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SUDDEN EXTINCTION OF A CRITICAL BRANCHING PROCESS IN A RANDOM ENVIRONMENT*

V. A. VATUTIN† AND V. WACHTEL‡

(*Translated by V. A. Vatutin*)

Abstract. Let T be the extinction moment of a critical branching process $Z = (Z_n, n \geqq 0)$ in a random environment specified by independent identically distributed probability generating functions. We study the asymptotic behavior of the probability of extinction of the process Z at moment $n \to \infty$, and show that if the logarithm of the (random) expectation of the offspring number of a particle belongs to the domain of attraction of a non-Gaussian stable law, then the extinction occurs at time moment T owing to a very unfavorable environment forcing the process, having at time moment $T-1$ an exponentially large population, to die out instantly. We also give an interpretation of the obtained results in terms of random walks in a random environment.

Key words. branching processes in random environment, random walk in random environment, local time, limit theorems, overshoot, undershoot, conditional limit theorems

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1. Introduction and main results. We consider a branching process in a random environment specified by a sequence of independent identically distributed random offspring generating functions

$$f_n(s) := \sum_{k=0}^{\infty} f_{nk} s^k, \qquad n \geqq 0. \tag{1}$$

Denoting by Z_n the number of particles in the process at time n, we define its evolution by the relations

$$Z_0 := 1,$$
$$\mathbf{E}\,[s^{Z_{n+1}} | f_0, f_1, \dots, f_n; Z_0, Z_1, \dots, Z_n] := (f_n(s))^{Z_n}, \qquad n \geqq 0.$$

Put $X_k := \log f'_{k-1}(1)$, $k \geqq 1$, and denote $S_0 := 0$, $S_n := X_1 + X_2 + \cdots + X_n$. Following [1] we call the process $Z := \{Z_n,\ n \geqq 0\}$ *critical* if and only if the random walk $S := \{S_n,\ n \geqq 0\}$ is oscillating, that is,

$$\limsup_{n\to\infty} S_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} S_n = -\infty$$

with probability 1. This means that the stopping time

$$T^- := \min\{k \geqq 1 \colon\ S_k < 0\}$$

is finite with probability 1 and, as a result (see [1]), the extinction moment

$$T := \min\{k \geqq 1 \colon\ Z_k = 0\}$$

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http://www.siam.org/journals/tvp/54-3/98428.html
†Steklov Mathematical Institute RAS, Gubkin St. 8, 19991, Moscow, Russia (vatutin@mi.ras.ru).
‡Mathematical Institute, University of Munich, Theresienstrasse 39, D–80333 Munich, Germany (wachtel@mathematik.uni-muenchen.de).

of the process Z is finite with probability 1. For this reason it is natural to study the asymptotic behavior of the survival probability $\mathbf{P}(T > n)$ as $n \to \infty$. This has been done in [1] as follows: If

$$\lim_{n\to\infty} \mathbf{P}(S_n > 0) =: \rho \in (0,1), \tag{2}$$

then (under some mild additional assumptions)

$$\mathbf{P}(T > n) \sim \theta \mathbf{P}(T^- > n) = \theta \, \frac{l(n)}{n^{1-\rho}}, \tag{3}$$

where $l(n)$ is a slowly varying function and θ is a known positive constant whose explicit expression is given by formula (4.10) in [1].[1]

A local version of (3) was obtained in [7], where it was established that if the offspring generating functions $f_n(s), n = 0, 1, \dots$, are linear fractional with probability 1 and (along with some other conditions) $\mathbf{E}\, X_n = 0$ and $\mathbf{D}X_n \in (0, \infty)$, then

$$\mathbf{P}(T = n) \sim \theta \mathbf{P}(T^- = n) \sim \frac{C}{n^{3/2}}. \tag{4}$$

The aim of the present paper is to complement (4) by the investigation of the asymptotic behavior of the probability $\mathbf{P}(T = n)$ as $n \to \infty$ in the case $\mathbf{D}X_n = \infty$. In addition, we consider the asymptotic behavior of the joint distribution of the random variables T and Z_{T-1} as $T \to \infty$.

Let

$$\begin{aligned}\mathcal{A} := \{0 < \alpha < 1,\ |\beta| < 1\} &\cup \{1 < \alpha < 2,\ |\beta| \leqq 1\} \\ \cup \{\alpha = 1,\ \beta = 0\} &\cup \{\alpha = 2,\ \beta = 0\}\end{aligned}$$

be a subset in $\mathbf{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable X we write $X \in \mathcal{D}(\alpha, \beta)$ if the distribution of X belongs to the domain of attraction of a stable law with the characteristic function

$$G_{\alpha,\beta}(t) := \exp\left\{ -c|t|^{\alpha}\left(1 - i\beta \frac{t}{|t|} \operatorname{tg} \frac{\pi\alpha}{2}\right)\right\}, \qquad c > 0, \tag{5}$$

and, in addition, $\mathbf{E}\, X = 0$ if this moment exists. Hence, there exists a sequence $\{c_n, n \geqq 1\}$ such that $c_n^{-1} S_n$ converges in distribution to the stable law whose characteristic function is specified by (5). Observe that if $X_n \stackrel{d}{=} X \in \mathcal{D}(\alpha, \beta)$, then (see, for instance, [11]) the quantity ρ in (2) is calculated by the formula

$$\rho = \begin{cases} \dfrac{1}{2} & \text{if } \alpha = 1, \\[2ex] \dfrac{1}{2} + \dfrac{1}{\pi\alpha} \operatorname{arctg}\left(\beta \operatorname{tg} \dfrac{\pi\alpha}{2}\right) & \text{otherwise.} \end{cases} \tag{6}$$

Introduce the following basic assumption.

Condition A. The random variables $X_n = \log f'_{n-1}(1)$, $n \geqq 1$, are independent copies of $X \in \mathcal{D}(\alpha, \beta)$ with $0 < \alpha < 2$ and $|\beta| < 1$.

Now we formulate our first result.

[1] We write $a_n \sim b_n$ if $\lim_{n\to\infty}(a_n/b_n) = 1$.

THEOREM 1. *Assume that the offspring generating functions are geometric, i.e.,*

$$f_{n-1}(s) := \frac{e^{-X_n}}{1 + e^{-X_n} - s}, \qquad n = 1, 2, \dots, \tag{7}$$

with X_n, $n \geqq 1$ satisfying Condition A. Then

$$\mathbf{P}(T = n) \sim \theta \mathbf{P}(T^- = n) \sim \theta(1 - \rho)\,\frac{l(n)}{n^{2-\rho}} \quad \textit{as } n \to \infty. \tag{8}$$

Remark 1. In the case of geometric offspring distributions one has an explicit formula for the conditional probability of the event $\{T = n\}$ given the environment $f_0, f_1, \dots, f_{n-1}$ in terms of an exponential functional of the associated random walk $\{S_k, k \geqq 0\}$ (see (49) in what follows). Thus, the analysis of the properties of the extinction probability $\mathbf{P}(T = n)$ in this case is reduced to the study of the expectation of a certain functional of the associated random walk.

We now turn to the joint distribution of T and the population size Z_{T-1}. Here we do not restrict ourselves to the case of geometric offspring reproduction laws. To formulate the respective result we set

$$\zeta(b) := e^{-2X_1} \sum_{k=b}^{\infty} k^2 f_{0k}, \qquad b = 0, 1, \dots,$$

and let $\Lambda := \{\Lambda_t,\ 0 \leqq t \leqq 1\}$ denote the meander of a strictly stable process with parameters α, β, i.e., a strictly stable Lévy process conditioned to stay positive on the time interval $(0, 1]$ (see [3] and [4] for details). Along with Λ consider a stochastic process $\widetilde{\Lambda} := \{\widetilde{\Lambda}_t,\ 0 \leqq t \leqq 1\}$ defined by

$$\mathbf{E}\,[\phi(\widetilde{\Lambda})] = \frac{\mathbf{E}\,[\Lambda_1^{-\alpha}\phi(\Lambda)]}{\mathbf{E}\,\Lambda_1^{-\alpha}} \text{ for any function } \phi \in D[0, 1],$$

where $D[0, 1]$ denotes the space of càdlàg functions on the unit interval.

THEOREM 2. *Assume that Condition A is valid and there exists $\delta > 0$ such that*

$$\mathbf{E}\,(\log^+ \zeta(b))^{\alpha+\delta} < \infty$$

for some $b \geqq 0$. Then, for every $x > 0$,

$$\lim_{n\to\infty} \frac{\mathbf{P}(Z_{n-1} > e^{xc_n};\ T = n)}{\mathbf{P}(T^- = n)} = \theta \mathbf{P}(\widetilde{\Lambda}_1 > x). \tag{9}$$

Remark 2. It is easy to see that $\zeta(2) \leqq 4$ for the geometric offspring distributions. Therefore, the statement of Theorem 2 holds in this case. Moreover, in view of (8),

$$\lim_{x\downarrow 0} \lim_{n\to\infty} \frac{\mathbf{P}(Z_{n-1} \leqq e^{xc_n};\ T = n)}{\mathbf{P}(T = n)} = 0 \tag{10}$$

provided that the conditions of Theorem 1 hold.

We now complement Theorem 2 by the following statement being valid for the geometric offspring distributions.

THEOREM 3. *Under the conditions of Theorem 1, as $n \to \infty$,*

$$\mathcal{L}\left(\frac{\log Z_{[(n-1)t]}}{c_n},\ 0 \leqq t \leqq 1 \,\middle|\, T = n\right) \Longrightarrow \mathcal{L}(\widetilde{\Lambda}_t,\ 0 \leqq t \leqq 1).$$

Here $\Longrightarrow$ denotes the weak convergence with respect to the Skorokhod topology in the space $D[0,1]$.

In fact, even a stronger result is valid. To formulate it we consider for integers $0 \leqq r \leqq n-1$ the rescaled generation size process $W^{r,n} = (W_t^{r,n}, 0 \leqq t \leqq 1)$ given by

$$W_t^{r,n} := e^{-S_{r+[(n-r-1)t]}} Z_{r+[(n-r-1)t]}, \qquad 0 \leqq t \leqq 1. \tag{11}$$

THEOREM 4. *Let $r_1, r_2, \ldots$ be a sequence of positive integers such that $r_n \leqq n-1$ and $r_n \to \infty$. Then, under the conditions of Theorem* 1,

$$\mathcal{L}\big(W^{r_n,n} \,|\, T = n\big) \Longrightarrow \mathcal{L}(W_t,\ 0 \leqq t \leqq 1) \quad \textit{as } n \to \infty;$$

here the limiting process is a stochastic process with a.s. constant paths, i.e., $\mathbf{P}(W_t = W \textit{ for all } t \in [0,1]) = 1$ *for some random variable W. Furthermore,*

$$\mathbf{P}(0 < W < \infty) = 1.$$

Combining Theorems 1, 2, and 3 shows, in particular, that

$$\lim_{n\to\infty} \mathbf{P}(Z_{n-1} > e^{xc_n} \,|\, T = n) = \mathbf{P}(\widetilde{\Lambda}_1 > x) \tag{12}$$

in the case when the offspring distributions are geometric. The last equality, along with Theorem 3, allows us to make the following nonrigorous description of the evolution of a critical branching process Z, being subject to the conditions of Theorem 1. If the process survives for a long time ($T = n \to \infty$), then $\log Z_{[(n-1)t]}$ grows, roughly speaking, as $c_n \widetilde{\Lambda}_t$ up to moment $n-1$, and then the process instantly dies out. In particular, $\log Z_{n-1}$ is of order c_n (compare with Corollary 1.6 in [1]). This may be interpreted as the development of the process in a favorable environment up to the moment $n-1$ and the sudden extinction of the population at time moment $T = n \to \infty$ because of a very unfavorable, even "catastrophic," environment at moment $n-1$. At the end of the paper we show that this phenomenon is in sharp contrast to the case $\mathbf{E}\, X_n = 0$, $\sigma^2 := \mathbf{D} X_n \in (0,\infty)$. Namely, if, additionally,

$$\mathbf{E}\,(1 - f_{00})^{-1} < \infty, \quad \mathbf{E}\, f_{00}^{-1} < \infty, \tag{13}$$

then

$$\lim_{N\to\infty} \limsup_{n\to\infty} \mathbf{P}(Z_{n-1} > N \,|\, T = n) = 0, \tag{14}$$

while (see Corollary 1.6 in [1])

$$\mathcal{L}\left(\frac{\log Z_{[(n-1)t]}}{\sigma\sqrt{n}},\ 0 \leqq t \leqq 1 \,\middle|\, Z_{n-1} > 0\right) \Longrightarrow \mathcal{L}(B_t^+,\ 0 \leqq t \leqq 1),$$

where $B^+ := \{B_t^+, 0 \leqq t \leqq 1\}$ is the Brownian meander.

These facts demonstrate that the phenomenon of "sudden extinction" in a favorable environment is absent for the case $\sigma^2 < \infty$. Moreover, one can say that in this case we observe a "natural" extinction of the population. Indeed, the extinction occurs at moment $T = n$ because of the small size of the population in the previous generation rather than under the pressure of the unfavorable environment.

In the present paper we deal with the annealed approach. As is shown in [9], one cannot see the phenomenon of "sudden extinction" under the quenched approach even if the conditions of Theorem 1 are valid. A "typical" trajectory of a critical branching process in a random environment under the quenched approach oscillates before the extinction. The process passes through a number of bottlenecks corresponding to the strictly descending moments of the associated random walk and dies in a "natural" way because of the small number of individuals in generation $T-1$, just as under the annealed approach for the case $\sigma^2 < \infty$ (see [9] for a more detailed discussion).

Another consequence of Theorem 2 is the following lower bound for $\mathbf{P}(T=n)$.

COROLLARY 1. *Under the conditions of Theorem* 2,

$$\liminf_{n\to\infty} \frac{\mathbf{P}(T=n)}{\mathbf{P}(T^-=n)} = \theta. \tag{15}$$

We conjecture that the relation $\mathbf{P}(T=n) \sim \theta\mathbf{P}(T^-=n)$ is valid for any critical branching processes in a random environment meeting the conditions of Theorem 2, i.e., without the assumption that the offspring distributions are geometric. With Theorem 2 in hand, one can easily infer that our conjecture is equivalent to the equality

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{\mathbf{P}(Z_{n-1} \leqq e^{\varepsilon c_n}; T=n)}{\mathbf{P}(T^-=n)} = 0.$$

But this is exactly the phenomenon of "sudden extinction" described above.

It is known that there is a natural correspondence between the critical (subcritical) branching processes in a random environment and the simple random walks in a random environment with zero (negative) drift. In particular, (8) admits an interpretation in terms of the following simple random walk $\{R_k, k \geqq 0\}$ in a random environment. The walk starts at point $R_0 = 0$ and has transition probabilities

$$q_n := \mathbf{P}(R_{k+1} = n-1 \,|\, R_k = n) = \frac{e^{-X_{n+1}}}{1+e^{-X_{n+1}}}, \tag{16}$$

$$p_n := \mathbf{P}(R_{k+1} = n+1 \,|\, R_k = n) = \frac{1}{1+e^{-X_{n+1}}}, \tag{17}$$

$n \in \mathbf{Z}$, where $\{X_n,\ n \in \mathbf{Z}\}$ are independent identically distributed random variables. Let

$$\chi := \min\{k > 0\colon\ R_k = -1\}$$

and let

$$\ell(n) := \sum_{0\leqq k\leqq \chi} 1\{R_k = n\}, \qquad n \geqq -1,$$

be the local time of the random walk in a random environment calculated for the first nonnegative excursion of this walk. Clearly, if

$$Z_n := \sum_{i=0}^{n} (-1)^i \ell(n-i-1), \qquad n \geqq 0,$$

then

$$\ell(n) = Z_{n+1} + Z_n, \qquad n \geqq 0.$$

One can show that if the sequence $S_n := \log(p_1/q_1) + \cdots + \log(p_n/q_n)$ is either oscillating or tends to $-\infty$ with probability 1, then $\{Z_n,\ n \geqq 0\}$ is, respectively, a critical or subcritical branching process in a random environment specified by the offspring generating functions

$$f_n(s) := \frac{q_n}{1 - p_n s}$$

(see [8] for more details). In particular, $T := \min\{j > 0\colon\ \ell(j) = 0\}$ is the extinction moment of the branching process. Clearly, if $\overline{R} := \max_{0 \leqq k < \chi} R_k$, then

$$\{\overline{R} = n - 1\} = \{T = n\}.$$

In these terms Theorem 1 and relation (12) are equivalent to the following statement.

THEOREM 5. *If q_n and p_n, specified by* (16) *and* (17), *are such that*

$$X_n := \log \frac{p_n}{q_n}, \qquad n \in \mathbf{Z},$$

satisfy Condition A, then, as $n \to \infty$,

$$\mathbf{P}(\overline{R} = n) \sim \theta \mathbf{P}(T^- = n).$$

In addition,

$$\mathbf{P}(\ell(n) > e^{xc_n} \,|\, \ell(n) > 0, \ell(n+1) = 0) \sim \mathbf{P}(\widetilde{\Lambda}_1 > x), \qquad x > 0,$$

and, moreover,

$$\mathcal{L}\left(\frac{\log \ell([nt])}{c_n},\ 0 \leqq t \leqq 1 \,\middle|\, \ell(n) > 0,\ \ell(n+1) = 0\right) \Longrightarrow \mathcal{L}(\widetilde{\Lambda}_t,\ 0 \leqq t \leqq 1).$$

Hence, the random walk in a random environment visits the maximal possible level for the first excursion many times provided the length χ of the excursion is big. This is essentially different from the case $\mathbf{E}\log(p_n/q_n) = 0, \mathbf{E}\log^2(p_n/q_n) < \infty$, where (compare with (14))

$$\lim_{N\to\infty} \limsup_{n\to\infty} \mathbf{P}(\ell(n) > N \,|\, \ell(n) > 0,\ \ell(n+1) = 0) = 0.$$

2. Some auxiliary results for random walks. Let us agree to denote by $C, C_1, C_2, \ldots$ positive constants which may be different from formula to formula.

It is known (see, for instance, [5, Chap. XVII, section 5]) that if $X \in \mathcal{D}(\alpha, \beta)$, then the scaling sequence

$$c_n := \min\{x > 0\colon\ \mathbf{P}(X > x) \leqq n^{-1}\}, \qquad n \geqq 1, \tag{18}$$

for S_n is regularly varying with index α^{-1}, i.e., there exists a function $l_1(n)$, slowly varying at infinity, such that

$$c_n = n^{1/\alpha} l_1(n). \tag{19}$$

Moreover, if $X \in \mathcal{D}(\alpha, \beta)$ with $\alpha \in (0, 2)$, then

$$\mathbf{P}(|X| > x) \sim \frac{1}{x^\alpha l_0(x)} \quad \text{as } x \to \infty, \tag{20}$$

where $l_0(x)$ is a function slowly varying at infinity and

$$\frac{\mathbf{P}(X < -x)}{\mathbf{P}(|X| > x)} \to q, \quad \frac{\mathbf{P}(X > x)}{\mathbf{P}(|X| > x)} \to p \quad \text{as } x \to \infty \tag{21}$$

with $p + q = 1$ and $\beta = p - q$ in (5). Besides,

$$\mathbf{P}(X < -c_n) \sim \frac{(2-\alpha)q}{\alpha n} \quad \text{as } n \to \infty \tag{22}$$

by (18) and (19).

2.1. Asymptotic behavior of overshoots and undershoots. In this subsection we prove some results concerning the asymptotic behavior of the distributions of overshoots and undershoots. We believe that these results are of independent interest.

Let $\tau^- := \min\{k \geqq 1 \colon\ S_k \leqq 0\}$.

Durrett [4] has shown that if $X \in \mathcal{D}(\alpha, \beta)$, then

$$\lim_{n\to\infty} \mathbf{P}(S_n \leqq x c_n \,|\, \tau^- > n) = \mathbf{P}(\Lambda_1 \leqq x) \quad \text{for all } x \geqq 0. \tag{23}$$

By minor changes of the proof of (23) given in [4], one can demonstrate that

$$\lim_{n\to\infty} \mathbf{P}(S_n \leqq x c_n \,|\, T^- > n) = \mathbf{P}(\Lambda_1 \leqq x) \quad \text{for all } x \geqq 0. \tag{24}$$

We now establish analogues of (23) and (24) either under the condition $\{\tau^- = n\}$ or under the condition $\{T^- = n\}$.

LEMMA 1. *If Condition A is valid, then, for any $u > 0$,*

$$\lim_{n\to\infty} \mathbf{P}(S_n \leqq -u c_n \,|\, \tau^- = n) = \lim_{n\to\infty} \mathbf{P}(S_n \leqq -u c_n \,|\, T^- = n) = \frac{\mathbf{E}\,(u + \Lambda_1)^{-\alpha}}{\mathbf{E}\,\Lambda_1^{-\alpha}}.$$

Proof. For a fixed $u > 0$ we have

$$\mathbf{P}(S_n \leqq -u c_n;\ \tau^- = n) = \mathbf{E}\,[\mathbf{P}(X \leqq -S_{n-1} - u c_n);\ \tau^- > n-1]. \tag{25}$$

Since, under the conditions of our lemma,

$$\frac{\mathbf{P}(X \leqq -x - u c_n)}{\mathbf{P}(X \leqq -c_n)} = \left(\frac{x}{c_n} + u\right)^{-\alpha} (1 + o(1)) \tag{26}$$

uniformly in $x \in [0, \infty)$, we may approximate for large n the right-hand side of (25) by the quantity

$$\mathbf{P}(X \leqq -c_n)\,\mathbf{E}\,[(S_{n-1}/c_n + u)^{-\alpha};\ \tau^- > n-1].$$

Using (22) and (23), we obtain

$$\mathbf{P}(S_n \leqq -u c_n;\ \tau^- = n) \sim \frac{q(2-\alpha)}{\alpha n}\,\mathbf{P}(\tau^- > n-1)\,\mathbf{E}\,(\Lambda_1 + u)^{-\alpha}. \tag{27}$$

Recall that, by Theorem 7 in [10],

$$\mathbf{P}(\tau^- = n) \sim (1-\rho)\,\frac{\mathbf{P}(\tau^- > n-1)}{n}. \tag{28}$$

Therefore,

$$\mathbf{P}(S_n \leqq -uc_n \,|\, \tau^- = n) \sim \frac{q(2-\alpha)\,\mathbf{E}\,(u+\Lambda_1)^{-\alpha}}{(1-\rho)\alpha}. \tag{29}$$

This finishes the proof of the first part of the lemma since

$$\mathbf{E}\,\Lambda_1^{-\alpha} = \frac{(1-\rho)\alpha}{q(2-\alpha)} \tag{30}$$

according to formula (109) in [10].

To demonstrate the second part it is sufficient to replace τ^- by T^- everywhere in the arguments above. The lemma is proved.

LEMMA 2. *If Condition A is valid, then, for any* $v > 0$,

$$\lim_{n\to\infty} \mathbf{P}(S_{n-1} \geqq vc_n \,|\, \tau^- = n) = \lim_{n\to\infty} \mathbf{P}(S_{n-1} \geqq vc_n \,|\, T^- = n) = \mathbf{P}(\widetilde{\Lambda}_1 \geqq v).$$

Proof. To establish the desired statement one should use the equality

$$\mathbf{P}(S_{n-1} \geqq vc_n \,|\, \tau^- = n) = \frac{\mathbf{P}(\tau^- > n-1)}{\mathbf{P}(\tau^- = n)} \times \mathbf{E}\,[\mathbf{P}(X \leqq -S_{n-1})\,1\{S_{n-1} \geqq vc_n\} \,|\, \tau^- > n-1],$$

a similar representation with τ^- replaced by T^-, asymptotic equality (26) with $u = 0$ and $x \geqq vc_n$, and the arguments similar to those applied to demonstrate Lemma 1.

Remark 3. It follows from Lemmas 1 and 2 that the passage from positive to negative (nonnegative) values just at moment n is possible only owing to a big negative jump of order c_n at this moment. More precisely, in this case the undershoot S_{n-1} and the overshoot $-S_n$ are of order c_n.

2.2. Expectations on the event $\{T^- = n\}$. Let

$$T_0 := 0, \quad T_{j+1} := \min(n > T_j\colon\ S_n < S_{T_j}), \qquad j \geqq 0,$$

be strictly descending ladder epochs of the random walk S. Clearly, $T^- = T_1$. Put $L_n := \min_{0\leqq k\leqq n} S_k$ and introduce the function

$$V(x) := \sum_{j=0}^{\infty} \mathbf{P}(S_{T_j} \geqq -x), \quad x > 0; \quad V(0) = 1; \quad V(x) = 0, \quad x < 0.$$

The fundamental property of the function $V(x)$ is the identity

$$\mathbf{E}\,[V(x+X);\, X + x \geqq 0] = V(x), \qquad x \geqq 0. \tag{31}$$

Denote by $\mathcal{F}$ the filtration consisting of the σ-algebras $\mathcal{F}_n$ generated by the random variables $S_0, \ldots, S_n$ and $Z_0, \ldots, Z_n$. By means of $V(x)$ we may specify a probability measure $\mathbf{P}^+$ as

$$\mathbf{E}^+[\psi(S_0, \ldots, S_n; Z_0, \ldots, Z_n)] := \mathbf{E}\,[\psi(S_0, \ldots, S_n; Z_0, \ldots, Z_n)V(S_n);\ L_n \geqq 0],$$

where ψ is an arbitrary measurable function on the respective space of arguments. One can check that, in view of (31), this measure is well defined (see [1] for more details).

We now formulate a statement related to the measure $\mathbf{P}^+$ which is a particular case of Lemma 2.5 in [1].

LEMMA 3 (see [1]). *Let condition* (2) *hold and let* ξ_k *be a bounded* $\mathcal{F}_k$*-measurable random variable. Then*

$$\lim_{n\to\infty} \mathbf{E}\,[\xi_k \,|\, T^- > n] = \mathbf{E}^+\xi_k.$$

More generally, let $\xi_1, \xi_2, \ldots$ *be a sequence of uniformly bounded random variables adopted to the filtration* $\mathcal{F}$ *such that*

$$\lim_{n\to\infty} \xi_n =: \xi_\infty \tag{32}$$

exists $\mathbf{P}^+$*-a.s. Then*

$$\lim_{n\to\infty} \mathbf{E}\,[\xi_n \,|\, T^- > n] = \mathbf{E}^+\xi_\infty.$$

We prove a "local" version of this lemma under the additional assumption $X \in D(\alpha,\beta)$. To this aim let $\widetilde{S} := \{\widetilde{S}_n, n \geqq 0\}$ be a probabilistic copy of $S = \{S_n,\ n \geqq\ 0\}$. Later on all variables and expectations related to $\widetilde{S}$ are supplied with the symbol $^\sim$. For instance, we set $\widetilde{L}_n := \min_{0\leqq k\leqq n} \widetilde{S}_k$.

LEMMA 4. *Let* $X \in D(\alpha,\beta)$ *with* $\alpha < 2$ *and* $\beta < 1$, *and let* ξ_k *be a bounded* $\mathcal{F}_k$*-measurable random variable. Then*

$$\lim_{n\to\infty} \mathbf{E}\,[\xi_k \,|\, T^- = n] = \mathbf{E}^+\xi_k.$$

More generally, let $\xi_1, \xi_2, \ldots$ *be a sequence of uniformly bounded random variables adopted to the filtration* $\mathcal{F}$ *such that the limit*

$$\lim_{n\to\infty} \xi_n =: \xi_\infty \tag{33}$$

exists $\mathbf{P}^+$*-a.s. Then*

$$\lim_{n\to\infty} \mathbf{E}\,[\xi_{n-1} \,|\, T^- = n] = \mathbf{E}^+\xi_\infty. \tag{34}$$

Moreover,

$$\lim_{n\to\infty} \mathcal{L}(\xi_{n-1} \,|\, T^- = n) = \mathcal{L}(\xi_\infty). \tag{35}$$

Proof. According to Lemma 2, for any fixed $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \mathbf{E}\left[\xi_k\, 1\left\{ \frac{S_{n-1}}{c_n} \notin [\varepsilon, \varepsilon^{-1}] \right\} \,\middle|\, T^- = n \right] \right| \\
&\quad \leqq C \limsup_{n\to\infty} \mathbf{P}\left(\frac{S_{n-1}}{c_n} \notin [\varepsilon, \varepsilon^{-1}] \,\middle|\, T^- = n \right) \leqq C\,\mathbf{P}(\Lambda_1 \notin [\varepsilon, \varepsilon^{-1}]),
\end{aligned} \tag{36}$$

and the last probability tends to zero as $\varepsilon \downarrow 0$. Further,

$$\begin{aligned}
&\mathbf{E}\left[\xi_k\, 1\left\{ \frac{S_{n-1}}{c_n} \in (\varepsilon, \varepsilon^{-1}) \right\};\ T^- = n \right] \\
&\quad = \mathbf{E}\left[\xi_k\, \mathbf{P}(X < -S_{n-1})\, 1\left\{ \frac{S_{n-1}}{c_n} \in [\varepsilon, \varepsilon^{-1}] \right\};\ T^- > n-1 \right].
\end{aligned}$$

Set $\psi_\varepsilon(x) := x^{-\alpha}\,1(\varepsilon \leqq x \leqq \varepsilon^{-1})$. Since

$$\mathbf{P}(X < -uc_n) \sim u^{-\alpha}\,\mathbf{P}(X < -c_n) \sim u^{-\alpha}\,\frac{q(2-\alpha)}{\alpha n}$$

uniformly in $u \in [\varepsilon, \varepsilon^{-1}]$, we have

$$\begin{aligned}
&\mathbf{E}\left[\xi_k\,\mathbf{P}(X < -S_{n-1})\,1\left\{\frac{S_{n-1}}{c_n} \in [\varepsilon, \varepsilon^{-1}]\right\};\ T^- > n-1\right]\\
&\qquad \sim \mathbf{P}(X < -c_n)\,\mathbf{E}\left[\xi_k\left(\frac{S_{n-1}}{c_n}\right)^{-\alpha} 1\left\{\frac{S_{n-1}}{c_n} \in [\varepsilon, \varepsilon^{-1}]\right\};\ T^- > n-1\right]\\
&\qquad = \frac{q(2-\alpha)}{\alpha n}\,\mathbf{E}\left[\xi_k\psi_\varepsilon\left(\frac{S_{n-1}}{c_n}\right);\ T^- > n-1\right]. \qquad (37)
\end{aligned}$$

Conditioning on $S_0, S_1, \ldots, S_k$ for $k < n-1$ gives

$$\begin{aligned}
&\mathbf{E}\left[\xi_k\psi_\varepsilon\left(\frac{S_{n-1}}{c_n}\right);\ T^- > n-1\right]\\
&\qquad = \mathbf{E}\left[\xi_k\widetilde{\mathbf{E}}\left[\psi_\varepsilon\left(\frac{\widetilde{S}_{n-k-1}}{c_n}\right);\ \widetilde{L}_{n-k-1} \geqq -S_k\right];\ T^- > k\right].
\end{aligned}$$

Using Lemmas 2.1 and 2.3 of [1], one can easily verify that

$$\begin{aligned}
&\mathbf{E}\left[\xi_k\,\widetilde{\mathbf{E}}\left[\psi_\varepsilon\left(\frac{\widetilde{S}_{n-k-1}}{c_n}\right);\ \widetilde{L}_{n-k} \geqq -S_{k-1}\right];\ T^- > k\right]\\
&\qquad \sim \mathbf{E}\,[\xi_k\mathbf{P}(\widetilde{L}_{n-k} \geqq -S_{k-1});\ T^- > k]\,\mathbf{E}\,[\psi_\varepsilon(\Lambda_1)]\\
&\qquad \sim \mathbf{E}\,[\xi_k V(S_{k-1});\ T^- > k-1]\,\mathbf{P}(T^- > n-k)\,\mathbf{E}\,[\psi_\varepsilon(\Lambda_1)]\\
&\qquad \sim \mathbf{E}^+[\xi_k]\,\mathbf{P}(T^- > n-1)\,\mathbf{E}\,[\psi_\varepsilon(\Lambda_1)].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbf{E}\left[\xi_k\psi_\varepsilon\left(\frac{S_{n-1}}{c_n}\right) 1\left\{\frac{S_{n-1}}{c_n} \in [\varepsilon, \varepsilon^{-1}]\right\} \Bigm| T^- = n\right]\\
&\qquad \sim \frac{\mathbf{P}(T^- > n-1)}{\mathbf{P}(T^- = n)}\,\frac{q(2-\alpha)}{\alpha n}\,\mathbf{E}^+[\xi_k]\,\mathbf{E}\,[\psi_\varepsilon(\Lambda_1)]. \qquad (38)
\end{aligned}$$

Clearly, $\mathbf{E}\,[\psi_\varepsilon(\Lambda_1)] \to \mathbf{E}\,\Lambda_1^{-\alpha}$ as $\varepsilon \to 0$. Combining these estimates with (30), (3), and the asymptotic relation

$$\mathbf{P}(T^- = n) \sim \frac{1-\rho}{n}\,\mathbf{P}(T^- > n-1) = (1-\rho)\,\frac{l(n)}{n^{2-\rho}}, \qquad (39)$$

established in Theorem 8 of [10], and recalling (36), we complete the proof of the first part of the lemma.

To show the second part we fix an $\varepsilon \in (0, 1)$ and write

$$\begin{aligned}
|\mathbf{E}\,[\xi_k - \xi_{n-1}\,|\,T^- = n]| &\leqq \mathbf{E}\left[|\xi_k - \xi_{n-1}|\,1\left\{\frac{S_{n-1}}{c_n} \notin [\varepsilon, \varepsilon^{-1}]\right\} \Bigm| T^- = n\right]\\
&\quad + \mathbf{E}\left[|\xi_k - \xi_{n-1}|\,1\left\{\frac{S_{n-1}}{c_n} \in [\varepsilon, \varepsilon^{-1}]\right\} \Bigm| T^- = n\right].
\end{aligned}$$

Similarly to (36),

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\mathbf{E}\left[|\xi_k-\xi_{n-1}|\,1\left\{\frac{S_{n-1}}{c_n}\notin[\varepsilon,\varepsilon^{-1}]\right\}\Big|\,T^-=n\right]$$
$$\leqq C\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\mathbf{P}\left(\frac{S_{n-1}}{c_n}\notin[\varepsilon,\varepsilon^{-1}]\,\Big|\,T^-=n\right)=0, \tag{40}$$

while, by analogy with (37) and (38),

$$\mathbf{E}\left[|\xi_k-\xi_{n-1}|\,1\left\{\frac{S_{n-1}}{c_n}\in[\varepsilon,\varepsilon^{-1}]\right\}\Big|\,T^-=n\right]$$
$$\leqq C\,\mathbf{E}\left[|\xi_k-\xi_{n-1}|\psi_\varepsilon\left(\frac{S_{n-1}}{c_n}\right)\Big|\,T^->n-1\right]$$
$$\leqq C\varepsilon^{-\alpha}\,\mathbf{E}\left[|\xi_k-\xi_{n-1}|\,|\,T^->n-1\right]. \tag{41}$$

We know by Lemma 3 that, given (33),

$$\lim_{k\to\infty}\lim_{n\to\infty}\mathbf{E}\left[|\xi_k-\xi_{n-1}|\,|\,T^->n-1\right]=\lim_{k\to\infty}\mathbf{E}^+|\xi_k-\xi_\infty|=0. \tag{42}$$

Combining (40)–(42) completes the proof of (34).

To prove (35) it is sufficient to observe that, by (34) and the dominated convergence theorem,

$$\lim_{n\to\infty}\mathbf{E}\left[e^{it\xi_{n-1}}\,|\,T^-=n\right]=\mathbf{E}^+e^{it\xi_\infty},\qquad t\in(-\infty,\infty).$$

Set $\mu_n:=\min\{k\geqq 0: S_k=L_n\}$.

LEMMA 5. *If $X\in\mathcal{D}(\alpha,\beta)$, then*

$$\limsup_{n\to\infty} nc_n\,\mathbf{E}\,e^{2L_n-S_n}<\infty$$

and

$$\limsup_{n\to\infty} nc_n\,\mathbf{E}\left[e^{S_n};\ \mu_n=n\right]<\infty. \tag{43}$$

Proof. By the factorization identity (see, for instance, Theorem 8.9.3 in [2]) applied with $\lambda=-1$ and $\mu=1$ to L_n instead of $M_n:=\max_{0\leqq k\leqq n}S_k$, we have

$$\sum_{n=0}^{\infty}r^n\,\mathbf{E}\left[e^{2L_n-S_n}\right]=\exp\left\{\sum_{n=1}^{\infty}\frac{r^n}{n}\left(\mathbf{E}\left[e^{-S_n};\ S_n\geqq 0\right]+\mathbf{E}\left[e^{S_n};\ S_n<0\right]\right)\right\}.$$

Since $X\in D(\alpha,\beta)$, the local limit theorem for asymptotically stable random walks implies

$$\mathbf{E}\left[e^{-S_n};\ S_n\geqq 0\right]+\mathbf{E}\left[e^{S_n};\ S_n<0\right]\leqq\frac{C}{c_n}. \tag{44}$$

Combining this with Theorem 6 in [6] gives

$$\limsup_{n\to\infty} nc_n\,\mathbf{E}\left[e^{2L_n-S_n}\right]<\infty,$$

proving the first statement of the lemma.

To prove the second it is sufficient to note that (see, for instance, Theorem 8.9.1 in [2])

$$\sum_{n=0}^{\infty} r^n \mathbf{E}\,[e^{S_n};\ \mu_n = n] = \exp\left\{\sum_{n=1}^{\infty} \frac{r^n}{n} \mathbf{E}\,[e^{S_n};\ S_n < 0]\right\},$$

and to use estimate (44) once again. The lemma is proved.

The previous lemma allows us to establish the following statement.

LEMMA 6. *If $X \in \mathcal{D}(\alpha,\beta)$ with $\alpha < 2$ and $\beta < 1$, then for every $\varepsilon > 0$ there exists a positive integer l such that*

$$\sum_{k=l}^{n-1} \mathbf{E}\,[e^{S_k};\ \mu_k = k]\,\mathbf{P}(T^- = n-k) \leqq \varepsilon\,\mathbf{P}(T^- = n)$$

for all $n \geqq l$.

Proof. By Lemma 5 and (39) we have, for any $\delta \in (0,1)$,

$$\begin{aligned}\sum_{k=l}^{n} \mathbf{E}\,[e^{S_k};\ \mu_k = k]\,\mathbf{P}(T^- = n-k) &\leqq \max_{n\delta \leqq j \leqq n} \mathbf{P}(T^- = j) \sum_{l \leqq k \leqq n(1-\delta)} \mathbf{E}\,[e^{S_k};\ \mu_k = k] \\ &\quad + \frac{C}{n(1-\delta)c_{n(1-\delta)}} \sum_{k \leqq n\delta} \mathbf{P}(T^- = k). \end{aligned} \tag{45}$$

On account of (39),

$$\max_{n\delta \leqq j \leqq n} \mathbf{P}(T^- = j) \leqq C\delta^{\rho-2}\,\frac{l(n)}{n^{2-\rho}} \leqq C_1\delta^{\rho-2}\,\mathbf{P}(T^- = n). \tag{46}$$

Using (6) it is not difficult to check that $1-\rho < \alpha^{-1}$ if Condition A holds. With this in view we have, by (19) and (39),

$$\frac{1}{nc_{n(1-\delta)}} \leqq \frac{C}{(1-\delta)^{1/\alpha} n^{1+1/\alpha} l_1(n)} \leqq \frac{C_1}{(1-\delta)^{1/\alpha} n^{1/\alpha+\rho-1}}\,\mathbf{P}(T^- = n). \tag{47}$$

Substituting (46) and (47) in (45) gives

$$\begin{aligned}\sum_{k=l}^{n} \mathbf{E}\,[e^{S_k};\ \mu_k = k]\,\mathbf{P}(T^- = n-k) \\ \leqq C_3\,\mathbf{P}(T^- = n)\left(\delta^{\rho-2}\sum_{k=l}^{\infty} \mathbf{E}\,[e^{S_k};\ \mu_k = k] + \frac{1}{(1-\delta)^{1+1/\alpha} n^{1/\alpha+\rho-1}}\right)\end{aligned}$$

for sufficiently large C_3. Recalling now (43), we complete the proof of the lemma by an appropriate choice of δ and l.

3. Proof of Theorem 1. Set

$$F_{m,n}(s) := f_m(f_{m+1}(\dots(f_{n-1}(s))\dots)), \quad m < n, \qquad F_{n,n}(s) := s.$$

Rewriting (7) as

$$\frac{1}{1-f_{n-1}(s)} = 1 + e^{-X_n}\,\frac{1}{1-s} \quad \text{for all } n \geqq 1,$$

one can easily get the representation

$$\frac{1}{1-F_{0,n}(s)} = 1 + e^{-S_1} + e^{-S_2} + \dots + e^{-S_{n-1}} + e^{-S_n}\,\frac{1}{1-s} \tag{48}$$

for all $n \geqq 1$. From this equality, setting

$$H_n := \left(\sum_{k=0}^{n} e^{-S_k}\right)^{-1}, \quad H_\infty := \lim_{n\to\infty} H_n,$$

we get

$$\mathbf{P}_f(Z_n > 0) := \mathbf{P}(Z_n > 0 \,|\, f_0, f_1, \dots, f_{n-1}) = 1 - F_{0,n}(0) = H_n$$

and

$$\mathbf{P}_f(T = n) := \mathbf{P}_f(Z_{n-1} > 0) - \mathbf{P}_f(Z_n > 0) = H_{n-1} H_n e^{-S_n}. \tag{49}$$

We split the expectation $\mathbf{E}\,[\mathbf{P}_f(T=n)] = \mathbf{P}(T=n)$ into two parts

$$\mathbf{P}(T = n) = \mathbf{E}\,[\mathbf{P}_f(T = n);\ \mu_n < n] + \mathbf{E}\,[\mathbf{P}_f(T = n);\ \mu_n = n]. \tag{50}$$

One can easily verify that $H_{n-1}H_n e^{-S_n} \leqq e^{2L_n - S_n}$ on the event $\{\mu_n < n\}$. From this bound and Lemma 5 we infer

$$\mathbf{E}\,[\mathbf{P}_f(T = n);\ \mu_n < n] \leqq \mathbf{E}\, e^{2L_n - S_n} \leqq \frac{C}{nc_n} \quad \text{for all } n \geqq 1. \tag{51}$$

Using estimate (47) with $\delta = 0$, we conclude

$$\mathbf{E}\,[\mathbf{P}_f(T = n);\ \mu_n < n] = o(\mathbf{P}(T^- = n)). \tag{52}$$

Consider now the expectation $\mathbf{E}\,[\mathbf{P}_f(T = n);\ \mu_n = n]$. Applying Lemma 5 once again, we see that

$$\mathbf{E}\,[\mathbf{P}_f(Z_n > 0);\ \mu_n = n] \leqq \mathbf{E}\,[e^{S_n};\ \mu_n = n] \leqq \frac{C}{nc_n} = o(\mathbf{P}(T^- = n)).$$

Thus,

$$\mathbf{E}\,[\mathbf{P}_f(T = n);\ \mu_n = n] = \mathbf{E}\,[\mathbf{P}_f(Z_{n-1} > 0);\ \mu_n = n] + o(\mathbf{P}(T^- = n)). \tag{53}$$

Since $\mathbf{P}_f(Z_{n-1} > 0) \leqq e^{\min_{0\leqq j\leqq n-1} S_j}$, we have by Lemma 6 that for any $\varepsilon > 0$ there exists l such that

$$\begin{aligned}
&\mathbf{E}\,[\mathbf{P}_f(Z_{n-1} > 0);\ \mu_{n-1} \geqq l,\ \mu_n = n] \\
&\quad = \sum_{k=l}^{n-1} \mathbf{E}\,[\mathbf{P}_f(Z_{n-1} > 0);\ \mu_{n-1} = k,\ \mu_n = n] \\
&\quad \leqq \sum_{k=l}^{n-1} \mathbf{E}\,[e^{S_k};\ \mu_{n-1} = k,\ \mu_n = n] \leqq \varepsilon\, \mathbf{P}(T^- = n)
\end{aligned} \tag{54}$$

for all $n \geqq l$. Denoting by $\{\tilde{f}_n,\ n \geqq 0\}$ a probabilistic and independent copy of $\{f_n,\ n \geqq 0\}$ we have, for any fixed $k < l$,

$$\begin{aligned}
&\mathbf{E}\,[\mathbf{P}_f(Z_{n-1} > 0);\ \mu_{n-1} = k,\ \mu_n = n] \\
&\quad = \mathbf{E}\,[1 - F_{0,n-2}(0);\ \mu_{n-1} = k,\ \mu_n = n] \\
&\quad = \mathbf{E}\,[1 - F_{0,k}(F_{k,n-2}(0));\ \mu_{n-1} = k,\ \mu_n = n] \\
&\quad = \mathbf{E}\,[1 - F_{0,k}(\widetilde{F}_{0,n-k-2}(0));\ \mu_k = k,\ \widetilde{T}^- = n-k] \\
&\quad = \mathbf{E}\,[1 - F_{0,k}(\widetilde{F}_{0,n-k-2}(0))\,1\{\mu_k = k\}\,|\,\widetilde{T}^- = n-k]\,\mathbf{P}(T^- = n-k).
\end{aligned}$$

By monotonicity of the extinction probability and Lemma 2.7 in [1],

$$\lim_{n\to\infty} \widetilde{F}_{0,n}(0) =: Q^+ < 1 \quad \mathbf{P}^+\text{-a.s.}$$

Hence, in view of (35) we get for any fixed $k < l$,

$$\begin{aligned}
&\mathbf{E}\left[(1 - F_{0,k}(\widetilde{F}_{0,n-k-2}(0)))\,1\{\mu_k = k\}\,|\,\widetilde{T}^- = n-k\right] \\
&\quad \sim \mathbf{E}\,[\mathbf{E}^+(1 - F_{0,k}(Q^+));\ \mu_k = k] > 0.
\end{aligned}$$

Using this relation, (39), and (54) it is not difficult to show that

$$\mathbf{E}\,[\mathbf{P}_f(Z_{n-1} > 0);\ \mu_n = n] \sim \theta\mathbf{P}(T^- = n), \tag{55}$$

where

$$\theta = \sum_{k=0}^{\infty} \mathbf{E}\,[\mathbf{E}^+(1 - F_{0,k}(Q^+));\ \mu_k = k] > 0. \tag{56}$$

It follows from (50), (52), (53), and (55) that

$$\mathbf{P}(T = n) \sim \theta\mathbf{P}(T^- = n).$$

It is easy to check that the expression for θ given by (56) is in complete agreement with formula (4.10) in [1]. This finishes the proof of Theorem 1.

4. Proofs for the general case.

Proof of Theorem 2. First, we obtain lower and upper bounds for the probability $\mathbf{P}(Z_1 = 0\,|\,Z_0 = k)$. It is easy to see that (recall (1))

$$f_{00} = \mathbf{P}(Z_1 = 0\,|\,Z_0 = 1; f_0) \geqq \max\left\{0,\ 1 - \sum_{k=1}^{\infty} k f_{0k}\right\}.$$

Therefore, for any fixed $\varepsilon \in (0, \frac{1}{2})$,

$$\begin{aligned}
\mathbf{P}(Z_1 = 0\,|\,Z_0 = k) &\geqq \mathbf{E}\,[(1 - e^{X_1})^k;\ X_1 < 0] \\
&\geqq (1 - k^{-1-\varepsilon})^k\,\mathbf{P}(X_1 \leqq -(1+\varepsilon)\log k).
\end{aligned} \tag{57}$$

To get an upper estimate we use the inequality $\mathbf{P}(Y > 0) \geqq (\mathbf{E}\,Y)^2/\mathbf{E}\,Y^2$, being valid for any nonnegative random variables with $\mathbf{E}\,Y > 0$, to conclude that

$$f_{00} \leqq 1 - \left(\frac{\sum_{k=1}^{\infty} k^2 f_{0k}}{(\sum_{k=1}^{\infty} k f_{0k})^2}\right)^{-1}.$$

Observing that

$$\frac{\sum_{k=1}^{\infty} k^2 f_{0k}}{(\sum_{k=1}^{\infty} k f_{0k})^2} \leqq \frac{b}{\sum_{k=1}^{\infty} k f_{0k}} + \zeta(b),$$

we get

$$f_{00} \leqq 1 - (be^{-X_1} + \zeta(b))^{-1} \leqq \exp\left\{ -\frac{1}{be^{-X_1} + \zeta(b)} \right\}.$$

This implies

$$\begin{aligned} \mathbf{P}(Z_1 = 0 \,|\, Z_0 = k) &\leqq \mathbf{E}\left[\exp\left\{ -\frac{1}{be^{-X_1} + \zeta(b)} \right\} \right] \\ &\leqq \mathbf{P}(X_1 \leqq -(1-\varepsilon)\log k) + \mathbf{P}(\zeta(b) > k^{1-\varepsilon}) + \exp\left\{ -\frac{k^{\varepsilon}}{b+1} \right\}. \end{aligned} \tag{58}$$

In view of the hypothesis $\mathbf{E}\,(\log^+ \zeta(b))^{\alpha+\delta} < \infty$ and the Markov inequality we have

$$\mathbf{P}(\zeta(b) > k^{1-\varepsilon}) \leqq C \log^{-\alpha-\delta} k. \tag{59}$$

Since the function $\mathbf{P}(X_1 < -x)$ is regularly varying at infinity with index $-\alpha$, estimates (57)–(59) imply

$$\begin{aligned} \frac{1}{(1+\varepsilon)^{\alpha}} &\leqq \liminf_{k\to\infty} \frac{\mathbf{P}(Z_1 = 0 \,|\, Z_0 = k)}{\mathbf{P}(X_1 < -\log k)} \\ &\leqq \limsup_{k\to\infty} \frac{\mathbf{P}(Z_1 = 0 \,|\, Z_0 = k)}{\mathbf{P}(X_1 < -\log k)} \leqq \frac{1}{(1-\varepsilon)^{\alpha}}. \end{aligned}$$

Letting $\varepsilon \to 0$ gives

$$\lim_{k\to\infty} \frac{\mathbf{P}(Z_1 = 0 \,|\, Z_0 = k)}{\mathbf{P}(X_1 < -\log k)} = 1.$$

Therefore,

$$\begin{aligned} \mathbf{P}(Z_{n-1} > e^{xc_n};\ T = n) &= \sum_{k > e^{xc_n}} \mathbf{P}(Z_{n-1} = k)\,\mathbf{P}(Z_1 = 0 \,|\, Z_0 = k) \\ &\sim \sum_{k > e^{xc_n}} \mathbf{P}(Z_{n-1} = k)\,\mathbf{P}(X \leqq -\log k) \\ &\sim \mathbf{E}\,[\mathbf{P}(X \leqq -\log Z_{n-1});\ \log Z_n > xc_n]. \end{aligned} \tag{60}$$

Since $\mathbf{P}(X \leqq -yc_n)/\mathbf{P}(X \leqq -c_n) \to y^{-\alpha}$ as $n \to \infty$ uniformly in $y \in [x, \infty)$, we have

$$\begin{aligned} &\mathbf{E}\,[\mathbf{P}(X \leqq -\log Z_{n-1});\ \log Z_{n-1} > xc_n] \\ &\quad\sim \mathbf{P}(X \leqq -c_n)\,\mathbf{E}\left[\left(\frac{\log Z_{n-1}}{c_n} \right)^{-\alpha};\ \log Z_{n-1} > xc_n \right] \\ &\quad\sim \mathbf{P}(X \leqq -c_n)\,\mathbf{P}(Z_{n-1} > 0) \\ &\qquad \times \mathbf{E}\left[\left(\frac{\log Z_{n-1}}{c_n} \right)^{-\alpha} 1\{\log Z_{n-1} > xc_n\} \,\middle|\, Z_{n-1} > 0 \right]. \end{aligned} \tag{61}$$

By Corollary 1.6 in [1],

$$\lim_{n\to\infty}\mathbf{P}\biggl(\frac{\log Z_{n-1}}{c_n}<x\,\Big|\,Z_{n-1}>0\biggr)=\mathbf{P}(\Lambda_1<x),\qquad x>0.$$

This and the dominated convergence theorem yield

$$\lim_{n\to\infty}\mathbf{E}\biggl[\biggl(\frac{\log Z_{n-1}}{c_n}\biggr)^{-\alpha}1\{\log Z_{n-1}>xc_n\}\,\Big|\,Z_{n-1}>0\biggr]=\mathbf{E}\,[\Lambda_1^{-\alpha}\,1\{\Lambda_1>x\}]. \tag{62}$$

Combining (60)–(62) and taking into account (22) and (30), we obtain

$$\begin{aligned}\mathbf{P}(Z_{n-1}>e^{xc_n};\ Z_n=0)&\sim(1-\rho)\,\frac{\mathbf{P}(Z_{n-1}>0)}{n}\,\frac{\mathbf{E}\,[\Lambda_1^{-\alpha}1\{\Lambda_1>x\}]}{\mathbf{E}\,[\Lambda_1^{-\alpha}]}\\&=(1-\rho)\,\frac{\mathbf{P}(Z_{n-1}>0)}{n}\,\mathbf{P}(\widetilde{\Lambda}_1>x).\end{aligned}$$

To complete the proof of Theorem 2 it remains to note that

$$(1-\rho)\,\frac{\mathbf{P}(Z_{n-1}>0)}{n}\sim(1-\rho)\,\frac{\theta\mathbf{P}(T^->n-1)}{n}\sim\theta\mathbf{P}(T^-=n)$$

in view of (3) and (39).

Proof of Theorem 3. Let ϕ be an arbitrary bounded continuous function from $D[0,1]$ and let

$$Z^{(n)}=\biggl\{\frac{\log Z_{[(n-1)t]}}{c_n},\ 0\leqq t\leqq 1\biggr\}.$$

As in the proof of Theorem 2, for any $x>0$,

$$\begin{aligned}&\sum_{k>e^{xc_n}}\mathbf{E}\,[\phi(Z^{(n)});\ Z_{n-1}=k]\,\mathbf{P}(Z_1=0\,|\,Z_0=k)\\&\sim\mathbf{P}(X\leqq -c_n)\,\mathbf{E}\biggl[\phi(Z^{(n)})\biggl(\frac{\log Z_{n-1}}{c_n}\biggr)^{-\alpha}1\{Z_{n-1}>e^{xc_n}\}\biggr]\\&\sim\theta\mathbf{P}(T^-=n)\,\mathbf{E}\biggl[\phi(Z^{(n)})\biggl(\frac{\log Z_{n-1}}{c_n}\biggr)^{-\alpha}1\{Z_{n-1}>e^{xc_n}\}\,\Big|\,Z_{n-1}>0\biggr]\\&\sim\mathbf{P}(T=n)\,\frac{\mathbf{E}\,[\phi(\Lambda)\Lambda_1^{-\alpha}\,1\{\Lambda_1>x\}]}{\mathbf{E}\,[\Lambda_1^{-\alpha}]}=\mathbf{P}(T=n)\,\mathbf{E}\,[\phi(\widetilde{\Lambda})\,1\{\widetilde{\Lambda}_1>x\}],\end{aligned} \tag{63}$$

where in the last step we have used Corollary 1.6 in [1].

On the other hand, according to (10),

$$\begin{aligned}&\sum_{0<k\leqq e^{xc_n}}\mathbf{E}\,[\phi(Z^{(n)});\ Z_{n-1}=k]\,\mathbf{P}(Z_1=0\,|\,Z_0=k)\\&\leqq\sup|\phi|\,\mathbf{P}(0<Z_{n-1}\leqq e^{xc_n};\ Z_n=0)=o(\mathbf{P}(T=n))\end{aligned} \tag{64}$$

as $x\downarrow 0$. Combining (63) and (64), we get

$$\lim_{n\to\infty}\mathbf{E}\,[\phi(Z^{(n)})\,|\,T=n]=\lim_{x\downarrow 0}\mathbf{E}\,[\phi(\widetilde{\Lambda})\,1\{\widetilde{\Lambda}_1>x\}]=\mathbf{E}\,[\phi(\widetilde{\Lambda})],$$

completing the proof of Theorem 3.

Proof of Theorem 4. By the equalities (52) and (53) we see that, as $n \to \infty$

$$\mathbf{P}(\{T = n\} \Delta \{Z_{n-1} > 0, \mu_n = n\}) = o(\mathbf{P}(T^- = n)),$$

where Δ is the symmetric difference of the respective events. Hence, for arbitrary bounded continuous function ϕ: $D[0,1] \to \mathbf{R}$ we have

$$\mathbf{E}\,[\phi(W^{r_n,n});\ T = n] = \mathbf{E}\,[\phi(W^{r_n,n});\ Z_{n-1} > 0,\ \mu_n = n] + o(\mathbf{P}(T^- = n)).$$

Moreover, it follows from formula (54) that for any $\varepsilon > 0$ there exists l such that

$$|\mathbf{E}\,[\phi(W^{r_n,n});\ Z_{n-1} > 0,\ \mu_{n-1} > l,\ \mu_n = n]| \leqq \varepsilon \mathbf{P}(T^- = n).$$

Thus, to prove Theorem 4 it suffices to find for each fixed k the asymptotics of the quantity

$$\mathbf{E}\,[\phi(W^{r_n,n});\ Z_{n-1} > 0,\ \mu_{n-1} = k,\ \mu_n = n]$$

as $n \to \infty$.

Let

$$\psi(z, s, r, n) := \mathbf{E}_z[\phi(e^{-s} W^{r,n});\ Z_{n-1} > 0,\ T^- = n],$$

where $\mathbf{E}_z[\,\cdot\,]$ means that the process starts at moment zero by z individuals. Clearly,

$$\begin{aligned} &\mathbf{E}\,[\phi(W^{r_n,n});\ Z_{n-1} > 0,\ \mu_{n-1} = k,\ \mu_n = n \,|\, \mathcal{F}_k] \\ &\qquad = \psi(Z_k, S_k, r_n - k, n - k)\, 1(Z_k > 0)\, 1(\mu_k = k). \end{aligned} \tag{65}$$

It follows from Proposition 3.1 in [1] that there exists a random variable W^+ satisfying the condition $\mathbf{P}(0 < W^+ < \infty) = 1$ such that for each $s \geqq 0$

$$\phi(e^{-s} W^{r_n,n})\, 1(Z_{n-1} > 0) \to \phi(e^{-s} W^+)\, 1(W^+ > 0) \quad \mathbf{P}^+\text{-a.s.}$$

This fact and Lemma 4 imply, as $n \to \infty$,

$$\psi(z, s, r, n) \sim \mathbf{E}_z^+[\phi(e^{-s} W^+)\, 1(W^+ > 0)]\, \mathbf{P}(T^- = n),$$

whence, using (65), we deduce that for each fixed k

$$\begin{aligned} &\mathbf{E}\,[\phi(W^{r_n,n});\ Z_{n-1} > 0,\ \mu_{n-1} = k, \mu_n = n] \\ &\qquad \sim \mathbf{E}\,[\mathbf{E}_{Z_k}^+[\phi(e^{-S_k} W^+)\, 1(W^+ > 0)];\ Z_k > 0,\ \mu_k = k]\, \mathbf{P}(T^- = n) \end{aligned}$$

as $n \to \infty$. As a result we get

$$\begin{aligned} &\mathbf{E}\,[\phi(W^{r_n,n});\ T = n] \sim \mathbf{P}(T^- = n) \\ &\qquad \times \sum_{k=0}^{\infty} \mathbf{E}\,[\mathbf{E}_{Z_k}^+[\phi(e^{-S_k} W^+)\, 1(W^+ > 0)];\ Z_k > 0,\ \mu_k = k] \end{aligned}$$

as $n \to \infty$. Hence

$$\begin{aligned} &\lim_{n\to\infty} \mathbf{E}\,[\phi(W^{r_n,n}) \,|\, T = n] \\ &\qquad = \sum_{k=0}^{\infty} \mathbf{E}\,[\mathbf{E}_{Z_k}^+[\phi(e^{-S_k} W^+)\, 1(W^+ > 0)];\ Z_k > 0,\ \mu_k = k]. \end{aligned}$$

Comparing the obtained relation with the proof of Theorem 3.1 in [1], we see that the distribution specified by the right-hand side of the last equality coincides with that one in [1]. Theorem 4 is proved.

Proof of Corollary 1. Letting $x \to 0$ in (9), we get

$$\liminf_{n\to\infty} \frac{\mathbf{P}(T=n)}{\mathbf{P}(T^-=n)} \geqq \theta.$$

Assuming that there exists $\varepsilon > 0$ such that

$$\mathbf{P}(T=n) \geqq (\theta+\varepsilon)\,\mathbf{P}(T^-=n)$$

for all $n \geqq N$ and summing this inequality over n from arbitrary $n_0 > N$ to ∞, we deduce

$$\mathbf{P}(T \geqq n_0) \geqq (\theta+\varepsilon)\,\mathbf{P}(T^- \geqq n_0)$$

for all $n_0 \geqq N$, which contradicts (3).

Proof of (14). Representation (48) implies

$$\mathbf{P}(Z_{n-1}=j) = \mathbf{E}\,[H_{n-1}^2 e^{-S_{n-1}}(1-H_{n-1}e^{-S_{n-1}})^{j-1}], \qquad j \geqq 1.$$

Hence, by Lemma 5,

$$\sup_{j\geqq 1} \mathbf{P}(Z_{n-1}=j) \leqq \mathbf{E}\,[H_{n-1}^2 e^{-S_{n-1}}] \leqq \mathbf{E}\,[e^{2L_{n-1}-S_{n-1}}] \leqq \frac{C}{nc_n} \leqq \frac{C_1}{\sigma n^{3/2}},$$

where in the last step we have used the relationship $c_n \sim \sigma\sqrt{n}$. Thus,

$$\mathbf{P}(Z_{n-1} > N;\ T=n) = \sum_{j=N+1}^{\infty} \mathbf{P}(Z_{n-1}=j)\,\mathbf{E}\,f_{00}^j \leqq \frac{C_1}{\sigma n^{3/2}} \sum_{j=N+1}^{\infty} \mathbf{E}\,f_{00}^j. \tag{66}$$

According to Theorem 1 in [7] the conditions $\sigma^2 < \infty$ and (13) yield $\mathbf{P}(T=n) \sim Cn^{-3/2}$. From this estimate, the first condition in (13), and (66) we get (14).

REFERENCES

[1] V. I. Afanasyev, J. Geiger, G. Kersting, and V. A. Vatutin, *Criticality for branching processes in random environment*, Ann. Probab., 33 (2005), pp. 645–673.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, UK, 1987.

[3] R. Doney, *Conditional limit theorems for asymptotically stable random walks*, Z. Wahrsch. Verw. Gebiete, 70 (1985), pp. 351–360.

[4] R. Durrett, *Conditioned limit theorems for some null recurrent Markov processes*, Ann. Probab., 6 (1978), pp. 798–827.

[5] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, John Wiley, New York, 1971.

[6] B. A. Rogozin, *Asymptotic behavior of the coefficients of functions of power series and Fourier series*, Siberian Math. J., 17 (1976), pp. 492–498.

[7] V. A. Vatutin and E. E. Dyakonova, *Critical branching processes in random environment: The probability of extinction at a given moment*, Discrete Math. Appl., 7 (1997), pp. 469–496.

[8] V. A. Vatutin and E. E. Dyakonova, *Branching processes in a random environment and bottlenecks in the evolution of populations*, Theory Probab. Appl., 51 (2007), pp. 189–210.

[9] V. A. Vatutin and A. E. Kyprianou, *Branching processes in random environment die slowly*, in Proceedings of the Fifth Colloquium on Mathematics and Computer Science (September 22–26, Blaubeuren, Germany), Discrete Math. Theor. Comput. Sci. Proc., AI, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2008, pp. 379–395. Available online at http://arxiv.org/abs/0804.1155

[10] V. A. Vatutin and V. Wachtel, *Local probabilities for random walks conditioned to stay positive*, Probab. Theory Related Fields, 143 (2009), pp. 177–217.

[11] V. M. Zolotarev, *Mellin–Stieltjes transforms in probability theory*, Theory Probab. Appl., 2 (1957), pp. 433–460.