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SUBSPACES OF SYMMETRIC MATRICES CONTAINING MATRICES WITH A MULTIPLE FIRST EIGENVALUE

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Let $\mathcal{U}$ be an $(r-1)(2n-r+2)/2$ dimensional subspace of $n \times n$ real valued symmetric matrices. Then $\mathcal{U}$ contains a nonzero matrix whose greatest eigenvalue is at least of multiplicity r, if $2 \leqq r \leqq n-1$. This bound is best possible. We apply this result to prove the Bohnenblust generalization of Calabi's theorem. We extend these results to hermitian matrices.

1. Introduction. Let $\mathcal{W}_n$ be the $n(n+1)/2$ dimensional vector space of all real valued $n \times n$ symmetric matrices. Let A belong to $\mathcal{W}_n$. Arrange the eigenvalues of A in decreasing order

$$\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots \geqq \lambda_n(A). \tag{1.1}$$

We say that $\lambda_1(A)$ is of multiplicity r if

$$\lambda_1(A) = \cdots = \lambda_r(A), \tag{1.2a}$$

$$\lambda_r(A) > \lambda_{r+1}(A). \tag{1.2b}$$

Let $\mathcal{U}$ be a subspace of $\mathcal{W}_n$ of dimension k. We consider the question of how large k has to be so that $\mathcal{U}$ must contain a nonzero matrix A which satisfies (1.2a) for a given r. The nontrivial case would be

$$2 \leqq r \leqq n-1. \tag{1.3}$$

Clearly for $r = n$ we must have $k = n(n+1)/2$ as $\mathcal{U}$ will contain the identity matrix I.

We now state our main result:

THEOREM 1. *Let $\mathcal{U}$ be a k dimensional subspace in the space $\mathcal{W}_n$ of $n \times n$ real valued matrices. Assume that an integer r satisfies the inequalities* (1.3).

If

$$k \geqq \kappa(r) \tag{1.4}$$

where

$$\kappa(r) = (r-1)(2n-r+2)/2, \qquad r = 1, 2, \cdots, n \tag{1.5}$$

then $\mathcal{U}$ contains a nonzero matrix A such that the greatest eigenvalue of A is at least of multiplicity r. The lower bound $\kappa(r)$ is best possible for $2 \leqq r \leqq n-1$.

Theorem 1 is proved in §2. In §3 we prove that Theorem 1 is equivalent to the following result due to Bohnenblust (cf. **[1]** and **[4]**). We denote as usual by (x, y) the inner product of the vectors x and y in $\mathbf{R}^n$, which is the underlying vector space for $\mathcal{W}_n$.

THEOREM 2 (Bohnenblust). *Let $\mathcal{V}$ be a subspace of dimension k in $\mathcal{W}_n$ and let $1 \leqq r \leqq n-1$. Assume that $\mathcal{V}$ has the following property:*

$$\sum_{i=1}^{r} (Ax_i, x_i) = 0 \textit{ for every } A \textit{ in } \mathcal{V} \tag{1.6}$$

implies that $x_i = 0$ for $i = 1, \cdots, r$. If

$$k < f(r+1) - \delta_{n,r+1}, \tag{1.7}$$

where

$$f(r) = r(r+1)/2, \tag{1.8}$$

then $\mathcal{V}$ contains a positive definite matrix.

In case $r = 1$, Bohnenblust's result reduces to the following theorem, known as the *Calabi theorem* **[2]**: Let $n \geqq 3$ and suppose that S_1 and S_2 are $n \times n$ symmetric matrices such that $(S_1x, x) = (S_2x, x) = 0$ implies $x = 0$. Then there exist real α_1 and α_2 such that $\alpha_1 S_1 + \alpha_2 S_2$ is positive definite.

Bohnenblust defines a subspace $\mathcal{V}$ with the property:

$$\sum_{i=1}^{r} (Ax_i, x_i) = 0 \text{ for every } A \neq 0 \text{ in } \mathcal{V} \text{ implies } x_1 = x_2 = \cdots = x_r = 0 \tag{1.9}$$

to be *jointly definite of degree r*. Thus, the equivalence of Theorems 1 and 2 relates the notion of a subspace which is jointly definite of degree r with that of a subspace containing a nonzero matrix whose largest eigenvalue has multiplicity r.

Finally, in §4 we prove that if we let $\mathcal{W}_n$ be the n^2 dimensional *real space* of all $n \times n$ hermitian matrices then Theorems 1 and 2 remain correct if $\kappa(r)$ and $f(r)$ are defined as follows

$$\kappa(r) = (r-1)(2n-r+1), \tag{1.10}$$

$$f(r) = r^2. \tag{1.11}$$

2. *Proof of Theorem* 1. We first establish a weaker form of Theorem 1 which will be needed for the proof of Theorem 1.

LEMMA 1. *Let* $1 \leqq r \leqq n$. *Let* $\mathcal{U}$ *be a* k*-dimensional subspace of* $\mathcal{W}_n$ *and assume that*

$$k \geqq 1 + \kappa(r). \tag{2.1}$$

Then there exists A *in* $\mathcal{U}$ *such that*

$$\lambda_1(A) = \cdots = \lambda_r(A) = 1. \tag{2.2}$$

Proof. For $r = 1$ (2.2) trivially holds. For $r = n$ (2.2) is also obvious as $1 + \kappa(n) = n(n+1)/2$. Suppose that the lemma holds for $r = p$. Next we construct A which satisfies (2.2) for $r = p + 1$. Let B^* satisfy

$$\lambda_1(B^*) = \cdots = \lambda_p(B^*) = 1, \qquad (p \geqq 1). \tag{2.3}$$

The existence of B^* follows from our assumptions. Assume that

$$1 > \lambda_{p+1}(B^*). \tag{2.4}$$

Otherwise B^* would satisfy (2.2) for $r = p + 1$. Let

$$B^*\xi_i = \lambda_i(B^*)\xi_i\,;\ (\xi_i, \xi_j) = \delta_{ij}, \qquad i, j = 1, \cdots, n. \tag{2.5}$$

Suppose that $A_1, \cdots, A_k$ form a basis for $\mathcal{U}$. Consider the system

$$\sum_{j=1}^{k} \alpha_j A_j \xi_i = 0, \qquad i = 1, \cdots, p. \tag{2.6}$$

We claim that (2.6) is equivalent to $\kappa(p+1) = \kappa(r)$ scalar equations. Indeed, we can assume $[\xi_1, \cdots, \xi_n]$ to be the standard basis in $\mathbf{R}^n$. Then each A_i is represented by an appropriate $n \times n$ symmetric matrix

$$A_i = (a^i_{\mu\nu}), \qquad i = 1, \cdots, k. \tag{2.7}$$

So (2.6) is equivalent to

$$\sum_{j=1}^{k} \alpha_j a_{\mu\mu} = 0, \qquad \mu = 1, \cdots, p, \tag{2.8a}$$

$$\sum_{j=1}^{k} \alpha_j a^j_{\mu\nu} = 0, \qquad \mu = 1, \cdots, p;\ \nu = \mu + 1, \cdots, n. \tag{2.8b}$$

Clearly (2.8a) and (2.8b) are a system of $\kappa(p+1) = p(2n-p+1)/2$ linear equations in the unknowns $\alpha_1, \cdots, \alpha_k$. As $k \geqq 1 + \kappa(p+1)$ we have a nontrivial solution of (2.6). Hence there exists $C \neq 0$ in $\mathcal{U}$ such that

$$C\xi_i = 0, \qquad i = 1, \cdots, p. \tag{2.9}$$

We can assume that

$$\lambda_1(C) > 0. \tag{2.10}$$

(Otherwise take $-C$). Consider the matrix

$$C(\alpha) = B^* + \alpha C. \tag{2.11}$$

Clearly, (2.3), (2.4) and (2.9) imply for $|\alpha|$ small enough

$$\lambda_1(C(\alpha)) = \cdots = \lambda_p(C(\alpha)) = 1, \tag{2.12a}$$

$$1 > \lambda_{p+1}(C(\alpha)). \tag{2.12b}$$

We claim that there exists α^* such that

$$\lambda_1(C(\alpha^*)) = \cdots = \lambda_{p+1}(C(\alpha^*)) = 1. \tag{2.13}$$

Otherwise we must have for all $\alpha > 0$ the conditions (2.12). But for a large positive α we have that $\lambda_1(C(\alpha)) = \alpha\lambda_1(C) + O(1)$. This contradicts (2.12a). Thus (2.13) holds. End of proof.

Thus, Theorem 1 shows that if we relax the condition that the largest eigenvalue of $A \neq 0$ of multiplicity r would be distinct from zero then for $2 \leqq r \leqq n-1$ the bound (2.1) can be reduced by 1. We will show later that the bound $\kappa(r) + 1$ is sharp.

LEMMA 2. *Let $2 \leqq r \leqq n$. Let $\mathcal{U}$ be a k-dimensional subspace of $\mathcal{W}_n$ and suppose that $k \geqq \kappa(r)$. Assume that for any nonzero A in $\mathcal{U}$ we have*

$$\lambda_1(A) > \lambda_r(A). \tag{2.14}$$

Let $\eta_1, \eta_2, \cdots, \eta_{r-1}$ be a set of $r-1$ arbitrary orthonormal vectors. Consider the system

$$(2.15) \qquad A\eta_i = \lambda\eta_i, \quad i = 1, 2, \cdots, r-1, \quad \text{and} \quad A \in \mathcal{U}.$$

Then there exists a nonzero matrix A_0 in $\mathcal{U}$ and a scalar λ_0 such that

$$(2.16) \qquad A_0\eta_i = \lambda_0\eta_i, \qquad i = 1, 2, \cdots, r-1,$$

and

$$(2.17) \qquad \lambda_0 = \lambda_1(A_0) = \cdots = \lambda_{r-1}(A_0).$$

Moreover, for any pair A and λ, where A belongs to $\mathcal{U}$, that satisfies (2.15), *there exists α such that*

$$A = \alpha A_0 \quad \text{and} \quad \lambda = \alpha\lambda_0.$$

Proof. From Lemma 1 we deduce the existence of $B^* \neq 0$ in $\mathcal{U}$ such that $\lambda_1(B^*) = \lambda_{r-1}(B^*) = 1$. Let $\xi_1, \cdots, \xi_{r-1}$ be $r-1$ orthonormal vectors corresponding to 1. We first prove the lemma in case that $\eta_i = \xi_i$, $i = 1, \cdots, r-1$. Suppose that there exists a matrix C in $\mathcal{U}$, linearly independent of B^*, such that $C\xi_i = \mu\xi_i$, $i = 1, \cdots, r-1$. We may assume that $\mu = 0$, for otherwise replace C by $C - \mu B^*$. As in the proof of Lemma 1 we define $C(\alpha) = B^* + \alpha C$ and may conclude that there exists α^* such that $\lambda_1(C(\alpha^*)) = \lambda_r(C(\alpha^*))$ holds. This contradicts (2.14). Thus $C = \beta B^*$ and since $\mu = 0$ we must have that $\beta = 0$. So for $\eta_i = \xi_i$, $i = 1, \cdots, r-1$ the lemma is proved.

Now let $\eta_1, \cdots, \eta_{r-1}$ be $r-1$ arbitrary orthonormal vectors. Since $r-1 < n$ it is easy to show that there exists a system $\xi_1(t), \cdots, \xi_{r-1}(t)$ of $r-1$ orthonormal vectors for $0 \leqq t \leqq 1$ which depends continuously on t and

$$(2.18) \qquad \xi_i(0) = \xi_i, \quad \xi_i(1) = \eta_i, \qquad i = 1, \cdots, r-1.$$

For any t, $0 \leqq t \leqq 1$, consider now the system

$$(2.19) \qquad A\xi_i(t) = \lambda\xi_i(t), \quad i = 1, \cdots, r-1, \text{ and } A \in \mathcal{U}.$$

As was shown in the proof of Lemma 1, this system is equivalent to $\kappa(r)$ linear equations. The number of variables is $k+1$, namely $\alpha_1, \cdots, \alpha_k, \lambda$ where $A = \sum_{i=1}^{k} \alpha_i A_i$ and k is the dimension of $\mathcal{U}$ ($A_1, A_2, \cdots, A_k$ form a basis for $\mathcal{U}$). The assumption $k \geqq \kappa(r)$ implies the existence of a nontrivial solution of (2.19). Clearly, if $A = 0$ then $\lambda = 0$, so we always have a nontrivial solution with respect to $\alpha_1, \cdots, \alpha_k$.

For $t=0$ it follows from (2.18) that the system (2.19) has rank $\kappa(r)$, whence $k=\kappa(r)$. Thus for $0 \leqq t \leqq \epsilon$ $(\epsilon>0)$ we would always have, up to scalar multiples, exactly one nontrivial solution $A(t)$ in $\mathcal{U}$ such that

$$A(t)\xi_i(t)=\lambda(t)\xi_i(t), \qquad i=1,\cdots,r-1. \tag{2.20}$$

We can choose $A(t)$ to be dependent continuously on t as long as the rank of the system (2.19) is $\kappa(r)$. Without any restriction we may assume that $\|A(t)\|=1$ for some matrix norm on $\mathcal{W}_n$. Since $\lambda(0)=\lambda_1(A(0))=\cdots=\lambda_{r-1}(A(0))$, the continuity of $A(t)$ for $0 \leqq t \leqq \epsilon$ and the assumption (2.14) imply

$$\lambda_1(A(t))=\lambda(t) \tag{2.21}$$

for $0 \leqq t \leqq \epsilon$. Suppose to the contrary that (2.15) has at least two linearly independent solutions. Let $0<t_0 \leqq 1$ be the first time that the system (2.19) has two linearly independent solutions. Thus $A(t)$ is continuous for $0 \leqq t < t_0$. Now (2.21) together with the assumption $\|A(t)\|=1$ implies the existence of $B \neq 0$ in $\mathcal{U}$ such that

$$B\xi_i(t_0)=\lambda_0\xi_i(t_0), \qquad i=1,\cdots,r-1, \tag{2.22}$$

and $\lambda_0=\lambda_1(B)=\cdots=\lambda_{r-1}(B)$. The condition (2.14) implies that $\lambda_1(B)>\lambda_r(B)$. By assumption we must have a solution C in $\mathcal{U}$, linearly independent of B, such that

$$C\xi_i(t_0)=\mu\xi_i(t_0), \qquad i=1,\cdots,r-1. \tag{2.23}$$

If $\mu=0$ then, as in the proof of Lemma 1, we deduce that there exists α^* such that $\lambda_1(C(\alpha^*))=\lambda_r(C(\alpha^*))$, where $C(\alpha)=B+\alpha C$. If $\mu \neq 0$ let $B_1=C(\alpha_1)$ where α_1 is chosen to be small enough such that $\lambda_1(B_1)>\lambda_r(B_1)$ and $\lambda_1(B_1)\neq 0$. Then as in the proof of Lemma 1 we may assume that $\mu=0$ and we again have the equality $\lambda_1(C(\alpha^*))=\lambda_r(C(\alpha^*))$. This contradicts (2.14). The proof is complete.

Proof of Theorem 1. Let $2 \leqq r \leqq n-1$. Assume to the contrary that any $A \neq 0$ in $\mathcal{U}$ satisfies the inequality (2.14). We then deduce the existence of a nonzero matrix in $\mathcal{U}$ such that

$$\lambda_1(C)>\lambda_2(C)=\cdots=\lambda_r(C)>\lambda_n(C). \tag{2.24}$$

For $r=2$ the condition (2.14) implies (2.24) for any $C \neq 0$. Let $3 \leqq r \leqq n-1$. Consider again the matrix B^* which satisfies $\lambda_1(B^*)=\cdots=\lambda_{r-1}(B^*)=1$. Let $\xi_1,\cdots,\xi_{r-1}$ be $r-1$ corresponding orthonormal eigenvectors. Let $\mathcal{U}'$ be a $\kappa(r)-1$ dimensional subspace of $\mathcal{U}$ which does not contain B^*. Consider the equation

$$C\xi_i = 0, \qquad i = 2, \cdots, r-1 \text{ and } C \in \mathcal{U}'. \tag{2.25}$$

Since U' is $\kappa(r)-1$ dimensional, (2.25) is equivalent to a linear system of $\kappa(r-1)$ equations in $\kappa(r)-1$ unknowns. Since we assumed that $3 \leqq r \leqq n-1$ it follows that $\kappa(r)-1 > \kappa(r-1)$, whence there exists a nonzero solution C of (2.25).

If $\lambda_2(C) = \cdots = \lambda_{n-1}(C) = 0$ then (2.24) clearly holds. Hence we may assume that $\lambda_1(C) \geqq \lambda_2(C) > 0$, and let $C(\alpha) = B^* + \alpha C$. It follows from (2.25) that $\lambda_1(B^*)$ is an eigenvalue of $C(\alpha)$ of multiplicity $r-2$ at least, for any α. But for α sufficiently large $\lambda_1(C(\alpha)) > \lambda_1(B^*)$ and $\lambda_2(C(\alpha)) > \lambda_1(B^*)$. Define

$$T = \{\alpha\colon \alpha \geqq 0,\ \lambda_1(C(\alpha)) > \lambda_1(B^*) \text{ and } \lambda_2(C(\alpha)) > \lambda_1(B^*)\}.$$

T is not empty, so define $\gamma = \inf\{\alpha\colon \alpha \in T\}$. We must have $\gamma > 0$, because of (2.14). The matrix $C(\gamma)$ satisfies (2.24).

Finally, we show that (2.14) leads to a contradiction. Let C be a matrix that satisfies (2.24). Let $\eta_1, \eta_2, \cdots, \eta_{r-1}$ be $r-1$ orthonormal eigenvectors corresponding to $\lambda_2(C) = \cdots = \lambda_r(C)$. By Lemma 2, there exists a matrix A in $\mathcal{U}$, $A \neq 0$, such that $\lambda_1(A) = \lambda_{r-1}(A)$ and $A\eta_i = \lambda_1(A)\eta_i$, $i = 1, 2, \cdots, r-1$. Moreover, by Lemma 2 $C = \alpha A$ for some $\alpha \neq 0$. But this contradicts (2.24). This contradiction proves that there exists a nonzero matrix in $\mathcal{U}$ satisfying the condition $\lambda_1(A) = \cdots = \lambda_r(A)$.

We now show that the bound $\kappa(r)$ is sharp. Consider the subspace $\mathcal{U}$ of $n \times n$ symmetric matrices $A = (a_{ij})$ of the form

$$a_{ij} = 0, \qquad i, j = 1, \cdots, n-r+1, \tag{2.26}$$

$$\sum_{i=n-r+2}^{n} a_{ii} = 0. \tag{2.27}$$

It is clear that the dimension of this subspace is $\kappa(r)-1$. We claim that there exists no $A \neq 0$ in $\mathcal{U}$ which satisfies $\lambda_1(A) = \lambda_r(A)$. Suppose to the contrary that such A exists. As $\operatorname{tr}(A) = 0$ and $A \neq 0$ we must have that $\lambda_1(A) > 0$. Consider the matrix $B = \lambda_1(A)I - A$. The assumption $\lambda_1(A) = \lambda_r(A)$ implies that the rank of B does not exceed $n-r$. From the conditions (2.26) we deduce that the principal minor $B\binom{1, \cdots, n-r+1}{1, \cdots, n-r+1} = \lambda_1(A)^{n-r+1} \neq 0$. So the rank of B is at least $n-r+1$. From the contradiction above we deduce the non-existence of $A \neq 0$ *in* $\mathcal{U}$ satisfying $\lambda_1(A) = \lambda_r(A)$. The proof of the theorem is completed.

REMARK 1. By modifying the example given in the proof of Theorem 1 we demonstrate that the bound $\kappa(r)+1$ which was given in Lemma 1 is sharp. Consider the $\kappa(r)$ dimensional subspace $\mathcal{U}$ given by the condition (2.26). Let $A \neq 0$ and $\lambda_1(A) = \lambda_r(A)$. The existence of such A follows from Theorem 1. Now let $B = \lambda_1(A)I - A$. Thus the rank of B does not exceed $n-r$. So $B\binom{1,\cdots,n-r+1}{1,\cdots,n-r+1} = \lambda_1(A)^{n-r+1} = 0$.

Theorem 1 shows that the situation described in Lemma 2 can only hold for $r = n$. Thus we have

COROLLARY 1. *Let $\mathcal{U}$ be a subspace of $\mathcal{W}_n$ of co-dimension* 1 (dim $\mathcal{U} = n(n+1)/2 - 1$). *Assume that $\mathcal{U}$ does not contain the identity matrix I. Then for any given $n-1$ orthonormal vectors $\eta_1, \cdots, \eta_{n-1}$ there exists a unique nonzero matrix A in $\mathcal{U}$ (up to a multiplication by positive scalar) such that*

$$\lambda_1(A) = \cdots = \lambda_{n-1}(A) > \lambda_n(A) \tag{2.28}$$

and the corresponding eigenspace for the eigenvalue $\lambda_1(A)$ is spanned by $\eta_1, \cdots, \eta_{n-1}$.

3. The equivalence of Theorems 1 and 2. We regard $\mathcal{W}_n$ as a real inner product space with the standard inner product $(A, B) = \mathrm{tr}(AB)$. Let

$$B\xi_i = \lambda_i(B)\xi_i, \quad (\xi_i, \xi_j) = \delta_{ij}, \qquad i, j = 1, \cdots, n. \tag{3.1}$$

Then by choosing $[\xi_1, \cdots, \xi_n]$ as a basis in $\mathbf{R}^n$ we obtain

$$\mathrm{tr}(AB) = \sum_{i=1}^{n} \lambda_i(B)(A\xi_i, \xi_i). \tag{3.2}$$

We need in the sequel the following well known lemma (cf. **[3]**).

LEMMA 3. *Let $\mathcal{U}$ be a subspace and $\mathcal{K}$ be a pointed closed convex cone in $\mathbf{R}^n$. Let $\mathcal{U}^\perp$ be the orthogonal complement of $\mathcal{U}$ and $\mathcal{K}^*$ the dual of $\mathcal{K}$ in $\mathbf{R}^n$. Then the following are equivalent*

(a) $\mathcal{U} \cap \mathcal{K} = \{0\}$.

(b) $\mathcal{U}^\perp \cap$ *interior* $\mathcal{K}^* \neq \emptyset$.

Now let $\mathcal{K}$ be the cone of positive semidefinite matrices in $\mathcal{W}_n$. It is a well known fact that $\mathcal{K}^* = \mathcal{K}$. Finally we remark that the functions $\kappa(r)$ and $f(r)$ defined by (1.5) and (1.8), respectively, satisfy the identity

$$\kappa(r) + f(n-r+1) = \dim \mathcal{W}_n, \qquad r = 1, \cdots, n. \tag{3.3}$$

(In case that $\mathcal{W}_n$ is the space of $n \times n$ hermitian matrices we use the Definitions (1.10) and (1.11).)

Theorem 1 *implies Theorem* 2. Suppose that the subspace $\mathcal{V}$ of $\mathcal{W}_n$ satisfies the assumptions of Theorem 2. By Lemma 3 it suffices to prove that

$$\mathcal{V}^\perp \cap \mathcal{K} = \{0\}. \tag{3.4}$$

Suppose this is not the case. It follows from (1.6) and (3.2) that $\mathcal{V}^\perp$ contains no nonzero positive semidefinite matrix of rank r or less. Let $d =$ dimension of $\mathcal{V}^\perp$. It follows from (1.7) and (3.3) that

$$d = \frac{n(n+1)}{2} - k > \frac{n(n+1)}{2} - f(r+1) + \delta_{n,r+1} = \kappa(n-r) + \delta_{n,r+1}. \tag{3.5}$$

Since $1 \leqq r \leqq n-1$ we have $1 \leqq n-r \leqq n-1$.

Suppose first that $\mathcal{V}^\perp$ contains a positive definite matrix. Since the assumptions and the conclusion of Theorem 2 remain valid under a congruence transformation, we may assume that $I \in \mathcal{V}^\perp$. If $r \leqq n-2$ then (3.5) and Theorem 1 imply that there exists a nonzero matrix in $\mathcal{V}^\perp$ such that $\lambda_1(A) = \lambda_{n-r}(A) > \lambda_n(A)$. Hence there exists a nonzero positive semidefinite matrix in $\mathcal{V}^\perp$ of the form $\alpha A + \beta I$ which has rank r or less, contrary to our assumption. If $r = n-1$ then $d \geqq 2$, by (3.5). Hence there exists A in $\mathcal{V}^\perp$ which is linearly independent of I. The matrix $\lambda_1(A)I - A$ is a nonzero positive semidefinite matrix of rank $n-1$ or less, contrary to our assumption.

It remains to consider the case that $\mathcal{V}^\perp$ contains no positive definite matrix. Let A_1 be a nonzero positive semidefinite in $\mathcal{V}^\perp$ of minimal rank q. Then $q \geqq r+1$. Hence we may assume that $1 \leqq r \leqq n-2$. We may also assume that

$$A_1 = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $A_1, A_2, \cdots, A_d$ be a basis for $\mathcal{V}^\perp$. Partition these matrices in the form

$$A_i = [A_i^{(1)}, A_i^{(2)}], \qquad i = 1, 2, \cdots, d,$$

where $A_i^{(1)}$ is of size $n \times q$. We claim that the matrices $A_2^{(2)}, \cdots, A_d^{(2)}$ are linearly dependent. Indeed, consider

$$\sum_{i=2}^{d} \alpha_i A_i^{(2)} = 0.$$

This leads to a linear system of $n(n+1)/2 - q(q+1)/2 = \kappa(n+1-q)$ equations in $d-1$ unknowns. By (3.5) $d-1 \geqq \kappa(n-r)$, so we get a nontrivial solution with the only possible exception being $q = r+1$ and $d-1 = \kappa(n-r)$. But in the latter case, if $A_2^{(2)}, \cdots, A_d^{(2)}$ are linearly independent, we may form a new basis for $\mathcal{V}^\perp$ that contains among its matrices the matrix A_1 and the matrices $B_1, B_2, \cdots, B_{n-q}$, where

$$B_i = \begin{bmatrix} B_{11}^i & 0 \\ 0 & E_{ii} \end{bmatrix}, \qquad i = 1, 2, \cdots, n-q.$$

Here E_{ii} is the matrix of order $n-q \times n-q$ all of whose entries are zero except the i, i entry which is 1. We can now form a positive definite matrix as a linear combination of $A_1, B_1, \cdots, B_{n-q}$, contrary to assumption. Hence $A_2^{(2)}, \cdots, A_d^{(2)}$ are linearly dependent.

Hence there exists a matrix B, $B = \sum_{i=2}^{d} \alpha_i A_i$, such that $b_{ij} = 0$ whenever $i > q$ or $j > q$. Clearly, there exists a linear combination of A_1 and B which is nonzero and positive semidefinite of rank $q-1$ or less. This contradicts the definition of q. Hence (3.4) is satisfied, completing the proof.

Theorem 2 *implies Theorem* 1. Assume that $2 \leqq r \leqq n-1$ and that $\mathcal{U}$ satisfies the assumptions of Theorem 1. Suppose that $\mathcal{U}$ contains no nonzero matrix A such that $\lambda_1(A) = \lambda_r(A)$. Then $I \notin \mathcal{U}$ and let $\mathcal{U}_1$ = linear space spanned by $\mathcal{U}$ and I. Clearly $\dim \mathcal{U}_1 \geqq \kappa(r) + 1$. Let $\mathcal{V} = \mathcal{U}_1^\perp$, so $\mathcal{U}_1 = \mathcal{V}^\perp$. The subspace $\mathcal{U}_1$ contains no nonzero positive semidefinite matrix of rank $n-r$ or less. Now (3.3) implies that $\dim \mathcal{V} < f(n-r+1)$. Since $n-r \leqq n-2$ we have that $\delta_{n,n+1-r} = 0$, so the subspace $\mathcal{V}$ satisfies the assumptions of Theorem 2. It follows that $\mathcal{V}$ contains a positive definite matrix. However, since I is in $\mathcal{U}_1$, from the fact that $\mathcal{V} = \mathcal{U}_1^\perp$ it follows that for any A in $\mathcal{V}$ we must have that $\operatorname{tr}(AI) = \operatorname{tr}(A) = 0$. Thus $\mathcal{V}$ could not contain a positive definite matrix. This contradiction implies the existence of $A \neq 0$ in $\mathcal{U}$ such that $\lambda_1(A) = \lambda_r(A)$.

4. Extensions and remarks. We now reformulate Theorems 1 and 2 in the case where $\mathcal{W}_n$ is the n^2 dimensional real space of $n \times n$ complex valued hermitian matrices.

THEOREM 3. *Let $\mathcal{U}$ be a k dimensional subspace in the space $\mathcal{W}_n$ of $n \times n$ complex valued hermitian matrices. Assume that an integer r satisfies the inequalities $2 \leqq r \leqq n-1$. If $k \geqq \kappa(r)$, where $\kappa(r) = (r-1)(2n-r+1)$, then $\mathcal{U}$ contains a nonzero matrix such that the greatest eigenvalue of A is at least of multiplicity r. The lower bound $\kappa(r)$ is best possible for $2 \leqq r \leqq n-1$.*

Proof. The proof of this theorem is identical with the proof of Theorem 1 except for the following detail. Let $\xi_1, \cdots, \xi_{r-1}$ be $r-1$ orthonormal vectors. Consider the system

$$A\xi_j = \lambda\xi_j, \qquad j = 1, \cdots, r-1, \tag{4.1}$$

where A belongs to $\mathcal{U}$. We claim that this system is equivalent to $\kappa(r)$ real valued equations. Indeed, if we complete the set $\xi_1, \cdots, \xi_{r-1}$ to a basis of orthonormal vectors $[\xi_1, \cdots, \xi_n]$ then, assuming this to be the standard basis, we obtain instead of (4.1):

$$a_{\mu\mu} = \lambda, \qquad \mu = 1, \cdots, r-1, \tag{4.2}$$

and

$$a_{\mu\nu} = 0, \quad \mu = 1, \cdots, r-1; \ \nu = \mu + 1, \cdots, n. \tag{4.3}$$

Since $A = (a_{ij})$, is hermitian, $a_{\mu\mu}$ is real. So (4.2) is equivalent to $r-1$ equations. Since $a_{\mu\nu}$ for $\mu \neq \nu$ is complex valued, (4.3) is equivalent to $(r-1)(2n-r)$ real equations. This fact explains the change of the value of $\kappa(r)$ in case that $\mathcal{W}_n$ is the space of hermitian matrices. End of proof.

Finally, we restate Bohnenblust's theorem for the hermitian case.

THEOREM 4 (Bohnenblust). *Let $\mathcal{V}$ be a subspace of dimension k in $\mathcal{W}_n$ and let $1 \leqq r \leqq n-1$. Assume that for any A in $\mathcal{V}$ the equality* (1.6) *implies that $x_i = 0$ for $i = 1, \cdots, r$. If the inequality* (1.7) *holds where $f(r) = r^2$, then $\mathcal{V}$ contains a positive definite matrix.*

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