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Sudarshan Diagonal Coherent State Representation: Developments and Applications

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Abstract: A review is presented of the early developments in quantum optical coherence. Some properties of the coherent states particularly their over completeness which led to the discovery of diagonal coherent state representation by Sudarshan will be discussed. We then consider some of the important applications of this diagonal coherent state representation.

Classical theory of optical coherence was developed by E. Wolf(1) by considering optical field as part of a stochastic process. Coherence functions were essentially correlations between field components at different space time points. The second order coherence function may thus be written as

$$\Gamma(\underline{r}_1,\underline{r}_2,\tau) = \left\langle V^*(\underline{r}_1,t)V(\underline{r}_2,t+\tau)\right\rangle_c , \tag{1a}$$

$$\left\langle(...)\right\rangle_c = \int (...)P(\{V\})d^2\{V\}, \tag{1b}$$

where V(**r**,t) is the analytic signal associated with the electromagnetic field component at the point **r** and at time t. The averages $\left\langle(...)\right\rangle_c$, are the stochastic averages over the given ensemble. Near about the same time L. Mandel(2) studied photo-electron counting statistics and derived the counting formula as a Poisson transform of the integrated light intensity distribution

$$P(n,T) = \int P(W)\frac{W^n}{n!}e^{-W}dW , \tag{2a}$$

$$W = \int_0^T I(t)\,dt . \tag{2b}$$

Most features of the photo-counting distribution were well understood by Mandel's formula. Several effects known at that time such as propagation of coherence, Brown-Twiss experiments on bunching of photons and others had been adequately explained by classical approach to light fluctuations. With the advent of lasers, a need arose for the quantum description of electromagnetic fields associated with arbitrary light beams. For an optical field described by a density operator ρ one then considers the analogous quantum coherence functions:

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$$\Gamma(\underline{r}_1,\underline{r}_2,\tau) = \left\langle E^{(-)}(\underline{r}_1,t)\, E^{(+)}(\underline{r}_2,t+\tau)\right\rangle_q \qquad (3)$$

with $\left\langle(...)\right\rangle_q = Tr[\rho(...)]$ and $E^{(-)}$ and $E^{(+)}$ being the creation and annihilation parts of the field operator E. If we take up the usual quantization of electromagnetic field and, for simplicity, restrict ourselves to one mode case only, we can use the harmonic oscillator number states to describe such fields. These states are the eigenstates of the number operator $a^{+}a$:

$$a^{+}a\left|n\right\rangle = n\left|n\right\rangle. \qquad (4)$$

The number operator being hermitian, the states $\left|n\right\rangle$ are orthogonal

$$\left\langle m \mid n\right\rangle = \delta_{mn} \qquad (5)$$

and form a complete set

$$\sum_n \left|n\right\rangle\left\langle n\right| = 1\,, \qquad (6)$$

so that an arbitrary state $\left|\psi\right\rangle$ can be expressed in the form

$$\left|\psi\right\rangle = \sum \psi_n \left|n\right\rangle \qquad (7)$$

with uniquely determined ψ_n given by

$$\psi_n = \left\langle n\middle|\psi\right\rangle. \qquad (8)$$

Glauber[3] realized the importance of using eigenstates of the annihilation operator

$$a\left|z\right\rangle = z\left|z\right\rangle, \qquad (9)$$

$$\left|z\right\rangle = e^{-\frac{1}{2}|z|^2} \sum \frac{z^n}{\sqrt{n!}}\left|n\right\rangle \qquad (10)$$

as the basis states in place of the number states. These so called coherent states are in fact the minimum uncertainty product states first introduced by Schroedinger[4]. Since a is not a hermitian operator, its eigenvalues are in general complex numbers $z = x + iy = r\, e^{i\theta}$ and the eigenstates are not orthogonal

$$\left\langle z'\middle|z\right\rangle \neq 0 \;\; for\; z \neq z'. \qquad (11)$$

However they form a complete set,

$$\frac{1}{\pi}\int |z\rangle\langle z| d^2z = 1 , \qquad (12)$$

where d^2z = dx dy = r dr dθ denotes integration over the complex z plain. An arbitrary state $|\psi\rangle$ may therefore be expressed linearly in terms of $|z\rangle$:

$$|\psi\rangle = \int \psi(z)|z\rangle d^2z \qquad (13)$$

with ψ (z) = $(1/\pi)\langle z|\psi\rangle$. Further since the states $|z\rangle$ are not linearly independent, the function ψ (z) is not unique. This assertion is readily verifiable from the obvious relation such as

$$\int z|z\rangle d^2z = 0 . \qquad (14)$$

Coherent states may also be used to express operators in the form [cf. Eq.(3) of ref. 3]

$$\rho = \int F(z,z')|z\rangle\langle z'| d^2z\, d^2z' \qquad (15)$$

and the function F(z,z') is doubly "non-unique". It was noted by Sudarshan(5) that these coherent states are not only complete but are in fact over complete. Sudarshan for the first time made use of this over completeness property and realized the possibility of a diagonal coherent state representation

$$\rho = \int \phi(z)|z\rangle\langle z|\ d^2z . \qquad (16)$$

The generalized function φ(z) in this expression is then unique. Thus for example the density operator ρ in the number representation

$$\rho = \sum \rho_{nn'}|n\rangle\langle n'| \qquad (17)$$

will have the diagonal coherent state representation φ(z), with $z = r\, e^{i\theta}$, given by [cf. Eq. (6) of ref. 5],

$$\phi(z) = \sum_{n,n'} \frac{\sqrt{n!\,n'!}\,\rho_{nn'}}{(n+n')!\ 2\pi r} e^{r^2 + i(n'-n)\theta} \left(-\frac{\partial}{\partial r}\right)^{n+n'} \delta(r) . \qquad (18)$$

This diagonal coherent state representation of the density matrix had far reaching applications. The expectation value of any normally ordered operator $a^{+\mu}\ a^{\nu}$ in the statistical state represented by this density operator takes on the form

$$\langle a^{+\mu}\ a^{\nu}\rangle = \int z^{*\mu}\ z^{\nu}\ \varphi(z)\, d^2z . \qquad (19)$$

This is the same as the expectation value of the complex classical function $z^{*\mu}\ z^{\nu}$ in a statistical state described by the probability distribution function φ(z). This property therefore enables the various coherence functions to appear in a similar way as in the Wolf- Mandel formulation. Eq. (16) may also be used to express $\rho_{n,n'}$ in terms of φ(z) and in particular we obtain for $\rho_{n,n}$ the expression

$$\rho_{n,n} = \langle n \mid \rho \mid n \rangle = \int \phi(z) e^{-|z|^2} \frac{|z|^{2n}}{n!} d^2 z \,. \tag{20}$$

Interpreting $|z|^2$ as the intensity W of the field, Eq. (20) reproduces Mandel's Poisson transform relation Eq. (2a).

Sudarshan also noted that the hermiticity of ρ implies that the distribution ϕ(z) is real. However the positivity of ρ does not imply positivity of ϕ and therefore *brings out a clear distinction between classical and quantum ensembles*. It may be emphasized that for all fields, the diagonal coherent state representation is formally as if it was a classical ensemble, but with the understanding that the ensemble probability distributions could be non-positive. With this non-positivity of the diagonal coherent state representation in general, one can deduce specific quantum effects such as anti-bunching, squeezing etc. Glauber's off-diagonal representation of ρ in terms of $|z\rangle\langle z'|$ as in Eq. (15) can in no way lead to this similarity or difference between the classical and quantum descriptions.

It is to be noted that ϕ(z) is unique, though there may be different equivalent forms for ϕ(z). Sudarshan's original expression for ϕ(z) as in Eq.(18) contains derivatives of delta function and as such appears to be singular. In a subsequent publication Mehta and Sudarshan[6] discussed the characteristic function approach to the diagonal coherent state representation, which gives a well defined and rigorous meaning to ϕ(z) as a generalized distribution. Consider the three characteristic functions defined as:

$$X_N(\alpha) = \left\langle e^{\alpha a^+} e^{-\alpha^* a} \right\rangle = \int \phi_N(z) e^{\alpha z^*} e^{-\alpha^* z} \, d^2 z; \tag{21a}$$

$$X_A(\alpha) = \left\langle e^{-\alpha^* a} e^{\alpha a^+} \right\rangle = \int \phi_A(z) e^{\alpha z^*} e^{-\alpha^* z} \, d^2 z; \tag{21b}$$

$$X_W(\alpha) = \left\langle e^{\alpha a^+ - \alpha^* a} \right\rangle = \int \phi_W(z) e^{\alpha z^*} e^{-\alpha^* z} d^2 z. \tag{21c}$$

Hear $\phi_N(z)$, $\phi_A(z)$ and $\phi_W(z)$ are the phase space distribution functions for the normal, anti-normal and Weyl's ordering rules respectively. The distribution $\phi_N(z)$ is essentially the diagonal coherent state representation ϕ(z), whereas $\phi_A(z)$, is given by

$$\phi_A(z) = (1/\pi) \langle z | \rho | z \rangle \,, \tag{22}$$

$$= \frac{1}{\pi} \int \varphi(z') e^{-|z-z'|^2} d^2 z' \,. \tag{23}$$

It may be readily seen that the characteristic functions X_N, X_A and X_W, are related as follows:

$$X_N(\alpha) = e^{\frac{1}{2}|\alpha|^2} X_W(\alpha) = e^{|\alpha|^2} X_A(\alpha) \,. \tag{24}$$

Since $\phi_A(z)$ is well behaved bounded positive real function, its Fourier transform X_A is also well behaved and bounded function. We thus find that ϕ(z) is the Fourier transform of $X_N(\alpha) = e^{|\alpha|^2} X_A(\alpha)$, and consequently it can be regarded as the limit of a sequence of tempered distributions[6]. In this sense it is a well defined generalized function. It is of interest to mention an alternate expression for $\phi(z)$. One readily obtains from Eq. (16) the relation

$$\langle -\alpha|\rho|\alpha\rangle e^{|\alpha|^2} = \int \phi(z)\exp(-|z|^2 - \alpha^* z + \alpha z^*)\, d^2 z, \tag{25}$$

which on taking the inverse Fourier transform gives[7]

$$\phi(z) = \frac{\exp(|z|^2)}{\pi^2}\int \langle -\alpha \mid \rho \mid \alpha\rangle \exp(|\alpha|^2 + \alpha^* z - \alpha z^*) d^2\alpha. \tag{26}$$

This form of $\phi(z)$ is particularly useful to obtain a well behaved (but equivalent) expression for it whenever possible. As an illustration, consider the pure coherent state $|\beta\rangle$ for which the density operator is given by $\rho = |\beta\rangle\langle\beta|$. Making use of Eq. (26) we readily obtain the expression $\phi(z) = \delta(z-\beta)$ for the diagonal coherent state representation for this density operator. It is possible to write a similar expression for the phase space distribution function $\phi_W(z)$ using coherent states[8]:

$$\varphi_W(z) = \frac{2\exp(2|z|^2)}{\pi^2}\int \langle -\alpha|\rho|\alpha\rangle \exp[2(\alpha^* z - \alpha z^*)] d^2\alpha \tag{27}$$

We observed that coherent states form an over complete set of basis states. It is possible that even a subset of coherent states may suffice to form a complete set. We illustrate this property by observing that the number state $|n\rangle$ may be written in a form[9]

$$|n\rangle = \frac{\sqrt{n!e}}{2\pi}\int_0^{2\pi} |e^{i\theta}\rangle e^{-in\theta} d\theta. \tag{28}$$

This completeness property may also be expressed in terms of the resolution of the identity

$$\frac{e}{2\pi}(a^+a)!\int_0^{2\pi} |e^{i\theta}\rangle\langle e^{i\theta}| d\theta = 1. \tag{29}$$

Relations (28) and (29) clearly demonstrates that even a very restricted sub-set of the coherent states, namely, those on the unit circle ($z = e^{i\theta}$) alone form a complete set. This set is in fact just complete and no more over complete. An arbitrary coherent state $|z\rangle$ may uniquely be expressed in terms of this sub-set as

$$|z\rangle = \frac{\sqrt{e}}{2\pi} e^{-\frac{1}{2}|z|^2}\int \frac{|e^{i\theta'}\rangle}{1 - ze^{-i\theta'}} d\theta'. \tag{30}$$

It is believed that the set of coherent states on any closed path in a complex plane would form a complete set.

It is felt necessary to comment on the Glauber's work on the so called 'P representation'.

In a later publication[10] Glauber discusses in much detail the representation of the density operator in terms of coherent states. In section 6 of this paper he develops the representation given by [cf. Eq. (15) above]

$$\rho = \int \aleph(\alpha,\alpha')\,|\alpha\rangle\langle\alpha'|\,d^2\alpha\, d^2\alpha' \,. \tag{31}$$

In section 7, he points out that not all fields require for their description density operators of quite so general a form. He mentions of a '*broad class of radiation fields*' for which it becomes possible to '*reduce*' the density operator to a considerably simpler form which he calls the P representation [Eq. (7.6) of ref. 10]:

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,d^2\alpha. \tag{32}$$

If a density operator is specified by means of the P representation, its matrix elements connecting the n-quantum states are given by [Eq. (7.12) of ref. 10]

$$< n\,|\,\rho\,|\,m > = (n!m!)^{-1/2}\int P(\alpha)\alpha^n(\alpha^*)^m e^{-|\alpha|^2}\,d^2\alpha. \tag{33}$$

However he never gives any method of inverting Eq. (32) even for such special fields which permit his P representation, nor does he mention any way of '*reducing*' his (α,α') representation given by Eq. (31) to that of the 'P representation'. It is thus clear from his discussion that he acknowledges the usefulness of his P representation, but fails to derive an explicit expression for it for any given density matrix. Yet in a discussion of Eq. (32), Glauber states: "In general, however, it is not possible to interpret the function P(α) as a probability distribution in any precise way since the projection operators $|\alpha\rangle\langle\alpha|$ with which it is associated are not orthogonal to one another for different values of α." He thus contradicts himself about the usefulness of P(α) even for cases when it is a well behaved positive definite function.

I wish to emphasize that in ref. 10, Glauber discusses the inversion of his expression [Eq. (5.6) of ref. 10], viz.,

$$T = \frac{1}{\pi^2}\int|\alpha\rangle\,\Im(\alpha^*,\beta)\langle\beta\,|\exp\{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\}\,d^2\alpha\, d^2\beta \tag{34}$$

to [Eq. (5.7) of ref. 10]

$$\Im(\alpha^*,\beta) = \langle\alpha|T|\beta\rangle\exp\{\frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2\}, \tag{35}$$

but he never discusses the inversion of his P representation [Eq. (7.6) of ref. 10]

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\,d^2\alpha\;, \tag{36}$$

even for the simple case of a pure coherent state. He never derives this expression. There is hardly any justification to say that in special cases his representation (31) '*reduces*' to the P representation (36), nor is there any reason whatsoever to dignify it as a '*representation*' at all. The only source from

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which Glauber gets his so called '*P-representation'* is Sudarshan's published work which he copies, changing ϕ to P and z to α. Despite this, his confusion about the representation is clearly evident between sections 6 and 7 of his Phys. Rev. paper (ref. 10). It is thus incontrovertible that the credit for formulating and discovering the diagonal coherent state representation must go to Sudarshan. It is truly ironic that although these facts are readily accessible, an expression which ought to be called '*Sudarshan's diagonal coherent state representation*' is dubbed '*Glauber's P representation*'. Nor is it correct even as compromise move , to call it the 'Glauber-Sudarshan representation'

Sudarshan's application of the over completeness of coherent states led to a unique expression for $\phi(z)$, the diagonal coherent state representation of the density operator. As such his work is the first and the only formulation of quantum coherence theory.

I will next give a couple of specific examples where the diagonal coherent state representation takes on negative values for some complex z and such a situation corresponds to a true quantum feature of the radiation field. From the Mandel's Poisson transform relation [Eqs. (2a) and (20)] one readily obtains the relation between the fluctuations of number of photons and that of the integrated intensity:

$$\left\langle \Delta n^2 \right\rangle = \left\langle n \right\rangle + \left\langle \Delta W^2 \right\rangle; \qquad (37a)$$

$$= \left\langle n \right\rangle + \left\langle \Delta(\left|z\right|^2)^2 \right\rangle. \qquad (37b)$$

It is evident that for positive $\phi(z)$ the second term on the right hand side of Eq.(37) is positive and we expect bunching of photons [$\left\langle \Delta n^2 \right\rangle \geq \left\langle n \right\rangle$] for such fields. This for example is the case for blackbody radiation for which $\left\langle \Delta n^2 \right\rangle = \left\langle n \right\rangle + \left\langle n \right\rangle^2$. On the other hand if $\phi(z)$ was negative for some values of z, we *may* expect the second term on the right hand side of Eq. (37) to be negative. In such cases there will be anti-bunching of photons [$\left\langle \Delta n^2 \right\rangle \leq \left\langle n \right\rangle$] and that is a true quantum feature. Such is a case for the number state where there are fixed number of photons and hence $<\Delta n^2>$ is zero. The second term on the right hand side of (37) is $-<n>$. Of course a negative $\phi(z)$ for some values of z will not *always* lead to anti-bunching of photons, but in such a case one should expect other quantum features associated with higher order correlations. A squeezed state (such as the squeezed vacuum) is another example of negative $\phi(z)$ and hence the case of a state having no classical analogue.

Finally I will consider briefly the relationship between operator ordering and coherent state representations. Very often one is interested in writing a given operator in a well-ordered form such as a normal, anti-normal or Weyl (completely symmetric) ordered form. Such ordering plays an important role in phase space representation of quantum mechanics, quantum c-number correspondence etc. In this context we note the following relations which may readily be verified. If $:f_N(a,a^+):$ is the normally ordered form of $F(a,a^+)$ then

$$\left\langle z \mid F(a,a^+) \mid z) \right\rangle = f_N(z,z^*) \qquad (38)$$

Similarly if "f_A (a,a^+)" is the anti-normal ordered form of $F(a,a^+)$, then

$$F(a,a^+) = \frac{1}{\pi} \int f_A(z,z^*) \left|z\right\rangle \left\langle z\right| d^2 z\,. \qquad (39)$$

These relations may effectively be used for operator ordering. As an illustration, let us assume that we want to write $\exp(-\lambda a^+ a)$ in a normally ordered form. We note that

$$\langle z|\exp(-\lambda a^+ a)|z\rangle = \sum \langle z|\exp(-\lambda a^+ a)|n\rangle\langle n|z\rangle$$
$$= \sum_n e^{-\lambda n}\left|\langle z|n\rangle\right|^2 = \exp(-|z|^2 + |z|^2 e^{-\lambda}). \quad (40)$$

Hence on making use of Eq. (38) we obtain the required normal ordered form:

$$\exp(-\lambda\, a^+ a) =: \exp[-a^+ a(1-e^{-\lambda})]:$$
$$= \sum \frac{[-(1-e^{-\lambda})]^n}{n!}(a^+)^n a^n. \quad (41)$$

Alternatively if we know the anti-normal ordered form of exp(-λa⁺a), viz.,

$$\exp(-\lambda a^+ a) = "\exp[-a^+ a(e^{-\lambda}-1)]"$$
$$= \sum \frac{[-(e^{-\lambda}-1)]^n}{n!} a^n (a^+)^n, \quad (42)$$

we may immediately write down, using Eq. (39), the diagonal coherent state representation for this operator:

$$\exp(-\lambda a^+ a) = \frac{1}{\pi}\int \exp[-|z|^2 (e^{-\lambda}-1)]\,|z\rangle\langle z|\,d^2 z. \quad (43)$$

One may use these techniques to obtain desired ordered forms of other operators in general.

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