Successive Compute-and-Forward

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***Abstract*—In prior work, we proposed the compute-and-forward framework for sending linear combinations of messages to relays. In this note, we extend the notion of successive interference cancellation to the compute-and-forward setting. We find that once a relay has decoded a linear combination, it can mix it with its channel output to create a new effective channel output. The resulting effective channel can be tuned so that it is more suitable for decoding a second linear combination than the original channel.**

I. Introduction

The classical approach to communication over a wireless relay network treats interference between transmitters as a nuisance to be avoided. Typically, each relay observes a noisy linear combination of the transmitted codewords and attempts to decode one (or more) of them while treating the others as additional noise. Recent efforts have revealed that interference can in fact be exploited if we are willing to expand the set of decoding possibilities to include linear combinations of codewords. One natural approach, sometimes referred to as compute-and-forward, is to employ a lattice codebook so that integer combinations of codewords are themselves codewords [1]–[5]. Relays are then free to select integer coefficients that match the channel coefficients as closely as possible, thus reducing the effective noise and increasing the achievable rates.

Under the classical approach, a relay can employ successive interference cancellation to remove decoded codewords from its channel observation. This decreases the effective noise encountered in the next decoding step. In this paper, we devise an analogous technique for the compute-and-forward framework. After decoding a linear combination, a relay can combine it with its channel observation to obtain a new effective channel that is even better for decoding the next linear combination. For ease of exposition, we will focus on the case where each relay wants to recover just two linear combinations.

Owing to space limitations, we do not attempt a full survey of the literature. We refer interested readers to [2] for additional references pertaining to compute-and-forward and to [6], [7] for surveys of the closely related topic of physical-layer network coding.

II. Problem Statement

Our setting is nearly identical to that of [2] and we reproduce some of the key definitions below. For ease of exposition, we will limit ourselves to real-valued channels and symmetric rates. We will denote addition and summation over $\mathbb{R}$ with $+$ and $\sum$, respectively, and use $\oplus$ and $\bigoplus$ to denote the same over $\mathbb{F}_p$.

Each transmitter (indexed by $\ell = 1, \dots, L$) has a length-k message that is drawn independently and uniformly over a prime-sized finite field, $\mathbf{w}_\ell \in \mathbb{F}_p^k$. An encoder, $\mathcal{E}_\ell : \mathbb{F}_p^k \to \mathbb{R}^n$, then maps the message into a length-n codeword, $\mathbf{x}_\ell = \mathcal{E}(\mathbf{w}_\ell)$, which must satisfy the usual power constraint $\|\mathbf{x}_\ell\|^2 \leq nP$. The message rate is $R = (k/n)\log p$.

Each relay (indexed by $m = 1, \dots, M$) observes a noisy linear combination of the codewords,

$$\mathbf{y}_m = \sum_{\ell=1}^{L} h_{m\ell}\mathbf{x}_\ell + \mathbf{z}_m \ , \tag{1}$$

where the $h_{m\ell} \in \mathbb{R}$ are the channel coefficients and $\mathbf{z}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ is i.i.d. Gaussian noise. Let $\mathbf{h}_m = [h_{m1} \ \cdots \ h_{mL}]^T$ denote the vector of channel coefficients. The goal is for each relay to recover *two linear combinations of the messages* of the form

$$\mathbf{u}_m^{(1)} = \bigoplus_{\ell=1}^{L} q_{m\ell}^{(1)}\mathbf{w}_\ell \qquad \mathbf{u}_m^{(2)} = \bigoplus_{\ell=1}^{L} q_{m\ell}^{(2)}\mathbf{w}_\ell \tag{2}$$

where the $q_{m\ell}^{(1)}, q_{m\ell}^{(2)} \in \mathbb{F}_p$ are finite field coefficients. To this end, each relay is equipped with a decoder, $\mathcal{D} : \mathbb{R}^n \to \mathbb{F}_p^k \times \mathbb{F}_p^k$, that produces estimates $\hat{\mathbf{u}}_m^{(1)}$ and $\hat{\mathbf{u}}_m^{(2)}$ of its desired linear combinations. We will say that the average probability of error is at most ϵ if

$$\mathbb{P}\Big(\bigcup_m \big\{\hat{\mathbf{u}}_m^{(1)} \neq \mathbf{u}_m^{(1)}\big\} \cup \big\{\hat{\mathbf{u}}_m^{(2)} \neq \mathbf{u}_m^{(2)}\big\}\Big) < \epsilon \ . \tag{3}$$

To map between the real-valued linear combination provided by the channel and the desired finite field linear combinations, we will need a bit of additional nomenclature. Specifically, we will refer to $\mathbf{u}_m^{(1)}$ as the linear combination with coefficient vector $\mathbf{a}_m = [a_{m1} \ \cdots \ a_{mL}]^T \in \mathbb{Z}^L$ if its finite field coefficients satisfy[1]

$$q_{m\ell}^{(1)} = [a_{m\ell}] \bmod p \ . \tag{4}$$

[1]This is a slight abuse of notation. More formally, we should explicitly define a mapping between $\mathbb{F}_p$ and $\{0, 1, \dots, p-1\}$. See [2, Definition 6] for more details.

Similarly, we will refer to $\mathbf{u}_m^{(2)}$ as the linear combination with coefficient vector $\mathbf{b}_m = [b_{m1} \ \cdots \ b_{mL}]^T \in \mathbb{Z}^L$ if

$$q_{m\ell}^{(2)} = [b_{m\ell}] \bmod p \ . \tag{5}$$

We will say that the computation rate region $\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m)$ is achievable if, for any $\epsilon > 0$ and n large enough, there exist encoders and decoders, such that all relays can recover their desired linear combinations with average probability of error ϵ so long as

$$R < \min_m \mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m) \ . \tag{6}$$

Note that the relays are free to choose which linear combinations to decode so long as (6) is satisfied.

III. Nested Lattice Codes

One key requirement of our scheme is that all integer combinations of codewords must be afforded protection against noise. Nested lattice codes are a natural fit for this purpose. A lattice Λ is a discrete subgroup of $\mathbb{R}^n$ with the property that if $\mathbf{t}_1, \mathbf{t}_2 \in \Lambda$ then $\mathbf{t}_1 + \mathbf{t}_2 \in \Lambda$. By construction, all of our lattices will contain the zero vector. A pair of lattices $\Lambda, \Lambda_{\text{FINE}}$ is nested if $\Lambda \subset \Lambda_{\text{FINE}}$.

A lattice quantizer is a function, $Q_\Lambda : \mathbb{R}^n \to \Lambda$, that maps vectors to the nearest lattice point in Euclidean distance,

$$Q_\Lambda(\mathbf{x}) = \arg\min_{\mathbf{t} \in \Lambda} \|\mathbf{x} - \mathbf{t}\| \ . \tag{7}$$

The fundamental Voronoi region is the subset of points in $\mathbb{R}^n$ that quantize to the zero vector, $\mathcal{V} = \{\mathbf{x} : Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$. The modulo operation returns the quantization error with respect to the lattice,

$$[\mathbf{x}] \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x}) \ , \tag{8}$$

and satisfies the distributive law,

$$\big[a[\mathbf{x}] \bmod \Lambda + b[\mathbf{y}] \bmod \Lambda\big] \bmod \Lambda = \big[a\mathbf{x} + b\mathbf{y}\big] \bmod \Lambda \ ,$$

for any $a, b \in \mathbb{Z}$.

A nested lattice code $\mathcal{C}$ is created by taking the set of fine lattice points that fall within the fundamental Voronoi region of the coarse lattice, $\mathcal{C} = \Lambda_{\text{FINE}} \cap \mathcal{V}$. Erez and Zamir have shown that their exist nested lattice codes that can approach the capacity of a point-to-point Gaussian channel [8].

IV. Compute-and-Forward

In [2], we proposed the compute-and-forward framework as a way of communicating linear combinations of messages. Our focus was on the case where each relay decodes a single linear combination with coefficient vector $\mathbf{a}_m$. Define

$$R_{\text{CF}}(\mathbf{h}, \mathbf{a}) \triangleq \frac{1}{2} \log^+ \left(\left(\|\mathbf{a}_m\|^2 - \frac{(\mathbf{h}_m^T \mathbf{a}_m)^2 P}{1 + P\|\mathbf{h}_m\|^2} \right)^{-1} \right)$$

where $\log^+(x) \triangleq \max(\log(x), 0)$.

Theorem 1 ([2, Theorem 2]): For any set of channel vectors $\mathbf{h}_m \in \mathbb{R}^L$ and coefficient vectors $\mathbf{a}_m \in \mathbb{Z}^L$, the following computation rate region is achievable:

$$\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m) = R_{\text{CF}}(\mathbf{h}_m, \mathbf{a}_m) \ . \tag{9}$$

We now provide a brief overview of the basic compute-and-forward encoding and decoding functions which will be useful in the proof of our main result. Using Construction A [8], [9], we select a pair of nested lattices $\Lambda \subset \Lambda_{\text{FINE}}$ that can approach the capacity of a point-to-point Gaussian channel. It can be shown that there is a one-to-one map ϕ between $\mathbb{F}_p^k$ and the nested lattice code $\mathcal{C}$ that preserves linearity (see [2, Lemma 5]). Using this mapping, the encoder chooses a lattice point $\mathbf{t}_\ell = \phi(\mathbf{w}_\ell)$. It then applies a dither $\mathbf{d}_\ell$ that is drawn independent and uniformly over $\mathcal{V}$ and transmits the result,

$$\mathbf{x}_\ell = [\mathbf{t}_\ell - \mathbf{d}_\ell] \bmod \Lambda \ . \tag{10}$$

Relay m observes $\mathbf{y}_m$ and has access to every dither[2]. It scales its observation by the minimum mean-squared error (MMSE) coefficient

$$\alpha_m = \frac{P\mathbf{h}_m^T \mathbf{a}_m}{1 + P\|\mathbf{h}_m\|^2} \tag{11}$$

and removes the dithers according to the desired coefficients $a_{m\ell}$. Afterwards, it quantizes the result onto the fine lattice and takes the modulus with respect to the coarse lattice,

$$\hat{\mathbf{v}}_m^{(1)} = \left[Q_{\Lambda_{\text{FINE}}} \Big(\alpha_m \mathbf{y}_m + \sum_{\ell=1}^{L} a_{m\ell} \mathbf{d}_\ell \Big) \right] \bmod \Lambda \ . \tag{12}$$

It can be shown that, with high probability, this is equal to

$$\mathbf{v}_m^{(1)} = \left[\sum_{\ell=1}^{L} a_{m\ell} \mathbf{t}_\ell \right] \bmod \Lambda \tag{13}$$

so long as

$$R < \min_m \frac{1}{2} \log^+ \left(\frac{P}{\alpha_m^2 + P\|\alpha_m \mathbf{h}_m - \mathbf{a}_m\|^2} \right) \ . \tag{14}$$

Finally, the relay applies the inverse map to get its estimate $\hat{\mathbf{u}}_m^{(1)} = \phi^{-1}(\hat{\mathbf{v}}_m^{(1)})$. Assuming that $\hat{\mathbf{v}}_m^{(1)} = \mathbf{v}_m^{(1)}$, it can be shown that $\hat{\mathbf{u}}_M^{(1)} = \mathbf{u}_m^{(1)}$ (see [2, Lemma 6]).

V. Successive Compute-and-Forward

Successive interference cancellation is a powerful technique for decoding several messages at a single receiver. Assume that, given the channel observation $\mathbf{y}_m$, a relay has correctly decoded $\mathbf{x}_i$. It can now completely remove the effect of $\mathbf{x}_i$ from its observation,

$$\mathbf{y}_m - h_{mi}\mathbf{x}_i = \sum_{\ell \neq i} h_{m\ell} \mathbf{x}_\ell + \mathbf{z}_m \ , \tag{15}$$

which reduces the interference and makes it easier to decode the next codeword.

As it turns out, we can employ a similar technique when decoding several linear combinations. Assuming the relay has decoded $\sum_\ell a_{m\ell} \mathbf{x}_\ell$, it can create a new effective channel

$$\mathbf{y}_m + \gamma_m \sum_{\ell=1}^{L} a_{m\ell} \mathbf{x}_\ell = \sum_{\ell=1}^{L} (h_{m\ell} + \gamma_m a_{m\ell}) \mathbf{x}_\ell + \mathbf{z}_m \ . \tag{16}$$

[2]These dithers can be replaced with deterministic sequences.

By adjusting the effective channel coefficients, we can make it easier for the relay to decode its second linear combination, and thus increase the computation rate region. Note that unlike successive interference cancellation, it is not always optimal to subtract the recovered linear combination. Below, we develop a successive computation scheme that follows the concept outlined above. We begin by showing that we can always recover the real sum of codewords if we have access to the modulo sum and the dithers.

Remark 1: In [2, Theorem 12], we described a limited version of successive computation. The key drawback is that this scheme only allows for integer-valued γ_m, owing to the fact that it works directly with the modulo sum of codewords.

As part of the compute-and-forward scheme, the relay recovers an estimate $\hat{\mathbf{v}}_m^{(1)}$ of the modulo linear combination of codewords $\mathbf{v}_m^{(1)}$ from (34). The lemma below shows that this modulo sum can be used to recover the real sum $\sum_\ell a_{m\ell}\mathbf{x}_\ell$ that is needed for successive computation.

Lemma 1: The relay can make an estimate $\hat{\mathbf{s}}_m$ of the real sum of codewords

$$\mathbf{s}_m = \sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell \tag{17}$$

with vanishing probability of error, $\lim_{n\to\infty}\mathbb{P}(\hat{\mathbf{s}}_m \neq \mathbf{s}_m) = 0$, so long as $R < \mathcal{R}(\mathbf{h}_m, \mathbf{a}_m)$.

Proof: Since $R < \mathcal{R}(\mathbf{h}_m, \mathbf{a}_m)$, we can use Theorem 1 to make an estimate $\hat{\mathbf{v}}_m^{(1)}$ that is equal to $\mathbf{v}_m^{(1)}$ with high probability. For the remainder of the proof, we will assume that this estimate is correct. First, the relay dithers this linear combination and takes the modulus with respect to Λ. This gives it access to a modulo combination of the dithered lattice points,

$$\left[\hat{\mathbf{v}}_m^{(1)} - \sum_{\ell=1}^{L} a_{m\ell}\mathbf{d}_\ell\right] \bmod \Lambda \tag{18}$$

$$= \left[\left[\sum_\ell a_{m\ell}\mathbf{t}_\ell\right] \bmod \Lambda - \sum_{\ell=1}^{L} a_{m\ell}\mathbf{d}_\ell\right] \bmod \Lambda \tag{19}$$

$$= \left[\sum_\ell a_{m\ell}(\mathbf{t}_\ell - \mathbf{d}_\ell)\right] \bmod \Lambda = \left[\sum_\ell a_{m\ell}\mathbf{x}_\ell\right] \bmod \Lambda \ .$$

It then subtracts this quantity from $\alpha_m \mathbf{y}_m$,

$$\mathbf{r}_m = \alpha_m\mathbf{y}_m - \left[\sum_\ell a_{m\ell}\mathbf{x}_\ell\right] \bmod \Lambda \tag{20}$$

$$= \sum_{\ell=1}^{L}\alpha_m h_{m\ell}\mathbf{x}_\ell + \alpha_m\mathbf{z}_m - \sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell + Q_\Lambda\left(\sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell\right)$$

$$= Q_\Lambda\left(\sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell\right) + \alpha_m\mathbf{z}_m + \sum_{\ell=1}^{L}(\alpha_m h_{m\ell} - a_{m\ell})\mathbf{x}_\ell \ ,$$

to get a quantized version of the desired sum $\mathbf{s}_m$ plus some effective noise with variance $\alpha_m^2 + P\|\alpha_m\mathbf{h}_m - \mathbf{a}_m\|^2$. To remove this noise, it applies the coarse lattice quantizer. This operation will be successful with high probability so long as the second moment of Λ exceeds the effective noise variance, i.e., $P > \alpha_m^2 + P\|\alpha_m\mathbf{h}_m - \mathbf{a}_m\|^2$. Assuming that $\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m) > 0$, this condition holds and we have that

$$Q_\Lambda(\mathbf{r}_m) = Q_\Lambda\left(\left[\sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell\right] \bmod \Lambda\right) \tag{21}$$

with high probability. Finally, since the relay knows the quantized sum as well as its quantization error, it can infer the desired real sum. Assuming that (21) holds, we have that

$$\hat{\mathbf{s}}_m = Q_\Lambda(\mathbf{r}_m) + \left[\sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell\right] \bmod \Lambda \tag{22}$$

$$= \sum_{\ell=1}^{L} a_{m\ell}\mathbf{x}_\ell = \mathbf{s}_m \ . \tag{23}$$

■

Using the real sum of the codewords, we can construct a successive compute-and-forward scheme. Define

$$R_{\text{SCF}}(\mathbf{h}, \mathbf{a}, \mathbf{b}) \triangleq \tag{24}$$

$$\frac{1}{2}\log^+\left(\left(\|\mathbf{b}\|^2 - \frac{(\mathbf{a}^T\mathbf{b})^2}{\|\mathbf{a}\|^2} - \frac{P\Big(\big(\mathbf{h} - \frac{\mathbf{a}^T\mathbf{h}}{\|\mathbf{a}\|^2}\mathbf{a}\big)^T\mathbf{b}\Big)^2}{1 + P\Big(\|\mathbf{h}\|^2 - \frac{(\mathbf{a}^T\mathbf{h})^2}{\|\mathbf{a}\|^2}\Big)}\right)^{-1}\right)$$

Theorem 2: For any set of channel vectors $\mathbf{h}_m \in \mathbb{R}^L$ and coefficient vectors $\mathbf{a}_m, \mathbf{b}_m \in \mathbb{Z}^L$, the following computation rate region is achievable:

$$\mathcal{R}(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m) = \max(R_{AB}, R_{BA}) \tag{25}$$

$$R_{AB} = \min\Big(R_{\text{CF}}(\mathbf{h}_m, \mathbf{a}_m), R_{\text{SCF}}(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m\Big) \tag{26}$$

$$R_{BA} = \min\Big(R_{\text{CF}}(\mathbf{h}_m, \mathbf{b}_m), R_{\text{SCF}}(\mathbf{h}_m, \mathbf{b}_m, \mathbf{a}_m\Big) \ . \tag{27}$$

Proof: Fix an $\epsilon > 0$. The expressions R_{AB} and R_{BA} correspond to the two possible decoding orders. We will prove that R_{AB} is achievable by first decoding the linear combination with coefficient vector $\mathbf{a}_m$ and then that with $\mathbf{b}_m$. The proof of R_{BA} follows identically by exchanging the role of $\mathbf{a}_m$ and $\mathbf{b}_m$.

We employ the same encoding framework as in Theorem 1. Relay m uses the same decoding framework to make an estimate $\hat{\mathbf{u}}_m^{(1)}$ of $\mathbf{u}_m^{(1)}$. For n large enough, this estimate is incorrect with probability at most $\epsilon/3$ if

$$R < \min_m R_{\text{CF}}(\mathbf{h}_m, \mathbf{a}_m) \ . \tag{28}$$

As a byproduct of successful decoding, the relay will obtain a correct estimate $\hat{\mathbf{v}}_m^{(1)}$ of $\mathbf{v}_m^{(1)}$. Using Lemma 1, it makes an estimate $\hat{\mathbf{s}}_m$ of $\mathbf{s}_m = \sum_\ell a_{m\ell}\mathbf{x}_\ell$ that has probability of error at most $\epsilon/3$ for n large enough. Below, we assume $\hat{\mathbf{s}}_m = \mathbf{s}_m$.

The relay removes the projection of $\hat{\mathbf{s}}_m$ onto $\mathbf{y}_m$ from $\mathbf{y}_m$ to get

$$\mathbf{r}_m = \mathbf{y}_m - \frac{\mathbf{a}_m^T\mathbf{h}_m}{\|\mathbf{a}_m\|^2}\hat{\mathbf{s}}_m \tag{29}$$

$$= \sum_{\ell=1}^{L}\left(h_{m\ell} - \frac{\mathbf{a}_m^T\mathbf{h}_m}{\|\mathbf{a}_m\|^2}a_{m\ell}\right)\mathbf{x}_\ell + \mathbf{z}_m \tag{30}$$

Define $g_{m\ell} = h_{m\ell} - \frac{\mathbf{a}_m^T \mathbf{h}_m}{\|\mathbf{a}_m\|^2} a_{m\ell}$ and $\mathbf{g}_m = [g_{m1} \ \cdots \ g_{mL}]^T$. It then forms a new effective channel observation

$$\tilde{\mathbf{y}}_m = \beta_m \mathbf{r}_m + \mu_m \hat{\mathbf{s}}_m \tag{31}$$

$$= \sum_{\ell=1}^{L} (\beta_m g_{m\ell} + \mu_m a_{m\ell}) \mathbf{x}_\ell + \beta_m \mathbf{z}_m \ . \tag{32}$$

and proceeds to decode the linear combination with coefficient vector $\mathbf{b}_m$ as in Theorem 1. Specifically, it forms the estimate

$$\hat{\mathbf{v}}_m^{(2)} = \left[Q_{\Lambda_{\text{FINE}}} \left(\tilde{\mathbf{y}}_m + \sum_{\ell=1}^{L} b_{m\ell} \mathbf{d}_\ell \right) \right] \bmod \Lambda \ . \tag{33}$$

It can be shown that, with probability of error at most $\epsilon/3$, this is equal to

$$\mathbf{v}_m^{(2)} = \left[\sum_{\ell=1}^{L} b_{m\ell} \mathbf{t}_\ell \right] \bmod \Lambda \tag{34}$$

so long as

$$R < \min_m \frac{1}{2} \log^+ \left(\frac{P}{\beta_m^2 + P\|\beta_m \mathbf{g}_m + \mu_m \mathbf{a}_m - \mathbf{b}_m\|^2} \right) \ . \tag{35}$$

Finally, the relay applies the inverse map to get its estimate $\hat{\mathbf{u}}_m^{(2)} = \phi^{-1}(\hat{\mathbf{v}}_m^{(2)})$.

It remains to solve for the β_m and μ_m that minimize the effective noise variance

$$f(\beta_m, \mu_m) = \beta_m^2 + P\|\beta_m \mathbf{g}_m + \mu_m \mathbf{a}_m - \mathbf{b}_m\|^2 \ . \tag{36}$$

This is a convex function whose global minimum is attained at

$$\beta_m^* = \frac{P \mathbf{g}_m^T \mathbf{b}_m}{1 + P\|\mathbf{g}_m\|^2} \tag{37}$$

$$\mu_m^* = \frac{\mathbf{a}_m^T \mathbf{b}_m}{\|\mathbf{a}_m\|^2} \ . \tag{38}$$

Plugging this back in, we find that

$$f(\beta_m^*, \mu_m^*) = P\|\mathbf{b}_m\|^2 - \frac{P(\mathbf{a}_m^T \mathbf{b}_m)^2}{\|\mathbf{a}_m\|^2} - \frac{P^2(\mathbf{g}_m^T \mathbf{b}_m)^2}{1 + P\|\mathbf{g}_m\|^2} \ .$$

Substituting this into (35), we get the desired condition

$$R < \min_m R_{\text{SCF}}(\mathbf{h}_m, \mathbf{a}_m, \mathbf{b}_m) \ . \tag{39}$$

By the union bound, the probability of error is at most ϵ. ■

Example 1: Consider a single relay with channel vector $\mathbf{h}_1 = [2 \ 1 \ 1]^T$ that wishes to decode the linear combinations with coefficient vectors $\mathbf{a}_1 = [1 \ 1 \ 1]^T$ and $\mathbf{b}_1 = [1 \ -1 \ -1]^T$ using Theorem 2. It is not possible to decode $\mathbf{b}_1$ first as $R_{\text{CF}}(\mathbf{h}_1, \mathbf{b}_1) = 0$. Decoding $\mathbf{a}_1$ first requires

$$R < \frac{1}{2} \log^+ \left(\frac{1 + 6P}{3 + 2P} \right) \ . \tag{40}$$

After recovering $\mathbf{a}_1$, the relay can adjust the channel and decode $\mathbf{b}_1$ so long as

$$R < \frac{1}{2} \log^+ \left(\frac{9}{24} + \frac{P}{4} \right) \ . \tag{41}$$

The example above demonstrates that successive compute-and-forward can make it possible to recover linear combinations that are not available via a direct application of the original compute-and-forward framework. In other words, the relay can first target a linear combination that is "easy" to decode and then use it to create a better effective channel for decoding the second linear combination.

From another perspective, successive compute-and-forward can be used to enlarge the computation rate region for decoding a single linear combination with coefficient vector $\mathbf{b}_m$. The relay should order all viable coefficient vectors (i.e., those satisfying $\|\mathbf{a}_m\|^2 \leq 1 + P\|\mathbf{h}_m\|^2$) by computation rate $R_{\text{CF}}(\mathbf{h}_m, \mathbf{a}_m)$ and set aside those $\mathbf{a}_m$ with rates larger than $R_{\text{CF}}(\mathbf{h}_m, \mathbf{b}_m)$. It can then calculate which pair $(\mathbf{a}_m, \mathbf{b}_m)$ offers the highest rate using Theorem 2. Finally, it applies successive compute-and-forward for this pair and keeps only the second equation. Example 1 demonstrates that this procedure does indeed enlarge the rate region.

VI. Generalizations and Extensions

Following the framework in [2], successive compute-and-forward can be generalized to include complex-valued channel models as well as unequal message rates. One can also envision extending this technique to the case where each relay may want more than two linear combinations. In this case, the linear combinations obtained thus far should be mixed together with the original channel observation to create a new effective channel for the next targeted linear combination.

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