_c$-closed $L_1(G)$ submodule of $L_\infty(G)$. Then $S = \mathrm{cl}_{\tau_c}(L_1(G) * S)$, and $S$ is left translation invariant.*

Since $AP(G) \subset R - \tau_c -$ AP, and due to the fact that $L_1(G) * AP(G) = AP(G)$, we have from lemma 2.2 $AP(G) = L_1(G) * AP(G) \subset L_1(G) * R - \tau_c - \mathrm{AP} \subset AP(G)$. Hence $L_1(G) * R - \tau_c - \mathrm{AP} = AP(G)$, and from lemma 2.2 we derive that $R - \tau_c - \mathrm{AP} = \mathrm{cl}_{\tau_c}(AP(G))$. Analogously, $L_1(G) * R - \tau_w - \mathrm{AP} = W(G)$, and $R - \tau_w - \mathrm{AP} = \mathrm{cl}_{\tau_c}(W(G))$. Moreover, both sets $R - \tau_c -$ AP and $R - \tau_w -$ AP are left invariant.

**Theorem 2.4.** *There exists a unique* TLIM *on* $R - \tau_c -$ AP.

Proof. There exists a unique LIM $m$ on $AP(G)$, and it is also a TLIM; hence $m$ is also continuous for the induced $\tau_c$-topology. Since $R - \tau_c - \mathrm{AP} = \mathrm{cl}_{\tau_c}(AP(G))$, there exists an extension of $m$ to a linear functional $M$ on $R - \tau_c -$ AP which is $\tau_c$-continuous; this extension is then necessarily unique. It remains to show that this extension $M$ is a left invariant mean on $R - \tau_c -$ AP. That $M(1) = 1$, $M(\bar{g}) = \overline{M(g)}$, and $M({}_a g) = M(g)$ for $g \in R - \tau_c -$ AP is readily verified using the definition of the $\tau_c$-topology and the properties of $m$. If $g \in R - \tau_c -$ AP and $g \geqq 0$ locally almost everywhere, choose an approximate identity $(e_\lambda)_{\lambda \in \Lambda}$ in $L_1(G)$ consisting of positive functions, and put $g_\lambda = e_\lambda * g$. Then each $g_\lambda$ belongs to $AP(G)$, $g_\lambda \geqq 0$, and $(g_\lambda)$ $\tau_c$-converges to $g$. Hence $M(g) \geqq 0$. Due to lemma 2.1, $M$ is a TLIM on $R - \tau_c -$ AP. ∎

Completely analogous to theorem 2.4 we may prove

**Theorem 2.5.** *There exists a unique* TLIM *on* $R - \tau_w -$ AP.

**Corollary 2.6.** *If $G$ is compact, there exists a unique* TLIM *on* $L_\infty(G)$.

Proof. Since for given $g$ in $L_\infty(G)$ the function $s \to g_s$ from $G$ to $L_\infty(G)$ is continuous for the $\tau_c$-topology on $L_\infty(G)$, any $g \in L_\infty(G)$ is $r - \tau_c$-a.p. when $G$ is compact, i.e. $R - \tau_c - \mathrm{AP} = L_\infty(G)$. The result then follows from theorem 2.4. ■

Remark 1. Since the LIM on $AP(G)$ or $W(G)$ is also right invariant, the same is ture for the TLIM on $R - \tau_c - \mathrm{AP}$ and $R - \tau_w - \mathrm{AP}$.

Let $G$ be an Abelian group with dual $\hat{G}$. A bounded measurable function $g$ on $G$ is called *almost periodic at the point* $\gamma_0 \in \hat{G}$ iff there exists a function $f$ in $L_1(G)$ such that $f * g$ is ($\| \ \|_\infty -$)almost periodic and $\hat{f}(\gamma_0) \neq 0$ (see Loomis [7], p. 364).

**Theorem 2.7.** *For Abelian G and* $g \in L_\infty(G)$ *we have*
$g \in R - \tau_c - \mathrm{AP}$ *iff g is almost periodic at each point of* $\hat{G}$.

Proof. By lemma 2.2 it is clear that any $g$ in $R - \tau_c - \mathrm{AP}$ is almost periodic at each point of $\hat{G}$.

To prove the converse implication we have to show that, given $g$ in $L_\infty(G)$ which is almost periodic at each point of $\hat{G}$, the function $f * g$ belongs to $AP(G)$ for each $f$ in $L_1(G)$. We use the notation of [7]; in particular, we denote by spg the spectrum of a bounded function $g$. Given $\varepsilon > 0$ and $f$ in $L_1(G)$, there exists a function $v$ in $L_1(G)$ such that $\hat{v}$ has compact support, and $\|f - v * f\|_1 < \varepsilon$; also $\mathrm{sp}(v * f) \subset \subset \mathrm{sp}v = \mathrm{supp}\,\hat{v}$. This means that there exists a net $(h_\lambda)_{\lambda \in \Lambda}$ in $L_1(G)$ such that $(h_\lambda)$ $\| \ \|_1$-converges to $f$, while each $h_\lambda$ has compact spectrum.

Since $(h_\lambda * g)$ is $\| \ \|_\infty$-convergent to $f * g$, this function will belong to $AP(G)$ as soon as each $h_\lambda * g$ is almost periodic. So it suffices to prove : given $f$ in $L_1(G)$ with compact spectrum, then the function $h = f * g$ is almost periodic. By [7] theorem 1, this will be the case iff $h$ is almost periodic at each point of $\hat{G}$. Given $\gamma_0 \in \hat{G}$, there exists a function $f_0$ in $L_1(G)$ such that $f_0 * g$ is almost periodic and $\hat{f}_0(\gamma_0) \neq 0$; then $f_0 * h = f * (f_0 * g)$, and this is almost periodic since $L_1(G) * AP(G) = AP(G)$. ■

## 3. THE EXTENT OF $R-\tau_w-\mathrm{AP}$

**Theorem 3.1.** *Let G be a non-compact σ-compact amenable group. Then the quotient space* $L_\infty(G)/_{R-\tau_w-\mathrm{AP}}$ *is nonseparable.*

Proof. Put $R - \tau_w - \mathrm{AP} \equiv A$ for short, and suppose that $L_\infty(G)/_A$ is separable. Then there exists a countable dense subset $\{[g_n]\}_{n=1}^\infty$ in $L_\infty(G)/_A$, where $[g_n] = g_n + A$, and $g_n \in L_\infty(G)$. Let $B$ be the linear span in $L_\infty(G)$ of the sequence $\{g_n\}_{n=1}^\infty$; then $A + B$ is dense in $L_\infty(G)$. Let $m$ be a TLIM on $L_\infty(G)$, and put $m(g_n) = \alpha_n$. If $M$ is also a TLIM on $L_\infty(G)$ such that $M(g_n) = \alpha_n$, then $M = m$; indeed, $M = m$ on $B$ by assumption, and $M = m$ on $A$ since $A$ has a unique TLIM; the result then follows from the denseness of $A + B$ in $L_\infty(G)$. Putting $C = \mathrm{cl}_w * P(G) \cap \cap \{\mathcal{M} \in \mathrm{TLIM} : \mathcal{M}(g_n) = \alpha_n\}$, we derive that $C$ is norm separable. According to [5, theorem 5], this is sufficient to conclude that $G$ would be compact. ■

**Corollary.** *If G is σ-compact and* $R - \tau_w - \mathrm{AP} = L_\infty(G)$, *then G is compact.*

Remark. The result of this last corollary is also true without the assumption that $G$ is $\sigma$-compact. Indeed, if $R - \tau_w - \mathrm{AP} = L_\infty(G)$, then $W(G) = L_1(G) * R - \tau_w - \mathrm{AP} = L_1(G) * L_\infty(G) = C_{ru}(G)$, where $C_{ru}(G)$ denotes the set of right uniformly continuous functions on $G$; hence $W(G)$ contains the set of functions on $G$ which are both left and right uniformly continuous. This is known to be a sufficient condition for the compactness of $G$.

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