SUCCESSIVE INTEGRATION AS A METHOD FOR FINDING LONG PERIOD CYCLES

By

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SYNOPSIS

A method is developed for finding the period of cycles in statistical data of longer period than can be found by ordinary periodogram method. It consists of computing the progressive summations of the deviations of the observed data from normal. (As a first approximation normal is taken as the mean of the observed data.) Then the progressive summations of these accumulated discrepancies are found, and so on to the third or fourth integration. This process rapidly "irons out" all chance and short period variations and leaves a smooth cycle whose approximate period is obvious. The objection that this is but an extension of the "quadrature" method, which makes cycles appear in data where none are present, is discussed and methods are presented for determining whether the cycle found is real or fictitious.

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The question of the search for hidden periodicities in statistical data is important for the civil engineer interested in stream flow, the meteorologist and agriculturist interested in rainfall or temperature, the business statistician and probably many others. The ordinary method of periodogram analysis as developed by Schuster[1] and discussed by Whittaker and Robinson[2] requires a period of record several times as long as the longest cycle considered. A century's record of annual rainfall at any given station or group of stations will enable us to test for the presence of cycles with periods of say from eight to twenty years, and possibly even to 33 or 50 years, but could not reveal a cycle of period of say between fifty and one hundred fifty years if

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1. Proc. Roy. Soc. 77 (1906) p. 136.
2. The Calculus of Observations, Chap. XIII, pp. 343-362.

one were present. For the latter purpose the following method has been developed.

Suppose the given data is of the form $y = a + b\cos\frac{k(x-c)}{p} \pm d$ where

y = any value of the dependent variable, as for example, the total inches of rainfall at a given station in any given year.

a = a constant, the normal value about which y fluctuates, as the normal rainfall in inches per year for the given station. If the data contains a straight line trend this term would be replaced by $(a + mx)$, where m measures the rate or slope of the trend.

b = a constant, the amplitude of the cycle.

k = 360, to give the angle in degrees, or 2π, to give the angle in radians.

x = the independent variable or serial number of the particular value y, as for example the date A. D. of the year whose rainfall is y.

c = a constant specifying the phase of the cycle, as for example the earliest date A. D. when the rainfall is known to have been a maximum.

p = the period of the cycle (in years, in this example).

d = a variable—the deviation of each observed value from the value given by the rest of the equation. This term takes care of all variations due to cycles of other periods, or to the form of this cycle not being that of the cosine curve, or the other variations which for want of further knowledge we must consider to be purely fortuitous.

If we subtract a, the normal value, from each of these values and take the progressive totals or first integral of the differences, we get a series of the form $\frac{bp}{2\pi}\sin\frac{k(x-c)}{p} + \Sigma d$ or $\frac{bp}{2\pi}\cos\frac{k(x-c-p/4)}{p} + \Sigma d$

The term Σd will in general be small, since the plus chance variations tend to balance the minus, as do the variations due to short period cycles which may be present. The second integration will give a series of the form $-b\left(\frac{p}{2\pi}\right)^2\cos\frac{k(x-c)}{p}$ or $b\left(\frac{p}{2\pi}\right)^2\cos\frac{k(x-c-p/2)}{p}$ if we neglect the sums of the Σd terms, which will, of course, tend to cancel out. The third integral will be of the form $b\left(\frac{p}{2\pi}\right)^3\cos\frac{k(x-c-3p/4)}{p}$

and the fourth integral of the form $b\left(\frac{p}{2\pi}\right)^4 \cos \frac{k(x-c-p)}{p}$. That is, each integration gives a cycle of the same period as existed in the original data, but advanced a quarter period in phase and with the amplitude multiplied by $\frac{p}{2\pi}$.

The way in which this method "irons out" chance variations and short period fluctuations is most amazing to one who has not tried it. Table I and Plate I illustrate the method as applied to the annual rainfall at Boston, Mass., for the years 1818-1928. Although this has been carried to the third integration to illustrate the method, it was unnecessary in this case, as the second integration gives a smooth curve. The first integration changed from minus to plus in 1865 (fractions of a year being neglected) and from plus to minus in 1912, giving a half period of 47 years. But by producing the first integration curve backward we see that it would pass through zero in 1814, which gives a whole period of 98 years. As a compromise, 96 years was taken as the value of p.

The Weather Bureau gives the rainfall to the hundredth of an inch, but it was found that the results were practically the same if the rainfall were taken to the nearest inch, which was done in this table. The mean rainfall for the 111 years was 43.45 and there seems to be no trend. (From *a priori* grounds, such as the relative constancy of flora, lake levels, etc., we are quite sure that the rainfall of New England has changed very slightly in the last few centuries, so that for a period of only 111 years we can assume the normal rainfall as constant. The matter could also be tested statistically by fitting a straight line to the data by least squares, but this was not done.) But the normal rainfall is not necessarily equal to the mean for the period. In fact, a preliminary trial indicated that the period of the most important cycle was a little less than 100 years, and that of the 111 years the portion in excess of a full cycle was below normal, therefore that the observed mean would be below normal. So 44 inches per year was taken as the first approximation to normal.

A second point in Table I which needs explanation is the initial value in each of the summation columns. A preliminary computation was made starting from zero. This was found to give a series of numbers averaging quite a little above zero. But the average of these values cannot be taken as their normal any more than in the case of the observed rainfall themselves. Averaging the maximum and minimum values gave + 46, therefore it was assumed that all values should be reduced 46 (accumulated inches) and a second trial was made using −46 as the value of the first integration for 1817. When the

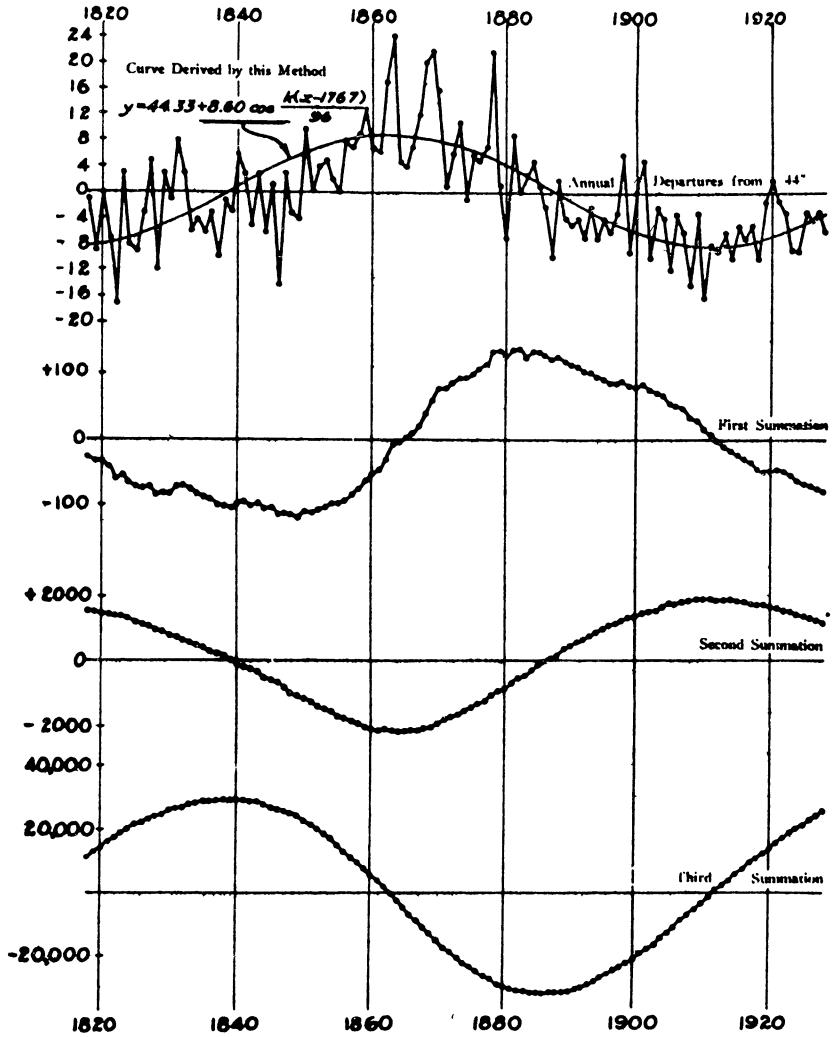
TABLE I

Search for a Long Period Cycle in the Precipitation Records of Boston, Mass.

Year	Inches of Rainfall	Excess or Deficiency From 44"	Summations		
			First	Second	Third
			- 26	1580	10400
1818	43	- 1	- 27	1553	11953
1819	35	- 9	- 36	1517	13470
1820	44	0	- 36	1481	14951
1821	37	- 7	- 43	1438	16389
1822	27	-17	- 60	1378	17767
1823	47	3	- 57	1321	19088
1824	36	- 8	- 65	1256	20344
1825	35	- 9	- 74	1182	21526
1826	41	- 3	- 77	1105	21526
1827	49	5	- 72	1033	23664
1828	32	-12	- 84	949	24613
1829	47	3	- 81	868	25481
1830	43	- 1	- 82	786	26267
1831	52	8	- 74	712	26979
1832	47	3	- 71	641	27620
1833	38	- 6	- 77	564	28184
1834	40	- 4	- 81	483	28667
1835	38	- 6	- 87	396	29063
1836	41	- 3	- 90	306	29369
1837	34	-10	- 100	206	29575
1838	43	- 1	- 101	105	29680
1839	41	- 3	- 104	1	29681
1840	49	5	- 99	- 98	29583
1841	47	3	- 96	- 194	29389
1842	39	- 5	- 101	- 295	29094
1843	47	3	- 98	- 393	28701
1844	38	- 6	- 104	- 497	28204
1845	46	2	- 102	- 599	27605
1846	30	-14	- 116	- 715	26890
1847	47	3	- 113	- 828	26062
1848	41	- 3	- 116	- 944	25118
1849	40	- 4	- 120	-1064	24054
1850	54	10	- 110	-1174	22880
1851	44	0	- 110	-1284	21596
1852	48	4	- 106	-1390	20206
1853	49	5	- 101	-1491	18715
1854	46	2	- 99	-1590	17125
1855	44	0	- 99	-1689	15436
1856	52	8	- 91	-1780	13656
1857	51	7	- 84	-1864	11792
1858	53	9	- 75	-1939	9853
1859	57	13	- 62	-2001	7852
1860	51	7	- 55	-2056	5796
1861	50	6	- 49	-2105	3691
1862	61	17	- 32	-2137	1554
1863	68	24	- 8	-2145	- 591
1864	49	5	- 3	-2148	- 2739
1865	48	4	1	-2147	- 4886
1866	51	7	8	-2139	- 7025
1867	56	12	20	-2119	- 9144
1868	64	20	40	-2079	-11223
1869	66	22	62	-2017	-13240
1870	60	16	78	-1939	-15179
1871	45	1	79	-1860	-17039

Year	Inches of Rainfall	Excess or Deficiency From 44"	Summations		
			First	Second	Third
1872	50	6	85	-1775	-18814
1873	55	11	96	-1679	-20493
1874	43	- 1	95	-1584	-22077
1875	50	6	101	-1483	-23560
1876	49	5	106	-1377	-24937
1877	51	7	113	-1264	-26201
1878	66	22	135	-1129	-27330
1879	45	1	136	- 993	-28323
1880	37	- 7	129	- 864	-29187
1881	53	9	138	- 726	-29913
1882	44	0	138	- 588	-30501
1883	35	- 9	129	- 459	-30960
1884	49	5	134	- 325	-31285
1885	45	1	135	- 190	-31475
1886	42	- 2	133	- 57	-31532
1887	34	-10	123	66	-31466
1888	46	2	125	191	-31275
1889	40	- 4	121	312	-30963
1890	39	- 5	116	428	-30535
1891	40	- 4	112	540	-29995
1892	37	- 7	105	645	-29350
1893	42	- 2	103	748	-28602
1894	37	- 7	96	844	-27758
1895	40	- 4	92	936	-26822
1896	38	- 6	86	1022	-25800
1897	41	- 3	83	1105	-24695
1898	50	6	89	1194	-23501
1899	35	- 9	80	1274	-22227
1900	44	0	80	1354	-20873
1901	49	5	85	1439	-19434
1902	34	-10	75	1514	-17920
1903	42	- 2	73	1587	-16333
1904	40	- 4	69	1656	-14677
1905	32	-12	57	1713	-12964
1906	41	- 3	54	1767	-11197
1907	38	- 6	48	1815	- 9382
1908	30	-14	34	1849	- 7533
1909	41	- 3	31	1880	- 5653
1910	28	-16	15	1895	- 3758
1911	36	- 8	7	1902	- 1856
1912	35	- 9	- 2	1900	44
1913	38	- 6	- 8	1892	1936
1914	34	-10	- 18	1874	3810
1915	39	- 5	- 23	1851	5661
1916	37	- 7	- 30	1821	7482
1917	39	- 5	- 35	1786	9268
1918	34	-10	- 45	1741	11009
1919	43	- 1	- 46	1695	12704
1920	46	2	- 44	1651	14355
1921	43	- 1	- 45	1606	15961
1922	41	- 3	- 48	1558	17519
1923	35	- 9	- 57	1501	19020
1924	35	- 9	- 66	1435	20455
1925	41	- 3	- 69	1366	21821
1926	40	- 4	- 73	1293	23114
1927	41	- 3	- 76	1217	24331
1928	38	- 6	- 82	1135	25466

PLATE I



Long Period Cycle of Rainfall Variation at Boston, Mass.

second integration was made from these figures it was found that the curve had a general downward slope, and from its slope a correction was estimated for the initial term of the first integration. The initial terms of the other columns were found in a similar manner. The values given are the result of several successive approximations. It is probable that by using a different combination of values for these initial numbers an equally good cycle of a slightly different period might be developed. It is claimed only that this is a method of finding the *approximate* period of long cycles. If the exact period were important we could, by least square methods, fit cycles of periods slightly less and slightly greater than that found, and see which gave the best fit.

The minimum value of the second summation was –2148 in 1864 and the maximum was +1902 in 1911. Therefore the double amplitude was 4050 and the single amplitude was 2025. Therefore $b\left(\frac{p}{2\pi}\right)^2 = 2025$ and $b = \frac{2025 \times 4\pi^2}{96^2} = 8.67$. A similar use of the values of the third integration gave an amplitude of $\frac{29681 + 31532}{2} = 30606.5$ and $b = \frac{30606.5 \times 8\pi^3}{96^3} = 8.60$.

As shown above the maxima of the second integration occur at about 1815 and 1911, therefore maxima of the rainfall itself should have occurred in 1767 and 1863 and c may be taken as 1767. Values of the term 8.60 cos $\frac{k(x-1767)}{96}$ were computed for each year. The first 96 terms will, of course, add to zero, but the last 15 terms add to –98.35, so that this will shift the 111-year average 0.89 inches below normal. As the average of the original data was 43.45, this indicates that the normal should be 43.45+0.89 = 44.34 and our equation becomes $y = 44.34 + 8.52 \cos \frac{k(x-1767)}{96} \pm d$.*

* It is not claimed that this equation as it stands can be used to forecast rainfalls at Boston. The data when examined by the periodogram method reveal several cycles of shorter periods, and even when corresponding terms are added to the formula the fortuitous variations average several inches per year. After this computation was made I found that C. F. Marvin had discussed this particular case (Monthly Weather Review, Aug., 1923, Vol. 51, pp. 383-390, "Concerning Normals, Secular Trends and Climatic Changes). By using data for Boston running back to 1750 (some actual and some "manufactured") he finds that straight line trends, rather than a cosine curve, best fit the data. He finds a normal annual rainfall of 40.06 from 1759 to 1849 and of 44.71 from 1849 to 1904, at which date the normal suddenly dropped again. The average rainfall from 1904 to 1928 inclusive was 37.80. Taking this as the normal for that period and the normals for the other two periods as given by Marvin, the average deviation from the normal for the 111 years was 5.46. The average deviation from the mean for the 111 years was 6.36. The average deviation from the formula derived above was 4.65. Adding four shorter cycles, we get the formula:

It will probably be urged against this method that the process of progressive summation often gives very misleading results, as has been pointed out by Bullock, Persons and Crum[1] and Simon Kuznets[2]. The latter gives, for example, 50 digits drawn at random, and the progressive totals of the deviations from the average, from which he deduces a pseudo cycle. Plate II shows that an even more striking cycle can be derived from this same random data by getting the second and third integrations. By following the same method as outlined above, a curve was deduced and drawn. The original data fit this curve with a mean deviation of 2.40, while they fit the average line with a mean deviation of 2.48. It may be urged that the cycle derived from the rainfall data is no more real than that derived from the chance data. But actually the cases are quite different, as we will proceed to show.

The test as to whether a time series contains cycles has been developed by Goutereau[3], Besson[4] and Woolard[5]. If the absolute values of the successive first differences of the series are averaged, this average is called the mean variability. The average of the absolue values of the differences of each value from the mean of all the values is called the mean deviation. Goutereau's Ratio, G, equals mean variation divided by mean deviation. For a random series of numbers whose distribution is Gaussian, the expected value of G will be $\sqrt{2}$. But if there is a cycle present, even if concealed by large chance varia-

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$$y = 44.22 + 8.60 \cos \frac{k(x-1767)}{96} + 2.58 \cos \frac{k(x-1773)}{47}$$

$$+ 2.70 \cos \frac{k(x-1798)}{34} + 2.56 \cos \frac{k(x-1811)}{18} + 1.88 \cos \frac{k(x-1810)}{10}$$

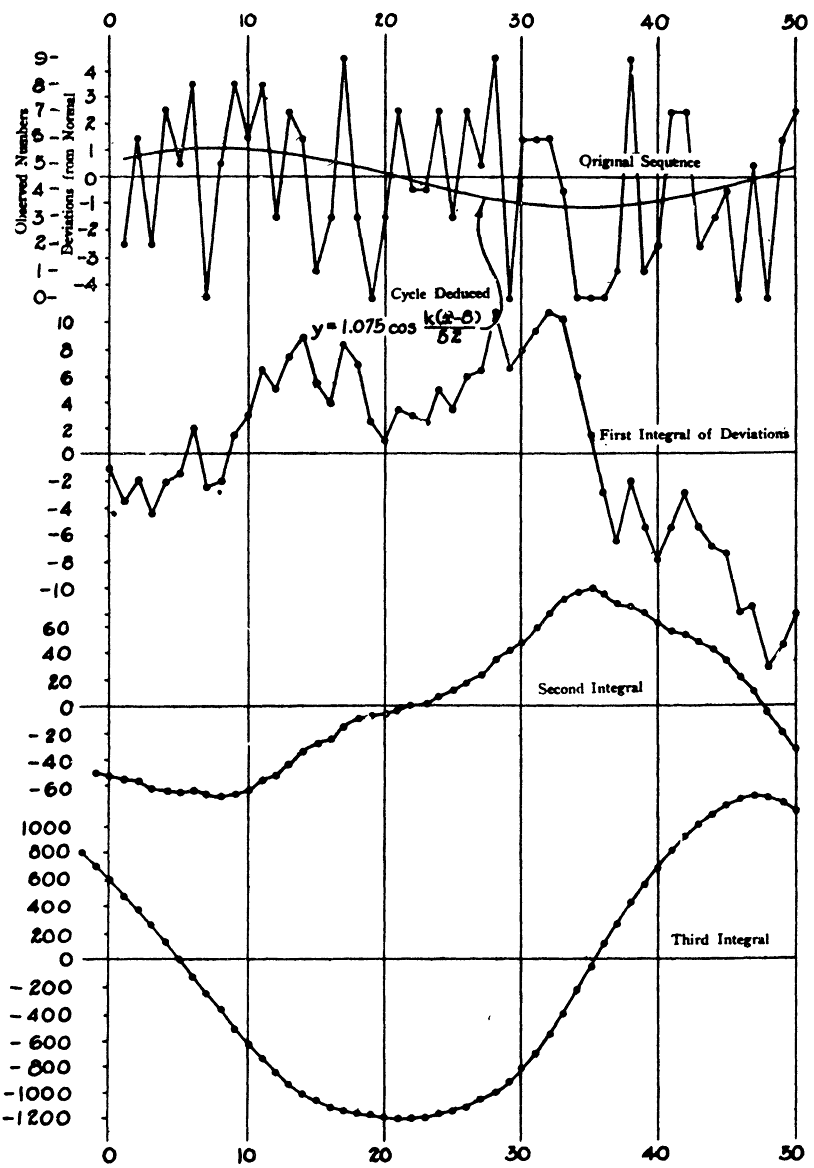
which gives an average deviation of 4.28. (Some of these figures were obtained by using the annual rainfall to the nearest inch and would be slightly different if the data to hundredths of an inch were used.) The reality of the four shorter cycles is very doubtful, but they produce a curve which fits the data much closer than the straight mean, or than Marvin's proposed normals, and somewhat closer than the simple 96 year cycle.

1. "A Reply to Karl Karsten's 'The Harvard Business Indexes—a New Interpretation'," Review of Economic Statistics, April, 1927, pp. 74-92.
2. "Random Events and Cyclical Oscillations," Journ. of the Amer. Statistical Assn.. Sept., 1919, pp. 258-275.
3. Sur la variabilite de la temperature, Annuaire de la Soc. Met. de France, 54, 122-127, 1906. Summarized by Edgar W. Wooland in Minthly Weather Review. Vol. 49 (1921), pp. 132-3.
4. "On the Comparison of Meteorological Data with Results of Chance," (Translated by E. W. Woolard) Monthly Weather Review, Vol. 48 (1920), pp. 89-94.
5. Edgar W. Woolard, "On the Mean Variability in Random Series," Monthly Weather Review (1925), pp. 107-111.

TABLE II

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
								- 65.0
							94.0	29.0
						-24.5	69.5	98.5
1	2	0	2		- 2.3	-26.8	42.7	141.2
2	6	1	7	5	2.7	-24.1	18.6	159.8
3	2	2	4	- 3	- 0.3	-24.4	- 5.8	154.0
4	7	3	10	6	5.7	-18.7	-24.5	129.5
5	5	3	8	- 2	3.7	-15.0	-39.5	90.0
6	8	4	12	4	7.7	- 7.3	-46.8	43.2
7	0	4	4	- 8	- 0.3	- 7.6	-54.4	- 11.2
8	5	4	9	5	4.7	- 2.9	-57.3	- 68.5
9	8	3	11	2	6.7	3.8	-53.5	-122.0
10	6	3	9	- 2	4.7	8.5	-45.0	-167.0
11	8	2	10	1	5.7	14.2	-30.8	-197.8
12	3	1	4	- 6	- 0.3	13.9	-16.9	-214.7
13	7	0	7	3	2.7	16.6	- 0.3	-215.0
14	6	- 1	5	- 2	0.7	17.3	17.0	-198.0
15	1	- 2	- 1	- 6	- 5.3	12.0	29.0	-169.0
16	3	- 3	0	1	- 4.3	7.7	36.7	-132.3
17	9	- 3	6	6	1.7	9.4	46.1	- 86.2
18	3	- 4	- 1	- 7	- 5.3	4.1	50.2	- 36.0
19	0	- 4	- 4	- 3	- 8.3	- 4.2	46.0	10.0
20	3	- 4	- 1	3	- 5.3	- 9.5	36.5	46.5
21	7	- 3	4	5	- 0.3	- 9.8	26.7	73.2
22	4	- 3	1	- 3	- 3.3	-13.1	13.6	96.8
23	4	- 2	2	1	- 2.3	-15.4	- 1.8	95.0
24	7	- 1	6	4	1.7	-13.7	-15.5	79.5
25	3	0	3	- 3	- 1.3	-15.0	-30.5	49.0
26	7	1	8	5	3.7	-11.3	-41.8	7.2
27	5	2	7	- 1	2.7	- 8.6	-50.4	- 43.2
28	9	3	12	5	7.7	- 0.9	-51.3	- 94.5
29	0	3	3	- 9	- 1.3	- 2.2	-53.5	-148.0
30	6	4	10	7	5.7	3.5	-50.0	-198.0
31	6	4	10	0	5.7	9.2	-40.8	-238.8
32	6	4	10	0	5.7	14.9	-25.9	-264.7
33	4	3	7	- 3	2.7	17.6	- 8.3	-273.0
34	0	3	3	- 4	- 1.3	16.3	8.0	-265.0
35	0	2	2	- 1	- 2.3	14.0	22.0	-243.0
36	0	1	1	- 1	- 3.3	10.7	32.7	-210.3
37	1	0	1	0	- 3.3	7.4	40.1	-170.2
38	9	- 1	8	7	3.7	11.1	51.2	-119.0
39	1	- 2	- 1	- 9	- 5.3	5.8	57.0	- 62.0
40	2	- 3	- 1	0	- 5.3	0.5	57.5	- 4.5
41	7	- 3	4	5	- 0.3	0.2	57.7	53.2
42	7	- 4	3	- 1	- 1.3	- 1.1	56.6	109.8
43	2	- 4	- 2	- 3	- 6.3	- 7.4	49.2	159.0
44	3	- 4	- 1	- 1	- 5.3	-12.7	36.5	195.5
45	4	- 3	1	2	- 3.3	-16.0	20.5	216.0
46	0	- 3	- 3	- 4	- 7.3	-23.3	- 2.8	213.2
47	5	- 2	3	6	- 1.3	-24.6	-27.4	185.8
48	0	- 1	- 1	- 4	- 5.3	-29.9	-57.3	128.5
49	6	0	6	7	1.7	-28.2	-85.5	43.0
50	7	1	8	2	3.7	-24.5	-110.0	- 67.0
Sums				92	91.4			
	214			-86	-91.4			
Sum of absolute values				178	182.8			

PLATE II



Showing How Apparent Cycles May Be Generated by Random Data.

tions superimposed on the cycle, the mean variation will be smaller than otherwise, and G will be less than 1.41. If the distribution is not Gaussian, the expected value of G will no longer be $\sqrt{2}$, but it will still be true that the presence of cycles will make G less than the expected value. Woodard, in the reference cited, gives a method for computing the expected value of G for a random order with any sort of distribution.

For the example of 50 random numbers used in Plate II, the mean deviation was 2.48 and the average variation was 3.45, making $G = 1.39$. The expected average deviation of numbers drawn from a universe containing equal numbers of each of the digits from 0 to 9 is 2.50 and the expected deviation by Woolard's method is 3.30, making the expected value of $G = 1.32$[1]. In the data for Boston rainfall, given in Table I, the mean deviation was 6.356 and the mean variability 6.54, or $G = 1.03$, while Woolard's method would give an expected value of 1.38 ± 0.11 for random succession. The distribution of the "universe" of which this is a sample is not Gaussian, but it is not *much* different from Gaussian, so that the true value of G is not far from 1.41. An investigation of a much larger sample of rainfall data, which the writer hopes to publish soon, gives $G = 1.386 \pm .027$. It is therefore quite certain that the departure of the value of G from the expected value for random numbers is not accidental but indicates that we have here a real cycle, while in the case of drawn numbers we had only an apparent one.

To test the operation of the method in a case where it was known that there were both chance and cyclic elements present, Table II was prepared. The first two columns give the same random numbers from which Plate II was plotted. Column (3) is an artificial cycle which approximates a sine curve of period 24 and amplitude 4.00. Column (4) gives the algebraic sum of (2) and (3), and column (5) the first differences of column (4). Column (6) gives the deviations of the values in column (4) from the mean (4.30). Columns (7), (8)

1. The sample of 50 drawings gave by Woolard's method an expected mean variation for random succession of $3.27 \pm .39$ or $G = 1.32 \pm .16$. Thus the observed value fell within the range of the probable error. But the agreement is often much closer. The results of a little experiment made by the writer are as follows. A pack of cards was thoroughly shuffled and the cards turns up and their value recorded in order (Ace = 1, Jack = 11, Queen = 12, King = 13). The mean deviation is forced in this case to be 42/13 or 3.231. The observed mean variation was 4.314, making $G = 1.335$, while the expected value of G as given by Woolard's method for this case of rectangular distribution is 1.333.

and (9) give the first, second and third integrals of these values. These figures are plotted on Plate III. The period of the cycle in the third integral curve is 21.5 and the amplitude is about 200. The original cycle which would give this as a third integral would be $4.30+5.00 \cos \frac{k(x-7.5)}{21.5}$. This curve is plotted on the upper figure of Plate III as a solid line. The cyclic amounts added to the random data are plotted (above or below the line of the mean, 4.30) as crosses and connected by a dotted line. The cycle that emerges from the process is not quite the one that went in—it has been combined with the pseudo-cycle which arises from the fortuitous variations—but it is still a fair approximation of the original cycle. We can then assume that cycles derived by this process from statistical data that contain real cycles will be approximations of true cycles.

Goutereau's ratio gives a means of determining whether real cycles are present. If there are none it is not necessary to search for them. If there are cycles present two courses are open to us. We may first construct a periodogram and find whether there are short cycles present. If not, we can assume that a long-period cycle determined by the method of this paper will be real. If there are short period cycles present, we can eliminate them and test the residue by Goutereau's ratio. If it still contains a cycle, we can assume that there is a real long-period cycle. The other procedure would be to first find the long-period cycle by the method of this paper. If the amplitude of the cycle is large, it is quite certainly a real cycle. If it is very small, it may perhaps be negligible, even if real, and is probably unreal. In doubtful cases the cycle deduced may be subtracted from the given data and the residue tested again by Goutereau's ratio. If G is markedly larger than it was in the original data, we may assume that the cycle is real.

Criteria as to the reality of a given cycle have been proposed by C. F. Marvin[1], H. W. Clough[2], Dinsmore Alter[3], and Sir Gilbert Walker[4], but they are adapted only to cycles obtained by means of the

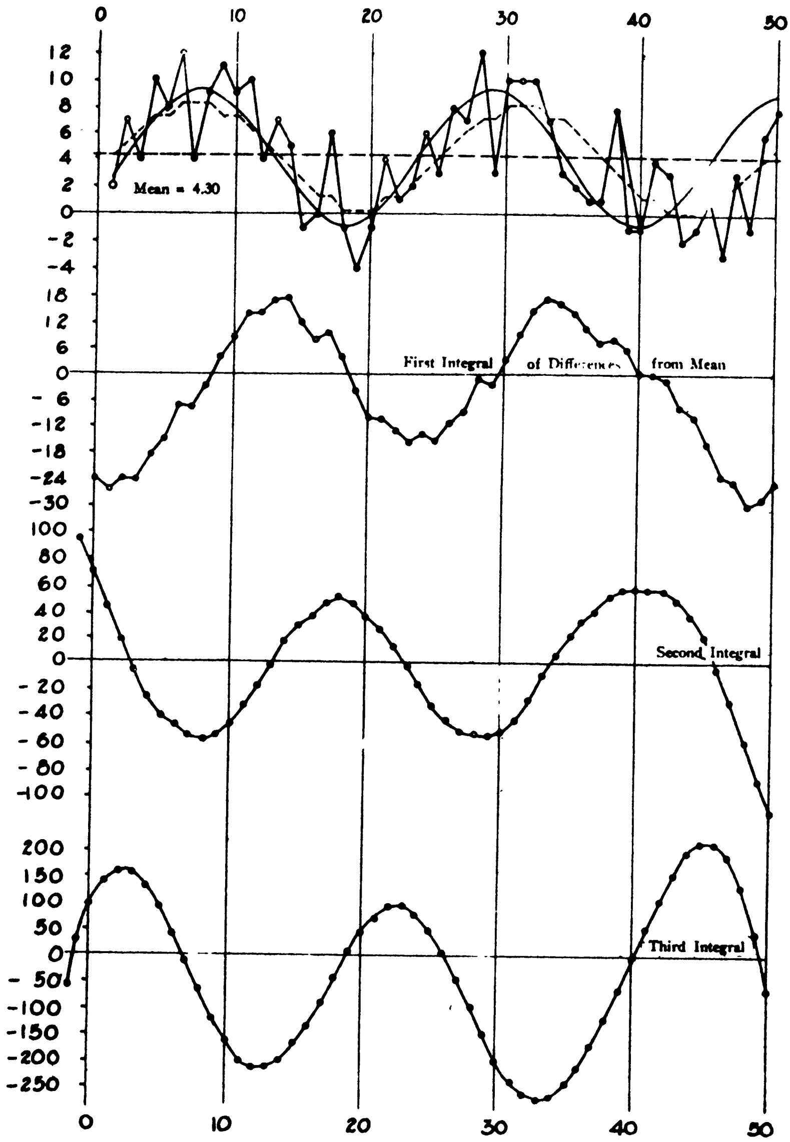
1. Theory and Use of the Periodocite, Monthly Weather Review, Vol. 49 (1921), pp. 115-124.

2. A Statistical Comparison of Meteorological Data with Data of Random Occurrence, Monthly Weather Review, Vol. 49 (1921), pp. 124-132.

3. The Criteria of Reality in the Periodogram, Monthly Weather Review, Vol. 54 (1926), pp. 57-58.

4. On Periodicity. Quart. Jour. Royal Met'l Soc., 51, No. 216, pp. 337-346.

PLATE III



Random Data of Plate II Combined with a Cycle (the Dashed Curve at the Top of This Plate).

periodogram and not very well even to those[1].

The question still remains as to how much less than the expected value for random succession G may be before we are to believe that a cycle is present. The writer would hazard 10 per cent as a rough guess. It is to be hoped that some master of mathematical statistics will give us before long a quantitative statement of, say, the relationship between the ratio of the observed mean variation and the expected mean variation for the same numbers arranged in random succession, and the probability of a cycle of given amplitude being present.

It should be added that the germ idea of this paper is a product of the fertile mind of the writer's colleague, Professor P. W. Ott.

Conclusions

(1) The method of successive integration of discrepancies will reveal the approximate period of long-period cycles if they are present.

(2) Even if no long-period cycle is present, the method will give a fictitious cycle, but there are tests by which the reality or falsity of the cycle can be investigated.

1. For example, Marvin's criterion depends only upon the standard deviations of the sums of the various columns of the tabulation as compared to the standard deviation of all the data, without reference to the order of the columns. Take, for example, the tabulation given on page 353 of Whitaker-Robinson's "Calculus of Observations." It is very evident that there is a real cycle. But suppose that another problem had yielded exactly the same columns of data but in a random order, say the fifth column, then the twentieth, then the third, etc. Are we to suppose that a cycle of period 24 days is equally probable in this case? This objection seems to the writer to make the method of Whitaker and Robinson much inferior to that of Schuster.