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Sufficient condition for Pareto optimization in Banach spaces

by

S. ROLEWICZ (Warszawa)

Abstract. The paper contains sufficient conditions for vector optimization in Banach spaces.

The aim of this paper is an extension of the results of [3] concerning sufficient conditions for vector optimization to Banach spaces.

Let X, Y_1, Y_2, Z, P be Banach spaces over reals. Let Y_1, Y_2, P be ordered. Let F, G_1, G_2, H be continuously differentiable operators mapping a domain $U \subset X$ into

$$F\colon U \to P,$$
$$G_1\colon U \to Y_1,$$
$$G_2\colon U \to Y_2,$$
$$H\colon U \to Z.$$

We are looking for a local Pareto minimum of the following problem

$$\text{(VP)}\qquad \begin{aligned} &F(x) \to \inf,\\ &G_1(x) \leqslant 0,\\ &G_2(x) \leqslant 0,\\ &H(x) = 0. \end{aligned}$$

We recall that $x_0 \in U$ is a *local Pareto minimum* of problem (VP) if there is a neighbourhood Q of x_0 such that for $x' \in Q$ satysfying constraints of problem (VP)

$$F(x) \leqslant F(x_0) \quad \text{implies} \quad F(x_0) \leqslant F(x).$$

THEOREM 1. *Suppose that*

(i) *there are continuous linear functionals*

$$\alpha \in P^*, \quad \lambda_1 \in Y_1^*, \quad \lambda_2 \in Y_2^*, \quad \gamma \in Z^*$$

such that

$$\alpha(\nabla F)+\lambda_1(\nabla G_1)+\lambda_2(\nabla G_2)+\gamma(\nabla H) = 0,$$

where ∇F, ∇G_1, ∇G_2, ∇H *are the differentials of* F, G_1, G_2, H *taken at the point* x_0 *(this is called a necessary condition of optimality of the Kuhn–Tucker type)*;

(ii) *the functionals* $\alpha, \lambda_1, \lambda_2$ *are uniformly positive, i.e., there are positive constant* C_α, C_1, C_2 *such that for* $p \geqslant 0$, $y_1 \geqslant 0$, $y_2 \geqslant 0$

$$\|p\| \leqslant C_\alpha \alpha(p),$$

$$\|y_1\| \leqslant C_1\lambda_1(y_1),$$

$$\|y_2\| \leqslant C_2\lambda_2(y_2);$$

(iii) *the constraints are active at* x_0, *i.e.,*

$$G_1(x_0) = 0, \quad G_2(x_0) = 0, \quad H(x_0) = 0;$$

(iv) ∇F *is a surjection on* P *and* $(\nabla G, \nabla G_2, \nabla H)$ *is a surjection on* $Y_1 \times Y_2 \times H$;

(v) *the space* $L_1 = \ker\nabla F$ *and the halfsubspace*

$$L_2 = \ker\nabla G_1 \cap \ker\nabla H \cap \{x\colon\ \nabla G_2(x) \leqslant 0\}$$

have a positive gap d, *i.e.,*

$$d = \max(\inf\{\|x-y\|,\ x \in L_1, y \in L_2, \|x\| = 1\},$$
$$\inf\{\|x-y\|, x \in L_1, y \in L_2, \|y\| = 1\}) > 0.$$

Then x_0 *is a local Pareto minimum of problem* (VP).

Proof. The prof will be conducted in 3 steps. In the first step we shall show that x_0 is a local minimum of the following scalar problem:

$$\text{(MPe)}\qquad \begin{aligned} &\alpha(F(x))+\beta\|F(x)-F(x_0)\|\to\inf,\\ &G_1(x) = 0,\\ &G_2(x) \leqslant 0,\\ &H(x) = 0 \end{aligned}$$

for all $\beta > 0$. In the second step, using results of [3] we shall show that x_0 is a local solution of the following problem

$$\text{(MP)}\qquad \begin{aligned} &\alpha(F(x))+\beta\|F(x)-F(x_0)\|\to\inf,\\ &G_1(x) \leqslant 0,\\ &G_2(x) \leqslant 0,\\ &H(x) = 0. \end{aligned}$$

The third step consists of the showing that for sufficiently small β each local solution of the problem (MP) is a local Pareto solution of problem (VP).

We shall start with the first step. By assumption (iv) and by the Ljusternik theorem [1] for each $\varepsilon > 0$ there is Q such that for $x \in Q$, such that $G_1(x) = 0$, $G_2(x) \leqslant 0$, $H(x) = 0$, there is $h \in L_2$ such that

$$\|x_0+h-x\| \leqslant \varepsilon\|h\|. \tag{1}$$

Basing ourselves on (v) and (iv), we find that there is a $k > 0$ such that

$$\|h\| \leqslant k\|\nabla F(h)\|. \tag{2}$$

Now we shall estimate $\alpha(F(x)-F(x_0))$ from below. By assumption (i), $\alpha(\nabla F(h)) \geqslant 0$. Thus by (1) and (2),

$$\alpha(F(x)-F(x_0)) \geqslant -\varepsilon\|h\| \geqslant -\varepsilon k\|\nabla F(h)\|.$$

By (1) and by the definition of a differential there is a neighbourhood $Q_1 \subset Q$ such that for $x \in Q_1$ satisfying the constraint there is an $h \in L_2$ such that (1) holds and

$$\|\nabla F(h)\| \leqslant 2\ \|(F(x)-F(x_0))\|.$$

Hence

$$\alpha(F(x))-\alpha(F(x_0)) \geqslant -2\varepsilon h\ \ \|F(x)-F(x_0)\|.$$

Taking $\beta > 2\varepsilon h$, we find that x_0 is a local solution of the problem (MPe).

Taking β sufficiently small, by Theorem 1 of [3] we conclude that x_0 is a local minimum of problem (MP). Here we use the fact that λ_1 is uniformly positive. To finish the proof we shall take $\beta < 1/C$. Then the function $\alpha(F(x))+\beta\|F(x)-F(x_0)\|$ is an ordering function at the point x_0, which means that $F(x) \leqslant F(x_0)$ implies that

$$\alpha(F(x))+\beta\|F(x)-F(x_0)\| \leqslant \alpha(F(x_0)).$$

In fact,

$$\alpha(F(x_0)-F(x)) \geqslant (1/C)\|F(x)-F(x_0)\| \geqslant \beta\|F(x)-F(x_0)\|.$$

Thus (see for example [4]) the fact that x_0 is a minimum of the function $\alpha(F(x))+\beta\|F(x)-F(x_0)\|$ implies that it is a local Pareto minimum of problem (VP).

References

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Commutative differential algebras with an algebraic element

by

L. BERG (Rostock)

Dedicated to Jan Mikusiński on the 70th birthday

Abstract. There are constructed commutative differential algebras containing an element h, which satisfies a polynomial equation. For the commutativity it is necessary that the polynomial possesses a double zero. In one case the algebras contain also an integral of h.

P. Antosik, J. Mikusiński and R. Sikorski in [2] suggested the study of associative differential algebras containing an element h with the properties

$$h = h^2 \tag{1}$$

and $h' \neq 0$. Since h should be interpreted as Heaviside's jump function, the derivative of h was denoted by

$$\delta = h'. \tag{2}$$

Article [3] gives a survey of such algebras. Afterwards article [4] was written, where (1) was replaced by

$$h = 3h^2-2h^3 \tag{3}$$

corresponding to the property $h(0) = 1/2$. However, the results obtained so far are not satisfactory; particulary, all differential algebras with (1) or (3) are noncommutative.

In what follows we construct commutative differential algebras (cf. [5]) where h satisfies other algebraic relations than (1) and (3). At the end of this article we list the references [8]–[16], which were omitted in [3] in printing (the numbering of references refers to this in paper [3]). In [12] one finds similar differential algebras without an algebraic relation for h.