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SUFFICIENT CONDITIONS FOR BIBO ROBUST STABILIZATION : GIVEN BY THE GAP METRIC

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Abstract

A relation between coprime fractions and the gap metric is presented. Using this result we provide some sufficient conditions for BIBO robust stabilization for a very wide class of systems. These conditions allow the plant and compensator to be disturbed simutaneously.

Key words: Robust stabilization; Gap metric; Coprime fraction.

1 Introduction

In a very real sense, almost all control system design problems are concerned with robust stabilization. One uses a mathematical model to design a controller that produces a stable feedback system when either the model or the physical system is in the loop. Mathematical design procedures often produce a high order controller, while the engineers prefer a low order one which can be easily manipulated. On the whole, in control system design it is necessary to consider the robust stabilization problem in which both plant and controller are subjected to uncertainties.

In order to investigate the system uncertainties involved in robust stability problem in a general sense, Vidyasagar[11] and

Zames et al.[13] proposed the graph topology and the gap topology respectively. Zhu[15] reformulated the two topologies in a very general setting.

It is known[6,7,11,12,13,15] that both the graph topology and the gap topology are the weakest topologies in which feedback stability is a robust property. More precisely, any plant P_2 can be stabilized by a controller which stabilizes plant P_1 if the plant P_2 is in a neighborhood of P_1 in the graph topology (or in the gap topology). So, these topologies are good measures of plants in the robust stability problem. We list some properties of the two topologies from [6,7,11,12,13,15] in the following :

1) The gap topology can be defined for a more general class of systems than the graph topology. If one confines them to the same plant set, they coincide.

2) Restricted to stable plants, the graph topology (or the gap topology) is identical to the norm topology.

3) Both the gap topology and the graph topology can be metrized.

4) The set of stable plants viewed as a subset of all plants (including stable plants and unstable plants) is open in the graph topology (or the gap topology).

5) A plant sequence (P_n) converges to P_0 in the graph topology (or in the gap topology) if and only if any controller C which stabilize P_0 also stabilizes P_n if n large enough , and the closed loop transfer matrix sequence $H(P_n,C_n)$(see Figure 2.1) converges to $H(P_0,C_0)$.

When one wants to apply these topologies to practical problems, the metric descriptions of the topologies are needed. Vidyasagar [11,12] designed a graph metric for lumped linear time invariant (LTI) systems, and by using this metric he offered a sufficient condition for robust stability. Callier et al. [3] extended this metric to single-input-single-output (SISO) distributed LIT systems, and Zhu [14] presented a graph metric for a class of multiple-input-multiple-output (MIMO) distributed LTI systems. Generally, it is difficulty to extend the definition of the graph metric to the distributed LTI systems because of the the spectral factorization problem is involved. Praagman [10] offered another graph metric which has a simple form and can be easily computed in the SISO case.

The gap metric can be defined for distributed LTI systems as well as lumped LTI systems. In this paper, we are going to give a sufficient condition for robust stability using the gap metric. This is a parallel work to the sufficient result given by Vidyasagar[12] in the graph metric. Our result depends upon a relation obtained in this paper between the coprime fraction and the gap metric. The concept of stability concerned in this paper is bounded-input-bounded-output (BIBO) stability. For lumped LTI system, BIBO stability is identical to the internal stability (or exponential stability), whereas for distributed LTI system this property is lost. The equivalence of BIBO stability and internal stability for a very wide class of infinite dimensional systems has been offered by Curtain [5].

This paper is organized in the following way: In section 2, we will introduce the framework as well as the definition of the gap metric. A relation between the gap metric and coprime fractions are presented in section 3. Finally, our main results, sufficient conditions for robust stability, are given in section 4.

2 Framework

In this section, we present the framework which was built in [15], and the definition of the gap metric.

Let H be an integral domain and F, which contains H, be a subset of the quotient field of H . Assume that X is a Hilbert space. Our framework is based on the following

Basic Assumption Each element $P \in F$ is a linear operator mapping X to X and this operator is bounded iff $P \in H$.

We consider F as the universe of the plants and H as the set of the stable plants. The following examples are given in order to demonstrate that the basic assumption is reasonable and including many important cases.

Example 1: Let H be the set of all proper rational functions without poles in the closed right half plan (RHP) and F be the set of all rational functions. The input and output space is chosen to be $H^2(C_+)$, the Hardy space. In this case, the basic assumption is satisfied.

Example 2: Take H to be $A_{-}(0)$, the algebra of the transfer functions studied by Callier and Desoer [1,2], take F to be B(0), and take X to be $H^2(C_+)$, Then the basic assumption holds.

As usual, let M(H) and M(F) denote the set of matrices with entries in H and F respectively. If necessary , we write $M(.)^{n\times m}$ to indicate the dimensions.

According to the basic assumption, each element $p\in H$ is a bounded operator mapping X to X. By [15, lemma 2.3], one can easily show that each element $f\in F$ is a closed operator mapping X to X. Consequently, each element $P\in M(H)^{n\times m}$ is a bounded operator mapping X^m to X^n and each element $P\in M(F)^{n\times m}$ is a closed operator mapping X^m to X^n.

We say that $P\in M(F)^{n\times m}$ has a right coprime fraction (r.c.f.) over the set of bounded operators, if there exist $N\in B(X^m, X^n)$ and $D\in B(X^m)$ such that

1) D is invertible;
2) There exist X and Y in the set of bounded operators, such that

$$XN + YD = I$$

3) $P = ND^{-1}$.

The left coprime fraction (l.c.f.) can be defined in the same way.

In this paper our result only holds for a subset R(F) of M(F) rather than M(F) itself, where R(F) consists of all elements in M(F) which has both right and left coprime fractions over the set of bounded operators.

Let us consider the standard feedback system in Figure 2.1, where P is the plant and C is the compensator. the closed loop transfer matrix is

$$H(P,C) := \begin{Bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{Bmatrix}$$

It is assumed that the system is well posed, so that the indicated inverse exists.

The feedback system is said to be stable iff $H(P,C) \in M(H)$.

Now we are in a position to define the gap metric. We know that each element P in M(F) is a closed operator mapping X^m to X^n. Denote the graph of P by G(P). Then G(P) is a closed subspace in $X^m \times X^n$. Let $\Pi(P)$ denote the orthogonal projection on the graph G(P). Then gap metric can be defined as

$$\delta(P_1,P_2) := \|\Pi(P_1) - \Pi(P_2)\| \qquad P_1,P_2 \in M(F)$$

The topology generated by the gap metric is called the gap topology.

From [4], one knows that if $P \in R(F)$, then

$$\Pi(P) = \begin{Bmatrix} D \\ N \end{Bmatrix} (D^*D+N^*N)^{-1} [D^*, N^*]$$

$$= I - \begin{Bmatrix} \tilde{N}^* \\ -\tilde{D}^* \end{Bmatrix} (\tilde{N}\tilde{N}^* + \tilde{D}\tilde{D}^*)^{-1} [\tilde{N}, \tilde{D}]$$

where (N,D) and $(\tilde{N},\tilde{D})$ are any right and left coprime fraction pair of P respectively, and D^* means the dual of D. (according to the basic assumption, D (or N etc.) is a bounded operator, so the dual exist.)

3 Gap metric and coprime fractions

The main purpose of this section is to dig out the relation between the gap metric and the coprime fractions. This relation plays an important role in our main result in the next section.

Now we start with the following lemma.

Lemma 3.1 Assume that $P \in R(F)^{m \times n}$, $D \in B(X^m)$

and $N \in B(X^m, X^n)$. If one regards P as an operator mapping X^m to X^n and denotes the graph of P by G(P), then

$$G(P) = \text{Range} \begin{bmatrix} D \\ N \end{bmatrix}$$

$$= \{ \ (Dz, \ Nz) : z \in X^m \ \} \qquad (3.1)$$

iff (N,D) is an r.c.f. pair of P.

Proof The sufficient part can be found in [12], here we just prove the necessary part.

Assume (D_1, N_1) is an r.c.f. pair of P and by the sufficient part, one knows that

$$G(P) = \{ \ (D_1 z, \ N_1 z) : z \in X^m \ \} \qquad (*)$$

Let X,Y be the operators such that

$$X N_1 + Y D_1 = I$$

Define

$$U := XN + YD$$

By (*) and (3.1), one knows that for every x in X^m there is a unique y in X^m such that

$$\begin{bmatrix} D \\ N \end{bmatrix} x = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} y$$

and vice versa. Equivalently the operator U: X^m-- X^m

$$Uy = x$$

is bijective. Consequently, U^{-1} as well as U is a bounded operator. As a result,

$$\begin{pmatrix} D \\ N \end{pmatrix} = \begin{pmatrix} D_1 \\ N_1 \end{pmatrix} U$$

Hence (N,D) is an r.c.f. pair of P. !

The next lemma is an alternative version of a result in Krasnosel'skii et al. [9,p206].

Lemma 3.2 Let $P_i \in R(F)^{n\times m}$ $(i=1,2)$. Then $\Pi(P_1)$ maps $G(P_2)$ bijectively onto $G(P_1)$ iff

$$\delta(P_1,P_2) < 1$$

Using lemma 3.1 and lemma 3.2, we can prove

Theorem 3.1 Let $P_i \in R(F)^{n\times m}$ $(i=1,2)$, and (N_1,D_1) be an r.c.f. pair of P_1. Define

$$\begin{pmatrix} D_2 \\ N_2 \end{pmatrix} := \Pi(P_2) \begin{pmatrix} D_1 \\ N_1 \end{pmatrix}$$

Then (N_2, D_2) is an r.c.f. pair of P_2 iff

$$\delta(P_1,P_2) < 1$$

Proof (sufficiency) By lemma 3.2, $\Pi(P_2)$ maps $G(P_1)$ onto $G(P_2)$ bijectively, therefore we know

$$G(P_2) = \{ (D_2 z, N_2 z) : z \in X^m \} \qquad (*)$$

From lemma 3.1, one knows that (D_2, N_2) is an r.c.f. pair of P_2.

(necessity) From the given condition and lemma 3.1 one can easily check that $\Pi(P_2)$ maps $G(P_1)$ onto $G(P_2)$ bijectively. Furthermore, according to lemma 3.2, one has

$$\delta(P_1, P_2) < 1$$!

Remark: The sufficient part of this result is also obtained by Vidyasagar[11]. But the proof is different from the one given here.

Now we turn our attention to the left coprime fraction and we wish to get the similar result as Theorem 3.1.

For a given plant $P \in R(F)^{n \times m}$, let $(\tilde{D},\tilde{N})$ be any l.c.f. pair of P, i.e.

1) $\tilde{D}$ is invertible;

2) there exist bounded operators X and Y such that

$$\tilde{N}X + \tilde{D}Y = I$$

3) $P = \tilde{D}^{-1}\tilde{N}$.

Define

$$T_P := \tilde{N}^*(-\tilde{D}^{*-1})$$

Then $(\tilde{N}^*,-\tilde{D}^*)$ is an r.c.f. pair of T_P, i.e.

1) $-\tilde{D}^*$ is invertible;

2) $X^*\tilde{N}^* + Y^*\tilde{D}^* = I$

3) $T_P = \tilde{N}^*(-\tilde{D}^{*-1})$

Remark: T_P is uniquely determined by P and independent of the choice of an l.c.f. pair of P.

One can readily prove the following lemma.

Lemma 3.3 Suppose that $P \in R(F)^{n \times m}$, $\tilde{D} \in B(X^n)$ and $\tilde{N} \in B(X^n, X^m)$. Then

$(\tilde{D},\tilde{N})$ is an l.c.f. pair of P

iff

$(\tilde{N}^*,-\tilde{D}^*)$ is an r.c.f. pair of T_P

Lemma 3.4 Let $P \in R(F)^{m\times n}$, $\tilde{D} \in B(X^n)$ and $\tilde{N} \in B(X^m, X^n)$. Then

$$G(P) = \mathrm{Ker}\ [\tilde{N}, -\tilde{D}]$$

$$= \{ (x,y) \in X^m \times X^n : \tilde{N}x - \tilde{D}y = 0 \}$$

iff
$(\tilde{D},\tilde{N})$ is an l.c.f. pair of P.

Proof One can easily check the sufficient part. To prove the necessity, we take one of the l.c.f. pair $(\hat{D},\hat{N})$. By the sufficiency, we know that

$$G(P) = \mathrm{Ker}\ [\hat{N}, -\hat{D}]$$

$$= \{ (x,y) \in X^m \times X^n : \hat{N}x - \hat{D}y = 0 \}$$

Hence

$$\mathrm{Ker}\ [\tilde{N}, -\tilde{D}] = \mathrm{Ker}\ [\hat{N}, -\hat{D}]$$

And

$$\mathrm{Ker}\ [\tilde{N}, -\tilde{D}]^{\perp} = \mathrm{Ker}\ [\hat{N}, -\hat{D}]^{\perp}$$

i.e.

$$\mathrm{Range}\begin{bmatrix} -\tilde{D}^* \\ \tilde{N}^* \end{bmatrix} = \mathrm{Range}\begin{bmatrix} -\hat{D}^* \\ \hat{N}^* \end{bmatrix}$$

Because the right hand side of the above equality is $G(T_P)$, and by lemma 3.1 we get that $(\tilde{N}^*, -\tilde{D}^*)$ is an r.c.f. pair of T_P. furthermore, by lemma 3.3 $(\tilde{D},\tilde{N})$ is an l.c.f. pair of P.

!

Lemma 3.5 $\delta(P_1, P_2) = \delta(T_{P1}, T_{P2})$

Proof By definition one can easily check this. We omit the details.

<u>Theorem 3.2</u> Suppose $P_i \in R(F)^{n\times m}(i=1,2)$, and $(\tilde{D}_1, \tilde{N}_1)$ is an l.c.f. pair of P_1. Define

$$\begin{pmatrix} \tilde{D} \\ \tilde{N} \end{pmatrix} = (\Pi(P_2))^{\perp} \begin{pmatrix} -\tilde{D}_1{}^* \\ \tilde{N}_1{}^* \end{pmatrix}$$

Then $(-\tilde{D}^*, \tilde{N}^*,)$ is an l.c.f. pair of P_2 iff

$$\delta(P_1, P_2) < 1$$

<u>Proof</u> Notice

$$1)\ (\Pi(P_2))^{\perp} = \Pi(T_{P2})$$

$$2)\ \delta(P_1, P_2) = \delta(T_{P1}, T_{P2})$$

And by lemma 3.3, it is equivalent to prove that $(\tilde{N},\tilde{D})$ is an r.c.f. pair of T_{P2} iff

$$\delta(T_{P1}, T_{P2}) < 1$$

This is the result of theorem 3.1. So the conclusion is true.

!

4 <u>Sufficient conditions for BIBO robust stability</u>

Now we are ready to state our main result. Let P_0 and C_0 in $R(F)^{n\times m}$ be the nominal plant and controller respectively with a stable closed loop transfer matrix $H(P_0,C_0)$. Take any r.c.f. pair (N_0,D_0) of P_0 and l.c.f. pair $(\tilde{D}_0,\tilde{N}_0)$ of C_0 respectively.

Denote

$$A_0 = \begin{pmatrix} D_0 \\ N_0 \end{pmatrix}$$

and

$$B_0 = [\tilde{D}_0, \tilde{N}_0]$$

Define

$$U_0 = B_0 \cdot A_0$$

It follows from [12] that $H(P_0,C_0)$ is stable iff U_0 is a bounded operator which maps X^m bijectively onto X^m.

Remark: Because we have assumed that $H(P_0,C_0)$ is stable, U_0 can be chosen as the identity.

Suppose that P,C in R(F) are the plant and controller considered to be disturbed from P_0 and C_0 respectively.

For the sake of convenience, denote

$$w = [\|A_0\| \; \|B_0\| \; \|U_0^{-1}\|]$$

Theorem 4.1 If

$$\delta(C,C_0)+\delta(P,P_0) < w^{-1} \tag{4.1}$$

then H(P,C) is stable.

Proof First, one can easily check that the right hand side of (4.1) is smaller than 1, According to theorem 3.1 and theorem 3.2, we can define an r.c.f. pair (N,D) of P and an l.c.f. pair $(\tilde{D},\tilde{N})$ of C respectively with

$$\begin{pmatrix} D \\ N \end{pmatrix} = \Pi(P) \begin{pmatrix} D_0 \\ N_0 \end{pmatrix}$$

and

$$\begin{pmatrix} -\tilde{D}^* \\ \tilde{N}^* \end{pmatrix} = \Pi(C)^{\perp} \begin{pmatrix} -\tilde{D}_0^* \\ \tilde{N}_0^* \end{pmatrix}$$

Denote

$$A := \begin{pmatrix} D \\ N \end{pmatrix}$$

and

$$B = [\tilde{D}, \tilde{N}]$$

then

$$\|BA - B_0A_0\|$$

$$= \|BA - B_0A + B_0A - B_0A_0\|$$

$$= \|(B - B_0)A + B_0(A - A_0)\|$$

$$\leq \|A\| \, \|(B - B_0)\| + \|(A - A_0)\| \, \|B_0\|$$

$$\leq \|A_0\| \, \|(\Pi(C)-\Pi(C_0))B_0)\| + \|(\Pi(P)-\Pi(P_0)\| \, \|A_0)\| \, \|B_0\|$$

$$= \|A_0\| \, \|B_0)\| \, \delta(C,C_0) + \|A_0)\| \, \|B_0\| \, \delta(P,P_0)$$

$$< \|U_0^{-1}\|^{-1}$$

Therefore, BA is invertible and the inverse is also a bounded operator. Consequently H(P,C) is stable.

!

We can also give a sufficient condition by using only r.c.f. pairs of both plant and controller.

As before, let P_0, C_0 in $R(F)^{n \times m}$ and $H(P_0,C_0)$ is stable. Assume that (N_{P0},D_{P0}) and (N_{C0},D_{C0}) are any r.c.f. pairs of P_0 and C_0 respectively. Denote

$$A_0 = \begin{pmatrix} -N_{P0} \\ D_{P0} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} D_{C0} \\ N_{C0} \end{pmatrix}$$

$$U_0 := [\, B_0, A_0 \,]$$

$$w := \max\{ \|A_0\| , \|B_0\| \}$$

and

$$m = w \|U_0^{-1}\|$$

It follows from [12] that $H(P_0,C_0)$ is stable iff U_0 is bijective.

As above, suppose P, C in $R(F)^{n\times m}$ to be the disturbed plant and controller respectively.

Theorem 4.2 If

$$\delta(C, C_0) + \delta(P, P_0) < m^{-1} \qquad (4.2)$$

then H(P,C) is stable.

Proof According to theorem 3.1, we can define an r.c.f. pair (N_P,D_P) of P and an r.c.f. pair (D_C,N_C) of C respectively with

$$\begin{pmatrix} D_P \\ -N_P \end{pmatrix} = \Pi(-P)\begin{pmatrix} D_{P0} \\ -N_{P0} \end{pmatrix}$$

and

$$\begin{pmatrix} D_C \\ N_C \end{pmatrix} = \Pi(C)\begin{pmatrix} D_{C0} \\ N_{C0} \end{pmatrix}$$

Denote

$$A := \begin{pmatrix} -N_P \\ D_P \end{pmatrix}$$

$$B := \begin{pmatrix} D_C \\ N_C \end{pmatrix}$$

and

$$U = [B, A]$$

then

$$\|U - U_0\|$$

$$=\|[B-B_0, A-A_0]\|$$

$$<\|B-B_0\|+\|A-A_0\|$$

$$<\|\Pi(C)-\Pi(C_0)\| \|B_0\|+\|\Pi(-P)-\Pi(-P_0)\| \|A_0\|$$

$$= \delta(C,C_0)\|B_0\|+\delta(-P,-P_0)\|A_0\|$$

$$= \delta(C,C_0)\|B_0\|+\delta(P,P_0)\|A_0\|$$

$$< [\delta(C,C_0)+\delta(P,P_0)]w$$

$$<\|U_0^{-1}\|^{-1}$$

Therefore, U is bijective. Consequently, H(P,C) is stable.

!

In the same way, we can also give another sufficient condition by using only l.c.f. pairs. For the techniques are the same we omit it.

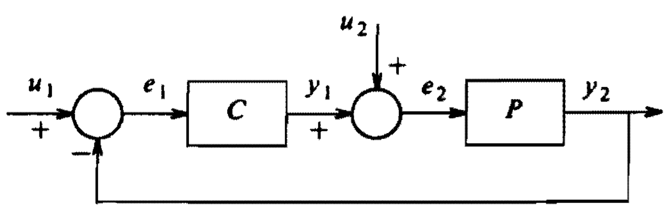

Figure 2.1 Feedback System

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