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Chapter 2

SUBSTRUCTURES OF FINITE CLASSICAL POLAR SPACES

***Jan De Beule*[*,†], *Andreas Klein*[*] and *Klaus Metsch*[‡]**

Abstract

We survey results and particular facts about (partial) ovoids, (partial) spreads, m-systems, m-ovoids, covers and blocking sets in finite classical polar spaces. `second revision, 6th januari`

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1 Finite classical polar spaces

The *finite classical polar spaces* are the geometries that are associated with non-degenerate sesquilinear and non-singular quadratic forms on vector spaces over a finite field. Given a projective space $\mathrm{PG}(d,q)$, then a polar space $\mathcal{P}$ in $\mathrm{PG}(d,q)$ consists of the projective subspaces of $\mathrm{PG}(d,q)$ that are totally isotropic with relation to a given non-degenerate sesquilinear form or that are totally singular with relation to a given non-singular quadratic form. The projective space $\mathrm{PG}(d,q)$ is called the *ambient projective space* of $\mathcal{P}$. In this article, with "polar space" we always refer to "finite classical polar space".

A projective subspace of maximal dimension in a polar space $\mathcal{P}$ is called a *generator*. One can prove (see [44], Theorem 26.1.2) that all generators have the same dimension $r-1$. We call r the *rank* of the polar space. A polar space of rank 1 only contains projective points. There exist five different types of finite classical polar spaces, which are, up to transformation of the coordinate system, described as follows:

[*]Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281 S22, B–9000 Gent, Belgium. E-mail: {jdebeule,klein}@cage.ugent.be.

[†]This author is a Postdoctoral Research Fellow of the Research Foundation – Flanders (FWO)

[‡]Universität Gießen, Mathematisches Institut, Arndtstraße 2, D–35392 Gießen, Germany. E-mail: Klaus.Metsch@math.uni-giessen.de

- The elliptic quadric $\mathrm{Q}^-(2n+1,q)$, $n \geq 1$, formed by all points of $\mathrm{PG}(2n+1,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2n-2}x_{2n-1} + f(x_{2n},x_{2n+1}) = 0$, where f is a homogeneous irreducible polynomial of degree 2 over $\mathbb{F}_q$.
- The parabolic quadric $\mathrm{Q}(2n,q)$, $n \geq 1$, formed by all points of $\mathrm{PG}(2n,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2n-2}x_{2n-1} + x_{2n}^2 = 0$.
- The hyperbolic quadric $\mathrm{Q}^+(2n+1,q)$, $n \geq 0$, formed by all points of $\mathrm{PG}(2n+1,q)$ which satisfy the standard equation $x_0x_1 + \cdots + x_{2n}x_{2n+1} = 0$.
- The symplectic polar space $\mathrm{W}(2n+1,q)$, $n \geq 0$, which consists of all points of $\mathrm{PG}(2n+1,q)$ together with the totally isotropic subspaces with respect to the standard symplectic form $\theta(x,y) = x_0y_1 - x_1y_0 + \cdots + x_{2n}y_{2n+1} - x_{2n+1}y_{2n}$.
- The hermitian variety $\mathrm{H}(n,q^2)$, $n \geq 1$, formed by all points of $\mathrm{PG}(n,q^2)$ which satisfy the standard equation $x_0^{q+1} + \cdots + x_n^{q+1} = 0$.

In the above list, the polar space of a given type has rank 1 for the smallest n that is allowed. Remark also that a quadric (also called an *orthogonal polar space*), and a hermitian variety, is determined completely by its point set, and can be described as above as a set of points whose coordinates satisfy an equation, which is of course derived from the sesquilinear or quadratic form.

Let Q be a point of a polar space $\mathcal{P}$. Then $Q^\perp$ is the set of points whose coordinates are orthogonal to Q with respect to the underlying sesquilinear or quadratic form[1], so $Q^\perp$ is the set of points of a hyperplane $T_Q(\mathcal{P})$, called the *tangent hyperplane at Q to $\mathcal{P}$*, and $Q^\perp \cap \mathcal{P}$ is necessarily the set of points of $\mathcal{P}$ that lie on a line through Q contained in $\mathcal{P}$. For any set A of points, $A^\perp := \cap_{P \in A} P^\perp$. The following result is fundamental in the theory of finite classical polar spaces.

Result 1.1. *Suppose that $\mathcal{P}_r$ is a finite classical polar space of rank $r \geq 2$. Then for any point P of $\mathcal{P}_r$, the set $P^\perp \cap \mathcal{P}_r$ is a cone with base $\mathcal{P}_{r-1}$ and vertex P, with $\mathcal{P}_{r-1}$ a finite classical polar space of rank $r-1$ of the same type as $\mathcal{P}_r$.*

From this theorem, it follows that the quotient space of a point P of $\mathcal{P}_r$, i.e. the set of all subspaces of $\mathcal{P}_r$ through P, is a polar space of rank $r-1$ of the same type as $\mathcal{P}_r$.

We define $\theta_i(q) := \frac{q^{i+1}-1}{q-1}$ for all integers $i \geq 0$, i.e. the number of points in $\mathrm{PG}(i,q)$.

Theorem 1.2. *The rank, the number of points, and the number of generators of all finite classical polar spaces are given in Table 1.*

Proof. We demonstrate the proof for $\mathrm{Q}^+(2n+1,q)$, the proofs for the other polar spaces are, mutatis mutandis, the same.

We prove the results by induction on n. For $n = 0$, the hyperbolic quadric $x_0x_1 = 0$ contains two points on a line and $2 = \frac{(q^0+1)(q^1-1)}{q^1-1}$. Formally, we set $|\mathrm{Q}^+(-1,q)| = 0$. Note that this definition fits with the general formula. Now suppose that $n \geq 1$. Take a line l of $\mathrm{Q}^+(2n+1,q)$. Then $l^\perp$ intersects $\mathrm{Q}^+(2n+1,q)$ in a cone over a $\mathrm{Q}^+(2n-3,q)$ which by induction has $a = q + 1 + q^2\frac{(q^{n-2}+1)(q^{n-1}-1)}{q-1}$ points.

For each point $P \notin l^\perp$ there exists exactly one point $R \in l$ with $R \in P^\perp$ or $P \in R^\perp$. Now $R^\perp$ intersects $\mathrm{Q}^+(2n+1,q)$ in a cone over a $\mathrm{Q}^+(2n-1,q)$ which by induction has

[1] When q is even, the quadratic form f determining $\mathcal{P}$, determines a possibly singular symplectic form σ. Two points P and Q are orthogonal with respect to f if, by definition, they are orthogonal with respect to σ.

polar space	rank	number of points	number of generators
$Q^-(2n+1,q)$	n	$(q^{n+1}+1)\theta_{n-1}(q)$	$(q^2+1)(q^3+1)\cdots(q^{n+1}+1)$
$Q(2n,q)$	n	$(q^n+1)\theta_{n-1}(q)$	$(q+1)(q^2+1)(q^3+1)\cdots(q^n+1)$
$Q^+(2n+1,q)$	$n+1$	$(q^n+1)\theta_n(q)$	$2(q+1)(q^2+1)\cdots(q^n+1)$
$W(2n+1,q)$	$n+1$	$(q^{n+1}+1)\theta_n(q)$	$(q+1)(q^2+1)\cdots(q^{n+1}+1)$
$H(2n,q^2)$	n	$(q^{2n+1}+1)\theta_{n-1}(q^2)$	$(q^3+1)(q^5+1)\cdots(q^{2n+1}+1)$
$H(2n+1,q^2)$	$n+1$	$(q^{2n+1}+1)\theta_n(q^2)$	$(q+1)(q^3+1)\cdots(q^{2n+1}+1)$

Table 1: Rank, number of points and number of generators of finite classical polar spaces

$b = 1+q\frac{(q^{n-1}+1)(q^n-1)}{q-1}$ points. Thus $|Q^+(2n+1,q)| = (q+1)(b-a)+a = \frac{(q^n+1)(q^{n+1}-1)}{q-1} = (q^n+1)\theta_n(q)$.

Now we count the number of generators, again using induction on n. For $n=0$ the 2 points of the hyperbolic quadric are its generators, hence it is a polar space of rank 1.

Now assume that $n \geq 1$, and that $Q^+(2n-1,q)$ is a polar space of rank n. Let P be a point of $Q^+(2n+1,q)$. Then $P^\perp$ intersects $Q^+(2n+1,q)$ in cone over a $Q^+(2n-1,q)$. Hence, by induction P lies on $2(q+1)(q^2+1)\cdots(q^{n-1}+1)$ generators. On the other hand, a generator of $Q^+(2n-1,q)$ contains $\frac{q^{n+1}-1}{q-1}$ points. Double counting gives for the number g of generators in $Q^+(2n+1,q)$ the equation

$$g\frac{q^{n+1}-1}{q-1} = |Q^+(2n-1,q)|2(q+1)(q^2+1)\cdots(q^{n-1}+1).$$

Solving the equation yields the number of generators. Finally, the dimension of the generators of $Q^+(2n+1,q)$ is one more than the dimension of the generators of $Q^-(2n-1,q)$. $\square$

It is well known (see e.g. [43]) that the generators of $Q^+(2n+1,q)$ fall into two equivalence classes, denoted by the sets $\mathcal{G}_1$ and $\mathcal{G}_2$. Recall that the rank of $Q^+(2n+1,q)$ is $n+1$. The following result is well known and can be found in [44] (Theorem 22.4.12 and its Corollary).

Result 1.3. *Let g_1 and g_2 be distinct generators of $Q^+(2n+1,q)$. If $n=2s$, then*

$$\dim(g_1\cap g_2) = \begin{cases} 0,2,4,\ldots,2s-2 & \text{if } g_1 \text{ and } g_2 \text{ belong to the same class} \\ -1,1,3,\ldots,2s-1 & \text{if } g_1 \text{ and } g_2 \text{ belong to a different class;} \end{cases}$$

and if $n=2s+1$, then

$$\dim(g_1\cap g_2) = \begin{cases} -1,1,3,\ldots,2s-1 & \text{if } g_1 \text{ and } g_2 \text{ belong to the same class} \\ 0,2,4,\ldots,2s & \text{if } g_1 \text{ and } g_2 \text{ belong to a different class.} \end{cases}$$

2 Morphisms of finite classical polar spaces

For q even, $Q(2n,q) \subseteq PG(2n,q)$ has a *nucleus*, i.e. a point $N \in PG(2n,q)\setminus Q(2n,q)$ contained in all tangent hyperplanes to $Q(2n,q)$. Projecting the elements of $Q(2n,q)$ from N yields a polar space isomorphic to $W(2n-1,q)$ (see e.g. [44]), so $Q(2n,q)$ and $W(2n-1,q)$ are isomorphic when q is even. The existence of this isomorphism implies that any result proved in one of these spaces, is also valid in the other one.

A *duality* δ between two rank 2 geometries $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is an incidence preserving map from $\mathcal{P}$ to $\mathcal{L}'$, $\mathcal{L}$ to $\mathcal{P}'$, and from $\mathcal{L}'$ to $\mathcal{P}$, $\mathcal{P}'$ to $\mathcal{L}$, such that δ^2 is the identity mapping. There exist dualities between different types of finite classical polar spaces of rank 2 [66].

- $\mathrm{Q}(4,q)$ is isomorphic to the dual of $\mathrm{W}(3,q)$. This means that interchanging the role of the points and generators of $\mathrm{Q}(4,q)$ yields an incidence geometry isomorphic to $\mathrm{W}(3,q)$, and vice versa. As a consequence, for q even, $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$ are self-dual.
- $\mathrm{Q}^-(5,q)$ is isomorphic to the dual of $\mathrm{H}(3,q^2)$.

Consider now $\mathrm{Q}^+(7,q)$ and define a rank 4 incidence geometry Ω as follows. $\Omega = (\mathcal{P}, \mathcal{L}, \mathcal{G}_1, \mathcal{G}_2)$, where $\mathcal{P}$ is the set of points of $\mathrm{Q}^+(7,q)$ and $\mathcal{L}$ is the set of lines of $\mathrm{Q}^+(7,q)$. An element $g_1 \in \mathcal{G}_1$ is incident with an element $g_2 \in \mathcal{G}_2$ if and only if $g_1 \cap g_2$ is a plane. Incidence between other elements is symmetrized containment. A *triality* of the geometry Ω is a map

$$\tau : \mathcal{L} \to \mathcal{L}, \mathcal{P} \to \mathcal{G}_1, \mathcal{G}_1 \to \mathcal{G}_2, \mathcal{G}_2 \to \mathcal{P}$$

preserving the incidence in Ω and such that τ^3 is the identity. Trialities of Ω exist [44].

The dualities and the triality described here, are used frequently to construct substructures of a polar space from different ones, as we will see in the next sections.

3 Ovoids, spreads and *m*-systems

“Ovoids” of polar spaces are inspired by ovoids of the projective space $\mathrm{PG}(3,q)$ (see e.g. [27]), and are defined for the first time in [75]. Also “spreads” occurred first in projective spaces, and are transferred to polar spaces.

Let $\mathcal{P}$ be a finite classical polar space of rank $r \geq 2$. An *ovoid* is a set $\mathcal{O}$ of points of $\mathcal{P}$, which has exactly one point in common with each generator of $\mathcal{P}$. A *spread* is a set $\mathcal{S}$ of generators of $\mathcal{P}$ which constitute a partition of the point set of $\mathcal{P}$.

Theorem 3.1. *An ovoid in* $\mathrm{Q}^-(2n-1,q)$*,* $\mathrm{Q}(2n,q)$*,* $\mathrm{Q}^+(2n+1,q)$ *or* $\mathrm{W}(2n-1,q)$ *has* q^n+1 *points. An ovoid of* $\mathrm{H}(2n,q^2)$ *or* $\mathrm{H}(2n+1,q^2)$ *has* $q^{2n+1}+1$ *points.*

A spread of $\mathrm{Q}^-(2n-1,q)$*,* $\mathrm{Q}(2n,q)$*,* $\mathrm{Q}^+(2n+1,q)$ *or* $\mathrm{W}(2n-1,q)$ *contains* q^n+1 *generators. A spread of* $\mathrm{H}(2n,q^2)$ *or* $\mathrm{H}(2n+1,q^2)$ *contains* $q^{2n+1}+1$ *generators.*

Proof. We demonstrate the proof for $\mathcal{P} = \mathrm{Q}^+(2n+1,q)$ as an example, the proof is analogous for the other polar spaces.

By Theorem 1.2, $\mathrm{Q}^+(2n+1,q)$ has $2(q+1)(q^2+1)\cdots(q^n+1)$ generators. By Result 1.1, the quotient space of a point is a $\mathrm{Q}^+(2n-1,q)$, hence, every point lies in $2(q+1)(q^2+1)\cdots(q^{n-1}+1)$ generators. Thus an ovoid must have $[2(q+1)(q^2+1)\cdots(q^n+1)]/[2(q+1)(q^2+1)\cdots(q^{n-1}+1)] = q^n+1$ elements.

By Theorem 1.2, $\mathrm{Q}^+(2n+1,q)$ has $(q^n+1)\theta_n(q)$ points. Each generator is a projective space of dimension n, that contains $\theta_n(q)$ points. Thus a spread must contain q^n+1 elements. $\square$

So in a polar space $\mathcal{P}$, the size of an ovoid equals the size of a spread, this number is denoted by $\mu_{\mathcal{P}}$.

3.1 Ovoids

Ovoids of finite classical polar spaces are rare, they seem to exist only in low rank, and for many polar spaces of high rank, a non-existence proof for ovoids is known. One important observation to show the non-existence of ovoids is the following lemma.

Lemma 3.2. *If $\mathcal{O}$ is an ovoid of a finite classical polar space $\mathcal{P}$ of rank $r \geq 3$, then $\mathcal{O}$ induces an ovoid of a finite classical polar space of the same type of rank $r-1$*

Proof. Consider any point $Q \notin \mathcal{O}$ of the polar space $\mathcal{P}$. The quotient space on Q is a polar space $\mathcal{P}'$ of rank $r-1$ of the same type. But each generator of $\mathcal{P}$ on Q contains exactly one point of $\mathcal{O}$, so $\mathcal{O}$ induces an ovoid of $\mathcal{P}'$. □

Hence, if the non-existence of ovoids is proved for a polar space of a certain type in some rank, the contraposition of Lemma 3.2 shows the non-existence in higher rank. In the very rare cases where an ovoid of a polar space of rank r is induced by an ovoid of a polar space of rank $r+1$, applying Lemma 3.2 is called "slicing". We now prove the non-existence of ovoids of $\mathrm{W}(3,q)$, q odd.

Lemma 3.3. *The polar space $\mathrm{W}(3,q)$ has ovoids if and only if q is even.*

Proof. If q is even, then $\mathrm{W}(3,q)$ is isomorphic to $\mathrm{Q}(4,q)$, and an embedded quadric $\mathrm{Q}^-(3,q)$ in $\mathrm{Q}(4,q)$ yields an ovoid of $\mathrm{W}(3,q)$. Conversely, suppose that $\mathcal{O}$ is an ovoid of $\mathrm{W}(3,q)$. Consider a line l of the ambient projective space $\mathrm{PG}(3,q)$ spanned by two points of $\mathcal{O}$. Since a generator of $\mathrm{W}(3,q)$ contains exactly one point of $\mathcal{O}$, the line l is not a generator of $\mathrm{W}(3,q)$. So $|l \cap \mathcal{O}| = c \geq 2$. We count the pairs $\{(P,Q) | P \in l, Q \in \mathcal{O} \setminus l\}$. For any point $P \in l \setminus \mathcal{O}$, the $q+1$ generators of $\mathrm{W}(3,q)$ on P each meet $\mathcal{O}$ in exactly one point, while on each point of $\mathcal{O} \setminus l$, there is exactly one generator of $\mathrm{W}(3,q)$ meeting l in a point not in $\mathcal{O}$. It follows that $(q+1-c)(q+1)+c = q^2+1$. This is a contradiction unless $c=2$. But then in the plane $P^\perp$ we see $q+1$ points of $\mathcal{O}$, which, together with P, constitute a set $\mathcal{H}$ of $q+2$ points such that each line of $P^\perp$ meets $\mathcal{H}$ in 0 or 2 points. So $\mathcal{H}$ is a hyperoval of $\mathrm{PG}(2,q)$, and q must be even. □

The non-existence of ovoids of $\mathrm{Q}^-(5,q)$, $\mathrm{W}(5,q)$ and $\mathrm{H}(4,q^2)$ can be proved using the same technique.

Corollary 3.4. *The polar spaces $\mathrm{Q}^-(2n+1,q)$, $\mathrm{H}(2n,q^2)$, $\mathrm{W}(2n+1,q)$, $n \geq 2$, and $\mathrm{Q}(2n,q)$, $n \geq 3$, q even, have no ovoid*

Proof. Use Lemma 3.2 and the result for $\mathrm{Q}^-(5,q)$, $\mathrm{W}(5,q)$ and $\mathrm{H}(4,q^2)$, and use the isomorphism between $\mathrm{W}(2n-1,q)$, q even and $\mathrm{Q}(2n,q)$, q even, for the last case. □

The non-existence of ovoids of $\mathcal{P} = \mathrm{Q}(8,q)$, q odd, is proved in [37] by associating a two-graph Γ to a hypothetical ovoid of $\mathcal{P}$. It is shown that Γ is regular, and using known relations between eigenvalues of the adjacency matrix of Γ, a contradiction follows rapidly. Lemma 3.2 closes the case $\mathrm{Q}(2n,q)$, q odd, $n \geq 4$.

Conditions for the non-existence of ovoids of $\mathrm{H}(2n+1,q^2)$ or $\mathrm{Q}^+(2n+1,q)$ are shown in [7, 64] by computing the p-rank of the incidence matrix of the points of a hypothetical ovoid and the hyperplanes of the ambient projective space. These conditions, shown in

polar space	existence and/or known examples	references
$\mathrm{Q}^-(2n+1,q)$	$n>1$: no	[77]
$\mathrm{Q}(4,q)$	q odd prime: yes, and every ovoid is classical	[4]
$\mathrm{Q}(4,q)$	$q=3^h$: $\mathrm{Q}(6,q)$ slices; 1 other example ($q=3^5$)	[45,66,82]; [67]
$\mathrm{Q}(4,q)$	q odd non prime: two infinite families	[45,82]
$\mathrm{W}(3,q)$	q even: classical, Tits ovoid for $q=2^{2h+1}$	[69]+ [75], [85]
$\mathrm{Q}(6,q)$	q even; $q>3$, q prime: no in both cases	[77]; [65]+ [4]
	$q=3^h$: two infinite families known	[9,45], [77]; [83]
$\mathrm{Q}(2n,q)$	$n\geq 4$: no (different proofs for q odd and even)	[37]; [77]
$\mathrm{Q}^+(3,q)$	several examples	[43]
$\mathrm{Q}^+(5,q)$	yes, equivalent with spreads of $PG(3,q)$	[43]
$\mathrm{Q}^+(7,q)$	$q=3^h$: known examples: from $\mathrm{Q}(6,q)$	
	$q=2^h$: 1 infinite family; 1 other example ($q=8$)	[45]; [29]
	$q=p^h$, $p\equiv 2 \mod 3$, p prime, h odd: yes	[45]
	$q\geq 5$ prime: yes	[11,63]
$\mathrm{Q}^+(2n+1,q)$	$q=p^h$, p prime, $p^n>\binom{2n+p}{2n+1}-\binom{2n+p-2}{2n+1}$: no	[30], [7]
$\mathrm{W}(2n+1,q)$	q odd $n=1$: no; all q, $n>1$: no	[77]
$\mathrm{H}(2n,q^2)$	$n\geq 2$: no	[77]
$\mathrm{H}(3,q^2)$	yes, see spreads of $\mathrm{Q}^-(5,q)$	[66,78,82]
$\mathrm{H}(5,4)$	no	[19]
$\mathrm{H}(2n+1,q^2)$	$q=p^h$, p prime, $p^{2n+1}>\binom{2n+p}{2n+1}^2-\binom{2n+p-1}{2n+1}^2$: no	[64]

Table 2: Existence and non-existence results on ovoids

Table 2, leave open an infinite number of cases. We mention that Dye [30] gave an upper bound on the size of partial ovoids of the polar spaces $\mathrm{Q}(2n,2)$, $\mathrm{Q}^+(2n+1,2)$ and $\mathrm{Q}^-(2n+1,2)$, which implies the non-existence of ovoids in some cases, in particular for $\mathrm{Q}^+(2n+1,2)$ for $n\geq 4$.

In [46], it is shown that the polar space $\mathrm{H}(2n+1,q^2)$ has no ovoids if $n>q^3$, and, similarly, in [17], that $\mathrm{Q}^+(2n+1,q)$ has no ovoids if $n>q^2$. This is weaker than the earlier known conditions, but the proofs only use geometrical and combinatorial arguments. Pushing a little bit further these arguments, it is shown in [19] that $\mathrm{H}(5,4)$ has no ovoid.

In [65], it is shown that $\mathrm{Q}(6,q)$, $q>3$, has no ovoids if all ovoids of $\mathrm{Q}(4,q)$ are elliptic quadrics. It is shown in [4] that this condition is satisfied for q odd prime. This leaves open the existence or non-existence of ovoids of $\mathrm{Q}(6,q)$ when $q=p^h$, p an odd prime, $h>1$, except for $p=3$, where ovoids are known to exist, see below.

Ovoids of $\mathrm{Q}(4,q)$ and $\mathrm{H}(3,q^2)$ can be constructed easily. The intersection with a hyperplane of the ambient projective space containing no generator, yields an ovoid. We call such ovoids *classical*. For $\mathrm{Q}(4,q)$, $\mathrm{H}(3,q^2)$ respectively, this is an elliptic quadric $\mathrm{Q}^-(3,q)$, a hermitian curve $\mathrm{H}(2,q^2)$ respectively. However, in $\mathrm{Q}(4,q)$ (q non-prime), and in $\mathrm{H}(3,q^2)$, also non-classical ovoids exist. It is shown in [69, 75] that ovoids of $\mathrm{W}(3,q)$, q even, are equivalent to ovoids of $\mathrm{PG}(3,q)$. So the Tits ovoid in $\mathrm{PG}(3,q)$, q even (see [85]) yields an ovoid of $\mathrm{W}(3,q)$, q even, and hence yields an ovoid of $\mathrm{Q}(4,q)$, q even, which is non-classical, [66]. For q odd non prime, infinite families of non-classical ovoids of $\mathrm{Q}(4,q)$

are known. Ovoids of $\mathrm{H}(3,q^2)$ are equivalent to spreads of $\mathrm{Q}^-(5,q)$, of which many non-classical examples are known, see Section 3.2.

The *Klein correspondence* is a bijective map from the line set of $\mathrm{PG}(3,q)$ to the point set of the polar space $\mathrm{Q}^+(5,q)$. Two lines of $\mathrm{PG}(3,q)$ have a point in common if and only if they are mapped to two points of $\mathrm{Q}^+(5,q)$ being contained in a common generator. Hence a spread of $\mathrm{PG}(3,q)$ is mapped to a set of q^2+1 points of $\mathrm{Q}^+(5,q)$ two by two not contained in a common generator, so constituting necessarily an ovoid. Since many different families of spreads of $\mathrm{PG}(3,q)$ are known (see e.g. [43]), there are many different examples of ovoids of $\mathrm{Q}^+(5,q)$. We mention that a regular spread of $\mathrm{PG}(3,q)$ corresponds to an elliptic quadric $\mathrm{Q}^-(3,q) \subset \mathrm{Q}^+(5,q)$.

Only two infinite families of ovoids of $\mathrm{Q}(6,q)$ are known, for $q=3^h$, $h \geq 1$. Embedding $\mathrm{Q}(6,q)$ as a hyperplane section in $\mathrm{Q}^+(7,q)$, it is easily observed that an ovoid of $\mathrm{Q}(6,q)$ induces an ovoid of $\mathrm{Q}^+(7,q)$, and all known ovoids of $\mathrm{Q}^+(7,q)$, $q=3^h$, arise from ovoids of $\mathrm{Q}(6,q)$. But several (infinite families of) ovoids $\mathrm{Q}^+(7,q)$, $q \neq 3^h$, are known, and all of them are not contained in a hyperplane section.

We now refer to Table 2 for an overview, including references.

3.2 Spreads

From the definition, it follows that ovoids of a polar space of rank 2 are spreads of the dual of $\mathcal{P}$. This immediately yields some examples of spreads in the rank two case. But we start with a construction result in the symplectic polar space $\mathrm{W}(2n+1,q)$.

Consider the projective space $\mathrm{PG}(d,q)$. When $(t+1) \mid (d+1)$, the multiplicative group of $\mathbb{F}_{q^{d+1}}$ can be partitioned by cosets of the multiplicative group of $\mathbb{F}_{q^{t+1}}$. Each such coset is a $\mathbb{F}_q$ vector space, so we find a partition of $\mathrm{PG}(d,q)$ by t-dimensional projective spaces. For $d=2n+1$ and $t=n$, we find a spread of $\mathrm{PG}(2n+1,q)$ consisting of n-dimensional subspaces. It is shown in [29] that there exists always a symplectic polarity ϕ of $\mathrm{PG}(2n+1,q)$ such that all n-dimensional subspaces of this spread are totally isotropic with relation to ϕ. This yields a spread of the polar space $\mathrm{W}(2n+1,q)$, $n \geq 1$, and, when q is even, a spread of the polar space $\mathrm{Q}(2n+2,q)$, $n \geq 1$. The same result is also shown in [76] for $n=2$, with a proof that is extendable to general n.

The polar space $\mathrm{Q}^+(4n+1,q)$, $n \geq 1$, has no spread, because by Result 1.3, at most two generators can be skew. Consider now $\mathrm{Q}(4n+2,q)$, $n \geq 1$, as a hyperplane intersection of $\mathrm{Q}^+(4n+3,q)$. Suppose that $\mathrm{Q}(4n+2,q)$ has a spread $\mathcal{S}$. Then each element $\pi \in \mathcal{S}$ is contained in two generators of $\mathrm{Q}^+(4n+3,q)$, one of each class, meeting in π. By Result 1.3, the set $\mathcal{S}'$ of all generators of one class, meeting $\mathrm{Q}(4n+2,q)$ in an element of $\mathcal{S}$, is a spread of $\mathrm{Q}^+(4n+3,q)$. Also, using hyperplane sections, the following proposition is easy to see.

Result 3.5. *[43]. If the polar space* $\mathrm{Q}^+(2n+1,q), n \geq 2; \mathrm{Q}(2n,q), n \geq 3; \mathrm{H}(2n+1,q^2), n \geq 2$*, respectively, has a spread, then the polar space* $\mathrm{Q}(2n,q), n \geq 2; \mathrm{Q}^-(2n-1,q), n \geq 3; \mathrm{H}(2n,q^2), n \geq 2$*, respectively, has a spread.*

It is shown in [66] that any spread of $\mathrm{PG}(3,q)$ gives rise to a spread of $\mathrm{Q}^-(5,q)$. Many spreads of $\mathrm{PG}(3,q)$ are known, so this gives rise to many spreads of $\mathrm{Q}^-(5,q)$, and, dually to ovoids of $\mathrm{H}(3,q^2)$. Finally, using the existence of a triality of $\mathrm{Q}^+(7,q)$, one observes easily that an ovoid of $\mathrm{Q}^+(7,q)$ is equivalent to a spread of $\mathrm{Q}^+(7,q)$. This has also consequences

polar space	existence and/or known examples	references
$\mathrm{W}(2n+1,q), n \geq 1$	yes; $n=1$: also see ovoids of $\mathrm{Q}(4,q)$	[29], [76]; [66]
$\mathrm{Q}(2n,q), n \geq 2$	q even: yes; $n=2$: also see ovoids of $\mathrm{Q}(4,q)$	Section 2; [66]
$\mathrm{Q}(6,q)$	all known examples: see spreads of $\mathrm{Q}^+(7,q)$	Result 3.5, [84]
$\mathrm{Q}^-(2n+1,q), n \geq 2$	q even: yes	Result 3.5
$\mathrm{Q}^-(5,q)$	yes, from spreads of $PG(3,q)$	[66]
$\mathrm{Q}^+(4n+3,q), n \geq 1$	q even: yes: see spreads of $\mathrm{Q}(4n+2,q)$ and Theorem 1.3	
$\mathrm{Q}^+(4n+1,q)$	no	[43]
$\mathrm{Q}^+(3,q)$	yes	[43]
$\mathrm{Q}^+(7,q)$	all known examples: see ovoids of $\mathrm{Q}^+(7,q)$	[84]
$\mathrm{Q}(4n,q)$	q odd: no	[75, 80]
$\mathrm{H}(2n+1,q^2)$	no	[77, 80]
$\mathrm{H}(4,4)$	no, unpublished computer result of A.E. Brouwer	

Table 3: Existence and non-existence results on spreads

for $\mathrm{Q}(6,q)$, since a spread of $\mathrm{Q}^+(7,q)$ induces, using a hyperplane section, a spread of $\mathrm{Q}(6,q)$. The non-existence of spreads of the polar spaces $\mathrm{Q}(4n,q)$, q odd and $n>1$, and $\mathrm{H}(2n+1,q^2)$, $n>1$ is proved for the first time in [80]. The proofs are purely geometric.

We refer now to Table 3 for an overview, including references.

3.3 *m*-Systems

Let $\mathcal{P}$ be a finite classical polar space of rank $r \geq 2$. A *partial m-system* of $\mathcal{P}$ is a set $\mathcal{M} = \{\pi_1, \dots, \pi_k\}$ of m-dimensional subspaces of $\mathcal{P}$, such that no generator of $\mathcal{P}$ containing π_i has any point in common with an element of $\mathcal{M} \setminus \{\pi_i\}$, for all elements $\pi_i \in \mathcal{M}$. If $|\mathcal{M}| = \mu_P$, then the partial m-system is called an *m-system*. Remark that for $m=0$, an m-system is an ovoid of $\mathcal{P}$, while for $m=r-1$, an m-system is a spread of $\mathcal{P}$.

This definition is given by Shult and Thas in [71]. Within the scope of this article, it is not possible to survey all existence and non-existence results of m-systems in a detailed way. Therefore, we will give information on particular facts and refer to existing surveys.

Field reduction is an appropriate way to construct m'-systems from m-systems. Consider the hermitian variety $\mathrm{H}(3,q^{2e})$, e odd, with associated hermitian form κ. With T the trace map from $\mathbb{F}_{q^{2e}}$ into $\mathbb{F}_{q^2}$, it is easy to check that the map $T \circ \kappa$ induces a hermitian form on $V(3e,q^2)$, so there is a geometry morphism from $\mathrm{H}(3,q^{2e})$ to $\mathrm{H}(3e-1,q^2)$, mapping points of $\mathrm{H}(3,q^{2e})$ to $(e-1)$-dimensional subspaces of $\mathrm{H}(3e-1,q^2)$. We have seen that $\mathrm{H}(3,q^{2e})$ has plenty of ovoids, and hence, $\mathrm{H}(3e-1,q^2)$ has plenty of $(e-1)$-systems.

The first examples of m-systems, those described in [71], are actually obtained by field reduction. Most known cases now are still found there, two cases are described in [72], and two cases are described in [39], and we refer to [81] for a survey. Morphisms of finite classical polar spaces based on field reduction are studied comprehensively in [33].

We discuss three sources of non-existence results on m-systems. The oldest results are due to Shult and Thas, who obtain non-existence results on m-systems comparable with the non-existence results on ovoids of Blokhuis and Moorehouse in [7, 64]. The following results are shown in [73], and are essentially based on the computation of the p-rank of an incidence matrix in two ways.

Result 3.6 (see [73])**.** *If the finite classical polar space $\mathcal{P}$ admits an m-system, then*

(i) *for* $\mathcal{P} = \mathrm{Q}^+(2n+1, 2^h)$, $2^n \leq \binom{2n+2}{m+1}$

(ii) *for* $\mathcal{P} = \mathrm{Q}(2n,q)$, $q = p^h$, *p an odd prime,* $p^n \leq \binom{\binom{2n+1}{m+1}+p-2}{p-1} - \binom{\binom{2n+1}{m+1}+p-4}{p-3}$

(iii) *for* $\mathcal{P} = \mathrm{Q}^+(2n+1,q)$, $q = p^h$, *p an odd prime,* $p^n \leq \binom{\binom{2n+2}{m+1}+p-2}{p-1} - \binom{\binom{2n+2}{m+1}+p-4}{p-3}$

(iv) *for* $\mathcal{P} = \mathrm{H}(2n+1,q^2)$, $q = p^h$, *p a prime,* $p^{2n+1} \leq \binom{\binom{2n+2}{m+1}+p-2}{p-1}^2 - \binom{\binom{2n+2}{m+1}+p-4}{p-3}^2$

Recently, Sin showed in [74] an upper bound on the number of elements of a partial m-system, using the p-rank approach of an incidence matrix in a more elaborate way. Let $N(n+1, r, p-1)$ be the number of monomials in $n+1$ variables of total degree r and with (partial) degree at most $p-1$ in each variable. This number is equal to the coefficient of x^r in $(1+x+\cdots+x^{p-1})^{n+1}$.

Result 3.7 (see [74])**.** *Let $\mathcal{M}$ be a partial m-system of a finite classical polar space $\mathcal{P}$ with ambient projective space* $\mathrm{PG}(n,q)$, $q = p^h$, *p prime. Then* $|\mathcal{M}| \leq 1 + N(n+1, (m+1)(p-1), p-1)^h$.

If the right hand side is smaller than $\mu_{\mathcal{P}}$, then this implies the non-existence of m-systems in $\mathcal{P}$. It is hard to compare both bounds in general. Both bounds imply non-existence of m-systems for polar spaces of "high" rank, but for given m and q, Result 3.7 implies non-existence often for lower rank than Result 3.6. A careful analysis is done in [74].

To describe the third non-existence result, we first have to go back to [71]. Suppose that $\mathcal{P} \in \{\mathrm{W}(2n+1,q), \mathrm{Q}^-(2n+1,q), \mathrm{H}(2n,q^2)\}$ and that $\mathcal{M}$ is an m-system of $\mathcal{P}$. The point set $\widetilde{\mathcal{M}}$ is the union of the elements of $\mathcal{M}$ as point sets. In [71], it is shown that $\widetilde{\mathcal{M}}$ is a two intersection set with respect to the hyperplanes of the ambient projective space. This implies that a strongly regular graph can be associated to $\mathcal{M}$. Hamilton and Mathon study this graph in [38] and compute its eigenvalues. This yields the following result.

Result 3.8. *m-Systems of* $\mathrm{W}(2n+1,q), \mathrm{Q}^-(2n+1,q), \mathrm{H}(2n,q^2)$ *do not exist for* $n > 2m+1$.

Hamilton and Mathon analyze their result and give examples for $\mathrm{W}(2n+1,q)$, q even and n odd, and $\mathrm{Q}^-(2n+1,q)$, n odd, showing that their bound is sharp in these cases. They also give an example that shows that their bound is better in some cases than the bound of [73]. Finally, the paper contains classification results for m-systems of $\mathrm{W}(2n+1,2), \mathrm{Q}^-(2n+1,2)$, and $\mathrm{Q}^+(2n+1,2)$ for $m = 1,2,3$, and 4, and applications.

A recent paper providing a general classification result is [6]. Bamberg and Penttila give a complete classification of m-systems admitting an insoluble transitive collineation group. There is no restriction on m, so their classification also holds for ovoids and spreads satisfying the condition. This paper also contains a detailed overview of some construction methods mentioned here, and a long list of references.

3.4 *m*-Ovoids

Let $\mathcal{P}$ be a finite classical polar space of rank $r \geq 2$. An *m-ovoid* is a set O of points of $\mathcal{P}$, which has exactly m points in common with each generator of $\mathcal{P}$. Thas defined m-ovoids of generalized quadrangles in [79]. Before this introduction, Segre studied already m-ovoids of $Q^-(5,q)$, but in the dual setting, i.e. as sets of lines of $H(3,q^2)$ covering each point m times. Segre proved that $m = \frac{q+1}{2}$, when q is odd, [70]. A line sets of $H(3,q^2)$ covering each point exactly $\frac{q+1}{2}$ times is also called a *hemisystem* of $H(3,q^2)$. Segre also gives an example of a hemisystem for $q = 3$, and it is only in [13] that hemisystems of $H(3,q^2)$ are constructed for all odd q. In [12], m-ovoids of $W(3,q)$ are constructed, for q odd and $m = \frac{q+1}{2}$ and for q even and $m \in \{2,\dots q-1\}$.

Up to our knowledge, the first systematic treatise of m-ovoids of polar spaces is [5]. In this paper, m-ovoids are treated in a more general framework, related to i-tight sets and intriguing sets of polar spaces. It is shown that m-ovoids of a polar space $\mathcal{P}$, with $\mathcal{P} \in \{H(2n,q^2), Q^-(2n+1,q), W(2n+1,q)\}$ have two intersection numbers with relation to hyperplanes of the ambient projective space. This gives rise to a strongly regular graph. Expressing that one of the parameters must be larger than 0, yields the lower bound on m. The following result is obtained in this way.

Result 3.9. *Let $\mathcal{P}$ be $H(2r,q^2), Q^-(2r+1,q), W(2r-1,q)$ respectively. If an m-ovoid of $\mathcal{P}$ exists, then $m \geq b$, with $b = \frac{(-3+\sqrt{9+4q^{2r+1}})}{2q^2-2}, \frac{(-3+\sqrt{9+4q^{r+1}})}{2q-2}, \frac{(-3+\sqrt{9+4q^r})}{2q-2}$ respectively.*

The above bounds are larger than 1 for $H(2r,q^2)$ and $Q^-(2r+1,q)$ for $r \geq 2$ and for $W(2r-1,q)$ for $r > 2$, for all q. Using a slicing argument that is in fact comparable with Lemma 3.2, the authors obtain the following result.

Result 3.10. *The following polar spaces do not admit a 2-ovoid: $W(2r-1,q)$, q odd and $r > 2$; $Q^-(2r+1,q)$, $r > 2$; $H(2r,q^2)$, $r > 2$; and $Q(2r,q)$, $r > 4$.*

Proof. Suppose that O is a 2-ovoid of the polar space $\mathcal{P}$ of rank r which is one of the mentioned examples. Consider any point $Q \in O$. Then the quotient space on Q is a polar space of rank $r-1$ of the same type. Since all generators of $\mathcal{P}$ on Q meet $O \setminus \{Q\}$ in exactly one point, O induces an ovoid in this quotient space. The result now follows from the non-existence of ovoids in the polar spaces mentioned. □

4 Partial ovoids and partial spreads

Let $\mathcal{P}$ be a finite classical polar space. A *partial ovoid* of $\mathcal{P}$ is a set O of points of $\mathcal{P}$ with the property that every generator of $\mathcal{P}$ contains at most one point of O. A partial ovoid is called *proper* if it is not an ovoid. A (proper) partial ovoid is called *maximal* if it is not contained in a partial ovoid of larger size. Clearly, a maximal proper partial ovoid is not an ovoid.

A *partial spread* of $\mathcal{P}$ is a set $\mathcal{S}$ of pairwise disjoint generators. A partial spread is called *proper* if it is not a spread. A proper partial spread is called *maximal* if it is not contained in a partial spread of larger size. Clearly, a maximal proper partial spread is not an spread.

Obviously, in the rank 2 case, (maximal) (proper) partial ovoids become (maximal) (proper) partial spreads in the dual space. After non-existence proofs for ovoids, spreads respectively, partial ovoids, partial spreads respectively, arise naturally, and then we are interested in an upper bound on their size. Secondly, we wish to derive a lower bound on the size in case of maximality. Finally, when ovoids, spreads respectively, exist, extendability of proper partial ovoids, proper partial spreads respectively, is studied.

polar space	lower bound	references
$W(2n+1,q)$	$q+1$ (sharp)	[10], [16]
$Q(4,q)$, q odd $Q(6,q)$, $q \in \{3,5,7\}$; $q \geq 9$ odd $Q(2n,q)$, $n \geq 4$, q odd; $Q(8,3)$	$1.419q$ $2q$; $2q-1$ $2q+1$; $2q$	[16, Theorem 2.2 (b)]
$Q^-(5,q)$, $q=2$; 3; $q \geq 4$ $Q^-(2n+1,q)$	6; 16; $2q+2$ $2q+1$	[16, Theorem 2.2 (c)]
$Q^+(2n+1,q)$, $n=2$; $n \geq 3$	$2q$; $2q+1$	[16, Theorem 2.2 (a)]
$H(3,q^2)$, q odd; even $H(2n+1,q^2)$, $n \geq 2$	$q^2+1+\frac{4}{9}q$; q^2+1 (sharp) q^2+q+1	[60]; [2] [15, Theorem 2.3]
$H(2n,q^2)$, $n=2$; $n \geq 3$	q^2+q+1	[62]; [15, Theorem 2.2]

Table 4: Lower bounds on the size of maximal partial ovoids

4.1 Partial ovoids

The first series of results we mention are based on the use of a combinatorial approach also found in [34], where Glynn derives a lower bound on the size of maximal partial spreads of $PG(3,q)$. Under the Klein correspondence, this is equivalent to a lower bound on the size of maximal partial ovoids of $Q^+(5,q)$. But not only the result translates, also the proof, and this proof can also be applied for partial ovoids of other polar spaces. This yields lower bounds on the size of maximal partial ovoids of $Q^+(2n+1,q)$, $n \geq 2$, $Q^-(2n+1,q)$, $n \geq 2$ and $Q(2n,q)$, $n \geq 3$ and q odd. A proof can be found in e.g. [16]. Lower bounds for other polar spaces obtained using a combinatorial approach, are also known. We refer to Table 4 for an overview.

Some upper bounds on the size of partial ovoids are derived from the non-existence proofs of ovoids. This complicates the situation when ovoids are known to exist, or when the existence or non-existence is not yet proved. Despite this complication, an upper bound on the size of maximal proper partial ovoids of $W(3,q)$, without any assumption on q, was obtained in [47]. Recall that $W(3,q)$ has ovoids if and only if q is even. In [80], an upper bound on the size of partial ovoids in $W(2n+1,q)$, $n \geq 2$, is obtained. In [16], this bound is improved for $n=2$, and using inductive arguments, this yields an upper bound for general n that is better than the one in [80]. The inductive argument is valid in all polar spaces, so we first describe now the low rank cases, and then give an overview of the inductive arguments.

The case $Q(4,q)$, q odd, seems to be much harder. Currently, it is only known that partial ovoids of size q^2 always extend to ovoids and that maximal proper partial ovoids of $Q(4,q)$ of size q^2-1, $q=p^h$, p odd, do not exist for $h>1$, and that examples are known for $q \in \{3,5,7,11\}$. The non-existence result is shown in [14], and the proof is also presented in [3, Corollary 6.9]. Furthermore, in [36], it is shown that if a maximal proper partial ovoid of $Q(4,q)$, q odd, of size $q^2+1-\delta$ exists, $\delta < \sqrt{q}$, then δ is even. Projection arguments and the results known on proper partial ovoids of $Q(4,q)$ for different values of q, yield an upper bound on the size of maximal proper partial ovoids of $Q(6,q)$ in [16].

A recent treatment of the case $Q^-(5,q)$ can be found in [15], where in fact the dual, i.e. partial spreads of $H(3,q^2)$, are considered, and which is described below.

Upper bounds on the size of maximal proper partial ovoids of $Q^+(5,q)$ are under the

polar space	upper bound	references
$\mathrm{W}(3,q)$	q^2-q+1	[47]
$\mathrm{W}(5,q)$	$1+\frac{q}{2}(\sqrt{5q^4+6q^3+7q^2+6q+1}-q^2-q-1)$	[16]
$\mathrm{Q}(4,q)$, q odd	q^2 (see description above)	
$\mathrm{Q}(6,q)$, $q>13$, q prime	q^3-2q+1	[16]
$\mathrm{Q}(8,q)$, q odd, q not a prime	$q^4-q\sqrt{q}$	[16]
$\mathrm{Q}^-(5,q)$	$\frac{1}{2}(q^3+q+2)$ (sharp for $q=2,3$)	[15] ([43])
$\mathrm{H}(3,q^2)$	q^3-q+1 (sharp)	[47]
$\mathrm{H}(5,q^2)$	$q^5+1-(q^2+\frac{1}{4}q-1)/\sqrt{2}$	[15]
$\mathrm{H}(4,q^2)$	$q^5-q^4+q^3+1$	[15]

Table 5: Upper bounds on the size of maximal proper partial ovoids in low rank polar spaces

polar space	recursion	references
$\mathrm{W}(2n+1,q)$	$x_{n,q}\leq 2+(q-1)x_{n-1,q}$	[16]
$\mathrm{Q}^-(2n+1,q)$	$x_{n,q}\leq 2+\frac{q^n+1}{q^{n-1}+1}(x_{n-1,q}-2)$	[46]
$\mathrm{Q}(2n,q)$	$x_{n,q}\leq 1+q(x_{n-1,q}-1)$	[16]
$\mathrm{Q}^+(2n+1,q)$	$x_{n,q}\leq 2+\frac{q^n-1}{q^{n-1}-1}(x_{n-1,q}-2)$	[16]
$\mathrm{H}(2n,q^2)$	$x_{n,q^2}\leq q^2x_{n-1,q^2}-q^2+1$	[15]
$\mathrm{H}(2n+1,q^2)$	$x_{n,q^2}\leq q^2x_{n-1,q^2}-q^2+1$	[15]

Table 6: Inductive bounds on the size of partial ovoids

Klein correspondence equivalent with upper bounds on the size of maximal proper partial spreads of $\mathrm{PG}(3,q)$. For q not prime and not a square, the best upper bound is found in [52]. A comprehensive survey, also including results for q square and for q prime, can be found in [61]. Improvements on parts of [61] can be found in [32]. Constructions of maximal partial spreads of $\mathrm{PG}(3,q)$ can e.g. be found in the series of papers [40–42].

The best upper bound on the size of maximal proper partial ovoids of $\mathrm{H}(3,q^2)$ is found in [47], where the dual case is considered, i.e. maximal proper partial spreads of $\mathrm{Q}^-(5,q)$, using geometrical arguments. An analogous result for $\mathrm{H}(5,q^2)$ is developed in [15], where also an upper bound on the size of partial ovoids of $\mathrm{H}(4,q^2)$ is obtained, which improves an earlier result of [36].

Suppose that $\mathcal{P}_r$ is a polar space of a given type of rank r. If it has no ovoid, and an upper bound on the size of a partial ovoid is known, then the argument used in Lemma 3.2 makes it possible to deduce an upper bound for a partial ovoid of a polar space $\mathcal{P}_{r+1}$. Inductive bounds described in [15] and [16] are presented in Table 6, where $x_{n,q}$ denotes the upper bound on the size of a partial ovoid in the corresponding classical finite polar space with ambient projective space $\mathrm{PG}(2n,q)$ or $\mathrm{PG}(2n+1,q)$.

4.2 Partial spreads

Partial spreads require a different treatment than partial ovoids. On the one hand, counting techniques like the one of Glynn mentioned above for maximal partial ovoids, applied in rank 2 to obtain lower bounds, yield, dualizing, lower bounds on the size of maximal partial spreads. On the other hand, inductive bounds are not possible for spreads, so arguments must be found for general rank.

We first mention results on lower bounds on the size of maximal partial spreads of polar spaces. It is shown in [15] that any maximal partial spread of a polar space $\mathcal{P}$ has at least $t+1$ elements, where $t+1$ is the number of lines through a point in the polar space $\mathcal{P}'$ of rank 2 of the same type as $\mathcal{P}$. For hyperbolic quadrics, this theorem yields a lower bound of 2, which is improved in [15] for $\mathrm{Q}^+(4n+3,q)$ to $q+1$. Better lower bounds for polar spaces of rank 2 can, if applicable, be found in Table 4, by applying duality. For $\mathrm{H}(4,q^2)$, the following result is known.

Result 4.1 (see [62, Theorem 2.2])**.** *A maximal partial spread of* $\mathrm{H}(4,q^2)$ *contains at least* $\lceil q^3+q\sqrt{q}-\frac{q}{2}-\frac{3}{8}\sqrt{q}+\frac{7}{8}\rceil$ *elements.*

As indicated, we start our overview of upper bounds with the case $\mathrm{H}(3,q^2)$. The proof relies on a geometric property of hermitian varieties that is useful in several cases.

Result 4.2 (see [80])**.** *Let* π_1*,* π_2 *and* π *be mutually skew generators of* $\mathrm{H}(2n+1,q^2)$*. Then the points of* π *that lie on a line of* $\mathrm{H}(2n+1,q^2)$*, meeting* π_1 *and* π_2*, form a hermitian variety* $\mathrm{H}(n,q^2)$ *in* π*.*

Theorem 4.3 (see [15])**.** *A partial spread of* $\mathrm{H}(3,q^2)$ *has at most* $\frac{1}{2}(q^3+q+2)$ *elements.*

Proof. Suppose that $\mathcal{S}$ is a partial spread of $\mathrm{H}(3,q^2)$ and that $|\mathcal{S}|=q^3+1-\delta$. Then the number of points of $\mathrm{H}(3,q^2)$ not covered by lines of $\mathcal{S}$ is $h=\delta(q^2+1)$. We call these points *holes*.

Consider triples (l_1,l_2,P), where l_1 and l_2 are different elements of $\mathcal{S}$ and where P is a hole. We will estimate how many of these triples have the property that the unique line of $\mathrm{PG}(3,q^2)$ on P that meets l_1 and l_2 is a line of $\mathrm{H}(3,q^2)$.

To do so, we consider a hole P. Then P lies on $q+1$ lines of $\mathrm{H}(3,q^2)$. If x_i, $i=1,\dots,q+1$, is the number of points on the i-th line on P covered by an element of $\mathcal{S}$, then we have $\sum x_i=|\mathcal{S}|$ and hence

$$\sum x_i(x_i-1)\geq(q+1)\frac{|\mathcal{S}|}{q+1}\left(\frac{|\mathcal{S}|}{q+1}-1\right).$$

So we find a lower bound on the number of triples, using that the number of holes equals $\delta(q^2+1)$.

Now choose a pair (l_1,l_2) of distinct spread elements. There are q^2+1 lines of $\mathrm{H}(3,q^2)$ that meet l_1 and l_2. These lines cover $(q^2+1)(q^2-1)$ points of $\mathrm{H}(3,q^2)$ not on l_1 and l_2. By Result 4.2, every line of $\mathcal{S}\backslash\{l_1,l_2\}$ contains $q+1$ of these points. Thus there are $(q^2+1)(q^2-1)-(|\mathcal{S}|-2)(q+1)$ holes. Together with the lower bound, this gives

$$|\mathcal{S}|(|\mathcal{S}|-1)\left[(q^4-1)-(|\mathcal{S}|-2)(q+1)\right]\geq(q^3+1-|\mathcal{S}|)(q^2+1)|\mathcal{S}|\left(\frac{|\mathcal{S}|}{q+1}-1\right).$$

After simplification, we obtain $|\mathcal{S}|\leq\frac{1}{2}(q^3+q+2)$. □

Remarkably, this bound is sharp for $q=2$ and $q=3$, [29, 31]. But for $q\geq 4$, we do not know whether this bound is sharp. In [15], this proof is presented for partial spreads of $\mathrm{H}(4n+3,q^2)$ and also yields for $n\geq 1$ an upper bound.

Result 4.2 has an analogon for hyperbolic quadrics and symplectic polar spaces.

Result 4.4 (see [48]). (i) *Let g_1, g_2 and g_3 be three mutually skew generators of* $\mathrm{Q}^+(4n+3,q)$*. Then the lines of g_1 that lie in a totally isotropic* 3*-space intersecting g_2, g_3 in a line, form a symplectic space* $\mathrm{W}(2n+1,q)$ *in g_1.*

(ii) *Let g_1, g_2 and g_3 be three pairwise skew generators of* $\mathrm{W}(2n+1,q)$*, $n \geq 2$. Let $\mathcal{P}$ be the set of points P in g_1 such that there exists a line in* $\mathrm{W}(2n+1,q)$ *through P intersecting g_2 and g_3.*
For q even and n even, $\mathcal{P}$ forms a pseudo-polarity of g_1.
For q even and n odd, $\mathcal{P}$ is either a pseudo-polarity or a symplectic polarity (depending on the relative position of g_1, g_2 and g_3).
For q odd and n even, $\mathcal{P}$ is a parabolic quadric in g_1.
For q odd and n odd, $\mathcal{P}$ is either an elliptic or hyperbolic quadric (depending on the relative position of g_1, g_2 and g_3).

In [48], these results are used to derive lower bounds on the size of maximal partial spreads in these polar spaces.

Vanhove [86] obtained very recently an upper bound on the size of partial spreads of $\mathrm{H}(4n+1,q^2)$. The proof relies on a remarkable link to *association schemes*, combinatorial structures consisting of a set Ω and a set of relations partitioning $\Omega \times \Omega$, with high regularity. In our case, if Ω is the set of generators of a polar space of rank d, and two generators g_1 and g_2 are i-related if the codimension of $g_1 \cap g_2$ in g_1 is i, then $(\Omega, (R_0, \ldots, R_d))$ is an association scheme. A partial spread of the polar space is a clique in the relation R_d of this association scheme.

The real algebra $\mathbb{R}\Omega$ can be decomposed orthogonally in $D+1$ subspaces V_i, all of them eigenspaces of the relations R_j of the association scheme. Define the matrix $P = (P_{ij})$, where P_{ij} is the eigenvalue of R_j for the eigenspace V_i. Then the *dual matrix of eigenvalues* is defined as $Q = |\Omega| P^{-1}$.

Let $\mathbf{a}$ be the inner distribution vector of any subset X of Ω. Then it is shown in e.g. [28] that every entry of $\mathbf{a}Q$ is non-negative. For X a clique in a relation R_j, this yields an upper bound on the size of X, which can be described only using the greatest and smallest eigenvalue of R_j [35]. Applied to this case, $q^{2n+1}+1$ is found as upper bound on the size of a partial spread. For other polar spaces, this method does not give non-trivial results.

Vanhove gives in [87] an alternative proof for this result, which is now purely geometric and based on a clever generalization of steps taken in [22]. This method gives some insight in case of equality, but this is not as far exploited yet as in [22], where it is shown that a partial spread attaining the upper bound, gives rise to a second partial spread of the same size.

Note that the upper bound $q^{2n+1}+1$ on the size of a partial spread in $\mathrm{H}(4n+1,q^2)$ is sharp. One sees easily that a spread of the symplectic polar space $\mathrm{W}(4n+1,q)$ embedded in $\mathrm{H}(4n+1,q^2)$ extends to a partial spread of $\mathrm{H}(4n+1,q^2)$. Maximality (proved earlier for $n=1$ in [1], and for general n in [51]) now follows from the upper bound.

Only for $\mathrm{Q}(4n,q)$, q odd, and $\mathrm{Q}^+(4n+1,q)$, it is proved, without further assumptions on q, that spreads do not exist. This is clear for $\mathrm{Q}^+(4n+1,q)$ by Result 1.3. An upper bound on the size of partial spreads of $\mathrm{Q}(4n,q)$, q odd, is proved in [36]. The upper bound is related to the size blocking sets of $\mathrm{PG}(2,q)$ (see e.g. [8]), and is obtained by analyzing the set of points of $\mathrm{Q}(4n,q)$ not covered by any element of the partial spread, and describing

polar space	upper bound	references
$Q(4n,q)$, q odd	$q^n+1-\delta$, $\delta \geq \varepsilon$, with $q+1+\varepsilon$ the size of the smallest non-trivial blocking set of $PG(2,q)$	[36]
$Q^+(4n+1,q)$	2	
$H(3,q^2)$	$\frac{1}{2}(q^3+q+2)$ (sharp for $q=2,3$)	[15] ([31,43])
$H(4n+1,q^2)$	$q^{2n+1}+1$	[22], [86], [87]
$H(4n+3,q^2)$	$q^{4n+3}-q^{3n+3}(\sqrt{q}-1)$	[15]

Table 7: Upper bounds on the size of partial spreads

this set using characterization results on multiple weighted blocking sets (minihypers) of the ambient projective space. Recent results on the latter objects can be found in [49].

Table 7 contains an overview of the cases where the non-existence of a spread is proved.

The existence of spreads of the polar space $Q(6,q)$ and $Q^+(7,q)$ is not known for all q. In this situation the difficulty is to find an upper bound on the size of a maximal partial spread, without any assumption on the existence of spreads. Using the results on (maximal) partial ovoids of $Q^+(7,q)$ and the triality map of $Q^+(7,q)$, the following result is derived in [16].

Result 4.5. *The polar space* $Q^+(7,q)$ *has no maximal proper partial spread of size* $q^3+1-\delta$ *with* $0<\delta<q+1$.

Embedding $Q(6,q)$ in $Q^+(7,q)$ as a hyperplane section, we find in [16] exactly the same result for $Q(6,q)$.

Upper bounds on the size of maximal proper partial spreads of $Q(4,q)$ and $Q^-(5,q)$ are found in Theorem 5.2 (a) and (b).

5 Covers and blocking sets

Let $\mathcal{P}$ be a classical finite polar space. A *cover* is a set $\mathcal{C}$ of generators such that every point of P lie in at least one generator of $\mathcal{C}$. A cover is *minimal* if it does not contain a smaller cover. A *blocking set* is a set $\mathcal{B}$ of points with the property hat every generator contains at least one point of $\mathcal{B}$. A blocking set is *minimal* if it does not contain a smaller blocking set.

If $\mathcal{P}$ has rank 2, then clearly a blocking set of $\mathcal{P}$ is mapped by a duality on a cover of the dual space of $\mathcal{P}$. So as in the ovoid-spread case, dualities, and other morphisms, can play a role in the construction of these objects from each other.

The study of blocking sets and covers is motivated in the same way as the study of partial ovoids and partial spreads. Non-existence of ovoids motivates the study of the sets of points blocking all generators. Existence of ovoids poses the question how large a blocking set must be if it does not contain an ovoid. The motivation for the study of covers is of course the same.

5.1 Covers

The study of covers is similar to the study of maximal partial spreads, but there are additional difficulties. We explain this with the following example.

Consider a minimal cover $\mathcal{C}$ (or maximal partial spread) of $\mathrm{Q}(4,q)$ (or $\mathrm{Q}^-(5,q)$) with $q^2+1\pm\delta$ (or $q^3+1\pm\delta$) lines. Let $w:\mathcal{P}\to\mathbb{N}$ be the function that assigns to every point of $\mathrm{Q}(4,q)$ (or $\mathrm{Q}^-(5,q)$) the number $w(P)$ of lines of $\mathcal{C}$ through P. Let $w'(P)=w(P)-1$ (or $w'(P)=1-w(P)$ if we start from a partial spread).

From now on, we work with the weight function w' and the arguments are the same for covers and spreads. The only difference is that in the case of partial spreads we know that w' has range $\{0,1\}$. Let π be a hyperplane, every line of $\mathcal{C}$ meed π either in 1 or $q+1$ points, so

$$\sum_{P\in\pi} w'(P)\equiv\delta \mod q\ .$$

This shows that for $1\le\delta<q$, the weight function w' defines a blocking set of the ambient projective space, completely contained in $\mathcal{P}$. For such blocking sets we have the following result.

Lemma 5.1 (see [47, Lemma 2.1])**.** *Consider in* $\mathrm{PG}(4,q)$ *a quadric that is a cone with vertex a point P over a non-degenerate elliptic quadric* $\mathrm{Q}^-(3,q)$*. Suppose that B is a set of at most $2q$ points contained in this quadric. If every solid of* $\mathrm{PG}(4,q)$ *meets B, then one of the following possibilities occurs:*

(a) Some line of the quadric is contained in B, or,

(b) $|B|>\frac{9}{5}q+1$, $P\in B$, and there exists a unique line l of the quadric that meets B in at least $1+\frac{1}{3}|B|$ points. This line has at most $|B|-1-q$ points in B.

Applying this lemma to the weight function w' shows immediately that for $\delta\le\frac{4}{5}q$, the corresponding blocking set contains a line. If w' has range $\{0,1\}$ (i.e. we start from a partial spread), then an extra argument shows that (b) does not occur, so one finds a line for all $\delta<q$. We may conclude the following result.

Theorem 5.2. *(a) Every partial spread of* $\mathrm{Q}(4,q)$ *of size $q^2+1-\delta$ ($\delta<q$) extends to a spread.*

(b) Every partial spread of $\mathrm{Q}^-(5,q)$ *of size $q^3+1-\delta$ ($\delta<q$) extents to a spread.*

(c) Let $\mathcal{C}$ be a cover of $\mathrm{Q}(4,q)$ *of size $q^2+1+\delta$ ($\delta<\frac{4}{5}q$). For every point P let $w'(P)+1$ be the number of lines of C through P. Then there exists δ (not necessarily different) lines $l_1,\dots,l_\delta$ of* $\mathrm{Q}(4,q)$ *such that $w'(P)$ is equal to the number of lines l_i through P.*

(d) Let $\mathcal{C}$ be a cover of $\mathrm{Q}^-(5,q)$ *of size $q^3+1+\delta$ ($\delta<\frac{4}{5}q$). For every point P let $w'(P)+1$ be the number of lines of $\mathcal{C}$ through P. Then there exists δ (not necessarily different) lines $l_1,\dots,l_\delta$ of* $\mathrm{Q}^-(5,q)$ *such that $w'(P)$ is equal to the number of lines l_i through P.*

In case of partial spreads, this theorem is exactly what we want. In case of covers, it is unclear if (some of) the lines $l_1,\dots,l_\delta$ belong to the cover. If yes, then the cover is not minimal, as the lines l_i consist entirely of multiple covered points. So, in order to transform (c) and (d) into a bound for minimal covers that are not spreads, one has to show that that all the lines l_i belong to the cover. For $\mathrm{Q}(4,q)$ this was done for small δ using a long and complicated algebraic argument.

Result 5.3 (see [47, Theorem 1.3])**.** *Let q be odd. Then a cover of $\mathrm{Q}(4,q)$ contains at least $q^2-q-\frac{3}{2}+\frac{\sqrt{8q^2+20q+25}}{2}\approx q^2+0.414q$ lines.*

For $\mathrm{Q}^-(5,q)$ this is however not possible as $\mathrm{Q}^-(5,q)$ has minimal covers of size q^3+2, constructed in the following example.

Example 5.4. *Consider a hermitian spread $\mathcal{S}$ of $\mathrm{Q}^-(5,q)$, that is a spread translating to a classical ovoid of $\mathrm{H}(3,q^2)$ under the duality between $\mathrm{Q}^-(5,q)$ and $\mathrm{H}(3,q^2)$. Using this duality, it is easy to see that such a spread is the union of q^2 reguli R_i through a common line. Let R_i^{opp} be the regulus opposite to R_i. Define $\mathcal{S}' := (\mathcal{S}\cup R_1^{opp})\setminus R_1$. Then $\mathcal{S}'$ is again a spread. But this procedure can be repeated, and now $\mathcal{S}'' := (\mathcal{S}'\cup R_2^{opp})\setminus R_2$ will be a minimal cover of size q^3+2. Clearly, one can construct minimal covers of any size in the range $q^3+2,\ldots,q^3+q^2$ using this method.*

This is quite typical for covers and blocking sets of finite polar spaces. Using arguments from the partial spread and partial ovoid case yield results similar to Theorem 5.2. Deciding if the extra lines (or points) are already inside the cover (or blocking set) is the hard part.

5.2 Blocking sets

Suppose that $\mathcal{P}_r$ is a polar space of rank r of a given type. In most cases where the non-existence of ovoids of $\mathcal{P}_{r-1}$ is proved, the smallest minimal blocking sets of $\mathcal{P}_r$ are known. To describe the examples, we introduce a *truncated cone*. Suppose that π is any subspace in $\mathrm{PG}(n,q)$, and O any point set contained in π', a subspace skew to π. The truncated cone π^*O, $O\subseteq\pi^\perp\setminus\pi$ is the set of all points on all lines connecting a point of π and a point of O, minus the points of π. Table 8 lists the smallest minimal blocking sets of polar spaces for which the non-existence of ovoids is proved, except for $\mathrm{W}(2n+1,q)$, q odd, and $\mathrm{H}(5,4)$. A d-dimensional subspace of $\mathrm{PG}(n,q)$ is denoted as π_d.

The result on blocking sets of the polar spaces $\mathrm{W}(2n+1,q)$, $n\geq 2$ is found by Metsch [58]. It classifies the smallest minimal blocking sets when q is even, and shows a lower bound on the size when q is odd. Apart from this lower bound, nothing is known. Independently, De Beule and Storme treated the case $n=2$ and q even in [24].

Another interesting open case is to determine the smallest minimal blocking sets of $\mathrm{Q}(2n,q)$, q odd, q not prime and $q\neq 3$. It is conjectured (see e.g. [65]) that $\mathrm{Q}(2n,q)$ has ovoids if and only if $q=3^h$, and it is expected that the smallest minimal blocking sets always are truncated cones $\pi^*_{n-3}O$, O an ovoid of $\mathrm{Q}(4,q)$, when $q\neq 3^h$.

In the spaces $\mathrm{Q}^+(2n+1,q)$, $q\in\{2,3\}$, $n\geq 4$, not only the smallest minimal blocking sets are known. In [26] and [23], the geometrical arguments used to study blocking sets enable to classify the two smallest minimal blocking sets of $\mathrm{Q}^+(2n+1,2)$, $n\geq 3$, and the three smallest minimal blocking sets of $\mathrm{Q}^+(2n+1,3)$, $n\geq 3$.

For $\mathrm{Q}(4,q)$, the smallest blocking sets are ovoids. Clearly, a truncated cone $\pi_0^*\mathcal{C}$, $\mathcal{C}$ a conic, is a minimal blocking set of $\mathrm{Q}(4,q)$ different from an ovoid. But up to now, for q even, minimal blocking sets different from an ovoid of size s, $s<q^2+1+\frac{q+4}{6}$, are excluded [68]. For q odd, q prime, only minimal blocking sets of size q^2+2 are excluded [20]. The smallest minimal blocking sets of $\mathrm{Q}(6,3)$ different from an ovoid are truncated cones π_0^*O, O an ovoid of $\mathrm{Q}(4,3)$ [25]. Blocking sets of $\mathrm{W}(3,q)$, q odd, are dually the same as covers

polar space	example	references
$W(2n+1,q)$, q even, $n>2$	$\pi^*_{n-2}O$, O an ovoid of $W(3,q)$	[58], [24]
$Q(2n,p)$, $p>3$ odd prime, $n>2$	$\pi^*_{n-3}Q^-(3,q)$	[18]
$Q^-(2n+1,q)$, $n\geq 2$	$\pi^*_{n-2}Q^-(3,q)$	[53]
$H(2n,q^2)$, $n\geq 2$	$\pi^*_{n-2}H(2,q^2)$	[21]
$Q(2n,3)$, $n\geq 4$	$\pi^*_{n-4}O$, O an ovoid of $Q(6,3)$	[25]
$Q^+(2n+1,q)$, $q\in\{2,3\}$, $n\geq 4$	$\pi^*_{n-4}O$, O an ovoid of $Q^+(7,q)$	[26], [23]

Table 8: Smallest minimal blocking sets

of $Q(4,q)$, q odd, so we refer to Result 5.3. Finally, minimal blocking sets of $H(3,q^2)$, dually minimal covers of $Q^-(5,q)$, are constructed with size in the range $q^3+2,\ldots,q^3+q^2$ in Example 5.4.

Let now $\mathcal{P}$ be a finite classical polar space of rank r. A *blocking set with respect to the s-dimensional spaces of $\mathcal{P}$* is a set of points of $\mathcal{P}$ blocking all s-dimensional spaces, $s\leq r-1$, contained in $\mathcal{P}$. When $s=r-1$, we are considering blocking sets. We have seen that in some cases the smallest blocking sets are truncated cones with base an ovoid of a polar space of low rank, so the existence or non-existence of ovoids, which is not completely known for all polar spaces, complicates the work. However, more can be done for blocking sets with respect to s-spaces for $1\leq s<r-1$. The basic observation is that s-dimensional spaces of $\mathcal{P}$ also are s-dimensional subspaces of the ambient projective space, and these are all blocked by a subspace of the ambient projective space of codimension s. In many cases, it can be shown that a blocking set with respect to s-dimensional subspaces of $\mathcal{P}$ can be constructed from an intersection of $\mathcal{P}$ with a subspace of the ambient projective space of codimension s . All results we describe here, are based on results found in the series of papers [50, 53–57, 59]. The following general result for quadrics is proved in [57].

Result 5.5. *Let Q be a non-degenerate quadric and d the dimension of its generators. Assume that $s<d$ when Q is not elliptic, and assume $s\leq d$ otherwise. Then the smallest (minimal) blocking sets with respect to the s-dimensional spaces of Q have the form $(T\setminus T^{\perp})\cap Q$ for a suitable subspace T of the ambient projective space of codimension s.*

The *suitability* of the subspace T refers to its intersection type with Q, which we will describe in detail below. The size of the constructed blocking set is dependent on the intersection type, hence it is not surprising that in some cases minimal blocking sets are obtained that are not the smallest. Result 5.5 leads to the classification of blocking sets with respect to s-dimensional spaces below a given size. Table 9 surveys known results for quadrics. Each line must be interpreted as: *a blocking set B of the space $\mathcal{P}$ with respect to its s-dimensional spaces, with size smaller than the given size, contains one of the given examples*. Only for $Q^-(2n+1,q)$ the shown result includes the result for the smallest minimal blocking sets with respect to its generators.

The following result and corollary for $H(2n+1,q^2)$ is proved in [59].

Result 5.6. *Consider $H(2n+1,q^2)$ and an integer s, $1\leq s<n$. Concerning the cardinalities of the minimal blocking sets of $H(2n+1,q^2)$ with respect to s-spaces, the sets $(T\setminus T^{\perp})\cap H(2n+1,q^2)$, T a subspace of the ambient projective space $PG(2n+1,q^2)$ of codimension*

polar space	dimension s	given size	structure
$Q^+(2n+1,q)$	$2 \leq s \leq n-1$	$(q^n+q^{s-2}+1)\theta_{n-s}(q)$	$\pi^*_{s-3}Q^-(2(n-s)+3,q)$
$Q^+(2n+1,q)$	1	$(q^{n-1}+1)\theta_{n-1}(q)$	$Q(2n,q)$ or $P^*Q^+(2n-1,q)$
$Q(2n,q)$	$1 \leq s \leq n-2$	$(q^n+q^{s-1}+1)\theta_{n-s-1}(q)$	$\pi^*_{s-2}Q^-(2(n-s)+1,q)$
$Q^-(2n+1,q)$	$2 \leq s \leq n-2$	$(q^{n+1}+q^s+1)\theta_{n-s-1}(q)$	$\pi^*_{s-1}Q^-(2(n-s)+1,q)$

Table 9: Blocking sets with respect to s-spaces

s, provide the two smallest cardinalities when $s \in \{1,2\}$ and the $s-2$ smallest cardinalities when $s \geq 3$.

Corollary 5.7. *The smallest blocking sets of* $H(2n+1,q^2)$ *with respect to s-spaces,* $1 \leq s < n$*, are truncated cones* $\pi^*_{s-2}H(2n+2-2s,q^2)$.

Consider the embedding of $H(2n,q^2)$ in $H(2n+1,q^2)$ as a hyperplane section. It is clear that a point set $B \subset H(2n,q^2)$ is a blocking set of $H(2n,q^2)$ with respect to s-spaces, $1 \leq s \leq n-1$, if and only if B is a blocking set of $H(2n+1,q^2)$ with respect to $(s+1)$-spaces. So by Corollary 5.7 we know the smallest blocking sets of $H(2n,q^2)$ with relation to s-spaces, $1 \leq s < n-1$. Recall that the case $s = n-1$ for $H(2n,q^2)$ is described in the fourth line of Table 8. Finally, the case $W(2n+1,q)$, q odd, is completely open. Even the smallest blocking sets with respect to lines of $W(2n+1,q)$, q odd, are not known.

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