

Sufficient Conditions for Stackelberg and Nash Strategies with Memory¹

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Abstract. Sufficiency conditions for Stackelberg strategies for a class of deterministic differential games are derived when the players have recall of the previous trajectory. Sufficient conditions for Nash strategies when the players have recall of the trajectory are also derived. The state equation is linear, and the cost functional is quadratic. The admissible strategies are restricted to be affine in the information available.

Key Words. Stackelberg differential games, Nash differential games, strategies with memory, sufficiency conditions for game strategies.

1. Introduction

Stackelberg and Nash differential games have received recently a lot of attention and have been studied by several researchers. The reader can find in Refs. 1 and 5 surveys of basic concepts, definitions, and results concerning Stackelberg and Nash games, respectively. These two types of games seem to be very promising in studying large-scale systems, hierarchical systems, or situations of conflict in an engineering, economic, or social context. The definitions of Stackelberg and Nash equilibrium can be found in the literature, but we repeat these definitions here for the sake of completeness of the presentation. Let U, V be two sets and J_1, J_2 two functions $J_i: U \times V \rightarrow R, i = 1, 2$. Consider the set-valued mapping T

$$T: U \rightarrow V, \quad u \xrightarrow{T} Tu \subseteq V,$$

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defined by

$$Tu = \{v \mid v = \arg \inf [J_2(u, \bar{v}); \bar{v} \in V]\}.$$

Clearly, $Tu = \emptyset$ if the infimum is not achieved. We also consider the minimization problem

$$\inf J_1(u, v), \quad \text{subject to } u \in U, \quad v \in Tu, \quad (1)$$

where we use the usual convention $J_1(u, v) = +\infty$, if $v \in Tu = \emptyset$.

Definition of a Stackelberg Equilibrium. A pair $(u^*, v^*) \in U \times V$ is called a Stackelberg equilibrium pair if (u^*, v^*) solves (1).

In Stackelberg games, it is standard to say that a leader chooses $u \in U$ and has cost J_1 and a follower chooses $v \in V$ and has cost J_2 .

Definition of a Nash Equilibrium. A pair $(u^*, v^*) \in U \times V$ is called a Nash equilibrium pair if (u^*, v^*) satisfies

$$\begin{aligned} J_1(u^*, v^*) &\leq J_1(u, v^*), & \text{for all } u \in U, \\ J_2(u^*, v^*) &\leq J_2(u^*, v), & \text{for all } v \in V. \end{aligned} \quad (2)$$

It is a known fact that, in Stackelberg and Nash differential games, the resulting trajectory and strategy values vary with the admissible strategy spaces. By *strategy spaces*, we mean the information available to each player together with a set of functions with this information as *domain*. These functions are actually the permissible ways in which the players are allowed to use that information. For example, open-loop strategies, where at each instant of time t the players have knowledge of the present time instant t and the initial condition $x(0)$, result in different equilibrium from the strategies where at each instant of time t the players have knowledge of $x(t)$, t , $x(0)$. In the latter case, the players might be restricted, in addition, to using only affine functions of $x(t)$. Most of the results available until now deal with cases where the current state or the initial state or both of them are the only available information to the players. A more general situation is to assume that, at each instant of time, each player knows something about the previous values of the state of the system and about the previous values of his and the other player's decisions. The first attempt to derive necessary conditions for zero-sum games where the strategies depend at each instant of time t on the part of the state trajectory between $t-r$ and t , where $r > 0$, appears to be in Ref. 2. In Refs. 3 and 4, the zero-sum case is considered where one player has a time lag information on the value of the state. In Ref. 4, a Hamilton-Jacobi theory is developed for such games.

In the present paper, we consider a continuous-time, two-player, deterministic differential game with a linear state equation and two quadratic cost functionals. We consider the case where the players have, at each instant of time, recall of previous values of the trajectory, i.e., they have memory. What they remember about the previous values of the trajectory is allowed to change with the elapse of time. In our model, a wide range of delayed information structures is included, from perfect recall of the previous trajectory to recall of only one previous value of the trajectory. Cases where information about the past strategy values is available to the players are also considered. We consider strategies affine in the available information and represent them by using Lebesgue–Stieltjes integrals. Both Stackelberg and Nash equilibrium concepts are considered, and sufficient conditions are developed for a particular, but quite interesting, class of problems. Particular emphasis is placed on the Stackelberg case.

In Ref. 7, the Stackelberg differential game is solved when the leader's information at time t is $x(t)$, $x(0)$, t , and he is not restricted to use a linear function of $x(t)$. It is shown there that the leader can in general restrict himself to strategies affine in $x(t)$ and that use of nonlinear strategies in $x(t)$ will not improve his cost. The arguments of Ref. 7 can be extended to the case where the leader's information at time t is $\{x(\theta), t_0 \leq \theta \leq t\}$ and one can show that the leader does not in general deteriorate his cost if he uses strategies affine in $\{x(\theta), t_0 \leq \theta \leq t\}$. Therefore, one is motivated to restrict *a priori* the strategy of the leader to be of the form

$$\int_{t_0}^t [d_s \eta(t, s)] x(s) + b(t),$$

in which case η and b are what the leader will actually choose. For given η and b , the follower solves his problem. Necessary and sufficient conditions for the follower's problem can be found in Refs. 10 and 11 (Theorem 5.2), respectively. On the other hand, the leader's problem is quite difficult, since his unknowns are η and b . It was also shown in Ref. 7 that the principle of optimality holds in Stackelberg games iff the leader's problem can be treated as a team problem for both leader and follower. This does not necessarily mean that $J_1 = J_2$. These remarks motivate us to study Stackelberg games where the solutions are linear in $\{x(\theta), t_0 \leq \theta \leq t\}$ and constitute a team solution for the leader's problem.

The structure of the paper is the following. In Section 2, we give an example of a Stackelberg game where the leader, by using previous values of the state, forces the follower to such a reaction that the leader's final cost is the same as it would have been if both leader and follower were striving to minimize the leader's cost. The main steps in solving this example serve as an

illustration of how a more general case should be analyzed. In Section 3, we derive sufficiency conditions for optimality for a control problem of a special form (of interest on its own), which are used in the next sections. In Section 4, we apply the results of Section 3 to a Stackelberg game where the leader has recall of the previous trajectory and the game is such that the solution of the Stackelberg game (u^*, v^*) minimizes the leader's cost over all admissible (u, v) , i.e., the leader's problem is actually treated as a team problem of both the leader and follower. In Section 5, we consider certain special cases and generalizations of the Stackelberg game of Section 4. In Section 6, we apply the results of Section 3 to a Nash game where the two players have perfect recall of the whole previous trajectory. Finally, we have a conclusions section and one appendix.

Notation. Let

$$C([t_0, t_f], R^n) = C_n$$

denote the Banach spaces of continuous function $\varphi: [t_0, t_f] \rightarrow R^n$, with norm

$$\|\varphi\| = \sup\{|\varphi(t)|; t \in [t_0, t_f]\},$$

where $|\cdot|$ denotes the usual Euclidean distance in R^n . Let

$$L_1([t_0, t_f], R^n) = L_{1,n}$$

denote the Banach space of Lebesgue integrable functions $\varphi: [t_0, t_f] \rightarrow R^n$, with norm

$$\|\varphi\| = \int_{t_0}^{t_f} |\varphi(t)| dt.$$

Let

$$L_\infty([t_0, t_f], R^n) = L_{\infty,n}$$

denote the Banach space of Lebesgue measurable functions which are almost everywhere bounded, with norm

$$\|\varphi\| = \text{ess sup}\{|\varphi(t)|, t \in [t_0, t_f]\}.$$

And let

$$\text{NBV}([t_0, t_f], R^n) = \text{NBV}$$

denote the Banach space of normalized functions of bounded variation, i.e., continuous from the right on (t_0, t_f) , zero at t_f , and

$$\|\varphi\| = \text{Var}(\varphi) \text{ for } \varphi \in \text{NBV}.$$

A norm in one of these spaces is denoted sometimes by $\|\cdot\|_C, \|\cdot\|_{L_1}, \|\cdot\|_{NBV}$. B^* denotes the conjugate space of a Banach space B . If $x^* \in B^*$ and $x \in B$, we write $\langle x^*, x \rangle = x^*(x)$. The prime denotes transpose for vectors and matrices.

2. Introductory Example

In this section, we provide an example of a Stackelberg differential game where the leader uses the previous values of the state in calculating his control values. The game considered is such that the leader, by using this type of strategy, forces the follower to such a reaction that the leader's optimal cost is the one that he would achieve if both leader and follower had as their common objective the minimization of the leader's cost; i.e., the leader's problem to minimize J_1 is actually treated as a team problem where the team is composed of both the leader and the follower. A similar idea occurs in Ref. 9. The strategies found provide a Stackelberg equilibrium pair, with the property above, for any x_0 . Also, the dependence of the leader's control values on previous state values is not trivial, in the sense that the same result (team solution of the leader's problem) cannot be achieved by strategies depending only on current state value information. We develop the example in such a way that the proof of the optimality of the indicated strategies is clear. Actually, we do not give only one example, but provide a way of constructing a whole class of Stackelberg games with the above properties.

Consider the following state equation and cost functionals

$$\dot{x} = 2x + u + v, \quad x(0) = x_0, \quad t \in [0, 1], \tag{3}$$

$$J_1 = 4x(1)^2 + \int_0^1 (6x^2 + u^2 + v^2) dt, \tag{4}$$

$$J_2 = 2x(1)^2 + \int_0^1 (qx^2 + rv^2) dt, \tag{5}$$

where x, u, v are scalar-valued. The solution of the problem

$$\begin{aligned} &\underset{u,v}{\text{minimize}} J_1, \\ &\text{subject to (3),} \end{aligned} \tag{6}$$

is

$$\bar{u} = -2kx, \quad \bar{v} = -2kx, \tag{7}$$

where k solves

$$-k = 3 + 4k - 4k^2, \quad k(1) = 2,$$

and is given explicitly by

$$k(t) = \frac{15 + \exp[8(t-1)]}{10 - 2 \exp[8(t-1)]} \tag{8}$$

We want to show that there exist q, r, l_1, l_2 , so that the problem

$$\begin{aligned} &\underset{v}{\text{minimize}} J_2, \\ &\text{subject to } \dot{x} = 2x + l_1x + l_2z + v, \quad x(0) = x_0, \\ &\qquad \qquad \qquad \dot{z} = x, \quad z(0) = 0, \end{aligned} \tag{9}$$

has the solution

$$v^* = \mu_1x + \mu_2z, \tag{10}$$

and that

$$l_1(t)x^*(t) + l_2(t)z^*(t) = -2k(t)x^*(t), \quad t \in [0, 1], \tag{11}$$

$$v^*|_t = \mu_1(t)x^*(t) + \mu_2(t)z^*(t) = -2k(t)x^*(t), \quad t \in [0, 1], \tag{12}$$

$$l_2(t) \neq 0, \quad t \in [0, 1], \tag{13}$$

for any x_0 , where x^* is the common optimal trajectory of the problems (9) and (6), since (11) and (12) will hold.

It is clear that, if conditions (11) and (12) are satisfied, then the pair

$$u = l_1(t)x(t) + l_2(t) \int_0^t x(\tau) d\tau, \quad v = -2k(t)x(t)$$

constitutes a Stackelberg equilibrium pair for the Stackelberg differential game associated with (3)–(5) and where

$U = \{u | \text{value of } u \text{ at time } t \text{ is given by } u(x, t), \text{ where } x_t \in C([0, t], R), x_t(\theta) = x(\theta) \text{ for all } \theta \in [0, t], u(x, t) \text{ is Frechet differentiable in } x_t \text{ and piecewise continuous in } t \in [0, 1]\},$

$V = \{v | v \text{ is a function of } x(t) \text{ and } t, \text{ at time } t, v(x, t) \text{ is continuous in } x \in R \text{ and piecewise continuous in } t \in [0, 1]\}.$

We set

$$\alpha(t) = 2 - 4k(t), \tag{14}$$

and thus the optimal trajectory for the problem (6) is

$$x^*(t) = \exp \left[\int_0^t \alpha(\tau) d\tau \right] \cdot x_0, \tag{15}$$

$$z^*(t) = \int_0^t \left(\exp \left[\int_0^\sigma \alpha(\sigma) d\sigma \right] \right) d\tau \cdot x_0.$$

The solution v^* of (9) is

$$v^* = -(1/r)(p_1x + p_2z), \tag{16}$$

where

$$\begin{aligned} -\dot{p}_1 &= 2(2 + l_1)p_1 + 2p_2 + q - (1/r)p_1^2, & p_1(1) &= 2, \\ -\dot{p}_2 &= l_2p_1 + (2 + l_1)p_2 + p_3 - (1/r)p_1p_2, & p_2(1) &= 0, \\ -\dot{p}_3 &= 2l_2p_2 - (1/r)p_2^2, & p_3(1) &= 0. \end{aligned} \tag{17}$$

Substituting (15) and (16) in (11) and (12), we obtain

$$p_1 = 2rk - p_2\varphi, \quad l_1 = -2k - l_2\varphi, \tag{18}$$

where

$$\varphi(t) = \left(\int_0^t \exp \left[\int_0^\tau \alpha(\sigma) d\sigma \right] d\tau \right) \left(\exp \left[\int_0^t \alpha(\tau) d\tau \right] \right). \tag{19}$$

Since $\varphi = z/x$, it is easy to see that

$$\dot{\varphi} = 2 - (2 - 4k)\varphi.$$

Substituting p_1, l_1 from (18) into (17), we obtain further

$$2l_2\varphi rk - (p_2 + \varphi p_3) - q + 4rk^2 + 6r - 2rk = 0, \tag{20}$$

$$-\dot{p}_2 = l_2(2rk - p_2\varphi) - (2 - 2k - l_2\varphi)p_2 + p_3 - (1/r)(2rk - p_2\varphi)p_2, \tag{21}$$

$$-\dot{p}_3 = 2l_2p_2 - (1/r)p_2^2, \tag{22}$$

$$2r(1)k(1) - p_2(1)\varphi(1) = 2, \tag{23}$$

$$p_2(1) = 0, \tag{24}$$

$$p_3(1) = 0. \tag{25}$$

From (21) and (22), setting

$$w \triangleq p_2 + \varphi p_3,$$

we obtain

$$\dot{w} = -2rkl_2 - (2 - 4k)w, \quad w(1) = 0. \tag{26}$$

Solving (20) for $p_2 + \varphi p_3$ and substituting into (26), we obtain finally the following system, equivalent to (20)–(25):

$$\begin{aligned} & l_2\varphi rk + l_2[rk + 2(1 - 2k)\varphi rk + (d/dt)(\varphi rk)] \\ & + [(d/dt)(2rk^2 + 3r - rk) + 2(1 - 2k)(2rk^2 + 6r - 2rk)] \\ & + [(1 - 2k)q - \frac{1}{2}\dot{q}] = 0, \end{aligned} \tag{27}$$

$$-\dot{p}_3 = 2l_2(w - \varphi p_3) - (1/r)(w - \varphi p_3)^2, \tag{28}$$

$$w = 2l_2\varphi rk - q + 4rk^2 + 6r - 2rk, \tag{29}$$

$$p_2 = w + \varphi p_3, \tag{30}$$

$$l_2(1) = \frac{1}{2}(-11 + 4\dot{r}(1) + q(1))\varphi(1)^{-1}, \tag{31}$$

$$p_3(1) = 0, \tag{32}$$

$$r(1) = \frac{1}{2}. \tag{33}$$

We can choose now r, q, l_2, l_1 so as to satisfy (27)–(33) and (18). We choose $r(t)$ to be a twice-differentiable function of $t \in [0, 1]$, with

$$r(t) > 0, \quad r(1) = \frac{1}{2},$$

and $q(t)$ to be a differentiable function of t . Obviously, q and r can be chosen so that the linear differential equation for l_2 [Eq. (27)] with initial condition (31) has the solution $l_2(t) \neq 0$. For example, let

$$r \equiv \frac{1}{2}, \quad q \equiv \text{constant} \neq 11.$$

Notice that the differential equation (27) for l_2 can be solved explicitly for l_2 as soon as r and q are specified, since φ and k are known. Nonetheless, since $\varphi(0) = 0$, the point $t = 0$ is a singular point of this differential equation. The singularity was sort of expected to appear, since (as it has been shown in Ref. 7) the leader's problem is singular with respect to the partial derivative $\partial(u(x(t), t)/\partial x$ of his control; and arguments similar to those in Ref. 7 can be used to show that this holds even for the case where u is allowed to be of the more general form $u(x, t)$. Notice also that the only essential restriction on the follower's cost, in order for the leader to achieve his team solution (allowing even $l_2 = 0$), is that $r(1) = \frac{1}{2}$.

If the leader were allowed to use a strategy $u(x, t)$ which is perhaps nonlinear in the current state $x(t)$, but he was not permitted to use previous values of the state, then it should again be true that

$$u(x^*(t), t) = -2k(t)x^*(t), \quad \text{for every } x_0,$$

i.e.,

$$u\left(\exp\left[\int_0^t a(\tau) d\tau\right]x_0, t\right) = -2k(t) \exp\left[\int_0^t a(\tau) d\tau\right]x_0, \quad \text{for all } x_0 \in R,$$

from which we obtain that u is linear in x . Therefore, we conclude that, for the given example, if the leader wishes to achieve his team solution (for any x_0) when he applies his Stackelberg strategy and cannot do that with a linear strategy in $x(t)$, he cannot do it with a nonlinear strategy in $x(t)$ either. Therefore, use of memory is his only way to achieve his team solution.

In the example presented here, the two crucial steps were the identifications (11), (12) and the use of the fact that the conditions (16), (17) are sufficient to characterize completely the optimal reaction of the follower to the leader's strategy

$$u = l_1(t)x(t) + l_2(t) \int_0^t x(\tau) d\tau.$$

Therefore, in order to generalize the procedure presented to cases where more general types of strategies are used by the leader, one should provide sufficient conditions for the problem faced by the follower, in addition to imposing identifications similar to (11) and (12). In the next section, we prove sufficiency conditions for a special type of control problem, which we will use later in guaranteeing the optimality of the follower's reaction, when the leader uses strategies represented as continuous linear functionals over the whole previous trajectory.

3. A Control Problem with State-Control Constraints

Consider the following problem [Problem (P)]:

$$\text{minimize } \bar{J} = \frac{1}{2} \left[x'(t_f)Fx(t_f) + \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t)) dt \right], \quad (34)$$

$$\text{subject to } \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \quad (35)$$

$$\int_{t_0}^{t_f} [d_s \eta(t, s)]x(s) + \int_{t_0}^{t_f} [d_s \eta_1(t, s)]u(s) = q(t), \quad (36)$$

$$u \in L_{\infty, m},$$

where the matrices $A, B, Q = Q' \geq 0, R = R' \geq 0$ are piecewise continuous functions of time, $x(t) \in R^n, u(t) \in R^m$, and where the interval $[t_0, t_f]$, the matrix $F = F' \geq 0$, and $q \in L_{1, k}$ are fixed. The solution $x(t)$ of (35) is assumed to be absolutely continuous, so that (35) holds almost everywhere with respect to the Lebesgue measure in $[t_0, t_f]$. The integrals in (36) should be interpreted as Lebesgue–Stieltjes integrals. The matrix-valued function

$$\eta(t, \theta), \eta: [t_0, t_f] \times R \rightarrow R^{k \times n}$$

is measurable in (t, θ) , normalized so that

$$\eta(t, \theta) = \begin{cases} 0, & \text{for } \theta \geq t_f, \\ \eta(t, t_0), & \text{for } \theta \leq t_0. \end{cases} \quad (37)$$

$\eta(t, \theta)$ is continuous from the left in θ on (t_0, t_f) , $\eta(t, \theta)$ has bounded variation in θ on $[t_0, t_f]$ for each t , and there is a $c \in L_{1,1}$, such that

$$\left\| \int_{t_0}^{t_f} [d_s \eta(t, s)] \varphi(s) \right\|_{L_1} \leq c(t) \|\varphi\|_C, \tag{38}$$

for all $t \in [t_0, t_f]$ and for all $\varphi \in C_n$. Exactly the same assumptions hold for

$$\eta_1 : [t_0, t_f] \times R \rightarrow R^{k \times m},$$

with c_1 replacing c in (38). η and η_1 are given for Problem (P). The dimension k is arbitrary but fixed.

Problem (P) is of interest to us, since we will use the results of this section in the next ones, where we will consider games with delayed information structure. Nonetheless, it is of interest on its own. It is worthy to point out that Problem (P) is of a quite general form, since for example Problem (P'),

$$\begin{aligned} &\text{minimize } \frac{1}{2} \left[x'(t_f) F x(t_f) + \int_{t_0}^{t_f} (y'(t) Q y(t) + u_1'(t) R(t) u_1(t)) dt \right], \\ &\text{subject to } x(t) = \int_{t_0}^{t_f} [d_s \eta^1(t, s)] x(s) + \int_{t_0}^{t_f} [d_s \eta^2(t, s)] \bar{u}(s), \\ &\qquad y(t) = \int_{t_0}^{t_f} [d_s \eta^3(t, s)] x(s), \tag{39} \\ &\qquad u_1(t) = \int_{t_0}^{t_f} [d_s \eta^4(t, s)] \bar{u}(s), \\ &\qquad x(t_0) = x_0, \end{aligned}$$

can be brought to the form of Problem (P) by introducing

$$\begin{aligned} u_2(t) &= \int_{t_0}^{t_f} [d_s \eta^1(t, s)] x(s), \\ u_3(t) &= \int_{t_0}^{t_f} [d_s \eta^2(t, s)] x(s), \tag{40} \\ u_4(t) &= y(t). \end{aligned}$$

Using (39), (40), Problem (P') can be written equivalently as

$$\text{minimize } \frac{1}{2} \left[x'(t_f) F x(t_f) + \int_{t_0}^{t_f} (u_4' Q u_4 + u_1' R u_1) dt \right], \tag{41}$$

subject to $\dot{x}(t) = u_2(t) + u_3(t)$,

$$u_1 = \int_{t_0}^{t_f} [d\eta^4]\bar{u}, \quad u_2 = \int_{t_0}^{t_f} [d\eta^1]x, \quad u_3 = \int_{t_0}^{t_f} [d\eta^2]x, \quad u_4 = \int_{t_0}^{t_f} [d\eta^3]x, \tag{42}$$

where the role of x and u in (35), (36), is played now by x and $(\bar{u}, u_1, u_2, u_3, u_4)$, respectively. Clearly, (42) is of the form (36).

In the following theorem, we give sufficiency conditions for optimality for Problem (P). The proof is carried out by reformulating Problem (P) as a constrained optimization problem in a Banach space and is given in the Appendix.

Theorem 3.1. Consider Problem (P), and assume that there exist functions

$$\mu : [t_0, t_f] \rightarrow R^n, \quad \lambda \in L_{\infty, k}, \quad x^* : [t_0, t_f] \rightarrow R^n, \quad u^* \in L_{\infty, m},$$

where μ is of bounded variation on $[t_0, t_f]$ and continuous from the right on (t_0, t_f) , and x^* is absolutely continuous, which satisfy (35), (36), and

$$-\int_t^{t_f} (R(\tau)u^*(\tau) + B'(\tau)\mu(\tau)) d\tau + \int_{t_0}^{t_f} \eta'_1(\tau, t)\lambda(\tau) d\tau = 0, \tag{43}$$

$$\mu(t) - \int_t^{t_f} (Q(\tau)x^*(\tau) + A'(\tau)\mu(\tau)) d\tau + \int_{t_0}^{t_f} \eta'(\tau, t)\lambda(\tau) d\tau = Fx(t_f). \tag{44}$$

Then, u^*, x^* solve Problem (P).

It is easy to see that, in the case $\eta, \eta_1 \equiv 0$, (43) and (44) reduce to

$$R(t)u^*(t) + B'(t)\mu(t) = 0, \\ -\dot{\mu}(t) = Q(t)x(t) + A'(t)\mu(t), \quad \mu(t_f) = Fx(t_f),$$

as should be expected.

Theorem 3.1 can be easily extended to the case where cross terms $u'Lx$ exist in the integrand of (34) and to cases where more general convex cost functionals (34) are considered.

4. A Stackelberg Game with Delayed Information

Consider the dynamic system

$$\dot{x}(t) = Ax(t) + B_1\bar{u}(t) + B_2\bar{v}(t), \quad x(t_0) = x_0, \quad t \in [t_0, t_f], \tag{45}$$

and the cost functionals

$$J_1 = \frac{1}{2} \left[x'(t_f)F_1x(t_f) + \int_{t_0}^{t_f} (x'(t)Q_1x(t) + \bar{u}'(t)R_{11}\bar{u}(t) + \bar{v}'(t)R_{12}\bar{v}(t)) dt \right], \quad (46)$$

$$J_2 = \frac{1}{2} \left[x'(t_f)F_2x(t_f) + \int_{t_0}^{t_f} (x'(t)Q_2x(t) + \bar{u}'(t)R_{21}\bar{u}(t) + \bar{v}'(t)R_{22}\bar{v}(t)) dt \right], \quad (47)$$

where the matrices $A, B, Q_i = Q'_i \geq 0, R_{ij} = R'_{ij} \geq 0$ are piecewise continuous functions of time over $[t_0, t_f]$ and R_{11}, R_{22}, R_{12} are nonsingular, for all $t \in [t_0, t_f]$. The matrices $F_i = F'_i \geq 0$ and the time interval $[t_0, t_f]$ are fixed.

Consider the Stackelberg game associated with (45)–(47). The admissible strategies of the leader are of the form

$$u(x, t) = \int_{t_0}^t [d_s \eta(t, s)] x(s), \quad (48)$$

where η is as in (37) and (38), so that $u(\cdot, t)$ is a continuous linear functional on $C([t_0, t], R^{m_1})$, for each $t \in [t_0, t_f]$. The admissible strategies of the follower are of the form $v(x, t), x \in R^n, t \in R$, where v is continuously differentiable in x and piecewise continuous in t . All the matrices in (45)–(47) are considered to be of appropriate dimensions. By x, \bar{u}, \bar{v} , we mean

$$x_t: [t_0, t] \rightarrow R^n, \quad x_t(\theta) = x(\theta), \quad \text{for all } \theta, t \in [t_0, t_f], \quad (49)$$

$$\bar{u}(t) = u(x, t), \quad \bar{v}(t) = v(x(t), t), \quad (50)$$

where $x(t)$ is the trajectory of (45) for given u and v . For each choice of u and v , the behavior of the dynamic system (45) and the values of J_1, J_2 are unambiguously defined, assuming that the solution of (45) exists over $[t_0, t_f]$. Actually, when the strategy (48) is considered, one might without loss of generality restrict η to be 0 for $s \geq t, t \in [t_0, t_f]$. The costs of the leader (J_1) and of the follower (J_2) are functions of u and v . We denote by U and V the sets of admissible strategies for the leader and follower, respectively. With these explanations, the Stackelberg game associated with (45)–(47) is clearly defined.

In the sequel, we single out a subclass of Stackelberg games with the nice property that the leader achieves the best possible outcome for himself; i.e., the leader's and follower's strategies constitute together an optimal control law for the control problem with cost functional $J_1(u, v)$ subject to the constraint of the state equation. A similar idea occurs in Ref. 9. The interest of the authors in Ref. 9 is to solve the Stackelberg game (discrete

time) when the leader's strategy depends on information about the present and the past values of the state. The procedure followed is the following. First, solve the leader's problem as a control problem with controls u, v . Let $(\bar{u}^*(t), \bar{v}^*(t), x^*(t))$ be the optimum control pair and trajectory, where $\bar{u}^*(t), \bar{v}^*(t)$ are piecewise continuous functions of time. Consider any function $\tilde{u} \in U$, such that

$$\tilde{u}(x^*, t) = \bar{u}^*(t), \quad \text{for all } t \in [t_0, t_f].$$

Second, solve the following inverse control problem: with $u = \tilde{u}$ in the follower's cost, and the state equation, minimize $J_2(\tilde{u}, v)$, and seek conditions so that v^* solves this problem, and the resulting optimal trajectory for this problem is again $x^*(t)$. So, if these conditions are assumed to hold *a priori*, then the pair (\tilde{u}, \bar{v}^*) constitutes a Stackelberg pair. One may derive conditions by solving the inverse control problem, where \tilde{u} depends only on $x(t)$, or on almost any subset of $\{x(\tau); t_0 < \tau \leq t\}$ for each t . One may also single out a whole class of Stackelberg problems where the inverse control problem does not have v^* as its solution, whatever is the \tilde{u} . For example, if

$$J_2 = \int_{t_0}^{t_f} \bar{v}'(t)\bar{v}(t) dt,$$

then v^* will be optimum iff $\bar{v}^*(t) \equiv 0$. It is trivial to exhibit now a class of J_1 's and A, B_1, B_2 , so that $\bar{v}^*(t) \neq 0$.

Consider the control problem

$$\begin{aligned} &\text{minimize } J_1, \\ &\text{subject to } \bar{u}, \bar{v} \text{ piecewise continuous functions of } t \text{ and (45)}. \end{aligned} \tag{51}$$

Then, (51) has the solution

$$\bar{u}^*(t) = -R_{11}^{-1}B_1'Kx(t), \quad \bar{v}^*(t) = -R_{12}^{-1}B_2'Kx(t), \tag{52}$$

where K is the continuous solution of

$$\begin{aligned} -\dot{K} &= KA + A'K + Q_1 - K[B_1R_{11}^{-1}B_1' + B_2R_{12}^{-1}B_2']K, \\ K(t_f) &= F_1, \quad t \in [t_0, t_f], \end{aligned} \tag{53}$$

which is assumed to exist. Let $\Phi(t, t_0)$ be the transition matrix of the resulting closed-loop system in (45), i.e.,

$$\begin{aligned} \partial\Phi(t, t_0)/\partial t &= (A - B_1R_{11}^{-1}B_1'K - B_2R_{12}^{-1}B_2'K)\Phi(t, t_0), \\ \Phi(t_0, t_0) &= I, \quad t \in [t_0, t_f]. \end{aligned} \tag{54}$$

Then, the optimal trajectory x^* and control values of \bar{u}^*, \bar{v}^* for (51) are

given by

$$x^*(t; t_0, x_0) = \Phi(t, t_0)x_0, \tag{55}$$

$$\bar{u}^*(t) = -R_{11}^{-1}B_1'K\Phi(t, t_0)x_0, \tag{56}$$

$$\bar{v}^*(t) = -R_{12}^{-1}B_2'K\Phi(t, t_0)x_0. \tag{57}$$

Let η be as in (37), (38), with $\eta(t, \theta) = 0$ for $\theta \geq t$, and let η satisfy the identity

$$\int_{t_0}^t [d_s\eta(t, s)]\Phi(s, t) \equiv -R_{11}^{-1}(t)B_1'(t)K(t), \quad t \in [t_0, t_f]. \tag{58}$$

If η satisfies (58), then

$$\bar{u}^*(t) = \int_{t_0}^t [d_s\eta(t, s)]\Phi(s, t_0)x_0 = \int_{t_0}^t [d_s\eta(t, s)]x^*(s). \tag{59}$$

Equation (58) characterizes all the η 's which result in the same $\bar{u}^*(t)$ [Eq. (56)], i.e., it provides a class of different representations of $\bar{u}^*(t)$ as a linear continuous functional of

$$x_t^* = \{x^*(\theta); t_0 \leq \theta \leq t\}.$$

This class of η 's is not empty, since for example

$$\bar{\eta}(t, \theta) = \begin{cases} 0, & \text{for } \theta \geq t, t \in (t_0, t_f], \\ -R_{11}^{-1}(t)B_1'(t)K(t), & \text{for } \theta < t, t \in (t_0, t_f], \\ \text{and for } \theta \leq t_0, t = t_0, \end{cases} \tag{60}$$

satisfies (58). For fixed t , the set of all $\eta(t, \cdot)$ which satisfy (58) is the hyperplane

$H_t = \{\eta(t, \cdot) | \eta(t, \cdot) \in \text{NBV}([t_0, t], R^{m_1 \times n}), \eta(t, \cdot) \text{ perpendicular to } \Phi(\cdot, t)\}$, shifted by $\eta(t, \cdot)$ from the origin in the dual space of $C([t_0, t], R^{n \times n})$. A useful class of η 's which satisfy (58) is given by

$$\eta(t, s) = \bar{\eta}(t, s) + H_0(t, s) + \sum_{i=1}^p A_i(t, s)d(s - \rho_i(t)), \tag{61}$$

where H_0 is absolutely continuous in s for each t , $A_i: [t_0, t_f] \times R \rightarrow R^{m_1 \times n}$ is continuous, $\rho_i: [t_0, t_f] \rightarrow R$ is continuous, $d(s) = 0$ for $s \leq 0$, $d(s) = 1$ for $s > 0$, and

$$\int_{t_0}^t [\partial H_0(t, s)/\partial s]\Phi(s, t) ds + \sum_{i=1}^p A_i(t, \rho_i(t))\Phi(\rho_i(t), t) \equiv 0, \quad \text{on } [t_0, t_f]. \tag{62}$$

Another η which satisfies (58) is

$$\tilde{\eta}(t, s) = \begin{cases} 0, & \text{for } \theta \geq t/2, t \in (t_0, t_f], \\ 0, & \text{for } \theta > t_0, t = t_0, \\ -R_{11}^{-1}(t)B_1'(t)K(t) \int_0^s \Phi(t, \sigma) d\sigma \cdot [2/(t-t_0)], & \text{for } \theta < t/2, t \in (t_0, t_f], \\ -R_{11}^{-1}(t_0)B_1'(t_0)K(t_0), & \text{for } \theta \leq t_0, t = t_0. \end{cases} \quad (63)$$

Notice that

$$\bar{u}^*(t) = \int_{t_0}^{t_0+(t-t_0)/2} [d_s \tilde{\eta}(t, s)]x^*(s); \quad (64)$$

i.e., only the first half of the trajectory up to time t is used in calculating $\bar{u}^*(t)$.

Theorem 4.1. Assume that there exists a function η^* as in (37), (38), with $\eta^*(t, \theta) = 0$ for $\theta \geq t$, and an $n \times n$ matrix function $P: [t_0, t_f] \rightarrow R^{n \times n}$ which satisfy

$$\int_{t_0}^t [d_s \eta^*(t, s)]\Phi(s, t) = -R_{11}^{-1}(t)B_1'(t)K(t), \quad t \in [t_0, t_f], \quad (65)$$

$$R_{22}^{-1}(t)B_2'(t)P(t) = R_{12}^{-1}(t)B_2'(t)K(t), \quad t \in [t_0, t_f], \quad (66)$$

$$P(t) + \int_t^{t_f} \{-A'(\tau)P(\tau) - Q_2(\tau) + \eta^{*'}(\tau, t)B_1'(\tau)P(\tau) + \eta^{*'}(\tau, t)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau)\}\Phi(\tau, t) d\tau = F_2\Phi(t_f, t), \quad t \in [t_0, t_f]. \quad (67)$$

Then, the pair

$$u^*(x, t) = \int_{t_0}^t [d_s \eta^*(t, s)]x(s), \quad (68)$$

$$v^*(x(t), t) = -R_{12}^{-1}B_1'(t)K(t)x(t) \quad (69)$$

constitutes an equilibrium pair for the Stackelberg game associated with (45)–(47) for any x_0 with strategy spaces U and V .

Proof. We set

$$\bar{\lambda}(t) = P(t)\Phi(t, t_0)x_0. \quad (70)$$

Then, the vector

$$\lambda(t) = (-1, \bar{\lambda}(t))'$$

and the control

$$v = -R_{22}^{-1} B_2' \bar{\lambda}(t) \tag{71}$$

satisfy the sufficiency conditions of Theorem 3.1 for the problem

$$\begin{aligned} &\text{minimize } J_2(u^*(x, t), v), \\ &\text{subject to } v \in V \text{ and (45),} \end{aligned} \tag{72}$$

where u is kept fixed equal to u^* . That the u^* in (68) is the leader's best reaction to v^* in (49) is an immediate consequence of the fact that the pair (68), (69) solves the problem (51). \square

The case where the leader's strategy is allowed to be of the form

$$\int_{t_0}^t [d_s \eta(t, s)] y(t, s),$$

where

$$y(t, x) = C(t, x)x(s), (d/d_s)C(t, s) = 0, \quad \text{a.e. } t_0 \leq s \leq t \leq t_f,$$

with $\eta(t, s) \cdot C(t, s)$ as in (37)–(38) can also be considered. The property

$$(d/d_s)C(t, s) = 0, \quad \text{a.e. } t_0 \leq s \leq t \leq t_f,$$

allows one to write

$$\int_{t_0}^t [d_s \eta(t, s)] y(t, s) = \int_{t_0}^t d_s [\eta(t, s) C(t, s)] x(s),$$

and thus to use directly Theorem 3.1. We only mention that, in this case, the leader has restricted memory and $\eta^* \cdot C$ should play the role of η^* in (65)–(69) in the corresponding sufficiency conditions.

For given η^* , (67) is an integral equation for $P(t)$. Since it has a Volterra kernel, if in addition it holds that $A'(\tau) - \eta^{*'}(\tau, t)' B_1(\tau)$ is bounded by some M for any $t_0 \leq \tau \leq t, t_0 \leq t \leq t_f$, then the Neumann series for (67) is always uniformly convergent and furnishes the unique solution of (67); see Ref. 13. If $\eta^*(t, s)$ is of the form

$$\eta^*(t, s) = \sum_{i=1}^k H_i'(t) \cdot H_i''(s), \quad H_i(t) \in R^{n \times m_1}, H_i(s) \in R^{n \times p}, \tag{73}$$

then (67) can be written as

$$\begin{aligned}
 P(t) + \int_t^{t_f} \{-A'(\tau) + \Sigma H^i(t)H_i(\tau)B_1'(\tau)\}P(\tau)\Phi(\tau, t) d\tau &= F_2\Phi(t_f, t) \\
 + \int_t^{t_f} \{Q_2(\tau) - \eta^{*'}(\tau, t)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau)\}\Phi(\tau, t) d\tau, &\quad (74)
 \end{aligned}$$

which is an integral equation for P with a kernel of finite rank; thus, its solution is of the form

$$P(t) = \Xi_0\Phi(t_0, t) + \sum_{i=1}^k H^i(t)\Xi_i\Phi(t_0, t), \quad (75)$$

where $\Xi_0, \Xi_1, \dots, \Xi_k$ are constant matrices which can be found as solutions of algebraic linear equations. In this case, checking (66) is easy as soon as the Ξ_i 's in (75) are found.

If $(B_2'KB_2)^{-1}$ exists over $[t_0, t_f]$ (it suffices that $\text{rank } B_2 = m_2$ and $F_1 > 0$), then (66) is equivalent to

$$P(t) = M(t) + Y(t), \quad M(t) = KB_2(B_2'KB_2)^{-1}R_{22}R_{12}^{-1}B_2'K, \quad (76)$$

$$B_2'(t)Y(t) = 0, \quad \text{on } [t_0, t_f], \quad (77)$$

and (67) can be transformed into an integral equation for Y .

Theorem 4.1 suggests that, for a Stackelberg game with given A, B_i, Q_i, R_{ij}, F_i , one may try to find η^* and P which satisfy (65)–(67) and then consider (68), (69) as a solution. Also, by solving (67) for Q_2 , one can exhibit a whole class of Stackelberg games with solution (68), (69), where $\eta^*, P, K, A, B_1, B_2, F_1, R_{11}, R_{12}, R_{22}$ are chosen so as to satisfy (53), (54), (65), (66), $F_2 = P(t_f)$, and R_{21} is chosen arbitrarily.

5. Special Cases and Generalizations

We first apply the results of Theorem 4.1 to two special cases.

Case (i). Let

$$\eta^* = \bar{\eta},$$

as in (60). Then, u^* in (68) assumes the form

$$u^*(x_i, t) = -R_{11}^{-1}B_1'Kx(t).$$

Equation (65) is satisfied and (67) simplifies to

$$-\dot{P}(t) = P(A - B_1R_{11}^{-1}B_1'K) + (A - B_1R_{11}^{-1}B_1'K)'P + Q_2 + KB_1R_{11}^{-1}R_{12}R_{11}^{-1}N_1'K - PB_2R_{22}^{-1}B_2'P, \quad P(t_f) = F_2. \quad (78)$$

If $(B_2'KB_2)^{-1}$ exists and is differentiable on $[t_0, t_f]$ and if $B_2, KR_{22}R_{12}^{-1}B_2'K$ are differentiable on $[t_0, t_f]$ and of constant rank, then all the R_{22}, Q_2, F_2, P with $R_{22} > 0, P > 0$ which satisfy (66) and (78) are given by (see Ref. 8)

$$R_{22} = V\Gamma V', \quad (79)$$

$$P = M + Y, \quad (80)$$

$$Q_2 = -\dot{P} - P(A - B_1R_{11}^{-1}B_1'K) - (A - B_1R_{11}^{-1}B_1'K)'P \quad (81)$$

$$-KB_1R_{11}^{-1}R_{12}R_{11}^{-1}B_1'K + PB_2R_{22}^{-1}B_2'P, \quad (82)$$

$$F_2 = Q_2(t_f), \quad (83)$$

where

$$B_2'KB_2 = V\Lambda V^{-1}, \quad \Lambda = \text{Jordan diagonal form}, \quad (84)$$

$$\Gamma\Lambda = \Lambda\Gamma, \quad \Gamma = \Gamma' > 0, \quad (85)$$

$$B'Y = 0, \quad Y = Y' \geq 0. \quad (86)$$

If Γ and Y do not satisfy $\Gamma > 0, Y \geq 0$, then one cannot conclude that $R_{22} > 0$ and $P \geq 0$, respectively. Y and R_{12} have to be chosen properly differentiable, so that P exists and is piecewise continuous. The above construction does not guarantee that $Q_2 \geq 0, F_2 \geq 0$.

Case (ii). Let

$$\eta^* = \eta_1 + \eta_2,$$

where

$$\eta_1(t, s) = \begin{cases} -R_{11}^{-1}(t)B_1'(t)L_1(t), & \text{for } s < t, t \in (t_0, t_f], \\ -R_{11}^{-1}(t_0)B_1'(t_0)L_1(t_0), & \text{for } s \leq t_0, \\ 0, & \text{for } s \geq t, t \in (t_0, t_f], \\ 0, & \text{for } s > t_0, t = t_0, \end{cases} \quad (87a)$$

$$\eta_2(t, s) = \begin{cases} (s-t)L_2(t), & \text{for } s < t, t \in [t_0, t_f], \\ 0, & \text{for } s \geq t, \end{cases} \quad (87b)$$

where L_1, L_2 are real-valued matrices. Then, u^* in (68) assumes the form

$$u^*(x_t, t) = -R_{11}^{-1}(t)B_1'(t)L_1(t)x(t) + L_2(t) \int_{t_0}^t x(s) ds, \quad (88)$$

and (65)–(67) simplify to

$$-R_{11}^{-1}(t)B_1'(t)L_1(t) + L_2(t) \int_{t_0}^t \Phi(s, t) ds = -R_{11}^{-1}(t)B_1(t)K(t), \quad (89)$$

$$R_{22}^{-1}(t)B_2'(t)P(t) = -R_{12}^{-1}(t)B_2'(t)K(t), \quad (90)$$

$$\begin{aligned}
 P(t) + \int_t^{t_f} \{ & -A'(\tau)P(\tau) - Q_2(\tau) - L_1'(\tau)B_1(\tau)R_{11}^{-1}(\tau)B_1'(\tau)P(\tau) \\
 & + (t-\tau)L_2'(\tau)B_1'(\tau)P(\tau) - L_1'(\tau)B_1(\tau)R_{11}^{-1}(\tau)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau) \\
 & + (t-\tau)L_2'(\tau)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau) \} \Phi(\tau, t) d\tau = F_2\Phi(t_f, t). \quad (91)
 \end{aligned}$$

Cases (i) and (ii) are special cases of the case considered in the previous section. We will consider now cases where the leader uses the previous strategy values as well. In the Stackelberg game considered in Section 4, the value of the leader's strategy at time t was allowed to depend on the previous trajectory $x_t = \{x(\theta); t_0 \leq \theta \leq t\}$. More generally, one may allow that the values $\bar{u}(t)$ of the admissible strategies of u of the leader at time t depend, not only on the previous values of x , but also on those of v . Assuming this dependence to be linear, we have

$$\bar{u}(t) = \int_{t_0}^t [d_s \eta_1(t, s)]x(s) + \int_{t_0}^t [d_s \eta_3(t, s)]v(s),$$

or more generally

$$q(t) = \int_{t_0}^t [d_s \eta_1(t, s)]x[s] + \int_{t_0}^t [d_s \eta_2(t, s)]\bar{u}(s) + \int_{t_0}^t [d_s \eta_3(t, s)]\bar{v}(s), \quad (92)$$

$q \in L_{1,k}$ fixed.⁴ The η_1, η_2, η_3 in (92) are as in (37), (38). So, for a given choice η_1, η_2, η_3 by the leader, the follower is faced with the problem

$$\text{minimize } \frac{1}{2} \left[x'(t_f)F_2x(t_f) + \int_{t_0}^{t_f} (x'(t)Q_2x(t) + \bar{u}'(t)R_{21}\bar{u}(t) + \bar{v}(t)R_{22}\bar{v}(t)) dt \right],$$

$$\text{subject to } \dot{x}(t) = Ax(t) + B_1\bar{u}(t) + B_2\bar{v}(t), \quad x(t_0) = x_0, \quad (93)$$

(92), and \bar{u}, \bar{v} piecewise continuous functions of time.

Theorem 3.1 can now be used to derive sufficient conditions for problem (93).

⁴ Notice that, in (92), $\bar{u}(t)$ depends on its own previous values. If $\bar{u}(t)$ was allowed to be any function of $x(\theta), \bar{v}(\theta), t_0 \leq \theta \leq t$, then the dependence of $\bar{u}(t)$ on its previous values would not buy the leader anything additional. But, if $\bar{u}(t)$ is restricted to depend on $x(\theta), \bar{v}(\theta), t_0 \leq \theta \leq t$, in a special form [like in (92); see also (94)–(98)], then allowing dependence of $\bar{u}(t)$ on its own previous values will benefit the leader.

A simple version of (92) is

$$\bar{u}(t) = \int_{t_0}^t [d_s \bar{\eta}_1(t, s)]x(s) + \int_{t_0}^t [d_s \bar{\eta}_2(t, s)]z(s) + L(t)\bar{v}(t), \quad (94)$$

where

$$\dot{z}(t) = \bar{A}_1(t)x(t) + \bar{A}_2(t)z(t) + \bar{B}_1(t)\bar{u}(t) + \bar{B}_2(t)\bar{v}(t), \quad z(t_0) = z_0, \quad (95)$$

and the matrices L, \bar{A}_i, \bar{B}_i are real-valued, piecewise continuous functions of time, and $z(t) \in R^l, l$ arbitrary. For the linear system (45) with quadratic costs (46), (47), we augment (95) to (45), set $\tilde{x} = (x'z)'$, and the system is

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{bmatrix} A & 0 \\ \bar{A}_1 & \bar{A}_2 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} B_1 \\ \bar{B}_1 \end{bmatrix} \bar{u}(t) + \begin{bmatrix} B_2 \\ \bar{B}_2 \end{bmatrix} \bar{v}(t), \\ &= \tilde{A}\tilde{x} + \tilde{B}_1\bar{u} + \tilde{B}_2\bar{v}, \end{aligned} \quad \tilde{x}(t_0) = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \quad (96)$$

with costs J_1, J_2 as in (46), (47) and with the strategy of the leader restricted to be of the form

$$u(x, t) = \int_{t_0}^t [d_s \tilde{\eta}(t, s)]\tilde{x}(s) + L(t)\bar{v}(t). \quad (97)$$

The results of Section 4 are directly applicable to (96) and (97), and the problem is to find $\tilde{\eta}, \bar{A}_i, \bar{B}_i, L, \tilde{P}$ so that (65)–(67) are satisfied where in (65)–(67) one should use $\tilde{A}, \tilde{B}_1, (\tilde{B}_2 + \tilde{B}_1L)$ in place of A, B_1, B_2 . As far as it concerns z_0 , it may be set arbitrarily equal to a constant or to a function of x_0 preferably linear. The choice of z_0 might affect not only the feasibility of (65)–(67) but the follower's optimum cost value as well. A simpler case of (97) is

$$\bar{u}(t) = \bar{L}_1x(t) + \bar{L}_2z(t) + L\bar{v}(t), \quad (98)$$

in which case the solution of the Stackelberg game is easy, since the leader's controls are actually $\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{L}_1, \bar{L}_2, L$, i.e., the leader plays open loop. Nonetheless, the leader's problem will be nonlinear, since his control multiplies the state $(x', z)'$.

6. A Nash Game with Delayed Information

Consider the Nash game associated with (45)–(47) where, at each instant of time t , both players have access to all the previous values of the state. The admissible strategies for both players are of the form

$$u(x, t) = \int_{t_0}^t [d_s \eta_1(t, s)]x(s) + b_1(t), \quad (99)$$

$$v(x, t) = \int_{t_0}^t [d_s \eta_2(t, s)]x(s) + b_2(t). \tag{100}$$

η_1 and η_2 are as in (37) and (38); $b_i(t)$ are piecewise continuous functions of time with appropriate dimensions. By x_t, \bar{u}, \bar{v} , we mean

$$\begin{aligned} \bar{u}(t) &= u(x_t, t), & \bar{v}(t) &= v(x_t, t), \\ x_t: [t_0, t] &\rightarrow \mathbb{R}^n, & x_t(\theta), & \text{ for all } \theta \in [t_0, t], \text{ for all } t \in [t_0, t_f]. \end{aligned} \tag{101}$$

In the next proposition, we give sufficient conditions for a pair of the form (99), (100) to constitute a Nash equilibrium pair. The first part of the proposition refers to a particular initial point x_0 , while the second part gives conditions similar to the coupled Riccati differential equations (see Ref. 1), which result in solutions in feedback form which are solutions for any initial point x_0 .

Proposition 6.1. (i) Assume that there exist η_1^*, η_2^* as in (37) and (38), b_1^*, b_2^* piecewise continuous, and $\mu_1, \mu_2: [t_0, t_f] \rightarrow \mathbb{R}^n$ of bounded variation which satisfy

$$\begin{aligned} \mu_i(t) - \int_t^{t_f} [A'(\tau)\mu_i(\tau) + Q_i(\tau)x(\tau)] d\tau + \int_t^{t_f} \eta_i^{*'}(\tau, t)[B_j(\tau)\mu_i(\tau) \\ + R_{ij}(\tau)R_{jj}^{-1}(\tau)B_j'(\tau)\mu_j(\tau)] d\tau = F_i x(t_f), \quad i \neq j, \quad i, j = 1, 2, \end{aligned} \tag{102}$$

$$b_i^*(t) + \int_{t_0}^t [d_s \eta_i^*(t, s)]x(s) = -R_{ii}^{-1}(t)B_i'(t)\mu_i(t), \quad i = 1, 2, \tag{103}$$

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) - B_1(t)R_{11}^{-1}(t)B_1'(t)\mu_1(t) - B_2(t)R_{22}^{-1}(t)B_2'(t)\mu_2(t), \\ x(t_0) &= x_0. \end{aligned} \tag{104}$$

Then, the strategies

$$u^*(x, t) = \int_{t_0}^t [d_s \eta_1^*(t, s)]x(s) + b_1^*(t), \tag{105}$$

$$v^*(x, t) = \int_{t_0}^t [d_s \eta_2^*(t, s)]x(s) + b_2^*(t) \tag{106}$$

constitute an equilibrium pair for the Nash game associated with (45)–(47), with admissible strategies (99), (100) and with $x(t_0) = x_0$.

(ii) Assume that there exist η_1^*, η_2^* as in (37) and (38) and matrix functions $P_1, P_2: [t_0, t_f] \rightarrow R^n \times R^n$ of bounded variation which satisfy

$$\begin{aligned}
 P_i(t) - \int_t^{t_f} (A'(\tau)P_i(\tau) + Q_i(\tau) \\
 + \eta_j^{*t}(\tau, t)[B_j(\tau)P_i(\tau) + R_{ij}(\tau)R_{jj}^{-1}(\tau)B_j'(\tau)P_j(\tau)])\Phi(\tau, t) d\tau \\
 = F_i\Phi(t_f, t), \quad i, j = 1, 2, i \neq j,
 \end{aligned} \tag{107}$$

$$\begin{aligned}
 \partial\Phi(t, t_0)/\partial t = [A(t) - B_1(t)R_{11}^{-1}(t)B_1'(t)P_1(t) \\
 - B_2(t)R_{22}^{-1}(t)B_2'(t)P_2(t)]\Phi(t, t_0), \quad \Phi(t_0, t_0) = I,
 \end{aligned} \tag{108}$$

$$\int_{t_0}^t [d_s\eta_i^*(t, s)]\Phi(s, t) = -R_{ii}^{-1}(t)B_i'(t)P_i(t), \quad i = 1, 2. \tag{109}$$

Then, the strategies

$$u^*(x_t, t) = \int_{t_0}^t [d_s\eta_1^*(t, s)]x(s), \tag{110}$$

$$v^*(x_t, t) = \int_{t_0}^t [d_s\eta_2^*(t, s)]x(s) \tag{111}$$

constitute an equilibrium pair for the Nash game associated with (45)–(47), with admissible strategies (99), (100) and for any $x_0 \in R^n$.

Proof. (i) If the second player plays (100), then (102), with $i = 1, j = 2$ and

$$\bar{u}(t) = -R_{11}^{-1}(t)B_2'(t)\mu_1(t), \tag{112}$$

constitute sufficient conditions for optimality, by Theorem 3.1, of \bar{u} for the control problem faced by the first player. In (102), the term $R_{22}^{-1}B_2'\mu_2$ is replaced by $-\int_{t_0}^t [d_s\eta_2^*]x$ in these sufficient conditions. Similar reasoning applies for the control problem faced by the second player when the first player plays (105).

(ii) We will first seek solutions μ_1, μ_2 of (102) which will work for any x_0 . Let

$$\mu_i(t) = P_i(t)\Phi(t, t_0)x_0, \tag{113}$$

where Φ is as in (108). Using (111) in (102) and (103), we obtain (107) and (109), where we considered $b_i \equiv 0$. It is clear now that, if (107) and (109) hold, the μ_i 's as in (109) satisfy (102)–(104).

The case where the players use strategies of the form

$$\begin{aligned} u(y_1, t) &= \int_{t_0}^t [d_s \eta_1(t, s)] y_1(t, s) + b_1(t), \\ v(y_2, t) &= \int_{t_0}^t [d_s \eta_2(t, s)] y_2(t, s) + b_2(t), \end{aligned} \tag{114}$$

where, for $i = 1, 2$,

$$\begin{aligned} y_i(t, s) &= C_i(t, s)x(s), \quad t_0 \leq s \leq t, \\ (d/d_s)C_i(t, s) &= 0, \quad \text{a.e. } t_0 \leq s \leq t \leq t_f, \\ y_i(t, s) &\in R^{n_i}, \end{aligned}$$

and

$$\tilde{\eta}_i(t, s) = \eta_i(t, s)C_i(t, s)$$

are as in (37) and (38), can also be considered. The strategies (114) correspond to the case where the i th player's information at time t is $\{C_i(t, s)x(s); t_0 \leq s \leq t\}$. We only mention that, in this case, $\eta_i^* C_i$ should play the role of η_i^* in the conditions of Proposition 6.1.

The results of Proposition 6.1 [see also Problem (P')] can be used to study the Nash game associated with (45)–(47) where the players use previous values of their opponent's strategy values. For example,

$$\begin{aligned} \bar{u}(t) &= \int_{t_0}^t [d_s \eta_{11}(t, s)] x(s) + \int_{t_0}^t [d_s \eta_{12}(t, s)] \bar{v}(s), \\ \bar{v}(t) &= \int_{t_0}^t [d_s \eta_{21}(t, s)] x(s) + \int_{t_0}^t [d_s \eta_{22}(t, s)] \bar{u}(s). \end{aligned}$$

Strategies of the form (94), (97), (98) can be considered for the Nash game, and the augmentation (95) and (96) may also be employed in this case. The procedure for studying sufficiency conditions for Nash games with such strategies should be obvious by now and we will not take it up here.

7. Conclusions

In this paper, we provided sufficient conditions for two strategies to constitute an equilibrium Stackelberg or Nash pair, when the players use previous values of the trajectory of the system and possibly previous values of their own or their opponent's strategies. The problem that we dealt with differs from those considered by Halanay in Ref. 2 and by Ciletti in Refs. 3

and 4. Halanay considers the zero-sum case only, and he allows the strategy values at time t to depend on the part of the trajectory between $t - \tau$ and t , where $\tau > 0$ is fixed. Ciletti considers also the zero-sum case and allows dependence of the strategy values at time t only on $x(t - \sigma)$ and the strategy values between $t - \sigma$ and t , where $\sigma > 0$ is fixed. The strategies that we considered were restricted to be affine in the data available. Existence and uniqueness conditions related to the sufficiency conditions proved here are not as yet known. Our results generalize trivially to the N -player case for a Nash game and to the one leader- N followers case for a Stackelberg game. Although, for the time being, our results are not accompanied by computationally efficient procedures, they are of importance since they provide value characterizations.

Appendix

Proof of Theorem 3.1. Consider the functions

$$\begin{aligned}
 H_1: R^n \times C_n \times L_{\infty,m} &\rightarrow C_n, \\
 H_2: R^n \times C_n \times L_{\infty,m} &\rightarrow L_{1,k}, \\
 H_3: R^n \times C_n \times L_{\infty,m} &\rightarrow R^n, \\
 J: R^n \times C_n \times L_{\infty,m} &\rightarrow R,
 \end{aligned}
 \tag{115}$$

defined for $(\xi, x, u) \in R^n \times C_n \times L_{\infty,m}$ by

$$\begin{aligned}
 H_1(\xi, x, u)(t) &= x(t) - \int_{t_0}^t A(\tau)x(\tau) d\tau - \int_{t_0}^t B(\tau)u(\tau) d\tau - x_0, \\
 H_2(\xi, x, u)(t) &= \int_{t_0}^{t_f} [d_s \eta(t, s)]x(s) + \int_{t_0}^{t_f} [d_s \eta_1(t, s)]u(s) - q(t), \\
 H_3(\xi, x, u) &= \xi - x_0 - \int_{t_0}^{t_f} A(\tau)x(\tau) d\tau - \int_{t_0}^{t_f} B(\tau)u(\tau) d\tau, \\
 J(\xi, x, u) &= \frac{1}{2} \left[\xi' F \xi + \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t)) dt \right].
 \end{aligned}
 \tag{116}$$

Clearly, H_1, H_3, J are well defined. To show that H_2 is well defined, it suffices to show that, if $u \in L_{\infty,m}$, then

$$\int_{t_0}^{t_f} [d_s \eta_1(t, s)]u(s) \in L_{1,k}.$$

Let

$$u \in L_{\infty,m}, \quad \|u\|_{L_{\infty}} = M.$$

Then, there exists a sequence $\{u_n\}_{n=1}^\infty$ of continuous functions

$$u_n: [t_0, t_f] \rightarrow R^m, \text{ such that } u_n(t) \rightarrow u(t) \text{ a.e.}$$

and

$$|u_n(t)| \leq M + 1, \quad \text{for all } t \in [t_0, t_f], \text{ for all } n;$$

see Theorem 3, page 106, Ref. 14. Since

$$y_n(t) = \int_{t_0}^{t_f} [d_s \eta_1(t, s)] u_n(s)$$

is measurable,

$$|y_n(t)| \leq (M + 1)m_1(t),$$

and thus $y_n \in L_{1,m}$. Since $u_n \rightarrow u$ a.e., by Egoroff's theorem we have that⁵

$$\text{for all } \epsilon > 0, \mu_l(A_n^c) \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

where

$$A_n^c = \{s : s \in [t_0, t_f], |u_n(s) - u(s)| \geq \epsilon\}.$$

The following holds:

$$\begin{aligned} \left| y_n(t) - \int_{t_0}^{t_f} [d_s \eta_1(t, s)] u(s) \right| &= \left| \int_{t_0}^{t_f} [d_s \eta_1(t, s)] (u_n(s) - u(s)) \right| \leq \left| \int_{A_n} \right| + \left| \int_{A_n^c} \right| \\ &\leq \epsilon \cdot c_1(t) + (2M + 1)c_1(t)\mu_l(A_n^c). \end{aligned}$$

Since c_1 is finite a.e., letting $n \rightarrow +\infty$, we obtain

$$\left| \lim y_n(t) - \int_{t_0}^{t_f} [d_s \eta_1(t, s)] u(s) \right| \leq \epsilon \cdot c_1(t), \quad \text{a.e. in } t \in [t_0, t_f],$$

where $\lim y_n(t)$ stands for either limsup or liminf. Since this inequality holds, for all $\epsilon > 0$, we conclude that

$$\int_{t_0}^{t_f} [d_s \eta_1(t, s)] u(s) = \lim y_n(t), \quad \text{a.e. in } [t_0, t_f]. \tag{117}$$

Since

$$|y_n(t)| \leq (M + 1)c_1(t)$$

and (117) holds, we conclude by Lebesgue's theorem that

$$\int_{t_0}^{t_f} [d_s \eta_1(t, s)] u(s) \in L_{1,k}.$$

⁵ μ_l denotes the Lebesgue measure on $[t_0, t_f]$.

Problem (P) can be written equivalently as

$$\begin{aligned} &\text{minimize } J(\xi, x, u), \\ &\text{subject to } H_i(\xi, x, u) = 0, \quad i = 1, 2, 3, \\ &(\xi, x, u) \in \mathbf{R}^n \times C_n \times L_{\infty, m} = \Omega. \end{aligned} \tag{118}$$

By Theorem 1, page 220, Ref. 15, we conclude that a sufficient condition for (ξ^*, x^*, u^*) to solve (118) is the existence of a $(\mu, \lambda, k) \in (C_n^*, L_{1, k}, \mathbf{R}^n)^*$, such that

$$\begin{aligned} &J(\xi^*, x^*, u^*) + \langle H_1(\xi^*, x^*, u^*), \mu \rangle + \langle H_2(\xi^*, x^*, u^*), \lambda \rangle \\ &+ \langle H_3(\xi^*, x^*, u^*), k \rangle \\ &\leq J(\omega) + \langle H_1(\omega), \mu \rangle + \langle H_2(\omega), \lambda \rangle + \langle H_3(\omega), k \rangle, \text{ for all } \omega \in \Omega. \end{aligned} \tag{119}$$

Since the function

$$\tilde{J}(\omega) = J(\omega) + \langle H_1(\omega), \mu \rangle + \langle H_2(\omega), \lambda \rangle + \langle H_3(\omega), k \rangle$$

is convex and Frechet differentiable, a necessary and sufficient condition for (119) to hold is that

$$\begin{aligned} &d\tilde{J}(\xi^*, x^*, u^*; \zeta, h, v) = 0, \\ &\text{for all } (\zeta, h, v) \in \mathbf{R}^n \times C_n \times L_{\infty, m}, \end{aligned} \tag{120}$$

where $d\tilde{J}$ denotes the Frechet differential. Straightforward calculations result in the following explicit form for (120):

$$(\xi'F + k')\zeta = 0, \quad \text{for all } \xi \in \mathbf{R}^n, \tag{121}$$

$$\begin{aligned} &\int_{t_0}^{t_f} x'(t)Q(t)h(t) dt + \int_{t_0}^{t_f} [d\mu'(t)]h(t) + \int_{t_0}^{t_f} \lambda'(t) \left(\int_{t_0}^{t_f} [d_s \eta(t, s)]h(s) \right) dt \\ &+ \int_{t_0}^{t_f} \mu'(t)A(t)h(t) dt - k' \int_{t_0}^{t_f} A(t)h(t) dt = 0, \quad \text{for all } h \in C_n, \end{aligned} \tag{122}$$

$$\begin{aligned} &\int_{t_0}^{t_f} u'(t)R(t)v(t) dt + \int_{t_0}^{t_f} \mu'(t)B(t)v(t) dt + \int_{t_0}^{t_f} \lambda'(t) \left(\int_{t_0}^{t_f} [d_s \eta_1(t, s)]v(s) \right) dt \\ &- k' \int_{t_0}^{t_f} B(t)v(t) dt = 0, \quad \text{for all } v \in L_{\infty, m}. \end{aligned} \tag{123}$$

Use of the unsymmetric Fubini theorem in Ref. 12 yields

$$\int_{t_0}^{t_f} \lambda'(t) \left(\int_{t_0}^{t_f} [d_s \eta(t, s)] h(s) \right) dt = \int_{t_0}^{t_f} \left[ds \left(\int_{t_0}^{t_f} \lambda'(t) \eta(t, s) dt \right) \right] h(s), \quad (124)$$

$$\int_{t_0}^{t_f} \lambda'(t) \left(\int_{t_0}^{t_f} [d_s \eta_1(t, s)] v(s) \right) dt = \int_{t_0}^{t_f} \left[ds \left(\int_{t_0}^{t_f} \lambda'(t) \eta_1(t, s) dt \right) \right] v(s). \quad (125)$$

Using (123) and (124) in (121) and (122), we obtain the sufficiency conditions (10), (11), where we replaced μ by $\mu - k$ and k by $-F\xi^* = -Fx(t_f)$. \square

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