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Successive-Minima-Type Inequalities

U. Betke, M. Henk, and J. M. Wills

Mathematisches Institut, Universität Siegen, Hölderlinstrasse 3,
W-5900 Siegen, Federal Republic of Germany

Abstract. We show analogues of Minkowski's theorem on successive minima, where the volume is replaced by the lattice point enumerator. We further give analogous results to some recent theorems by Kannan and Lovász on covering minima.

1. Introduction

Throughout this paper E^d denotes the d-dimensional euclidean space and the set of all convex bodies—compact convex sets—in E^d is denoted by $\mathscr{K}^d$. Further, $\mathscr{K}_0^d$ denotes the 0-symmetric convex bodies, i.e., $K \in \mathscr{K}^d$ with $K = -K$, and a convex body $K \in \mathscr{K}^d$ is called strictly convex if the segment $\overline{xy}$ intersects the interior of K for all $x, y \in K$, $x \neq y$. The set of lattices $\mathbb{L} \subset E^d$ with lattice determinant $\det(\mathbb{L}) > 0$ is denoted by $\mathscr{L}^d$, and the lattice of all points with integral coordinates in E^d is denoted by $\mathbb{Z}^d$. The kth coordinate of a point $x \in E^d$ is denoted by x_k, and $\lfloor \alpha \rfloor$ ($\lceil \alpha \rceil$) denotes, for $\alpha \in \mathbb{R}$, the largest (smallest) integer $\leq \alpha$ ($\geq \alpha$).

The ith successive minimum $\lambda_i(K, \mathbb{L})$, $1 \leq i \leq d$, for $K \in \mathscr{K}_0^d$, $\dim(K) = d$, with respect to a lattice $\mathbb{L} \in \mathscr{L}^d$, is defined by

$$\lambda_i(K, \mathbb{L}) = \min\{\lambda \in \mathbb{R} \mid \lambda > 0, \dim(\lambda K \cap \mathbb{L}) \geq i\}.$$

Between the volume V and the successive minima Minkowski established, for $K \in \mathscr{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathscr{L}^d$, the following relations (see p. 28 of [EGH], p. 123 of [GL], and [M]):

$$(\lambda_1(K, \mathbb{L}))^d V(K) \leq 2^d \det(\mathbb{L}), \tag{1.1}$$

$$\lambda_1(K, \mathbb{L}) \times \cdots \times \lambda_d(K, \mathbb{L}) V(K) \leq 2^d \det(\mathbb{L}), \tag{1.2}$$

$$\lambda_1(K, \mathbb{L}) \times \cdots \times \lambda_d(K, \mathbb{L}) V(K) \geq \frac{2^d}{d!} \det(\mathbb{L}). \tag{1.3}$$

All these inequalities are tight. The theorem on successive minima of Minkowski (1.2) is a deep result in geometry of numbers with many applications and is an improvement of (1.1) since $\lambda_1(K, \mathbb{L}) \leq \cdots \leq \lambda_d(K, \mathbb{L})$.

There are several analogues of these results, e.g., by Mahler, Weyl, and Hlawka (see [EGH], [GL], and [H]). In the main part of our paper we give some analogues of (1.1), (1.2), and (1.3) where V is replaced by the lattice point enumerator

$$G(K, \mathbb{L}) = \operatorname{card}(K \cap \mathbb{L}).$$

The results yield in particular a generalization of the following inequalities by Minkowski [M, p. 79] which are closely related to (1.1). For $K \in \mathscr{K}_0^d$, $\mathbb{L} \in \mathscr{L}^d$, $\dim(K) = d$, with $\lambda_1(K, \mathbb{L}) = 1$, it holds that

$$G(K, \mathbb{L}) \leq 3^d, \tag{1.4}$$

if, in addition, K is strictly convex, then

$$G(K, \mathbb{L}) \leq 2^{d+1} - 1. \tag{1.5}$$

The covering minima introduced by Kannan and Lovász [KL] form another sequence of numbers associated with a convex body and a lattice. For $K \in \mathscr{K}^d$, $\dim(K) = d$, and $\mathbb{L} \in \mathscr{L}^d$, the ith covering minimum $\mu_i(K, \mathbb{L})$, $1 \leq i \leq d$, is defined by

$$\mu_i(K, \mathbb{L}) = \min\{t \in \mathbb{R} \,|\, tK + \mathbb{L} \text{ meets every } (d-i)\text{-dimensional affine subspace}\}.$$

For example, the last covering minimum $\mu_d(K, \mathbb{L})$ is the classical inhomogeneous minimum (see p. 98 of [GL]) and $(\mu_1(K, \mathbb{L}))^{-1}$ is called the $\mathbb{L}$-width of K.

Kannan and Lovász [KL] showed several analogies and relations between the λ_i and the μ_i. In particular they proved that there are constants α_d, $\beta_d > 0$ only depending on d, such that, for $K \in \mathscr{K}^d$ and $\mathbb{L} \in \mathscr{L}^d$,

$$G(K, \mathbb{L}) \geq \left(\left\lfloor \frac{\alpha_d}{\mu_1(K, \mathbb{L})} \right\rfloor\right)^d - 1, \tag{1.6}$$

and, if $K \in \mathscr{K}_0^d$,

$$G(K, \mathbb{L}) \geq \left\lfloor \left(\frac{\beta_d}{\mu_1(K, \mathbb{L})} - d\right)^d \right\rfloor. \tag{1.7}$$

Here we obtain an analogous result where the lattice point enumerator is replaced by the volume.

2. Lattice Points and Successive Minima

In analogy to (1.1) we have

Theorem 2.1. *Let* $K \in \mathcal{K}_0^d$, $\dim(K) = d$, *and* $\mathbb{L} \in \mathcal{L}^d$. *Then*

$$G(K, \mathbb{L}) \leq \left(\left\lfloor \frac{2}{\lambda_1(K, \mathbb{L})} + 1 \right\rfloor\right)^d, \tag{2.1}$$

if K, *in addition, is strictly convex, then*

$$G(K, \mathbb{L}) \leq 2\left(\left\lceil \frac{2}{\lambda_1(K, \mathbb{L})} \right\rceil\right)^d - 1. \tag{2.2}$$

None of these inequalities can be improved.

Remark. Obviously inequality (2.2) is an improvement of (2.1) only if $2/\lambda_1(K, \mathbb{L})$ is an integer.

Proof. It suffices to prove the theorem for the standard lattice $\mathbb{Z}^d$, since $G(K, \mathbb{Z}^d) = G(AK, A\mathbb{Z}^d)$ and $\lambda_i(K, \mathbb{Z}^d) = \lambda_i(AK, A\mathbb{Z}^d)$ for every linear map A with $\det(A) \neq 0$.

Let $p = \lfloor 2/\lambda_1(K, \mathbb{Z}^d) + 1 \rfloor$. First suppose that there are two lattice points $g = (g_1, \ldots, g_d)^T$, $h = (h_1, \ldots, h_d)^T$, $g \neq h$, in K with

$$g_i \equiv h_i \mod p, \qquad i = 1, \ldots, d. \tag{2.3}$$

By the convexity of K and from $p > 2/\lambda_1(K, \mathbb{Z}^d)$ it follows that the lattice point

$$\left(\frac{g_1 - h_1}{p}, \ldots, \frac{g_d - h_d}{p}\right)^T = \frac{1}{2}\left(\frac{2}{p} g\right) + \frac{1}{2}\left(-\frac{2}{p} h\right) \tag{2.4}$$

belongs to $(\mathbb{Z}^d \backslash \{0\}) \cap \operatorname{int}(\lambda_1(K, \mathbb{Z}^d)K)$.

This is a contradiction to the definition of $\lambda_1(K, \mathbb{Z}^d)$ and so there exist no lattice points $g, h \in K$, $g \neq h$, satisfying (2.3). Hence each lattice point $g \in K$ corresponds uniquely to a representation $(\bar{g}_1, \ldots, \bar{g}_d)$ where $\bar{g}_i$ denotes the residue class with respect to p of the ith coordinate of g. There are at most p^d such representations, so we get (2.1). For the cube $C_q^d = \{x \in E^d \,|\, |x_i| \leq q,\ 1 \leq i \leq d\}$, $q \in \mathbb{N}$, it follows that $G(C_q^d, \mathbb{Z}^d) = (2q + 1)^d = (\lfloor 2/\lambda_1(C_q^d, \mathbb{Z}^d) + 1 \rfloor)^d$ and this shows that (2.1) cannot be improved.

For the proof of (2.2) let $p = \lceil 2/\lambda_1(K, \mathbb{Z}^d) \rceil$ and $g, h \in K \cap \mathbb{Z}^d$, $g \neq h$, such that (2.3) holds. From $2/p \leq \lambda_1(K, \mathbb{Z}^d)$ it follows that the lattice point (2.4) lies in the boundary of $\lambda_1(K, \mathbb{Z}^d)K$. By the strict convexity of K this implies $g = -h$. So (as above) each pair $g, -g$ with $g \in K \cap (\mathbb{Z}^d \backslash \{0\})$ corresponds uniquely to a residue class vector $(\bar{g}_1, \ldots, \bar{g}_d)^T$ which shows (2.2). To show that (2.2) is tight, we construct a standard example. Let $C^d = \{x \in E^d | 0 \le x_i \le 1,\ 1 \le i \le d\}$ and let P be the 0-symmetric polytope

$$P = \operatorname{conv}\{C^d,\ -C^d\}.$$

We have $G(P, \mathbb{Z}^d) = 2^{d+1} - 1$ and $\lambda_1(P, \mathbb{Z}^d) = 1$. With elementary considerations, the existence of a strictly 0-symmetric convex body K (in fact, of infinitely many) follows with $P \subset K$, $G(K) = G(P)$, and $\lambda_i(K, \mathbb{Z}^d) = 1$, $i = 1, \dots, d$. This shows that (2.2) cannot be improved. □

Let us remark that, for $\lambda_1(K, \mathbb{Z}^d) = 1$, inequalities (2.1) and (2.2) become Minkowski's inequalities (1.4) and (1.5). In the case $d = 2$ we can improve (2.1) and (2.2) in the following way:

Theorem 2.2. *Let* $K \in \mathscr{K}_0^2$, $\dim(K) = 2$, *and* $\mathbb{L} \in \mathscr{L}^2$. *Then*

$$G(K, \mathbb{L}) \le \left\lfloor \frac{2}{\lambda_1(K, \mathbb{L})} + 1 \right\rfloor \cdot \left\lfloor \frac{2}{\lambda_2(K, \mathbb{L})} + 1 \right\rfloor, \tag{2.5}$$

if K, *in addition, is strictly convex, then*

$$G(K, \mathbb{L}) \le 2\left\lceil \frac{2}{\lambda_1(K, \mathbb{L})} \right\rceil \cdot \left\lceil \frac{2}{\lambda_2(K, \mathbb{L})} \right\rceil - 1. \tag{2.6}$$

None of these inequalities can be improved.

Remark. Again, in general, inequality (2.6) is not an improvement of (2.5).

Proof. It obviously suffices to prove the theorem for the lattice $\mathbb{Z}^2$. Let z^1, z^2 be linearly independent lattice points with $z^i \in \lambda_i(K, \mathbb{Z}^2)K$, $i = 1, 2$, and such that the segment $\overline{z^1 z^2}$ is free of other lattice points. Then the triangle $\operatorname{conv}\{0, z^1, z^2\}$ contains no other lattice points except 0, z^1, z^2, and so z^1, z^2 are a basis of $\mathbb{Z}^2$ (see p. 20 of [GL]). Hence we may assume (see p. 22 of [GL])

$$z^1 = (1, 0)^T \quad \text{and} \quad z^2 = (0, 1)^T.$$

Now we have, for each $x = (x_1, x_2)^T \in K$,

$$\lambda_2(K, \mathbb{Z}^2)|x_1| \le 1 \quad \text{or} \quad \lambda_2(K, \mathbb{Z}^2)|x_2| \le 1; \tag{2.7}$$

otherwise the lattice point $(x_1/|x_1|,\ x_2/|x_2|)^T$ would belong to the interior of $\operatorname{conv}(\{\pm z^1, \pm z^2, \lambda_2(K, \mathbb{Z}^2)x\}) \subset \lambda_2(K, \mathbb{Z}^2)K$ which contradicts the definition of the second successive minimum.

Now let $p_i = \lfloor 2/\lambda_i(K, \mathbb{Z}^2) + 1 \rfloor$, $i = 1, 2$, and let $f\colon E^2 \to E^2$ be the linear map with

$$f((x_1, x_2)^T) = \left(\frac{2}{p_1} x_1, \frac{2}{p_2} x_2\right)^T.$$

With $2/p_i < \lambda_i(K, \mathbb{Z}^2)$ we get from (2.7)

$$f(K) \cap \mathbb{Z}^2 = \{0\}. \tag{2.8}$$

Let $g = (g_1, g_2)^T$, $h = (h_1, h_2)^T$, $g \neq h$, be two lattice points of K with

$$g_i \equiv h_i \mod p_i, \qquad i = 1, 2. \tag{2.9}$$

By the convexity of $f(K)$ it follows that the lattice point

$$\left(\frac{g_1 - h_1}{p_1}, \frac{g_2 - h_2}{p_2}\right)^T = \tfrac{1}{2}f(g) + \tfrac{1}{2}f(-h) \tag{2.10}$$

belongs to $f(K) \cap (\mathbb{Z}^2\backslash\{0\})$ which contradicts (2.8).

Hence there are no two lattice points of K with property (2.9) and so each lattice point $g \in K$ corresponds uniquely to a representation $(\bar{g}_1, \bar{g}_2)$ where $\bar{g}_i$ denotes the residue class with respect to p_i of the ith coordinate of g. There are at most $p_1 p_2$ such representations, so we get (2.5).

For the proof of (2.6) let $p_i = \lceil 2/\lambda_i(K, \mathbb{Z}^2)\rceil$, $i = 1, 2$, and let g, h, and $f(K)$ be as above. From $2/p_i \leq \lambda_i(K, \mathbb{Z}^2)$ and (2.7) it follows that $\operatorname{int}(f(K)) \cap \mathbb{Z}^2 = \{0\}$. Hence the lattice point (2.10) belongs to the boundary of K. With the strict convexity of $f(K)$ this implies $g = -h$ and as in the proof of (2.2) we get (2.6).

The examples in the proof of Theorem 2.1 show that both inequalities are tight. □

On account of Minkowski's theorem on successive minima (1.2) we conjecture that Theorem 2.2. can be generalized to

Conjecture 2.1. *Let $K \in \mathscr{K}_0^d$, $\dim(K) = d$, and $\mathbb{L} \in \mathscr{L}^d$. Then*

$$G(K, \mathbb{L}) \leq \prod_{i=1}^{d} \left\lfloor \frac{2}{\lambda_i(K, \mathbb{L})} + 1 \right\rfloor,$$

if K, in addition, is strictly convex, then

$$G(K, \mathbb{L}) \leq 2\left(\prod_{i=1}^{d} \left\lceil \frac{2}{\lambda_i(K, \mathbb{L})} \right\rceil\right) - 1.$$

The following proposition shows that inequalities of this type exist.

Proposition 2.1. *Let $K \in \mathscr{K}_0^d$, $\dim(K) = d$, and $\mathbb{L} \in \mathscr{L}^d$. Then*

$$G(K, \mathbb{L}) \leq \prod_{i=1}^{d} \left(\frac{2i}{\lambda_i(K, \mathbb{L})} + 1\right).$$

Proof. Let $\lambda_1(K, \mathbb{L}), \ldots, \lambda_j(K, \mathbb{L}) \leq 1$, $\lambda_{j+1}(K, \mathbb{L}), \ldots, \lambda_d(K, \mathbb{L}) > 1$, and let $z^1, \ldots, z^j$ be j linearly independent lattice points with $z^i \in (\lambda_i(K, \mathbb{L})K) \cap \mathbb{L}$, $1 \leq i \leq j$. Further, let L be the linear subspace spanned by $z^1, \ldots, z^j$ and let $\bar{K} = K \cap L$ and $\bar{\mathbb{L}} = \mathbb{L} \cap L$. We clearly have $\lambda_i(K, \mathbb{L}) = \lambda_i(\bar{K}, \bar{\mathbb{L}})$, $1 \leq i \leq j$, and with Blichfeldt's theorem [GL, p. 62] and (1.2) for $\bar{K}$, $\bar{\mathbb{L}}$ it follows that

$$G(K, \mathbb{L}) = G(\bar{K}, \bar{\mathbb{L}}) \leq j! \cdot \frac{V_j(\bar{K})}{\det \bar{\mathbb{L}}} + j \leq \prod_{i=1}^{j} \left(\frac{2i}{\lambda_i(K, \mathbb{L})} + 1 \right),$$

with equality only for $j = 1$. □

On the other hand, Conjecture 2.1. is in a sense stronger than Minkowski's second theorem because it is easy to derive the latter from the former:

Proposition 2.2. *If an inequality*

$$G(K, \mathbb{L}) \leq \prod_{i=1}^{d} \left(\frac{2}{\lambda_i(K, \mathbb{L})} + c_i \right), \qquad c_i \in \mathbb{R},$$

holds for all $\mathbb{K} \in \mathscr{K}_0^d$ *and all* $\mathbb{L} \in \mathscr{L}^d$, *then*

$$\frac{V(K)}{\det(\mathbb{L})} \leq \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \mathbb{L})}.$$

Proof. Let $K \in \mathscr{K}_0^d$. Then we have for $\mu \in \mathbb{R}$, $\mu \neq 0$, $\lambda_i(K, \mu\mathbb{L}) = \mu\lambda_i(K, \mathbb{L})$. Further, we have, by elementary properties of the Riemann integral,

$$\frac{V(K)}{\det(\mathbb{L})} = \lim_{\mu \to 0} \mu^d G(K, \mu\mathbb{L}) \leq \lim_{\mu \to 0} \prod_{i=1}^{d} \mu \left(\frac{2}{\lambda_i(K, \mu\mathbb{L})} + c_i \right) = \prod_{i=1}^{d} \frac{2}{\lambda_i(K, \mathbb{L})}.$$ □

The lower bound (1.3) is much easier to prove than Minkowski's theorem (1.2). The same seems to hold for the case of lattice points as we have as a satisfactory general lower bound:

Theorem 2.3. *Let* $K \in \mathscr{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathscr{L}^d$, *and* $\lambda_1(K, \mathbb{L}) \leq 1$. *Then*

$$G(K, \mathbb{L}) \geq \frac{2^d}{d!} \left(1 - \frac{\lambda_1(K, \mathbb{L})}{2} \right)^d \prod_{i=1}^{d} \frac{1}{\lambda_i(K, \mathbb{L})}. \tag{2.11}$$

In general the constants cannot be improved.

Proof. Again it suffices to prove the theorem for the lattice $\mathbb{Z}^d$. For convenience we write $v_i = 1/\lambda_i(K, \mathbb{Z}^d)$, $1 \leq i \leq d$. Let $z^1, \ldots, z^d$ be d linearly independent points in $\mathbb{Z}^d$ with $v_i z^i \in K$, $1 \leq i \leq d$, and let $Q \subset K$ be the cross-polytope with vertices

$\pm v_i z^i$, $1 \le i \le d$. Denoting by e^i the ith coordinate unit vector and by P the cross-polytope with vertices $\pm v_i e^i$, $1 \le i \le d$, we have $G(K, \mathbb{Z}^d) \ge G(Q, \mathbb{Z}^d) \ge G(P, \mathbb{Z}^d)$ and $V(P) = 2^d/d!\,(v_1 \times \cdots \times v_d)$. Hence it is only necessary to prove

$$G(P, \mathbb{Z}^d) \ge \left(1 - \frac{1}{2v_1}\right)^d V(P). \tag{2.12}$$

Let $\rho = 1 - 1/2v_1$ and let E_j denote the plane spanned by $e^1, \ldots, e^j$, $1 \le j \le d$. The total orthogonal complement of the plane E_j is denoted by $E_j^\perp$. We show by induction that, for each $z^j \in \rho P \cap \mathbb{Z}^d \cap E_j^\perp$,

$$G(P \cap (z^j + E_j), \mathbb{Z}^d) \ge V^j(\rho P \cap (z^j + E_j)), \tag{2.13}$$

where V^j denotes the j-dimensional volume. For $j = 1$ we have

$$V^1(\rho P \cap (z^1 + E_1)) = V^1(P \cap (z^1 + E_1)) - 1.$$

As for any segment S we have $G(S, \mathbb{Z}^d) \ge V^1(S) - 1$ the assertion is proved. Now let $z^{j+1} \in \rho P \cap \mathbb{Z}^d \cap E_{j+1}^\perp$ and let $\eta \in \mathbb{R}$ be the maximal number such that $\eta e^{j+1} + z^{j+1} \in \rho P$. Then we have

$$\begin{aligned} G(P \cap (z^{j+1} + E_{j+1}), \mathbb{Z}^d) &\ge \sum_{i=-\lfloor\eta\rfloor}^{\lfloor\eta\rfloor} G(P \cap (ie^{j+1} + z^{j+1} + E_j), \mathbb{Z}^d) \\ &\ge \sum_{i=-\lfloor\eta\rfloor}^{\lfloor\eta\rfloor} V^j(\rho P \cap (ie^{j+1} + z^{j+1} + E_j)) \\ &= V^j(\rho P \cap (z^{j+1} + E_j)) \sum_{i=-\lfloor\eta\rfloor}^{\lfloor\eta\rfloor} \left(1 - \frac{|i|}{\eta}\right)^j. \end{aligned}$$

It follows that

$$G(P \cap (z^{j+1} + E_{j+1}), \mathbb{Z}^d) \ge 2V^j(\rho P \cap (z^{j+1} + E_j))\left(\sum_{i=0}^{\lfloor\eta\rfloor} \frac{f(i) + f(i+1)}{2}\right), \tag{2.14}$$

where f is the function given by

$$f(x) = \begin{cases} (1 - x/\eta)^j & \text{for } x \in (-\infty, \eta], \\ 0 & \text{for } x \in [\eta, \infty). \end{cases}$$

Now f is convex on $[0, \infty)$ and from this we obtain, by elementary properties of the integral,

$$\sum_{i=0}^{\lfloor\eta\rfloor} \frac{f(i) + f(i+1)}{2} \ge \int_0^\eta f(x)\,dx = \frac{\eta}{j+1}.$$

Along with (2.14) it follows that

$$G(P \cap (z^{j+1} + E_{j+1}), \mathbb{Z}^d) \geq \frac{2\eta}{j+1} V^j(\rho P \cap (z^{j+1} + E_j))$$
$$= V^{j+1}(\rho P \cap (z^{j+1} + E_{j+1})),$$

which proves (2.13). In particular, for $j = d$, (2.13) is equivalent to (2.12).

It remains to show that (2.11) cannot be improved in general. To this end we consider the regular crosspolytope $P^d = \operatorname{conv}(\{\pm e^1, \ldots, \pm e^d\})$. From Ehrhart's theorems (see p. 135 of [GL]) it follows that, for $k \in \mathbb{N}$,

$$G(kP^d, \mathbb{Z}^d) = \sum_{i=0}^{d} k^i G_i(P^d), \qquad G(kP^d, \mathbb{Z}^d)^0 = \sum_{i=0}^{d} k^i (-1)^{d-i} G_i(P^d), \tag{2.15}$$

where $G(P^d, \mathbb{Z}^d)^0$ denotes the number of lattice points in the interior of P^d and $G_i(P^d)$ are constants. Especially we have $G_d(P^d) = V(P^d) = 2^d/d!$. Next we observe

$$G(kP^d, \mathbb{Z}^d) = G(kP^d, \mathbb{Z}^d)^0 + G(kP^{d-1}, \mathbb{Z}^d)^0 + G(kP^{d-1}, \mathbb{Z}^d)$$

and thus we obtain, from (2.15), $G_{d-1}(P^d) = 2^{d-1}/(d-1)!$. On the other hand, (2.11) yields

$$G(kP^d, \mathbb{Z}^d) \geq \left(1 - \frac{1}{2k}\right)^d V(kP^d) = \frac{2^d}{d!} k^d \sum_{i=0}^{d} \binom{d}{i} \left(\frac{-1}{2k}\right)^i.$$

Comparing the coefficients of k^d and k^{d-1} with the coefficients of $G(kP^d, \mathbb{Z}^d)^0$ in (2.15) we see that we can do no better than in the theorem. □

Minkowski's inequalities (1.2) and (1.3) appear to be much more symmetric than Theorems 2.2. and 2.3. By a slight weakening of Theorem 2.3. we obtain a corollary, which up to the Gauss-brackets is completely symmetric to Theorem 2.2 in the same way as (1.2) is to (1.3):

Corollary 2.1. *Let* $K \in \mathscr{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathscr{L}^d$, *and* $\lambda_d(K, \mathbb{L}) \leq 2$. *Then*

$$G(K, \mathbb{L}) \geq \frac{1}{d!} \prod_{i=1}^{d} \left(\frac{2}{\lambda_i(K, \mathbb{L})} - 1 \right).$$

In general the constants cannot be improved.

Proof. On account of $\lambda_1(K, \mathbb{L}) \leq \cdots \leq \lambda_d(K, \mathbb{L})$ the assertion follows from (2.11). □

In the proof of Theorem 2.3, (2.12) appears to have some interest of its own, as it relates volume, lattice number, and successive minima for cross-polytopes.

Thus there is the natural question for a formula of this kind, which holds for all 0-symmetric convex bodies, and for a corresponding upper bound. Certainly (2.12) is not true for all $K \in \mathscr{K}_0^d$ as the class of open boxes with edges parallel to the coordinate axes shows. However, this class suggests:

Conjecture 2.2. *Let* $K \in \mathscr{K}_0^d$, $\dim(K) = d$, $\mathbb{L} \in \mathscr{L}^d$, *and* $\lambda_d(K, \mathbb{L}) \le 2$. *Then*

$$\frac{V(K)}{\det(\mathbb{L})} \prod_{i=1}^{d} \left(1 - \frac{\lambda_i(K, \mathbb{L})}{2}\right) \le G(K, \mathbb{L}).$$

3. Covering Minima

For the volume of a convex body we have the following lower bound with respect to the covering minima:

Theorem 3.1. *Let* $K \in \mathscr{K}^d$ *and* $\mathbb{L} \in \mathscr{L}^d$. *Then there is a constant* τ_d, *only depending on* d, *with* $0 < \tau_d \le (d!)^{-1}$ *and*

$$(\mu_1(K, \mathbb{L}))^d V(K) \ge \tau_d \cdot \det(\mathbb{L}). \tag{3.1}$$

Proof. Since $V(K) = V(A^{-1}K) \cdot |\det(A)|$ and $\mu_i(K, \mathbb{L}) = \mu_i(A^{-1}K, A^{-1}\mathbb{L})$ for every linear map A with $\det(A) \ne 0$, it suffices to prove the theorem for the lattice $\mathbb{Z}^d$. For a convex body $K \in \mathscr{K}^d$ Kannan and Lovász [KL] proved

$$\mu_1(K, \mathbb{Z}^d) = (\lambda_1((K - K)^*, \mathbb{Z}^d))^{-1},$$

where $(K - K)^*$ denotes the polar body of the difference body $K - K$ of K. So

$$\mu_1(K, \mathbb{Z}^d)^d V(K) = (\lambda_1((K - K)^*, \mathbb{Z}^d))^{-d} V(K). \tag{3.2}$$

From Roger's and Shephard's theorem on the difference body (see p. 32 of [GL]) and Bourgain's and Milman's theorems on the polar body (see p. 31 of [EGH], and [KL]) we have, with a constant c_1,

$$\binom{2d}{d} V(K) \ge V(K - K) \ge \left(\frac{c_1}{d}\right)^d V((K - K)^*)^{-1}. \tag{3.3}$$

From (3.2) and (3.3) we obtain, with (1.1) and

$$\binom{2d}{d}^{-1} \left(\frac{c_1}{d}\right)^d = 2^d \tau_d,$$

$$\mu_1(K, \mathbb{Z}^d)^d V(K) \ge 2^d \tau_d (\lambda_1((K - K)^*, \mathbb{Z}^d))^{-d} V((K - K)^*)^{-1} \ge \tau_d.$$

The regular cross-polytope shows that $\tau_d \le (d!)^{-1}$. □

The constants α_d, β_d in (1.6), (1.7) and τ_d in (3.1) are not best possible. We conjecture

Conjecture 3.1. *Let $K \in \mathcal{K}^d$ and $\mathbb{L} \in \mathcal{L}^d$. Then*

$$(\mu_1(K, \mathbb{L}))^d V(K) \geq \frac{\det(\mathbb{L})}{d!}.$$

From $\mu_1(K, \mathbb{L}) \leq \cdots \leq \mu_d(K, \mathbb{L})$ and (2.6) it follows that

$$\mu_1(K, \mathbb{L}) \times \cdots \times \mu_d(K, \mathbb{L}) V(K) \geq \tau_d \cdot \det(\mathbb{L}),$$

i.e., an analogue of Minkowski's theorem on successive minima (1.2), although not with best constant. As a direct consequence of a result by Nosarzewska, Hadwiger, and Wills we obtain, for the surface area F and the lattice $\mathbb{Z}^d$,

Proposition 3.1. *Let $K \in \mathcal{K}^d$, $\dim(K) = d$, and $1 \leq i \leq d$. Then*

$$\mu_i(K, \mathbb{Z}^d) V(K) < \tfrac{1}{2} F(K). \tag{3.4}$$

None of these inequalities can be improved.

Proof. For lattice-point-free $K \in \mathcal{K}^d$ with respect to the standard lattice $\mathbb{Z}^d$ Nosarzewska ($d = 2$), Wills ($d = 3, 4$), and Hadwiger (general d) (see p. 282 of [GL] and p. 22 of [EGH]) proved $V(K) < \frac{1}{2}F(K)$. From this it follows, for general $K \in \mathcal{K}^d$, that

$$\mu_d(K, \mathbb{Z}^d) V(K) < \tfrac{1}{2} F(K).$$

On account of $\mu_i(K, \mathbb{Z}^d) \leq \mu_d(K, \mathbb{Z}^d)$, $1 \leq i \leq d$, this shows (3.4). Now let $q \in \mathbb{N}$, $q \geq 3$, and

$$Q_q = \left\{ x \in E^d \,\middle|\, |x_i| \leq q,\ 1 \leq i \leq d-1,\ |x_d| \leq \frac{1}{2} - \frac{1}{q} \right\}.$$

Then $\mu_1(Q_q, \mathbb{Z}^d) > 1$, $V(Q_q) < \frac{1}{2}F(Q_q)$, and

$$\lim_{q \to \infty} \mu_1(Q_q, \mathbb{Z}^d) V(Q_q) F(Q_q)^{-1} = \tfrac{1}{2},$$

hence none of the inequalities can be improved. □

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