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SUFFICIENT CONDITIONS FOR THE OSCILLATION OF n-TH ORDER NONLINEAR DELAY DIFFERENTIAL EQUATION

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We consider the equation

$$u^{(n)}(t) + p(t) \left|u(\tau(t))\right|^{\alpha} \operatorname{sign} u(\tau(t)) = 0\,, \tag{1}$$

where

(i) $0 \leqq p(t) \in C_{[t_0,\infty)}$, $p(t)$ is not identically zero in any neighborhood $O(\infty)$,

(ii) $\tau(t) \in C_{[t_0,\infty)}$, $\tau(t) \leqq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$,

(iii) $n \geqq 2$, $\alpha > 0$.

Without mentioning them again, we shall assume the validity of conditions (i), (ii) and (iii) throughout the paper.

Suppose that there exist solutions of the equation (1) on an interval of the form $[b, \infty)$ where $b \geqq t_0$. In the sequel we shall use the term "solution" only to denote a solution which exists on $[b, \infty)$ where $b \geqq t_0$. Moreover, we shall exclude from our considerations solutions of (1) with the property that $u(t) \equiv 0$ for $t \geqq T_1 \geqq t_0$.

A solution $u(t)$ of (1) is called *oscillatory* for $t \geqq t_0$ if there exists an infinite sequence of points $\{s_i\}_{i=1}^{\infty}$ such that $u(s_i) = 0$ and $s_i \to \infty$ for $i \to \infty$. A solution $u(t)$ of (1) is called *nonoscillatory* if there exists a number $T_2 \geqq t_0$ such that $u(t) \neq 0$ for $t \geqq T_2$. A nonoscillatory solution is said to be *strongly monotone*, if it tends monotonically to zero together with its first $n - 1$ derivatives as $t \to \infty$.

The purpose of this paper is to give sufficient conditions for all solutions of (1) to be oscillatory in the case n is even and for every solution of (1) to be either oscillatory or strongly monotone when n is odd.

We begin our considerations with some preliminary lemmas.

Lemma 1. *Let* $u(t) \in C^n_{[T,\infty)}$ *and let either*

$$u(t) > 0\,, \quad u^{(n)}(t) \leqq 0 \quad \textit{for} \quad t \geqq t_1 \geqq T\,,$$

or

$$u(t) < 0\,, \quad u^{(n)}(t) \geqq 0 \quad for \quad t \geqq t_1 \geqq T\,.$$

Let $u^{(n)}(t)$ be not identically zero in any neighborhood $O(\infty)$. Then

1) *there exists a number $t_2 \geqq t_1$ such that the functions $u^{(j)}(t), j = 1, 2, \ldots, n - 1$ are of a constant sign on $[t_2, \infty)$,*

2) *there exists a number $k \in \{1, 3, 5, \ldots, n - 1\}$ if n is even, or $k \in \{0, 2, 4, \ldots, n - 1\}$ if n is odd, such that*

$$(2) \qquad u(t)\, u^{(j)}(t) > 0 \quad for \quad j = 0, 1, 2, \ldots, k \quad and \quad t \geqq t_2\,,$$

$$(-1)^{k+j}\, u(t)\, u^{(j)}(t) > 0 \quad for \quad j = k + 1, k + 2, \ldots, n - 1 \quad and \quad t \geqq t_2\,,$$

3) *either*

$$(3) \qquad \operatorname{sign} u(s) = \operatorname{sign} \lim_{t\to\infty} u^{(j)}(t) \quad for \quad j = 0, 1, 2, \ldots, q \quad and \quad s \geqq t_2\,,$$

$$\lim_{t\to\infty} u^{(j)}(t) = 0 \quad for \quad j = q + 1, q + 2, \ldots, n - 1\,,$$

where $q = k$ if $u(s) \lim_{t\to\infty} u^{(k)}(t) > 0$, and $q = k - 1$ if $k > 0$ and $\lim_{t\to\infty} u^{(k)}(t) = 0$, or

$$(4) \qquad \lim_{t\to\infty} u^{(j)}(t) = 0 \quad for \quad j = 0, 1, 2, \ldots, n - 1$$

if $k = 0$ and $\lim_{t\to\infty} u^{(k)}(t) = \lim_{t\to\infty} u(t) = 0$.

Proof of this lemma may be found in [1].

Let us denote $\gamma(t) = \sup\{s \geqq t_0 \mid \tau(s) \leqq t\}$ for $t \geqq t_0$.

We see that $t \leqq \gamma(t)$ and $\tau(\gamma(t)) = t$. Another property of the function $\gamma(t)$ is given in the following lemma.

Lemma 2. *For every t such that $t_0 \leqq t < \infty$, the value $\gamma(t)$ is finite.*

Proof of Lemma 2 may be found in [1].

Theorem 1. *Let $\alpha > 1$ and let the equation (1) have an unbounded nonoscillatory solution. Then*

$$\limsup_{t\to\infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x)\, dx = 0\,.$$

Proof. Let $u(t)$ be a nonoscillatory unbounded solution of (1). We may assume that $u(t) > 0$ for $t \geqq T \geqq t_0$ (the case $u(t) < 0$ is treated similarly). Then (see (ii)) there exists $t_1 \geqq T$ such that $u(\tau(t)) > 0$ for $t \geqq t_1$. Now, by (1), we have $u^{(n)}(t) \leqq 0$ for $t \geqq t_1$, but $u^{(n)}(t)$ is not identically zero in any neighborhood $O(\infty)$. It is clear that the function $u(t)$ satisfies the conditions of Lemma 1. So we can use the assertions of this lemma.

Now suppose that (3) holds true. Integrating (1) successively $n - q - 1$ times from $t(\geqq t_2)$ to ∞ we have

$$u^{(n-1)}(t) \geqq \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx\,,$$

$$-u^{(n-2)}(t) \geqq \int_t^\infty (x - t)\, p(x)\, u^\alpha(\tau(x))\, dx\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

(5) $$(-1)^{n-q}\, u^{(q+1)}(t) \geqq \frac{1}{(n-q-2)!} \int_t^\infty (x - t)^{n-q-2}\, p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Integrating (5) from v_1 to $v_2 (t_2 < v_1 < v_2)$ we obtain

(6) $$(-1)^{n-q}\, u^{(q)}(v_2) - (-1)^{n-q}\, u^{(q)}(v_1) \geqq$$

$$\geqq \frac{1}{(n-q-1)!} \int_{v_1}^{v_2} (x - v_1)^{n-q-1}\, p(x)\, u^\alpha(\tau(x))\, dx\, +$$

$$+ \frac{1}{(n-q-1)!} \int_{v_2}^\infty [(x - v_1)^{n-q-1} - (x - v_2)^{\prime -q-1}]\, p(x)\, u^\alpha(\tau(x))\, dx\,.$$

It is easy to verify by induction that

(7) $$(v_2 - v_1)^{n-q-1} \leqq (x - v_1)^{n-q-1} - (x - v_2)^{n-q-1}$$

for $v_1 < v_2 \leqq x$.

Therefore, taking into account (7), we obtain from (6) that

(8) $$(-1)^{n-q}\, u^{(q)}(v_2) - (-1)^{n-q}\, u^{(q)}(v_1) \geqq$$

$$\geqq \frac{1}{(n-q-1)!} \int_{v_1}^{v_2} (x - v_1)^{n-q-1}\, p(x)\, u^\alpha(\tau(x))\, dx\, +$$

$$+ \frac{1}{(n-q-1)!} (v_2 - v_1)^{n-q-1} \int_{v_2}^\infty p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Let $n - q$ be even. Then 2) implies that k is odd if n is even and k is even if n is odd. For this reason $n - k$ is always odd. Thus $q \neq k$. But then $q = k - 1$ (by Lemma 1). Now (2) yields

$$u^{(q+1)}(t) = u^{(k)}(t) > 0 \quad \text{for} \quad t \geqq t_2$$

and

$$u^{(q)}(t) = u^{(k-1)}(t) > 0 \quad \text{for} \quad t \geqq t_2\,.$$

Hence $u^{(q)}(t)$ is an increasing and positive function if $t \geqq t_2$. It is clear that the first integral on the right-hand side of (8) is nonnegative. Therefore from (8) we have

(9) $$u^{(q)}(v_2) \geqq \frac{1}{(n-q-1)!} (v_2 - v_1)^{n-q-1} \int_{v_2}^\infty p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Let $q > 0$. Integrating (9) from v_1 to t ($t_2 \leqq v_1 < t$) we get

$$u^{(q-1)}(t) - u^{(q-1)}(v_1) \geqq \frac{1}{(n-q)!}\left[(t - v_1)^{n-q} \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx + \right.$$

$$\left. + \int_{v_1}^t (v_2 - v_1)^{n-q}\, p(v_2)\, u^\alpha(\tau(v_2))\, dv_2 \right].$$

The second integral on the right-hand side of this inequality is nonnegative and $u^{(q-1)}(v_1) > 0$ (by (2)). Thus we can write

$$(10) \qquad u^{(q-1)}(t) \geqq \frac{1}{(n-q)!}(t - v_1)^{n-q} \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Analogously, integrating (10) $q - 1$ times from v_1 to t we obtain

$$(11) \qquad u^{(q-2)}(t) \geqq \frac{1}{(n-q-1)!}(t - v_1)^{n-q+1} \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx\,,$$

....

$$u(t) \geqq \frac{1}{(n-1)!}(t - v_1)^{n-1} \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Note that for $q = 0$, the inequality (11) reduces to the inequality (9) (just replace v_2 by t). Therefore, for $n - q$ even, we can consider the case $q > 0$ together with the case $q = 0$.

We know that $\gamma(t) \geqq t$ and $\tau(x) \geqq t$ if $x \geqq \gamma(t)$. Since, by the assumption of the theorem, $u(t)$ is an unbounded positive function, we conclude by Lemma 1 that $u'(t) > 0$ for $t \geqq t_2$. Hence $u(\tau(x)) \geqq u(t)$ for $x \geqq \gamma(t)$ and from (11) we have

$$u(t) \geqq \frac{u^\alpha(t)}{(n-1)!}(t - v_1)^{n-1} \int_{\gamma(t)}^\infty p(x)\, dx$$

and also

$$(12) \qquad \frac{u(t)}{u^\alpha(t)} \geqq \frac{(t - v_1)^{n-1}}{(n-1)!} \int_{\gamma(t)}^\infty p(x)\, dx\,.$$

The last inequality directly implies that

$$\limsup_{t \to \infty} t^{n-1} \int_{\gamma(t)}^\infty p(x)\, dx = 0$$

and the theorem is proved in the case when (3) holds and $n - q$ is an even number.

Again, let (3) hold true. Now let $n - q$ be odd. Then, by Lemma 1, we obtain $q = k$. Thus from (2) we have

$$u^{(q+1)}(t) = u^{(k+1)}(t) < 0 \quad \text{for} \quad t \geqq t_2$$

and

$$u^{(q)}(t) = u^{(k)}(t) > 0 \quad \text{for} \quad t \geqq t_2\,.$$

Hence $u^{(q)}(t)$ is a positive decreasing function for $t \geqq t_2$. It is easy to see that the second integral on the right-hand side of the inequality (8) is positive. Therefore (8) yields

$$u^{(q)}(v_1) - u^{(q)}(v_2) \geqq \frac{1}{(n-q-1)!} \int_{v_1}^{v_2} (x - v_1)^{n-q-1} p(x) u^{\alpha}(\tau(x)) \, dx$$

and for $v_2 \to \infty$,

$$u^{(q)}(v_1) \geqq \frac{1}{(n-q-1)!} \int_{v_1}^{\infty} (x - v_1)^{n-q-1} p(x) u^{\alpha}(\tau(x)) \, dx . \tag{13}$$

Let $q > 0$. Integrating (13) from v_3 to t $(t_2 < v_3 < t)$ we obtain

$$u^{(q-1)}(t) - u^{(q-1)}(v_3) \geqq \frac{1}{(n-q)!} \int_{v_3}^{t} (x - v_3)^{n-q} p(x) u^{\alpha}(\tau(x)) \, dx + \tag{14}$$

$$+ \frac{1}{(n-q)!} \int_{t}^{\infty} [(x - v_3)^{n-q} - (x - t)^{n-q}] \, p(x) u^{\alpha}(\tau(x)) \, dx .$$

It is evident (by (7)) that $(t - v_3)^{n-q} \leqq (x - v_3)^{n-q} - (x - t)^{n-q}$ for $v_3 < t \leqq x$. The first integral on the right-hand side of (14) is nonnegative, and by (2), $u^{(q-1)}(v_3) = u^{(k-1)}(v_3) > 0$. Thus replacing v_3 by v_1 in (14) we obtain

$$u^{(q-1)}(t) \geqq \frac{1}{(n-q)!} (t - v_1)^{n-q} \int_{t}^{\infty} p(x) u^{\alpha}(\tau(x)) \, dx . \tag{15}$$

The inequality (15) coincides with the inequality (10). Consequently, continuing as above we successively get the inequalities (11), (12) and the assertion of the theorem.

Now let $q = 0$, $n - q = n$ being odd. Since $q = k$, the inequality (2) gives $u'(t) < 0$ for each $t \geqq t_2$ but this is a contradiction with the assumption that $u(t)$ is a positive unbounded solution of (1).

Now consider the case when conditions (3) are not satisfied. Then conditions (4) are satisfied but this is again a contradiction with the assumption that $u(t)$ is a positive unbounded nonoscillatory solution of (1). The proof is complete.

Remark. From the proof of Theorem 1 we obtain the following consequence: Let $\alpha = 1$ and let the equation (1) have an unbounded nonoscillatory solution. Then

$$\limsup_{t \to \infty} t^{n-1} \int_{\gamma(t)}^{\infty} p(x) \, dx \leqq (n-1)\,!$$

Theorem 2. *Let* $\alpha > 0$ *and*

$$\limsup_{t \to \infty} t^{n-1} \int_{t}^{\infty} p(x) \, dx = \infty . \tag{16}$$

Then, for n even, all nonoscillatory solutions of (1) *are unbounded, while, for n odd, every nonoscillatory solution of* (1) *is either unbounded or strongly monotone.*

Proof. The proof is easy modification of that of Theorem 1. Let $u(t)$ be a non-oscillatory solution of (1). We may assume that $u(t) > 0$ for $t \geqq T \geqq t_0$. Then $u(\tau(t)) > 0$ for $t \geqq t_1 = \gamma(T)$. Now, by (1), we have $u^{(n)}(t) \leqq 0$ for $t \geqq t_1$, but $u^{(n)}(t)$ is not identically zero in any neighborhood $O(\infty)$. It is clear that this function $u(t)$ satisfies the conditions of Lemma 1. So we can use the assertions of this lemma.

Now suppose that (3) holds true. As well as in the proof of Theorem 1 we successively get the inequalities (5), (6), (7) and (8).

Let $n - q$ be even. Then, as in the corresponding part of the proof of Theorem 1, $q = k - 1$ and we get (9).

Let $q > 0$. Then from (9) we obtain (10) and (11). We see that for $q = 0$, the inequality (11) reduces to the inequality (9) (just replace v_2 by t). Therefore, for $n - q$ even, we can consider the case $q > 0$ together with the case $q = 0$. Note that for $q \geqq 0$ we have $k \geqq 1$. Thus by (2), $u'(t) > 0$ for $t \geqq t_2$. Since $x \geqq t \geqq t_2 \geqq t_1 = = \gamma(T)$ in (11) and $u(t)$ is an increasing function for $t \geqq t_2$, we get (from (11))

$$u(t) \geqq \frac{u^\alpha(t_2)}{(n-1)!}(t - v_1)^{n-1} \int_t^\infty p(x)\,\mathrm{d}x \tag{17}$$

for $t \geqq \gamma(t_2)$.

From the last inequality, by (16), we have

$$\lim_{t \to \infty} u(t) = \infty \tag{18}$$

and the theorem is proved in the case when (3) holds and $n - q$ is an even number.

Again, let (3) hold true. Now let $n - q$ be odd. Then $q = k$. Continuing as in the corresponding part of the proof of Theorem 1 we get (13), further, for $q > 0$ we get (14) and (15) which coincides with the inequality (10). From (10) we obtain (11) and (17) as above. Finally, from (17) we obtain (18).

Now let $q = 0$. Then $n - q = n$ is odd. Since $q = k$, (2) implies $u'(t) < 0$ for each $t \geqq t_2$. This means that $u(t)$ is a positive decreasing function for $t \geqq t_2$ and, by (3), $\lim_{t\to\infty} u(t) = c$ $(c > 0)$. We shall show that the condition (16) is not satisfied in this case.

We know that

$$c < u(t) \leqq u(t_2) \quad \text{for} \quad t \geqq t_2\,,$$

and for $t \geqq \gamma(t_2)$ we have

$$c < u(t) \leqq u(\tau(t)) \leqq u(t_2)\,.$$

These inequalities imply that

$$c^\alpha\, p(t) < p(t)\, u^\alpha(\tau(t)) = -u^{(n)}(t)$$

or

$$c^\alpha\, p(t) < -u^{(n)}(t)\,.$$

Integrating the last inequality n times from t to ∞ $(t \geqq \gamma(t_2))$ and using (3), we obtain

$$\frac{c^\alpha}{(n-1)!}\int_t^\infty (x-t)^{n-1}\, p(x)\,\mathrm{d}x \leqq u(t) - c$$

whence

$$\int_t^\infty (x-t)^{n-1}\, p(x)\,\mathrm{d}x \leqq \frac{(n-1)!\, u(t_2)}{c^\alpha}\,,$$

because $c > 0$ and $u(t) \leqq u(t_2)$.

Without loss of generality we can assume that $t > 0$. Then

$$(2t)^{n-1}\int_{2t}^\infty p(x)\,\mathrm{d}x \leqq 2^{n-1}\int_{2t}^\infty (x-t)^{n-1}\, p(x)\,\mathrm{d}x \leqq$$

$$\leqq 2^{n-1}\int_t^\infty (x-t)^{n-1}\, p(x)\,\mathrm{d}x \leqq 2^{n-1}\,.\,\frac{(n-1)!\, u(t_2)}{c^\alpha}\,,$$

whence

$$s^{n-1}\int_s^\infty p(x)\,\mathrm{d}x \leqq 2^{n-1}\,.\,\frac{(n-1)!\, u(t_2)}{c^\alpha}\,,$$

where $s = 2t$. From this we see that

$$\limsup_{s\to\infty} s^{n-1}\int_s^\infty p(x)\,\mathrm{d}x < \infty\,.$$

This completes the proof of the theorem if (3) holds and $n - q$ is an odd number.

Again, let $u(t)$ be a nonoscillatory solution of (1). Consider the case when the conditions (3) are not satisfied. But then the conditions (4) are satisfied and we see that in this case the solution $u(t)$ is strongly monotone. On the other hand, we know (by Lemma 1) that the conditions (4) are satisfied only if $k = 0$ and, further, that $k = 0$ only if n is odd. This completes the proof of the theorem.

Combining what has been said with Theorems 1 and 2 we arrive at the following result:

Theorem 3. *Let* $\alpha > 1$. *If*

$$\limsup_{t\to\infty} t^{n-1}\int_{\gamma(t)}^\infty p(x)\,\mathrm{d}x > 0 \tag{19}$$

and

$$\limsup_{t\to\infty} t^{n-1}\int_t^\infty p(x)\,\mathrm{d}x = \infty\,, \tag{20}$$

then every solution of (1) *is oscillatory if* n *is even, and is either oscillatory or strongly monotone if* n *is odd.*

Proof. On the contrary, let $u(t)$ be a nonoscillatory solution of (1). Moreover, let $u(t)$ be not strongly monotone if n is odd. Then, by (19) and Theorem 1, $u(t)$ must be bounded. On the other hand, by (20) and Theorem 2, $u(t)$ must be unbounded if n is even, and $u(t)$ must be either unbounded or strongly monotone if n is odd. This is a contradiction and the theorem is proved.

References

[1] *Ohriska, J.:* The argument delay and oscillatory properties of differential equation of n-th order. Czech. Math. J., *29* (*104*) 1979, 268—283.

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