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Subtleties in Robust Stability of Discrete-time Piecewise Affine Systems

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Abstract— In this paper we consider (inherent) robustness of discrete-time piecewise affine (PWA) systems. We demonstrate, via examples, that globally exponentially stable discrete-time PWA systems may have no robustness. More precisely, we show that the exponential stability property cannot prevent that *arbitrarily small* additive disturbances keep the state trajectory far from the origin. Mathematically speaking, this means that the system is not input-to-state stable with respect to arbitrarily small disturbances. The non-robustness property is related to the absence of a continuous Lyapunov function. These results indicate that one should be careful with existing stability analysis and synthesis methods for PWA systems that rely on discontinuous Lyapunov functions, as no robustness might be present. However, as the search for Lyapunov functions for discrete-time PWA systems often employs discontinuous Lyapunov functions (e.g. piecewise quadratic ones), robustness tests based on discontinuous Lyapunov functions are needed. Such tests are proposed in this article.

Index Terms— Hybrid systems, Piecewise affine systems, Discrete-time, Stability, Robustness.

I. INTRODUCTION

Discrete-time piecewise affine (PWA) systems are a powerful modeling class for the approximation of hybrid and nonlinear dynamics [1], [2]. The modeling capability of discrete-time PWA systems has already been shown in several applications, including switched power converters [3], direct torque control of three-phase induction motors [4], applications in automotive systems [5] and systems biology [6]. Therefore, there is an increasing interest in developing efficient tools for stability analysis and stabilizing controller synthesis for discrete-time PWA systems, as it is illustrated by several articles on this topic, see, for example, [7]–[10], to mention just a few.

In the case when asymptotic stability is established via a *continuous Lyapunov function*, such as a common quadratic Lyapunov function (i.e. $V(x) = x^\top P x$ for some matrix $P > 0$), it is well known [11] that inherent robustness to perturbations is achieved. This is a desirable property, as nominally stable closed-loop systems are always affected by perturbations in practice. However, it is also well known [8] that there exist discrete-time PWA systems that do not admit a common quadratic Lyapunov function, but they admit a piecewise quadratic (PWQ), possibly discontinuous Lyapunov function. As inherent robustness is not necessarily guaranteed when

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stability is established via discontinuous Lyapunov functions, it is important to answer the following question:

- Is it possible that nominally asymptotically stable (or even stronger, exponentially stable) discrete-time PWA systems can have no robustness to *arbitrarily small* perturbations?

By no robustness we mean that in spite of the asymptotic stability property, the system is not input-to-state stable (ISS) [12], [13] with respect to *arbitrarily small* additive disturbances.

One of the contributions of this article is answering the above question. We present two examples, a one-dimensional discrete-time PWA system and a two-dimensional discrete-time PWA system that are globally exponentially stable (notice that the existence of a discontinuous Lyapunov function is then guaranteed by Lemma 4 in [14]). For both examples, we show that arbitrarily small perturbations can keep the state trajectory far from the origin. Furthermore, via the results of [11], we then establish that:

- There exist globally exponentially stable discrete-time PWA systems that admit a discontinuous Lyapunov function, but not a continuous one.

Previous results on stability of discrete-time PWA systems [7]–[10] only indicated that continuous Lyapunov functions may be more difficult to find than discontinuous ones, but until now it was not shown that there exist exponentially stable PWA systems for which a continuous Lyapunov function does not exist. This is the case for the examples presented in this paper.

These results issue a warning regarding nominally stabilizing state-feedback synthesis methods for PWA systems [7]–[10], including those of model predictive control (MPC) [15]–[18]. These synthesis methods lead to a stable PWA closed-loop system and often rely on discontinuous Lyapunov functions. For example, in MPC the most natural candidate Lyapunov function is the value function corresponding to the MPC cost, which is generally discontinuous when PWA systems are used as prediction models [18]. As such, these controllers may result in closed-loop systems that are not ISS when arbitrarily small perturbations affect the system, which is always happening in practice. Therefore, in the case of discrete-time PWA systems for which a continuous Lyapunov function is not known, but a discontinuous Lyapunov function is available, robustness tests based on discontinuous Lyapunov functions are needed. In this paper we will present robustness tests based on discontinuous Lyapunov functions.

II. PRELIMINARIES

A. Nomenclature and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. Let $\|\cdot\|$ denote an arbitrary norm on $\mathbb{R}^n$ and let $|\cdot|$ denote the absolute value of a real number. For a sequence $\{z_p\}_{p \in \mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$ let $\|\{z_p\}_{p \in \mathbb{Z}_+}\| := \sup\{\|z_p\| \mid p \in \mathbb{Z}_+\}$. For a sequence $\{z_p\}_{p \in \mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$, $z_{[k]}$ denotes the truncation of $\{z_p\}_{p \in \mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k]} = \{z_p\}_{p \in \mathbb{Z}_{[0,k]}}$.

For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\partial \mathscr{S}$ the boundary, by $\text{int}(\mathscr{S})$ the interior and by $\text{cl}(\mathscr{S})$ the closure of $\mathscr{S}$. For two arbitrary sets $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^n$, let $\mathscr{S} \sim \mathscr{P} := \{x \in \mathbb{R}^n \mid x + \mathscr{P} \subseteq \mathscr{S}\}$ and $\mathscr{S} \oplus \mathscr{P} := \{x + y \mid x \in \mathscr{S}, y \in \mathscr{P}\}$ denote their Pontryagin difference and Minkowski sum, respectively. For any $\mu > 0$ we define $\mathscr{B}_\mu := \{x \in \mathbb{R}^n \mid \|x\| \leq \mu\}$. Let $\mathscr{B} := \mathscr{B}_1$ denote the closed unit ball in $\mathbb{R}^n$. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces.

The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as:

$$\|x\|_p := \begin{cases} \left(|x_1|^p + \ldots + |x_n|^p\right)^{\frac{1}{p}}, & p \in \mathbb{Z}_{[1,\infty)} \\ \max_{i=1,\ldots,n} |x_i|, & p = \infty, \end{cases}$$

where x_i, $i = 1,\ldots,n$ is the i-th component of x. For a matrix $Z \in \mathbb{R}^{m \times n}$ let $\|Z\|_p := \sup_{x \neq 0} \frac{\|Zx\|_p}{\|x\|_p}$, $p \in \mathbb{Z}_{\geq 1}$, denote its induced matrix norm.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}$ ($\varphi \in \mathscr{K}$) if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}_\infty$ ($\varphi \in \mathscr{K}_\infty$) if $\varphi \in \mathscr{K}$ and it is radially unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}\mathscr{L}$ ($\beta \in \mathscr{K}\mathscr{L}$) if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot,k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty}\beta(s,k) = 0$.

B. Input-to-state stability

To study robustness, we will employ the input-to-state stability (ISS) framework [12], [13]. Consider the discrete-time perturbed nonlinear system:

$$x_{k+1} = g(x_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}^{d_v}$ is an unknown disturbance input and $g : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is a nonlinear, possibly discontinuous function. For simplicity of notation, we assume that the origin is an equilibrium for (1) and zero disturbance input, meaning that $g(0,0) = 0$. We use the notation x_k to denote the solution of (1) at time $k \in \mathbb{Z}_+$, obtained from initial condition x_0 at time $k = 0$.

Definition II.1 A set $\mathscr{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ is called a *robustly positively invariant (RPI) set with respect to* $\mathbb{V}$ *for* system (1) if for all $x \in \mathscr{P}$ it holds that $g(x,v) \in \mathscr{P}$ for all $v \in \mathbb{V}$.

Next, we introduce the notion of global input-to-state stability for the discrete-time system (1), as defined in [13].

Definition II.2 The perturbed system (1) is *globally input-to-state stable (ISS)* if there exist a $\mathscr{K}\mathscr{L}$-function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$-function $\gamma(\cdot)$ such that, for each initial condition $x_0 \in \mathbb{R}^n$ and all $\{v_p\}_{p \in \mathbb{Z}_+}$ with $v_p \in \mathbb{R}^{d_v}$ for all $p \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) \quad \text{for all} \quad k \in \mathbb{Z}_{\geq 1}.$$

In this paper we will also use the following local ISS notion.

Definition II.3 Let $\mathbb{X}$ and $\mathbb{V}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^{d_v}$, respectively, with $0 \in \text{int}(\mathbb{X})$. We call system (1) *ISS in* $\mathbb{X}$ *for disturbances in* $\mathbb{V}$ if there exist a $\mathscr{K}\mathscr{L}$-function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$-function $\gamma(\cdot)$ such that, for each $x_0 \in \mathbb{X}$ and all $\{v_p\}_{p \in \mathbb{Z}_+}$ with $v_p \in \mathbb{V}$ for all $p \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) \quad \text{for all} \quad k \in \mathbb{Z}_{\geq 1}.$$

Throughout this article we will employ the following sufficient conditions for analyzing ISS of discrete-time systems.

Theorem II.4 [13], [19] *Let* $\alpha_1, \alpha_2, \alpha_3, \sigma \in \mathscr{K}_\infty$ *and let* $\mathbb{V}$ *be a subset of* $\mathbb{R}^{d_v}$ *that contains the origin. Let* $\mathbb{X}$ *with* $0 \in \text{int}(\mathbb{X})$ *be a RPI set with respect to* $\mathbb{V}$ *for system* (1) *and let* $V : \mathbb{X} \to \mathbb{R}_+$ *be a function with* $V(0) = 0$. *Consider the following inequalities:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2a}$$
$$V(g(x,v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|). \tag{2b}$$

If inequalities (2) *hold for all* $x \in \mathbb{X}$ *and all* $v \in \mathbb{V}$, *then system* (1) *is* ISS *in* $\mathbb{X}$ *for disturbances in* $\mathbb{V}$. *Furthermore, if inequalities* (2) *hold for all* $x \in \mathbb{R}^n$ *and all* $v \in \mathbb{R}^{d_v}$, *then system* (1) *is* globally ISS.

Definition II.5 A function $V(\cdot)$ that satisfies the hypothesis of Theorem II.4 is called an *ISS Lyapunov function*.

Note the following aspects regarding Theorem II.4:

(i) the hypothesis of Theorem II.4 allows that both $g(\cdot,\cdot)$ and $V(\cdot)$ are discontinuous. The hypothesis *only* requires continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$;

(ii) if the inequalities (2) are satisfied for $\alpha_1(s) = as^\lambda$, $\alpha_2(s) = bs^\lambda$, $\alpha_3(s) = cs^\lambda$, for some $a, b, c, \lambda > 0$, then the hypothesis of Theorem II.4 implies exponential stability of system (1) with zero disturbance input.

C. Lyapunov functions

To make a distinction with respect to Lyapunov functions in the classical sense, which are only required to have a non-negative one step forward difference, in this section we will introduce various types of Lyapunov functions for the unperturbed system corresponding to (1), i.e. $x_{k+1} = g(x_k, 0)$, $k \in \mathbb{Z}_+$. Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a positively invariant set for $x_{k+1} =$

$g(x_k, 0)$ with $0 \in \text{int}(\mathbb{X})$, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $V(0) = 0$, and consider the inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \qquad \forall x \in \mathbb{X}, \tag{3a}$$

$$V(g(x, 0)) - V(x) \leq 0, \qquad \forall x \in \mathbb{X}, \tag{3b}$$

$$V(g(x, 0)) - V(x) < 0, \qquad \forall x \in \mathbb{X} \setminus \{0\}, \tag{3c}$$

$$V(g(x, 0)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{X}. \tag{3d}$$

Definition II.6 A function $V(\cdot)$ that satisfies (3a) and (3b) is called a *Lyapunov function*. A function $V(\cdot)$ that satisfies (3a) and (3c) is called a *strict Lyapunov (SL) function*. A function $V(\cdot)$ that satisfies (3a) and (3d) is called a *uniformly strict Lyapunov (USL) function*.

Notice that a USL function can also be defined by replacing (3d) with the intermediate property

$$V(g(x, 0)) - V(x) \leq -\delta(x), \qquad \forall x \in \mathbb{X}, \tag{4}$$

where $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous and positive definite function. However, it can be shown that given such a USL function one can always find a new USL function that satisfies (3d), using ideas from [20]. Also, in the case when $g(\cdot, 0)$ and $V(\cdot)$ are *continuous* it can be proven that SL functions and USL functions that satisfy (4) are equivalent.

D. Robust global asymptotic stability

To make use of robustness results from [11] we need to define the robust global asymptotic stability property (RGAS) for systems of the form:

$$x_{k+1} = h(x_k), \quad k \in \mathbb{Z}_+, \tag{5}$$

where $h : \mathbb{R}^n \to \mathbb{R}^n$ with $h(0) = 0$ is an arbitrary nonlinear, possibly discontinuous function. For a continuous function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ we define a perturbed version of (5) as follows:

$$x_{k+1} \in h_\delta(x_k) := \{h(x_k + \delta(x_k)v) + \delta(x_k)v \mid v \in \mathscr{B}\}, \quad k \in \mathbb{Z}_+. \tag{6}$$

Let $\mathscr{S}_\delta(x_0)$ denote the set of all solutions of (6) corresponding to initial state x_0 at time $k = 0$.

Definition II.7 We call system (5) RGAS if there exist a $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous and positive definite function and a $\beta_\delta \in \mathcal{KL}$ such that for every $x_0 \in \mathbb{R}^n$ and all solutions $x_k^\delta \in \mathscr{S}_\delta(x_0)$ it holds that $\|x_k^\delta\| \leq \beta_\delta(\|x_0\|, k)$, for all $k \in \mathbb{Z}_{\geq 0}$.

E. Discrete-time PWA systems

In this paper we focus on nominal and perturbed discrete-time, possibly discontinuous, PWA systems of the form

$$x_{k+1} = G(x_k) := A_j x_k + f_j \qquad \text{if} \quad x \in \Omega_j, \tag{7a}$$

$$\tilde{x}_{k+1} = g(\tilde{x}_k, v_k) := A_j \tilde{x}_k + f_j + v_k \quad \text{if} \quad x \in \Omega_j, \tag{7b}$$

where $A_j \in \mathbb{R}^{n \times n}$, $f_j \in \mathbb{R}^n$ for all $j \in \mathscr{S}$ and $\mathscr{S} := \{1, 2, \ldots, s\}$ is a *finite set* of indices. The collection $\{\Omega_j \mid j \in \mathscr{S}\}$ defines a partition of $\mathbb{R}^n$, meaning that $\cup_{j \in \mathscr{S}} \Omega_j = \mathbb{R}^n$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\text{int}(\Omega_i) \neq \emptyset$ for all $i \in \mathscr{S}$. Each Ω_j is assumed to be a polyhedron, which is not necessarily closed. Let $\mathscr{S}_0 := \{j \in \mathscr{S} \mid 0 \in \text{cl}(\Omega_j)\}$ and let $\mathscr{S}_1 := \{j \in$
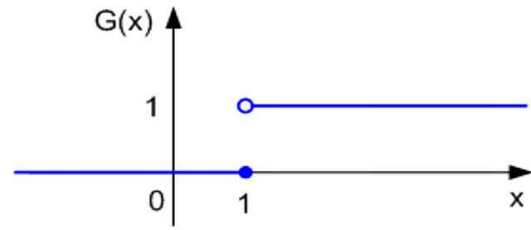

Fig. 1. A one-dimensional PWA system with no robustness.

$\mathscr{S} \mid 0 \notin \text{cl}(\Omega_j)\}$, so that $\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1$. For the origin to be an equilibrium in the Lyapunov sense in (7a) we assume as in [8] that $f_j = 0$ for all $j \in \mathscr{S}_0$.

III. EXAMPLES OF NON-ROBUST PWA SYSTEMS

First, we present a simple one-dimensional example inspired by [11].

Example 1: Consider the discontinuous nominal and perturbed PWA systems given by (7) with $v_k \in \mathscr{B}_\mu = \{v \in \mathbb{R} \mid |v| \leq \mu\}$ for some small positive parameter $\mu > 0$, $j \in \mathscr{S} := \{1, 2\}$, $k \in \mathbb{Z}_+$, and where $A_1 = A_2 = 0$, $f_1 = 0$, $f_2 = 1$ and the partition is given by $\Omega_1 = \{x \in \mathbb{R} \mid x \leq 1\}$, $\Omega_2 = \{x \in \mathbb{R} \mid x > 1\}$. Figure 1 shows the values of the function $G(x)$.

One can easily observe that any solution x_k at time $k \in \mathbb{Z}_+$ of system (7a) starting from an initial condition $x_0 \in \mathbb{R}$ satisfies $|x_k| \leq |x_0|$ (even $|x_k| < |x_0|$ when $x_0 \neq 0$) and converges exponentially to the origin. Moreover, any trajectory x_k reaches the origin in 2 discrete-time steps or less. Furthermore, it can be proven that $V(x) := \sum_{i=0}^\infty x_i^2$ is a USL function, where x_i denotes the solution of system (7a) at time $i \in \mathbb{Z}_+$, obtained from an initial condition $x_0 := x \in \mathbb{R}$. For example, since $V(x) = \sum_{i=0}^\infty x_i^2 = x_0^2 + x_1^2$ for any $x_0 = x \in \mathbb{R}$, it holds that $V(G(x)) - V(x) \leq -\alpha_3(|x|)$ for all $x \in \mathbb{R}$, where $\alpha_3(s) := s^2$. An explicit expression for $V(\cdot)$ is:

$$V(x) = \sum_{i=0}^\infty x_i^2 = x_0^2 + x_1^2 = \begin{cases} x^2 + 1 & \text{if } x > 1 \\ x^2 & \text{if } x \leq 1, \end{cases}$$

which shows that $V(\cdot)$ is discontinuous at $x = 1$.

Now consider the case where $v_k = \mu > 0$ for all $k \in \mathbb{Z}_+$ in (7b). Then, the origin of the perturbed system (7b) corresponding to the nominal system (7a) is not ISS, as $x_k = 1 + \mu$ is an equilibrium of (7b) to which all trajectories with initial conditions $x_0 \in \mathbb{R}_{>1}$ converge. Hence, no matter how small $\mu > 0$ is taken, the system (7b) is not ISS in $\mathbb{R}$ for disturbances in $\mathscr{B}_\mu$. $\square$

The following conclusions can be drawn from this example: globally exponentially stable (GES) discontinuous discrete-time systems can be non-robust and, the existence of a discontinuous USL function does not guarantee robustness for discontinuous systems.

Note that the GES PWA system of Example 1 also admits a *continuous* SL function, i.e. $V(x) := |x|$, which satisfies $V(G(x)) - V(x) < 0$ for all $x \neq 0$. However, $V(x) = |x|$ is not a USL function, as for any $\alpha_3 \in \mathcal{K}_\infty$ it holds

that $\lim_{x\downarrow 1}(V(G(x)) - V(x)) = \lim_{x\downarrow 1}(1 - x) = 0 > -\alpha_3(1)$. Hence, the existence of a continuous SL function does not necessarily guarantee any robustness for discontinuous systems.

Furthermore, we show via an example that for discontinuous systems a continuous SL function does not even imply global convergence, necessarily.

Example 2: Consider the PWA system (7a) with $j \in \mathscr{S} := \{1, 2\}$, $A_1 = f_1 = 0$, $A_2 = 0.5$, $f_2 = 0.5$ and the partition is given by $\Omega_1 = \{x \in \mathbb{R} \mid x \le 1\}$, $\Omega_2 = \{x \in \mathbb{R} \mid x > 1\}$. One can easily check that $\lim_{k\to\infty} x_k = 1$ for any $x_0 \in \mathbb{R}_{>1}$ and thus, this system is not globally asymptotically stable (GAS). Consider the function $V(x) := |x|$. Clearly, for $x \in \Omega_1 \setminus \{0\}$ we have $V(G(x)) - V(x) = -V(x) < 0$ and, for $x \in \Omega_2$ we have $V(G(x)) - V(x) = 0.5|x+1| - V(x) < |x| - V(x) = 0$. □

Hence, the system of Example 2 admits a continuous SL function but the trajectories do not converge to the origin globally. This indicates that SL functions which are not USL functions are virtually useless for discrete-time discontinuous systems.

The next example shows a *constrained* 2D PWA system that is exponentially stable but it has no robustness.

Example 3: Consider the discontinuous nominal and perturbed PWA systems

$$x_{k+1} = A_j x_k + f_j \qquad \text{if} \quad x_k \in \Omega_j, \qquad (8a)$$

$$\tilde{x}_{k+1} = A_j \tilde{x}_k + f_j + v_k \quad \text{if} \quad x_k \in \Omega_j, \qquad (8b)$$

with $v_k \in \mathscr{B}_\mu = \{v \in \mathbb{R}^2 \mid \|v\| \le \mu\}$ for some $\mu > 0$, $j \in \mathscr{S} := \{1, \ldots, 9\}$, $k \in \mathbb{Z}_+$, and where

$$A_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } j \neq 7; \quad A_7 = \begin{bmatrix} 0.35 & 0.6062 \\ 0.0048 & -0.0072 \end{bmatrix};$$

$$f_1 = -f_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}; \quad f_3 = f_4 = f_5 = f_6 = \begin{bmatrix} 0 \\ -1 \end{bmatrix};$$

$$f_7 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad f_8 = \begin{bmatrix} 0.4 \\ -0.1 \end{bmatrix}; \quad f_9 = \begin{bmatrix} -0.4 \\ -0.1 \end{bmatrix}.$$

The system state takes values in the set $\mathbb{X} := \cup_{j \in \mathscr{S}} \Omega_j$, where the regions Ω_j are polyhedra (the exact representations are omitted due to space limitations), as shown in Figure 2. The state trajectories[1] of system (8a) obtained for the initial states $x_0 = [0.2\ 3.6]^\top \in \Omega_2$ (square dotted blue line) and $x_0 = [0.2\ 3.601]^\top \in \Omega_1$ (circle dotted blue line) are plotted in Figure 2. □

Proposition III.1 *The following statements hold:*

(i) *The function* $V(x) := \|x_{10}\|_\infty + \sum_{i=0}^9 \|Qx_i\|_\infty$, *where* $Q = 0.04 I_2$ *and* x_i *is the solution of system* (8a) *obtained at time* $i \in \mathbb{Z}_{[0,10]}$ *from initial condition* $x_0 := x \in \mathbb{X}$, *is a* discontinuous USL function *for system* (8a);

(ii) *The PWA system* (8a) *is exponentially stable in* $\mathbb{X}$;

(iii) *For any small positive parameter* $\mu > 0$ *the PWA system* (8b) *is not ISS in* $\mathbb{X}$ *for disturbances in* $\mathscr{B}_\mu$.

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[1]Note that the regions Ω_1 and Ω_2 are such that for all $x \in \partial\Omega_1 \cap \partial\Omega_2$ the dynamics $x_{k+1} = A_2 x_k + f_2$ is active, i.e. $\partial\Omega_1 \cap \partial\Omega_2 \subseteq \Omega_2$.
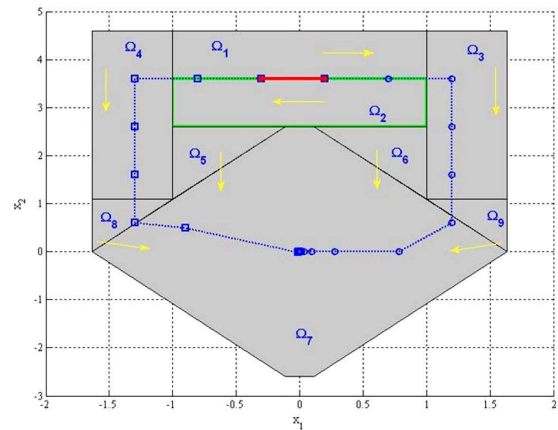

Fig. 2. A two-dimensional PWA system with no robustness: nominal state trajectories (square and circle dotted *blue* lines) and perturbed state trajectory (star solid *red* line).

Remark III.2 While the disturbance signal used in Example 1 does not have a particular structure, a specific disturbance signal was employed in Example 3 to destroy ISS. However, in practice there is often still some structure in the disturbances (for example, time delays in embedded systems or cyclic sensor/encoder errors), which makes such a situation not highly unlikely to happen.

IV. ROBUSTNESS TESTS BASED ON DISCONTINUOUS USL FUNCTIONS

In this section we consider the case when a discontinuous USL function $V(\cdot)$ is available for the PWA system $x_{k+1} = G(x_k) = A_j x_k + f_j$, if $x_k \in \Omega_j$, $k \in \mathbb{Z}_+$, while a continuous USL function is not known. We consider discontinuous Lyapunov functions $V : \mathbb{R}^n \to \mathbb{R}_+$, with $V(0) = 0$, of the form:

$$V(x) := V_i(x) \quad \text{if} \quad x \in \Gamma_i, \quad i \in \mathscr{J}, \qquad (9)$$

where[2] *for each* $i \in \mathscr{J}$, $V_i : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous function on $\mathbb{R}^n$. In (9), $\{\Gamma_i \mid i \in \mathscr{J}\}$, with $\mathscr{J} := \{1, \ldots, \overline{M}\}$ a finite set of indices, denotes a partition of $\mathbb{R}^n$, where the regions Γ_i, $i \in \mathscr{J}$, are convex sets.

Note that the class of Lyapunov functions of the form (9) captures a wide range of frequently used Lyapunov functions for PWA systems, such as PWQ, PWA or piecewise polynomial functions, including the value functions that arise in model predictive control of PWA systems.

The goal is now to establish either *robustness* (i.e. ISS in a desired domain of attraction for bounded and sufficiently small disturbances) or *non-robustness* (i.e. ISS is destroyed by arbitrarily small disturbances for at least one initial condition inside the desired domain of attraction) of system (7), despite the fact that a continuous USL function is not known. We propose to achieve this goal by using the available information on the discontinuous USL Lyapunov function (9).

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[2]In fact, the results developed in this section only require that, for each $i \in \mathscr{J}$, $V_i(\cdot)$ is defined on $\mathrm{cl}(\Gamma_i) \oplus \mathscr{B}_\mu$ and continuous, where $\mu > 0$ denotes a small positive parameter.

The first idea consists in examining the trajectory of the PWA system (7a) with respect to the set of states at which $V(\cdot)$ may be discontinuous. Let $\mathscr{P} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ be a compact RPI set for system (7b) with respect to $\mathscr{B}_\mu$, i.e.

$$\mathscr{R}_1(\mathscr{P}) \subseteq \mathscr{P} \sim \mathscr{B}_\mu,$$

where $\mathscr{R}_1(\mathscr{P}) := \{G(x) \mid x \in \mathscr{P}\}$ is the one-step reachable set for system (7a) from states in $\mathscr{P}$. Let $\mathbb{X}_D \subset \mathscr{P}$ denote the set of all states in $\mathscr{P}$ at which $V(\cdot)$ is not continuous. If one can verify that the state x_k of (7a) is $\mu > 0$ distance away from the set $\mathbb{X}_D$ for all $x_0 \in \mathscr{P}$ and all $k \in \mathbb{Z}_{\geq 1}$, then it can be proven that ISS is achieved for additive disturbances in $\mathscr{B}_\mu$, as formulated in the following result.

Theorem IV.1 *Let $\mathscr{P} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ be a compact RPI set for system (7b) with respect to $\mathscr{B}_\mu$. Suppose that the PWA system (7a) is globally asymptotically stable and it admits a discontinuous USL function of the form (9). Furthermore, suppose that there exists a $\mu > 0$ such that*

$$d(x, \mathbb{X}_D) > \mu \quad \text{for all} \quad x \in \mathscr{R}_1(\mathscr{P}), \tag{10}$$

where $d(x, \mathbb{X}_D) := \inf_{y \in \mathbb{X}_D} \|x - y\|$. Then, the PWA system (7b) is ISS in $\mathscr{P}$ for disturbances in $\mathscr{B}_\mu$.

The constant μ can be calculated as follows:

$$\mu = \min_{j \in \mathscr{S}} \left\{ \min_{y \in \Omega_j \cap \mathscr{P}, \, \bar{y} \in \mathbb{X}_D} \|(A_j y + f_j) - \bar{y}\| \right\}. \tag{11}$$

If the set $\mathbb{X}_D$ is the union of a finite number of convex subsets (or of a finite number of polyhedra), a solution to the optimization problem (11) can be obtained by solving a finite number of convex optimization problems (or a finite number of LP problems). After solving (11), if one obtains a strictly positive μ, then $\mu > 0$ can be considered as a measure of the (worst case) inherent robustness of system (7).

The sufficient condition (10) can be relaxed, as follows.

Proposition IV.2 *Let $\mathscr{P} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ be a compact RPI set for system (7b) with respect to $\mathscr{B}_\mu$. Suppose that the PWA system (7a) is globally asymptotically stable and it admits a discontinuous USL function of the form (9). Furthermore, suppose that there exists $\alpha_3 \in \mathscr{K}_\infty$ such that*

$$\max_{i \in \mathscr{I}} V_i(G(x)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{R}^n. \tag{12}$$

Then, the PWA system (7b) is ISS in $\mathscr{P}$ for disturbances in $\mathscr{B}_\mu$.

Notice that the above result is simply based on a stronger, more conservative extension of the stabilization conditions from [7]–[10], as it requires that the Lyapunov function is decreasing irrespective of which dynamics might be active at the next step.

The sufficient condition (12) can be significantly relaxed, as follows. Consider the set of states

$$\mathscr{Z} := \{x \in \mathscr{P} \mid [G(x) \oplus \mathscr{B}_\mu] \cap \mathbb{X}_D \neq \emptyset\}$$

and let

$$\mathscr{I}(x) :=$$
$$\{i \in \mathscr{J} \mid x \in \mathscr{Z}, \, G(x) \notin \Gamma_i, \, [G(x) \oplus \mathscr{B}_\mu] \cap \Gamma_i \neq \emptyset\}.$$

Theorem IV.3 *Let $\mathscr{P} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathscr{P})$ be a compact RPI set for system (7b) with respect to $\mathscr{B}_\mu$. Suppose that the PWA system (7a) is globally asymptotically stable and it admits a discontinuous USL function of the form (9). Furthermore, suppose that $\mathscr{Z} \neq \emptyset$ and there exists a $\mu > 0$ and a $\mathscr{K}$-function $\tilde{\alpha}_3(\cdot)$ such that*

$$\max_{i \in \mathscr{I}(x)} V_i(G(x)) - V(x) \leq -\tilde{\alpha}_3(\|x\|), \quad \forall x \in \mathscr{Z}. \tag{13}$$

Then, the PWA system (7b) is ISS in $\mathscr{P}$ for disturbances in $\mathscr{B}_\mu$.

Remark IV.4 The results of Theorem IV.1 and Theorem IV.3 require that a RPI set is known for the PWA system (7b) with respect to disturbances in $\mathscr{B}_\mu$. This can be avoided by taking $\mathscr{P} = \mathbb{R}^n$, which is an RPI set for any disturbance. In this case, however, one needs to assume that for each function $V_i(\cdot)$, $i \in \mathscr{J}$, there exists a function $\sigma_i \in \mathscr{K}_\infty$ such that $|V_i(x) - V_i(y)| \leq \sigma_i(\|x - y\|)$ for all $x, y \in \mathbb{R}^n$. Then, the ISS results of Theorem IV.1 and Theorem IV.3 hold in $\mathbb{R}^n$ for disturbances in $\mathscr{B}_\mu$.

V. ROBUSTNESS TESTS BASED ON THE RGAS PROPERTY

The tests presented so far can only be employed to establish if a discontinuous USL function can be used as a candidate ISS Lyapunov function. However, if both tests fail, one cannot say anything about robustness (or non-robustness) as the conditions of Theorem II.4 are only sufficient. A different method for checking robustness of discontinuous discrete-time nonlinear systems was presented in [11]. This method can establish robustness of a discontinuous discrete-time system by checking nominal stability for an upper semicontinuous set-valued regularization of the dynamics.

Non-robustness of a PWA system for which the proposed tests fail can be established via the method of [11] as follows. Recall the discontinuous nonlinear system (5) and its perturbed version (6), which include the unperturbed PWA system (7a) and the perturbed PWA system (7b), respectively, as a particular subclass. According to [11], a regularization of the discrete-time dynamics (5) is defined as the difference inclusion $x_{k+1} \in F(x_k)$, $k \in \mathbb{Z}_+$, with:

$$F(x) := \bigcap_{\rho > 0} \text{cl}(h(x \oplus \mathscr{B}_\rho)).$$

Note that $F(\cdot)$ is a set-valued mapping in general. By Theorem 14 of [11] the following statements are equivalent:

1) The origin is GAS for the regularization $x_{k+1} \in F(x_k)$ of system (5);
2) The origin is RGAS for system (5);
3) System (5) admits a continuous USL function.

Therefore, as observed in [11], it is sufficient to find an initial condition for which the state trajectory is not converging to

the origin for the regularization of (7a) (i.e. $x_{k+1} \in F(x_k) = \bigcap_{\rho>0} \mathrm{cl}(G(x_k \oplus \mathscr{B}_\rho)))$ to establish non-robustness.

However, this test may be difficult to implement as one has no criterion for selecting an initial condition. That is why it is useful to first perform the robustness tests of Theorem IV.1 and Theorem IV.3. Then, if these tests fail, the set of states $\mathscr{X}$ contains suitable candidates for initial conditions that may result in non-robustness.

For example, the tests of Theorem IV.1 and Theorem IV.3 fail for the PWA system of Example 3 and the discontinuous USL function defined in Proposition III.1, and they reveal the state $x = [0.2\,3.6]^\top$. The evaluation of the set-valued map $F(x)$ at $x_0 = [0.2\,3.6]^\top$ and $G(x_0) = [-0.3\,3.6]^\top$ yields:

$$F(x_0) = \left\{ [-0.3\,3.6]^\top, [0.7\,3.6]^\top \right\} = \left\{ G(x_0), [0.7\,3.6]^\top \right\}$$
$$F(G(x_0)) = \left\{ [-0.8\,3.6]^\top, [0.2\,3.6]^\top \right\} = \left\{ [-0.8\,3.6]^\top, x_0 \right\}$$

Therefore, $x_k = x_0$, $k = 0, 2, 4, \ldots$, and $x_k = G(x_0)$, $k = 1, 3, 5, \ldots$ is a limit cycle of $x_{k+1} \in F(x_k)$ for initial condition $x_0 = [0.2\,3.6]^\top$. This implies that the origin is not attractive for $x_{k+1} \in F(x_k)$, $k \in \mathbb{Z}_+$, and initial state x_0 and thus, it is not GAS. Hence, from 1) $\Leftrightarrow$ 2) it follows that the PWA system of Example 3 is not RGAS (and hence, non-robust) for the considered initial state.

Furthermore, 1) $\Leftrightarrow$ 3) implies that the PWA system of Example 3 does not admit a continuous USL function. Therefore, there exist exponentially stable PWA systems that do not admit a *continuous USL function*.

VI. CONCLUSIONS

We have considered robust stability of discrete-time PWA systems, as this is a crucial property for many practical applications. Via examples, we have shown that globally exponentially stable discrete-time PWA systems can be non-robust (i.e. not ISS or RGAS) when arbitrarily small perturbations are present. The main points of the paper can be summarized as follows (for brevity, let $\Rightarrow$ and $\not\Rightarrow$ stand for "implies" and "does not necessarily imply", respectively).

The following statements are true for *discontinuous (nonlinear) hybrid systems*:

- The existence of a continuous SL function $\not\Rightarrow$ global convergence (attractivity) - Example 2;
- GES $\not\Rightarrow$ the existence of a continuous USL function - Example 1, [11];
- The existence of a continuous or discontinuous USL function $\Rightarrow$ GAS - [19];
- GAS $\Rightarrow$ the existence of a discontinuous USL function - [14];
- RGAS $\Rightarrow$ the existence of a continuous USL function - [11];
- The existence of a discontinuous USL function $\not\Rightarrow$ robustness (ISS or RGAS) - Example 1, Example 3.

Several robustness tests based on discontinuous Lyapunov functions were presented. These tests can be employed to establish robustness of nominally asymptotically stable PWA systems in the case when a continuous USL function is not known, but a discontinuous USL function is available.

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