

## Successive derivatives and finite expansions involving the $H$ -function of one and more variables

by C. M. JOSHI (Udaipur) and N. L. JOSHI (Nathadwara)

**Abstract.** Certain results including the successive derivatives of the  $H$ -function of one and more variables are established. These remove the limitations of Ławrynowicz's (1969) formulas and as a result extend the results of Skibiński [13] and various other authors. As an application some finite expansion formulas are also established, which reduce to hypergeometric functions of one and more variables that are of common interest.

**1. Introduction.** Fox's  $H$ -function which is defined by a Mellin–Barnes type contour integral (for details see [12]) as

$$(1.1) \quad H[z] = H_{p,q}^{m,n} \left[ z \left| \begin{array}{l} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{array} \right. \right] = \frac{1}{2\pi\omega} \int_L \phi(\xi) z^\xi d\xi,$$

where

$$\phi(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j \xi)} \cdot \frac{\prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)},$$

has been extended to the multivariate  $H$ -function by Srivastava and Panda [16] in the form of the Mellin–Barnes type contour integral

$$(1.2) \quad H[z_1, \dots, z_r] = H_{p,q;p_1,q_1;\dots;p_r,q_r}^{0,n;m_1,n_1;\dots;m_r,n_r} \cdot \left[ \begin{array}{l} z_1 \\ \vdots \\ z_r \end{array} \left| \begin{array}{l} ((a_p; \alpha_p^{(1)}; \dots; \alpha_p^{(r)})) : ((c_{p_1}^{(1)}, \gamma_{p_1}^{(1)}); \dots; (c_{p_r}^{(r)}, \gamma_{p_r}^{(r)})) \\ ((b_q; \beta_q^{(1)}; \dots; \beta_q^{(r)})) : ((d_{q_1}^{(1)}, \delta_{q_1}^{(1)}); \dots; (d_{q_r}^{(r)}, \delta_{q_r}^{(r)})) \end{array} \right. \right]$$

---

1991 *Mathematics Subject Classification*: 33C40, 33C45, 33C50.

*Key words and phrases*:  $H$ -function of several variables, differential operator, expansion formulas, Appell functions, Lauricella functions, Kampé de Fériet function, generalized hypergeometric functions.

$$= \left( \frac{1}{2\pi\omega} \right)^r \int_{L_1} \dots \int_{L_r} \Psi(\xi_1, \dots, \xi_r) \prod_{i=1}^r \{\phi_i(\xi_i) z_i^{\xi_i}\} d\xi_1 \dots d\xi_r,$$

where

$$\Psi(\xi_1, \dots, \xi_r) = \frac{\prod_{j=1}^n \Gamma\left(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i\right)}{\prod_{j=n+1}^p \Gamma\left(a_j - \sum_{i=1}^r \alpha_j^{(i)} \xi_i\right) \prod_{j=1}^q \Gamma\left(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i\right)}$$

and for  $i = 1, \dots, r$ , we have

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)}.$$

Moreover,  $\omega^2 = -1$  and  $z_i (\neq 0)$  is a complex variable,  $z_i = \exp\{\xi_i(\log |z_i| + \omega \arg z_i)\}$ , where  $\log |z_i|$  represents the natural logarithm of  $|z_i|$ ; it is single-valued for  $|\arg z_i| < \pi$ .

By  $((b_q; \beta_q^{(1)}, \dots, \beta_q^{(r)}))$ , we mean a  $q$ -member array

$$(b_1; \beta_1^{(1)}, \dots, \beta_1^{(r)}), \dots, (b_q; \beta_q^{(1)}, \dots, \beta_q^{(r)})$$

etc. Further, the parameters  $a, b, c$  and  $d$  are complex numbers and the associated coefficients  $\alpha, \beta, \gamma$  and  $\delta$  are positive real numbers such that the poles of the integrand are separated.

The contour  $L_i$  in the  $\xi_i$ -plane runs from  $-\omega\infty$  to  $+\omega\infty$  along the imaginary axis, indented if necessary to ensure that the poles of  $\Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i)$  ( $j = 1, \dots, m_i$ ) lie to the right of the contour and those of  $\Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i)$  ( $j = 1, \dots, n$ ) and  $\Gamma(1 - c_j^{(i)} + \delta_j^{(i)} \xi_i)$  ( $i = 1, \dots, n_i$ ) are to the left of the contour. The positive integers  $n, p, q, n_i, m_i, p_i, q_i$  are constrained by the inequalities:

$$0 \leq n \leq p, \quad 0 \leq q, \quad 1 \leq m_i \leq q_i \quad \text{and} \quad 0 \leq n_i \leq p_i.$$

The multiple Mellin–Barnes contour integral (1.2) converges absolutely under the following conditions:

- (i)  $\lambda_i = \sum_{j=1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} < 0$ ,
- (ii)  $\mu_i = \sum_{j=1}^n \alpha_j^{(i)} - \sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0$  and

$$(iii) \quad |\arg z_i| < \frac{1}{2}\mu_i\pi.$$

Conditions (i) and (ii) can be relaxed to  $\lambda_i \leq 0$  and  $\mu_i \geq 0$  (see Joshi and Joshi [6]). But then the integral exists in a shrunken region. For example, when  $\lambda_i = 0$ , the region of convergence reduces to the interior of the circle in the  $\xi_i$ -plane with radius  $D_i^{-1}$ , where

$$D_i = \frac{\prod_{j=1}^p (\alpha_j^{(i)})^{\alpha_j^{(i)}} \prod_{j=1}^{p_i} (\gamma_j^{(i)})^{\gamma_j^{(i)}}}{\prod_{j=1}^q (\beta_j^{(i)})^{\beta_j^{(i)}} \prod_{j=1}^{q_i} (\delta_j^{(i)})^{\delta_j^{(i)}}},$$

with the parameters satisfying

$$q_i = \frac{1}{2}(p + p_i - q - q_i) - \operatorname{Re} \left( \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + \sum_{j=1}^{p_i} c_j^{(i)} - \sum_{j=1}^{q_i} d_j^{(i)} \right) < 0,$$

and for  $\mu_i = 0$  the region of convergence shrinks to the open interval  $(0, D_i^{-1})$  on the real axis in the  $\xi_i$ -plane, with the same condition on parameters as above.

**2. Removal of limitations in Ławrynowicz's formulas.** The successive differentiation formulas cited in Mathai and Saxena [9] (see also Srivastava *et al.* [17]) are:

$$(2.1) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma b_1/\beta_1} H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{matrix} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{matrix} \right. \right] \right\} \\ = \left( \frac{-\sigma}{\beta_1} \right)^r x^{-r-\sigma b_1/\beta_1} H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{matrix} ((a_p, \alpha_p)) \\ (r + b_1, \beta_1), (b_2, \beta_2), \dots, (b_q, \beta_q) \end{matrix} \right. \right],$$

$$(2.2) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma b_q/\beta_q} H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{matrix} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{matrix} \right. \right] \right\} \\ = \left( \frac{-\sigma}{\beta_q} \right)^r x^{-r-\sigma b_q/\beta_q} H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{matrix} ((a_p, \alpha_p)) \\ (b_1, \beta_1), \dots, (b_{q-1}, \beta_{q-1}), (r + b_q, \beta_q) \end{matrix} \right. \right],$$

$$(2.3) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma(1-a_1)/\alpha_1} H_{p,q}^{m,n} \left[ x^{-\sigma} \left| \begin{matrix} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{matrix} \right. \right] \right\} \\ = \left( \frac{-\sigma}{\alpha_1} \right)^r x^{-r-\sigma(1-a_1)/\alpha_1} \\ \times H_{p,q}^{m,n} \left[ x^{-\sigma} \left| \begin{matrix} (-r + a_1, \alpha_1), (a_2, \alpha_2), \dots, (a_p, \alpha_p) \\ ((b_q, \beta_q)) \end{matrix} \right. \right],$$

$$(2.4) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma(1-a_p)/\alpha_p} H_{p,q}^{m,n} \left[ x^{-\sigma} \left| \begin{matrix} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{matrix} \right. \right] \right\}$$

$$= \left( \frac{-\sigma}{\alpha_p} \right)^r x^{-r-\sigma(1-a_p)/\alpha_p} \\ \times H_{p,q}^{m,n} \left[ x^{-\sigma} \left| \begin{array}{l} (a_1/\alpha_1), \dots, (a_{p-1}, \alpha_{p-1}), (-r + a_p, \alpha_p) \\ ((b_q, \beta_q)) \end{array} \right. \right].$$

These formulas are stated to be true when  $\sigma = \beta_1$ ,  $\sigma = \beta_q$ ,  $\sigma = \alpha_1$  and  $\sigma = \alpha_p$  respectively. These restrictions are not essential and can be removed, as will be clear from what follows.

Consider for example the L.H.S. of (2.1):

$$\left( \frac{1}{2\pi\omega} \right) \int_L \phi(\xi) \frac{d^r}{dx^r} (x^{-\sigma b_1/\beta_1 + \sigma\xi}) d\xi \\ = \left( \frac{1}{2\pi\omega} \right) \int_L \phi(\xi) \{ (-\sigma b_1/\beta_1 + \sigma\xi) \dots (-\sigma b_1/\beta_1 + \sigma\xi - r + 1) x^{-\sigma b_1/\beta_1 + \sigma\xi - r} \} d\xi.$$

Ławrynowicz writes the product

$$(-\sigma b_1/\beta_1 + \sigma\xi)(-\sigma b_1/\beta_1 + \sigma\xi - 1) \dots (-\sigma b_1/\beta_1 + \sigma\xi - r + 1)$$

in the form

$$\left( -\frac{\sigma}{\beta_1} \right)^r (b_1 - \beta_1\xi)(b_1 - \beta_1\xi + \beta_1/\sigma) \dots (b_1 - \beta_1\xi + (r-1)\beta_1/\sigma)$$

and then interprets it as a product resulting by imposing the restriction  $\sigma = \beta_1$ . But this is not necessary, since obviously it can be put in the form

$$(-1)^r (\sigma b_1/\beta_1 - \sigma\xi)(\sigma b_1/\beta_1 - \sigma\xi + 1) \dots (\sigma b_1/\beta_1 - \sigma\xi + r - 1) \\ = (-1)^r (b_1/\beta_1 - \sigma\xi)_r = (-1)^r \frac{\Gamma(\sigma b_1/\beta_1 - \sigma\xi + r)}{\Gamma(\sigma b_1/\beta_1 - \sigma\xi)}.$$

Similarly in the case of (2.2), (2.3) and (2.4), the restriction can be removed.

Thus after removal of limitations Ławrynowicz's formulas take the following forms:

$$(2.5) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma b_1/\beta_1} H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{array}{l} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{array} \right. \right] \right\} \\ = (-1)^r x^{-r-\sigma b_1/\beta_1} H_{p+1,q+1}^{m+1,n} \left[ x^\sigma \left| \begin{array}{l} ((a_p, \alpha_p)), (\sigma b_1/\beta_1, \sigma) \\ (r + \sigma b_1/\beta_1, \sigma), ((b_q, \beta_q)) \end{array} \right. \right],$$

$$(2.6) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma b_q/\beta_q} H_{p,q}^{m,n} \left[ x^\sigma \left| \begin{array}{l} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{array} \right. \right] \right\} \\ = (-1)^r x^{-r-\sigma b_q/\beta_q} H_{p+1,q+1}^{m,n+1} \left[ x^\sigma \left| \begin{array}{l} (\sigma b_q/\beta_q, \sigma), ((a_p, \alpha_p)) \\ ((b_q, \beta_q)), (r + \sigma b_q/\beta_q, \sigma) \end{array} \right. \right],$$

$$(2.7) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma(1-a_1)/\alpha_1} H_{p,q}^{m,n} \left[ x^{-\sigma} \left| \begin{array}{l} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{array} \right. \right] \right\}$$

$$= (-1)^r x^{-r-\sigma(1-a_1)/\alpha_1} \\ \times H_{p+1,q+1}^{m,n+1} \left[ x^{-\sigma} \left| \begin{array}{l} (1-r-\sigma(1-a_1)/\alpha_1, \sigma), ((a_p, \alpha_p)) \\ ((b_q, \beta_q)), (1-\sigma(1-a_1)/\alpha_1, \sigma) \end{array} \right. \right],$$

and

$$(2.8) \quad \frac{d^r}{dx^r} \left\{ x^{-\sigma(1-a_p)/\alpha_p} H_{p,q}^{m,n} \left[ x^{-\sigma} \left| \begin{array}{l} ((a_p, \alpha_p)) \\ ((b_q, \beta_q)) \end{array} \right. \right] \right\} \\ = (-1)^r x^{-r-\sigma(1-a_p)/\alpha_p} H_{p+1,q+1}^{m+1,n} \left[ x^\sigma \left| \begin{array}{l} ((a_p, \alpha_p)), (1-r-\sigma(1-a_p)/\alpha_p, \sigma) \\ (1-\sigma(1-a_p)/\alpha_p, \sigma), ((b_q, \beta_q)) \end{array} \right. \right].$$

It follows from the foregoing discussion that the result of Srivastava *et al.* [17], which is stated without any restriction, is wrong and that formula (2.5) above gives its correct version.

**3. The successive derivative formulas.** The operator  $T_{k,q}$  was defined by Joshi and Prajapat [7] as

$$T_{k,q} = x^q \left( k + x \frac{d}{dx} \right),$$

$k$  and  $q$  being integer constants. We now define two new operators as

$$(3.1) \quad T(s, k : cx + d) = (cx + d)^s \left\{ k + (cx + d) \frac{d}{dx} \right\},$$

$$(3.2) \quad T_1(s, k : cx + d) = (cx + d)^s \left\{ k + (cx + d) \frac{\partial}{\partial x} \right\},$$

where  $c, d, s$  are real numbers and  $k$  may be a real or complex number. From the definition it is clear that

$$(3.3) \quad T(s, k : cx + d)(cx + d)^\alpha = (k + c\alpha)(cx + d)^{\alpha+s},$$

$$(3.4) \quad T^N(s, k : cx + d)(cx + d)^\alpha = (cs)^N \frac{\Gamma\left(\frac{k + c\alpha}{cs} + N\right)}{\Gamma\left(\frac{k + cx}{cs}\right)} (cx + d)^{\alpha+Ns}$$

and

$$(3.5) \quad T^N(0, k : cx + d)(cx + d)^\alpha = (k + c\alpha)^N (cx + d)^\alpha.$$

Now assuming  $u$  to be a constant, we have

$$T^N(s, k : cx + d) \{ (cx + d)^\alpha H[u(cx + d)^\sigma] \} \\ = \frac{1}{2\pi\omega} \int_L u^\xi \phi(\xi) \{ T^N(s, k : cx + d)(cx + d)^{\alpha+\sigma\xi} \} d\xi$$

$$= \begin{cases} \frac{1}{2\pi\omega} \int_L u^\xi \phi(\xi) (k + c\alpha + c\sigma\xi)^N (cx + d)^{\alpha + \sigma\xi} d\xi & \text{for } s = 0, \\ \frac{1}{2\pi\omega} \int_L u^\xi \phi(\xi) (cs)^N \\ \times \frac{\Gamma\left(\frac{k + c\alpha + c\sigma\xi}{cs} + N\right)}{\Gamma\left(\frac{k + c\alpha + c\sigma\xi}{cs}\right)} (cx + d)^{\alpha + \sigma\xi + Ns} d\xi & \text{for } s \neq 0. \end{cases}$$

Thus interpreting the R.H.S., we have

$$(3.6) \quad T^N(s, k : cx + d) \{(cx + d)^\alpha H[u(cx + d)^\sigma]\} \\ = (cs)^N (cx + d)^{\alpha + Ns} \\ \times H_{p+1, q+1}^{m, n+1} \left[ u(cx + d)^\sigma \left| \begin{matrix} \left(1 - \frac{k + c\alpha}{cs} - N, \frac{\sigma}{s}\right); \dots \\ \dots; \left(1 - \frac{k + c\alpha}{cs}, \frac{\sigma}{s}\right) \end{matrix} \right. \right], \quad s \neq 0,$$

$$(3.7) \quad T^N(s, k : cx + d) \{(cx + d)^\alpha H[u(cx + d)^\sigma]\} \\ = (cx + d)^\alpha \\ \times H_{p+N, q+N}^{m, n+N} \left[ u(cx + d)^\sigma \left| \begin{matrix} (-k - c\alpha, c\sigma), \dots, (-k - c\alpha, c\sigma), \dots \\ \dots, (1 - k - c\alpha, c\sigma), \dots, (1 - k - c\alpha, c\sigma) \end{matrix} \right. \right], \\ s = 0.$$

Proceeding in a similar manner, the two and  $r$ -variable analogues of this result are

$$(3.8) \quad T_1^N(s_1, k_1 : c_1x_1 + d_1) T_1^N(s_2, k_2 : c_2x_2 + d_2) \{(c_1x_1 + d_1)^{\alpha_1} (c_2x_2 + d_2)^{\alpha_2} \\ \times H[u_1(c_1x_1 + d_1)^{\sigma_1}, u_2(c_2x_2 + d_2)^{\sigma_2}]\} \\ = (c_1s_1)^N (c_2s_2)^N (c_1x_1 + d_1)^{\alpha_1 + Ns_1} (c_2x_2 + d_2)^{\alpha_2 + Ns_2} \\ \times H_{p, q; p_1+1, q_1+1; \dots; p_r+1, q_r+1}^{0, n; m_1, n_1+1; \dots; m_r, n_r+1} \left[ \begin{matrix} u_1(c_1x_1 + d_1)^{\sigma_1} \\ u_2(c_2x_2 + d_2)^{\sigma_2} \\ \left| \dots : \left(1 - \frac{k_1 + c_1\alpha_1}{c_1s_1} - N, \frac{\sigma_1}{s_1}\right); \left(1 - \frac{k_2 + c_2\alpha_2}{c_2s_2} - N, \frac{\sigma_2}{s_2}\right); \dots \right. \\ \left. \dots : \dots; \left(1 - \frac{k_1 + c_1\alpha_1}{c_1s_1}, \frac{\sigma_1}{s_1}\right); \dots; \left(1 - \frac{k_2 + c_2\alpha_2}{c_2s_2}, \frac{\sigma_2}{s_2}\right) \right] \end{matrix} \right]$$

and

$$\begin{aligned}
 (3.9) \quad & T_1^N(s_1, k_1 : c_1 x_1 + d_1) \dots T_1^N(s_r, k_r : c_r x_r + d_r) \prod_{j=1}^r \{(c_j x_j + d_j)^{\alpha_j} \\
 & \times H[u_1(c_1 x_1 + d_1)^{\sigma_1}, \dots, u_r(c_r x_r + d_r)^{\sigma_r}]\} \\
 & = \prod_{i=1}^r (c_i s_i)^N (c_i x_i + d_i)^{\alpha_i + N s_i} \\
 & \times H_{p,q;p_1+1,q_1+1;\dots;p_r+1,q_r+1}^{0,n;m_1,n_1+1;\dots;m_r,n_r+1} \left[ \begin{array}{c} u_1(c_1 x_1 + d_1)^{\sigma_1} \\ \vdots \\ u_r(c_r x_r + d_r)^{\sigma_r} \end{array} \right. \\
 & \left. \begin{array}{l} \dots : \left(1 - \frac{k_1 + c_1 \alpha_1}{c_1 s_1} - N, \frac{\sigma_1}{s_1}\right), \dots; \dots; \left(1 - \frac{k_r + c_r \alpha_r}{c_r s_r} - N, \frac{\sigma_r}{s_r}\right); \dots \\ \dots : \dots; \left(1 - \frac{k_1 + c_1 \alpha_1}{c_1 s_1}, \frac{\sigma_1}{s_1}\right); \dots; \left(1 - \frac{k_r + c_r \alpha_r}{c_r s_r}, \frac{\sigma_r}{s_r}\right); \dots \end{array} \right].
 \end{aligned}$$

For  $s_i = 0$ , we can give corresponding formulas similar to (3.6) above.

Again by using (3.3) and interpreting the result, we get the following formulas:

$$\begin{aligned}
 (3.10) \quad & T^N(s, k : at + b) \{(at + b)^\alpha H[z(at + b)^\sigma]\} \\
 & = (as)^N (at + b)^{\alpha + Ns} H_{p+1,q+1}^{m,n+1} \left[ z(at + b)^\sigma \left| \begin{array}{l} \left(1 - \frac{k + a\alpha}{as} - N, \frac{\sigma}{s}\right), \dots \\ \dots, \left(1 - \frac{k + a\alpha}{as}, \frac{\sigma}{s}\right) \end{array} \right. \right],
 \end{aligned}$$

$$\begin{aligned}
 (3.11) \quad & T^N(s, k : at + b) \{(at + b)^\alpha H[z_1(at + b)^{\sigma_1}; z_2(at + b)^{\sigma_2}]\} \\
 & = (as)^N (at + b)^{\alpha + Ns} \\
 & \times H_{p+1,q+1;\dots}^{0,n+1;\dots} \left[ \begin{array}{l} z_1(at + b)^{\sigma_1} \\ z_2(at + b)^{\sigma_2} \end{array} \left| \begin{array}{l} \left(1 - \frac{k + a\alpha}{as} - N, \frac{\sigma_1}{s}, \frac{\sigma_2}{s}\right), \dots \\ \dots, \left(1 - \frac{k + a\alpha}{as}, \frac{\sigma_1}{s}, \frac{\sigma_2}{s}\right) \end{array} \right. \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.12) \quad & T^N(s, k : at + b) \{(at + b)^\alpha H[z_1(at + b)^{\sigma_1}; \dots; z_r(at + b)^{\sigma_r}]\} \\
 & = (as)^N (at + b)^{\alpha + Ns} \\
 & \times H_{p+1,q+1;\dots}^{0,n+1;\dots} \left[ \begin{array}{l} z_1(at + b)^{\sigma_1} \\ \vdots \\ z_r(at + b)^{\sigma_r} \end{array} \left| \begin{array}{l} \left(1 - \frac{k + a\alpha}{as} - N; \frac{\sigma_1}{s_1}, \dots, \frac{\sigma_r}{s_r}\right), \dots \\ \dots, \left(1 - \frac{k + a\alpha}{as}; \frac{\sigma_1}{s_1}, \dots, \frac{\sigma_r}{s_r}\right) : \dots \end{array} \right. \right].
 \end{aligned}$$

**4. Particular cases.** In (3.5), let us set  $c = 1$ ,  $d = 0$ ,  $k = 0$ . Then on replacing one by one  $\alpha$  by  $-\sigma(b_1/\beta_1)$ ;  $\alpha$  by  $-\sigma(b_q/\beta_q)$ ;  $\alpha$  by  $-\sigma(1 - a_1/\alpha_1)$ ,  $\sigma$  by  $-\sigma$ ,  $\alpha$  by  $-\sigma(1 - a_p/\alpha_p)$ ,  $\sigma$  by  $-\sigma$ , we shall respectively get formulas (2.5) to (2.8). These correspond to Ławrynowicz's formulas for  $\sigma = \beta_1$ ,  $\sigma = \beta_q$ ,  $\sigma = \alpha_1$  and  $\sigma = \alpha_p$ . Again for the same set of values of  $c$ ,  $d$ ,  $k$ ,  $s$ , this formula will yield the result of Mathai and Saxena [9] for  $\alpha = 0$ , that of Skibiński [13] for  $\alpha = 0$ ,  $\sigma = -\delta$ , and that of Anandani [1] for  $\alpha = \lambda$ .

The particular cases of interest of (3.8) when  $s = 0$  and  $s = -1$  are the two results due to Gupta and Jain [4], when  $c = 1$ ,  $d = 0$ ,  $k = 0$ , and a result of Oliver and Kalla [10] when  $s = -1$ ,  $k = 0$ ,  $\alpha = 0$ .

Formula (3.9) on the other hand corresponds to two results of Goyal [3], for  $s = 0$  and  $s = -1$ , and that of Rakesh [12] for  $s = -2$ , when the other parameters involved are given the values  $a = 1$ ,  $b = 0$ ,  $k = 0$ . In the case  $k = 0$ ,  $s = -1$ , we get the result of Raina [11].

**5. Applications.** A particular case of (3.6) for  $c = 1$ ,  $d = 0$  gives

$$(5.1) \quad T^N(s, k : x) \{x^\alpha H[ux^\sigma]\} \\ = s^N x^{\alpha + Ns} H_{p+1, q+1}^{m, n+1} \left[ ux^\sigma \left| \begin{matrix} (1 - (k + \alpha)/s - N, \sigma/s), \dots \\ \dots, (1 - (k + \alpha)/s, \sigma/s) \end{matrix} \right. \right].$$

By using (3.4), we have

$$(5.2) \quad T^N(s, k : x) \{x^{\beta_1 + \beta_2} H[ux^\sigma]\} \\ = s^N x^{\beta_1 + \beta_2 + Ns} \sum_{M=0}^N \binom{N}{M} \frac{\Gamma\left(\frac{k + \beta_1}{s} + N - M\right)}{\Gamma\left(\frac{k + \beta_1}{s}\right)} \\ \times H_{p+1, q+1}^{m, n+1} \left[ ux^\sigma \left| \begin{matrix} \left(1 - \frac{\beta_2}{s} - M, \frac{\sigma}{s}\right), \dots \\ \dots, \left(1 - \frac{\beta_2}{s}, \frac{\sigma}{s}\right) \end{matrix} \right. \right].$$

Thus, if  $\alpha = \beta_1 + \beta_2$ , from (5.1) and (5.2), we get

$$(5.3) \quad H_{p+1, q+1}^{m, n+1} \left[ ux^\sigma \left| \begin{matrix} \left(1 - \frac{k + \alpha}{s} - N, \frac{\sigma}{s}\right), \dots \\ \dots, \left(1 - \frac{k + \alpha}{s}, \frac{\sigma}{s}\right) \end{matrix} \right. \right] \\ = \sum_{M=0}^N \binom{N}{M} \frac{\Gamma\left(\frac{k + \beta_1}{s} + N - M\right)}{\Gamma\left(\frac{k + \beta_1}{s}\right)} H_{p+1, q+1}^{m, n+1} \left[ ux^\sigma \left| \begin{matrix} \left(1 - \frac{\beta_2}{s} - M, \frac{\sigma}{s}\right), \dots \\ \dots, \left(1 - \frac{\beta_2}{s}, \frac{\sigma}{s}\right) \end{matrix} \right. \right].$$



Various such summation results are available in the literature. It is stated in [16] that all summation formulas that give finite and infinite series expansions of the  $H$ -function in terms of the  $H$ -function itself can be expressed as equation (2.11.1) of [15] (see also R. N. Jain [5]), derived by using Gauss's, Kummer's, Saalshutz's, Dixon's, Watson's theorem and a known theorem (Slater [14], p. 243, III eq. 10). It appears that our formula (5.3) and its multivariable analogue do not conform to this general form, and therefore may be considered as a new type of finite series representation for the  $H$ -function.

Similarly, comparing (3.8)–(3.11) for  $c = 0$ ,  $d = 0$  with their alternative forms obtained using Leibniz's theorem, we get the finite expansion formulas:

$$\begin{aligned}
 (5.4) \quad & H_{p,q;p_1+1,q_1+1;\dots;p_r+1,q_r+1}^{0,n;m_1,n_1+1;\dots;m_r,n_r+1} \\
 & \cdot \left[ \begin{array}{c} u_1 x_1^{\sigma_1} \\ \vdots \\ u_r x_r^{\sigma_r} \end{array} \middle| \dots : \left( 1 - \frac{k_1 + \alpha_1}{s_1} - N, \frac{\sigma_1}{s_1} \right), \dots; \left( 1 - \frac{k_r + \alpha_r}{s_r}, \frac{\sigma_r}{s_r} \right), \dots \right. \\
 & \left. \dots : \dots, \left( 1 - \frac{k_1 + \alpha_1}{s_1}, \frac{\sigma_1}{s_1} \right); \dots; \dots, \left( 1 - \frac{k_r + \alpha_r}{s_r}, \frac{\sigma_r}{s_r} \right) \right] \\
 & = \sum_{M_1, \dots, M_r=0}^N \prod_{i=1}^r \left\{ \binom{N}{M_i} \frac{\Gamma((k_i + \beta_i^{(i)})/s_i + N - M_i)}{\Gamma((k_i + \beta_i^{(i)})/s_i)} \right\} \\
 & \times H_{p,q;p_1+1,q_1+1;\dots;p_r+1,q_r+1}^{0,n;m_1,n_1+1;\dots;m_r,n_r+1} \\
 & \cdot \left[ \begin{array}{c} u_1 x_1^{\sigma_1} \\ \vdots \\ u_r x_r^{\sigma_r} \end{array} \middle| \dots : \left( 1 - \frac{\beta_2^{(1)}}{s_2} - M_1, \frac{\sigma_1}{s_1} \right), \dots, \left( 1 - \frac{\beta_2^{(r)}}{s_r} - M_r, \frac{\sigma_r}{s_r} \right), \dots \right. \\
 & \left. \dots : \dots, \left( 1 - \frac{\beta_2^{(1)}}{s_1}, \frac{\sigma_1}{s_1} \right); \dots; \dots, \left( 1 - \frac{\beta_2^{(r)}}{s_r}, \frac{\sigma_r}{s_r} \right) \right]
 \end{aligned}$$

provided  $\alpha_i = \beta_1^{(i)} + \beta_2^{(i)}$ ,  $i = 1, \dots, r$ ;

$$\begin{aligned}
 (5.5) \quad & H_{p+1,q+1;\dots}^{0,n+1;\dots} \left[ \begin{array}{c} z_1 t_1^{\sigma_1} \\ z_2 t_2^{\sigma_2} \end{array} \middle| \left( 1 - \frac{k + \alpha}{s} - N; \frac{\sigma_1}{s_1}, \frac{\sigma_2}{s_2} \right), \dots : \dots; \dots \right. \\
 & \left. \dots, \left( 1 - \frac{k + \alpha}{s}; \frac{\sigma_1}{s_1}, \frac{\sigma_2}{s_2} \right) : \dots; \dots \right] \\
 & = \sum_{M=0}^N \binom{N}{M} H_{p,q;p_1+1,q_1+1;p_2+1,q_2+1}^{0,n;m_1,n_1+1;m_2,n_2+1} \\
 & \cdot \left[ \begin{array}{c} z_1 t_1^{\sigma_1} \\ z_2 t_2^{\sigma_2} \end{array} \middle| \dots : \left( 1 - \frac{k + \beta_1}{s} + m - N, \frac{\sigma_1}{s_1} \right), \dots, \left( 1 - \frac{\beta_2}{s} - M, \frac{\sigma_2}{s} \right), \dots \right. \\
 & \left. \dots : \dots, \left( 1 - \frac{k + \alpha}{s} - N, \frac{\sigma_1}{s_1} \right), \dots; \dots, \left( 1 - \frac{\beta_2}{s_2}, \frac{\sigma_2}{s_2} \right) \right]
 \end{aligned}$$

and

$$\begin{aligned}
(5.6) \quad & H_{p+1, q+1: \dots}^{0, n+1: \dots} \left[ \begin{array}{c} z_1 t_1^{\sigma_1} \\ \vdots \\ z_r t_r^{\sigma_r} \end{array} \middle| \left( 1 - \frac{k + \alpha}{s} - N; \frac{\sigma_1}{s_1}, \dots, \frac{\sigma_r}{s_r} \right), \dots : \dots; \dots \right] \\
&= \sum_{M=0}^N \binom{N}{M} H_{p+2, q+2: \dots}^{0, n+2: \dots} \\
&\cdot \left[ \begin{array}{c} z_1 t_1^{\sigma_1} \\ \vdots \\ z_r t_r^{\sigma_r} \end{array} \middle| \dots : \left( 1 - \frac{k + \beta_1}{s} + M - N; \frac{\sigma_1}{s}, \dots, \frac{\sigma_{r-1}}{s}, 0 \right), \dots; \left( 1 - \frac{\beta_2}{s} - M, \frac{\sigma_r}{s} \right), \dots \right. \\
&\quad \left. \dots : \dots, \left( 1 - \frac{k + \beta_1}{s}; \frac{\sigma_1}{s}, \dots, \frac{\sigma_{r-1}}{s}, 0 \right); \dots; \dots; \left( 1 - \frac{\beta_2}{s}, \frac{\sigma_r}{s} \right) \right].
\end{aligned}$$

Iteration then leads to

$$\begin{aligned}
(5.7) \quad & H_{p+A, q+A: \dots}^{0, n+A: \dots} \left[ \begin{array}{c} z_1 \prod_{i=1}^A t_i^{\sigma_{i1}} \\ \vdots \\ z_r \prod_{i=1}^A t_i^{\sigma_{ir}} \end{array} \middle| \left( \left( 1 - \frac{k_A + \alpha_A}{s_A} - N; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{rA}}{s_A} \right) \right), \dots : \dots \right. \\
&\quad \left. \dots, \left( \left( 1 - \frac{k_A + \alpha_A}{s_A}; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{rA}}{s_A} \right) \right) : \dots \right] \\
&= \sum_{M_1, \dots, M_A=0}^N \prod_{i=1}^A \binom{N}{M_i} H_{p+A, q+A: p_1, q_1; \dots; p_r, q_r}^{0, n+A: m_1, n_1; \dots; m_r, n_r} \\
&\quad \left[ \begin{array}{c} z_1 \prod_{i=1}^A t_i^{\sigma_{i1}} \\ \vdots \\ z_r \prod_{i=1}^A t_i^{\sigma_{ir}} \end{array} \middle| \dots : \left( \left( 1 - \frac{k_A + \beta_1^{(A)}}{s_A} + M_A - N; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{(r-1)A}}{s_A}, 0 \right) \right); \dots; \left( \left( 1 - \frac{\beta_2^{(A)}}{s_A} - M_A; \frac{\sigma_{rA}}{s_A} \right) \right), \dots \right. \\
&\quad \left. \dots : \dots, \left( \left( 1 - \frac{k_A + \beta_1^{(A)}}{s_A}; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{(r-1)A}}{s_A}, 0 \right) \right); \dots; \dots; \left( \left( 1 - \frac{\beta_2^{(A)}}{s_A}, \frac{\sigma_{rA}}{s_A} \right) \right) \right]
\end{aligned}$$

and

$$(5.8) \quad H_{p+A, q+A: \dots}^{0, n+A: \dots} \left[ \begin{array}{c} z_1 \prod_{i=1}^A t_i^{\sigma_{i1}} \\ \vdots \\ z_r \prod_{i=1}^A t_i^{\sigma_{ir}} \end{array} \middle| \left( \left( 1 - \frac{k_A + \alpha_A}{s_A} - N; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{rA}}{s_A} \right) \right), \dots : \dots \right. \\
\quad \left. \dots, \left( \left( 1 - \frac{k_A + \alpha_A}{s_A}; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{rA}}{s_A} \right) \right) : \dots \right]$$

$$\begin{aligned}
 &= \sum_{M_1, \dots, M_A=0}^N \prod_{i=1}^A \binom{N}{M_i} H_{p+2A, q+2A: \dots}^{0, n+2A: \dots} \left[ \begin{array}{c} z_1 \prod_{i=1}^A t_i^{\sigma_{i1}} \\ \vdots \\ z_r \prod_{i=1}^A t_i^{\sigma_{ir}} \end{array} \right] \\
 &\left[ \begin{array}{l} \dots : \left( \left( 1 - \frac{k_A + \beta_1^{(A)}}{s_A} + M_A - N; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{r_1 A}}{s_A}, 0, \dots, 0 \right), \right. \\ \qquad \qquad \qquad \left. \left( \left( 1 - \frac{\beta_2^{(A)}}{s_A} - M_A; 0, \dots, 0, \frac{\sigma_{(r_1+1)A}}{s_A}, \dots, \frac{\sigma_{rA}}{s_A} \right), \dots : \dots \right. \right. \\ \left. \left. \dots, \left( \left( 1 - \frac{k_A + \beta_1^{(A)}}{s_A}; \frac{\sigma_{1A}}{s_A}, \dots, \frac{\sigma_{r_1 A}}{s_A}, 0, \dots, 0 \right), \left( \left( 1 - \frac{\beta_2^{(A)}}{s_A}, 0, \dots, 0, \frac{\sigma_{(r_1+1)A}}{s_A}, \dots, \frac{\sigma_{rA}}{s_A} \right) \right) \right) \right]
 \end{aligned}$$

provided  $\alpha_i = \beta_1^{(i)} + \beta_2^{(i)}$ ,  $\sigma^i / s^i > 0$  ( $i = 1, \dots, A$ ).

Alternatively these expansion formulas can also be obtained directly by using the identity  $T^N(s, k : x)(x^{\alpha+\beta}) = T^N(s, k : x)(x^\alpha \cdot x^\beta)$ , i.e.

$$\begin{aligned}
 &s^N x^{\alpha+\beta+N s} \frac{\Gamma\left(\frac{k+\alpha+\beta}{s} + N\right)}{\Gamma\left(\frac{k+\alpha+\beta}{s}\right)} \\
 &= s^N x^{\alpha+\beta+N s} \sum_{M=0}^N \binom{N}{M} \frac{\Gamma\left(\frac{k+\alpha}{s} + N - M\right)}{\Gamma\left(\frac{k+\alpha}{s}\right)} \cdot \frac{\Gamma\left(\frac{\beta}{s} + M\right)}{\Gamma\left(\frac{\beta}{s}\right)}
 \end{aligned}$$

(see also Joshi and Prajapat [7]).

**6. Expansion of hypergeometric functions.** The results established in the preceding section enable us to derive expansion formulas for Appell functions in terms of products of Gauss hypergeometric functions of the type considered earlier by Burchnall and Chaundy [2] and their multiple variable analogues. Indeed, if we assume  $n = p = q = 0$  and specialize the parameters, for example in (5.6), as

- (i)  $k = 0, s = 1, \sigma_1 = 1, \sigma_2 = 1,$
- (ii)  $m_1 = 1, n_1 = r_1, p_1 = r_1, q_1 = s_1, (c_j^{(1)}, \delta_j^{(1)}) = (1 - b_j, 1),$   
 $j = 1, \dots, r_1, (d_1^{(1)}, \delta_1^{(1)}) = (0, 1)$  and  $(d_j^{(1)}, \delta_j^{(1)}) = (1 - c_j, 1), j = 2, \dots, s_1,$
- (iii)  $m_2 = 1, n_2 = r_2, p_2 = r_2, q_2 = s_2 + 1, (c_j^{(2)}, \delta_j^{(2)}) = (1 - b'_j, 1),$   
 $j = 1, \dots, r_2, (d_1^{(2)}, \delta_1^{(2)}) = (0, 1)$  and  $(d_j^{(2)}, \delta_j^{(2)}) = (1 - c'_j, 1), j = 2, \dots, s_2,$

and replace  $z_1$  by  $-z_1$ ,  $z_2$  by  $-z_2$  and  $t$  by 1, we get

$$\begin{aligned} & \frac{\Gamma(\alpha + N)}{\Gamma(\alpha)} F_{1:s_1;s_2}^{1:r_1;r_2} \left( \begin{matrix} a : (b_{r_1}); (b_{r_2}); \\ c : (c_{s_1}); (c_{s_2}); \end{matrix} z_1, z_2 \right) \\ &= \sum_{M=0}^N \binom{N}{M} \frac{\Gamma(\beta_1 + N - M)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2 + M)}{\Gamma(\beta_2)} \\ & \quad \times {}_{r_1+1}F_{s_1+1} \left( \begin{matrix} \beta_1 + N - M, (b_{r_1}); \\ \beta_1, (c_{s_1}); \end{matrix} z_1 \right) \\ & \quad \times {}_{r_2+1}F_{s_2+1} \left( \begin{matrix} \beta_2 + M, (b_{r_2}); \\ \beta_2, (c_{s_2}); \end{matrix} z_2 \right). \end{aligned}$$

If we put  $\alpha + N = a$  and  $\alpha = a - c$ , we have

$$\begin{aligned} (6.1) \quad & \frac{\Gamma(a)}{\Gamma(c)} F_{1:s_1;s_2}^{1:r_1;r_2} \left( \begin{matrix} a + N : (b_{r_1}); (b_{r_2}); \\ c : (c_{s_1}); (c_{s_2}); \end{matrix} z_1, z_2 \right) \\ &= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1 + a - c - M)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2 + M)}{\Gamma(\beta_2)} \\ & \quad \times {}_{r_1+1}F_{s_1+1} \left( \begin{matrix} \beta_1 + a - c - M, (b_{r_1}); \\ \beta_1, (c_{s_1}); \end{matrix} z_1 \right) \\ & \quad \times {}_{r_2+1}F_{s_2+1} \left( \begin{matrix} \beta_2 + M, (b_{r_2}); \\ \beta_2, (c_{s_2}); \end{matrix} z_2 \right), \end{aligned}$$

provided

- (i)  $a - c > 0$  is a positive integer,  $c = \beta_1 + \beta_2$ ,
  - (ii)  $|z_1| < 1$ ,  $|z_2| < 1$ ,
  - (iii)  $\theta_1 \equiv \frac{1}{2}(r_1 - s_1) - \operatorname{Re} \left( \sum_{j=1}^{r_1} b_j - \sum_{j=1}^{s_1} c_j \right) < 0$ ,
- $$\theta_2 \equiv \frac{1}{2}(r_2 - s_2) - \operatorname{Re} \left( \sum_{j=1}^{r_2} b_j - \sum_{j=1}^{s_2} c_j \right) < 0,$$

which is of a different form than that of eq. (242) of Srivastava and Karlsson [15].

For particular values  $r_1 = 1$ ,  $s_1 = 0$ ,  $r_2 = 1$ ,  $s_2 = 0$ ,  $b_1 = b$ ,  $b'_1 = b'$  this takes the form

$$(6.2) \quad \frac{\Gamma(a)}{\Gamma(c)} F_1(a, b, b'; c; z_1, z_2)$$

$$\begin{aligned}
&= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1 + a - c - M)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2 + M)}{\Gamma(\beta_2)} \\
&\quad \times {}_2F_1 \left( \begin{matrix} \beta_1 + a - c - M, b; \\ \beta_1; \end{matrix} z_1 \right) {}_2F_1 \left( \begin{matrix} \beta_2 + M, b'; \\ \beta_2; \end{matrix} z_2 \right)
\end{aligned}$$

and the condition of convergence reduces to

- (i)  $a - c$  is a positive integer,  $c = \beta_1 + \beta_2$ ,
- (ii)  $|z_1| < 1$ ,  $|z_2| < 1$  and
- (iii)  $1/2 - \operatorname{Re}(b - c) < 0$  and  $1/2 - \operatorname{Re}(b' - c') < 0$ .

Proceeding in a similar manner from (5.7) to (5.11), we have respectively:

$$\begin{aligned}
(6.3) \quad &\frac{\Gamma(a)}{\Gamma(c)} F_D^{(r)}(a, b_1, \dots, b_r; c; z_1, \dots, z_r) \\
&= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1 + a - c)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2 + m)}{\Gamma(\beta_2)} \\
&\quad \times F_D^{(r-1)}(\beta_1 + a - c - m, b_1, \dots, b_r; c, z_1, \dots, z_{r-1}) \\
&\quad \times {}_2F_1(\beta_2 + m, b_r; \beta_2; z_r),
\end{aligned}$$

provided  $a - c$  is a positive integer and  $c = \beta_1 + \beta_2$ ;

$$\begin{aligned}
(6.4) \quad &\frac{\Gamma(a)}{\Gamma(c)} F_D^{(r)}(a, b_1, \dots, b_r; c; z_1, \dots, z_r) \\
&= \sum_{M=0}^{a-c} \binom{a-c}{M} \frac{\Gamma(\beta_1 + a - c)}{\Gamma(\beta_1)} \cdot \frac{\Gamma(\beta_2 + m)}{\Gamma(\beta_2)} \\
&\quad \times F_D^{(r_1)}(\beta_1 + a - c - m, b_1, \dots, b_{r_1}; \beta_1; z_1, \dots, z_{r_1}) \\
&\quad \times F_D^{(r-r_1)}(\beta_2 + m, b_{r_1+1}, \dots, b_r; \beta_2; z_{r_1+1}, \dots, z_r);
\end{aligned}$$

$$\begin{aligned}
(6.5) \quad &\prod_{i=1}^n \frac{\Gamma(a_i)}{\Gamma(c_i)} F_{n:q_1; q_2}^{n:p_1; p_2} \left( \begin{matrix} (a_n) : (b_p^{(1)}); (b_p^{(2)}); \\ (c_n) : (c_q^{(1)}); (c_q^{(2)}); \end{matrix} z_1, z_2 \right) \\
&= \sum_{m_1=0}^{a_1-c_1} \dots \sum_{m_n=0}^{a_n-c_n} \prod_{i=1}^n \binom{a_i - c_i}{m_i} \frac{\Gamma(\beta_1^{(i)} + a_i - c_i - m_i)}{\Gamma(\beta_1^{(i)})} \cdot \frac{\Gamma(\beta_2^{(i)} + m_i)}{\Gamma(\beta_2^{(i)})} \\
&\quad \times {}_{p_1+n}F_{q_1+n} \left( \begin{matrix} (\beta_1^{(n)} + a_n - c_n - m_n), (b_{p_1}); \\ (\beta_1^{(n)}), (c_{q_1}); \end{matrix} z_1 \right) \\
&\quad \times {}_{p_2+n}F_{q_2+n} \left( \begin{matrix} (\beta_2^{(n)} + m_n), (b_{p_2}); \\ (\beta_2^{(n)}), (c_{q_2}); \end{matrix} z_2 \right)
\end{aligned}$$

provided  $a_i - c_i$  is a positive integer and  $c_i = \beta_1^{(i)} + \beta_2^{(i)}$ ,  $i = 1, \dots, n$ ;

$$\begin{aligned}
 (6.6) \quad & \prod_{i=1}^n \frac{\Gamma(a_i)}{\Gamma(c_i)} F_{n:p_1; \dots; p_r}^{n:q_1; \dots; q_r} \left( (a_n) : (b_{p_1}^{(1)}); \dots; (b_{p_r}^{(r)}); z_1, \dots, z_r \right) \\
 & = \sum_{m_1=0}^{a_1-c_1} \dots \sum_{m_n=0}^{a_n-c_n} \prod_{i=1}^n \binom{a_i - c_i}{m_i} \frac{\Gamma(\beta_1^{(i)} + a_i - c_i - m_i)}{\Gamma(\beta_1^{(i)})} \cdot \frac{\Gamma(\beta_2^{(i)} + m_i)}{\Gamma(\beta_2^{(i)})} \\
 & \quad \times {}_{p_r+n}F_{q_r+n} \left( (\beta_2^{(n)} + m_n), (b_{p_r}); z_r \right) \\
 & \quad \times F_{n:p_1; \dots; p_{r-1}}^{n:q_1; \dots; q_{r-1}} \left( (\beta_1^{(n)} + a_n - c_n - m_n), (b_{p_1}); \dots; (b_{p_{r-1}}^{r-1}); z_1, \dots, z_r \right) \\
 & \quad \times F_{n:p_1; \dots; p_{r-1}}^{n:q_1; \dots; q_{r-1}} \left( (\beta_1^{(n)}), (c_{q_1}); \dots; (c_{q_{r-1}}); z_1, \dots, z_r \right)
 \end{aligned}$$

provided  $c_i = \beta_1^{(i)} + \beta_2^{(i)}$ ,  $i = 1, \dots, n$ , and  $a_i - c_i$  are positive integers.

The above results can be combined to yield interesting identities but for reasons of brevity we omit the details.

### References

- [1] P. Anandani, *On the derivative of H-function*, Rev. Roumaine Math. Pures Appl. 15 (1970), 189–191.
- [2] J. L. Burchnall and T. W. Chaundy, *On Appell's hypergeometric functions*, Quart. J. Math. 11 (1940), 249–270.
- [3] S. P. Goyal, *The H-function of two variables*, Kyungpook Math. J. 15 (1975), 117–131.
- [4] K. C. Gupta and U. C. Jain, *On the derivatives of H-function*, Proc. Nat. Acad. Sci. India Sect. A 38 (1968), 189–192.
- [5] R. N. Jain, *General series involving H-function*, Proc. Cambridge Philos. Soc. 65 (1969), 461–465.
- [6] C. M. Joshi and N. L. Joshi, *Reinvestigation of conditions of convergence of the H-function of two variables*, submitted for publication.
- [7] C. M. Joshi and M. L. Prajapat, *On some results concerning generalized H-function of two variables*, Indian J. Pure Appl. Math. 8 (1977), 103–116.
- [8] J. Ławrynowicz, *Remarks on the preceding paper of P. Anandani*, Ann. Polon. Math. 21 (1969), 120–123.
- [9] A. M. Mathai and R. K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*, Wiley Eastern, New Delhi, 1978.
- [10] M. L. Oliver and S. L. Kalla, *On the derivative of Fox's H-function*, Acta Mexican Ci. Tecn. 5 (1971), 3–5.
- [11] R. K. Raina, *On the repeated differentiation of H-function of two variables*, Vijnana Parishad Anusandhan Patrika 21 (1978), 221–228.
- [12] S. L. Rakesh, *On the derivatives of the generalised Fox's H-function of two variables*, ibid. 18 (1975), 17–25.
- [13] P. Skibiński, *Some expansion theorems for the H-function*, Ann. Polon. Math. 23 (1970), 125–138.

- [14] L. F. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, Cambridge, 1966.
- [15] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Horwood, Chichester, 1985.
- [16] H. M. Srivastava and R. Panda, *Some expansion theorems and generating relations for the  $H$ -function of several complex variables II*, *Comment. Math. Univ. St. Paul.* 25 (1975), 169–197.
- [17] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The  $H$ -Functions of One and Two Variables with Applications*, South Asian Publ., New Delhi–Madras, 1982.

Department of Mathematics and Statistics  
M. L. Sukhadia University  
Udaipur 313001, India

S. M. B. Government College  
Nathadwara, Rajasthan, India

*Reçu par la Rédaction le 6.1.1995*  
*Révisé le 30.6.1996*