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SUFFICIENT CONDITIONS FOR UNIVALENCE OF A GENERAL INTEGRAL OPERATOR

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Abstract. In this paper, univalence of a certain integral operator and some interesting properties involving the integral operators on the classes of complex order are obtained. Relevant connections of the results, which are presented in this paper, with various other known results are also pointed out.

1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{1}$$

which are analytic in the open disc $\mathcal{U} = \{z \in \mathbb{C} : | z |< 1\}$ and $\mathcal{S}$ be the class of function $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

Let the function $\phi(a, c; z)$ be given by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \qquad (c \neq 0, -1, -2, \ldots : z \in \mathcal{U}),$$

where $(x)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)(x+2) \cdots (x+n-1) & \text{if } n \in N = \{1, 2, , \ldots\}. \end{cases}$$

Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer [5] introduced a linear operator $L(a, c)f$, which is defined by the following Hadamard product (or convolution):

$$L(a, c)f(z) := \phi(a, c; z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n.$$

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We note that

$$L(a,a)f(z) = f(z), \quad L(2,1)f(z) = zf^{'}(z), \quad L(n+1,1)f(z) = D^n f(z),$$

where $D^n f$ is the Ruscheweyh derivative of f.

Let $\mathcal{S}^*_\gamma(a,c;b)$ denote the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\text{(2)} \quad \text{Re}\left\{1 + \frac{1}{b}\left(\frac{z(L(a,c)f(z))^{'}}{L(a,c)f(z)} - 1\right)\right\} > \gamma, \quad (0 \le \gamma < 1,\ z \in \mathcal{U};\ b \in \mathbb{C} \setminus \{0\}).$$

If $a = c$, the class $\mathcal{S}^*_\gamma(a,a;b) = \mathcal{S}^*_\gamma(b)$ is the well-known starlike functions of complex order b. When $a = 2$ and $c = 1$ the class $\mathcal{S}^*_\gamma(2,1;b) = \mathcal{C}_\gamma(b)$ is precisely the class of convex functions of complex order b. For details regarding these subclasses and various other classes we refer to [6, 7, 8].

Definition. For $n \in \mathbb{N} \cup \{0\}$, $i \in \{1,2,3,\ldots,n\}$, $\alpha_i \in \mathbb{C}$. We now define the integral operator $F_\alpha(a,c;z) : \mathcal{A}^n \longrightarrow \mathcal{A}$

$$\text{(3)} \qquad F_\alpha(a,c;z) = \int_0^z \left(\frac{L(a,c)f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{L(a,c)f_n(t)}{t}\right)^{\alpha_n} dt,$$

where $f_i \in \mathcal{A}$ and $L(a,c)f$ is the Carlson-Shaffer operator.

Remark 1.1. It is interesting to note that the integral operator $F_\alpha(a,c;z)$ generalizes many operators which were introduced and studied recently. Here we list a few of them.

1. When $a = c$, the operator $F_\alpha(a,a;z)$ reduces to an integral operator

$$\text{(4)} \qquad F_\alpha(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt.$$

   introduced and studied by D. Breaz and N. Breaz in [2].

2. When $a = 2$ and $c = 1$, the integral operator $F_\alpha(2,1;z)$ reduces to an operator

$$\text{(5)} \qquad G_\alpha(z) = \int_0^z \left(f_1^{'}(t)\right)^{\alpha_1} \cdots \left(f_n^{'}(t)\right)^{\alpha_n} dt, \quad (\alpha_i \ge 0),$$

   recently introduced and studied by D. Breaz, S. Owa, and N. Breaz in [4].

3. For the choice of the parameters $a = n+1$ and $c = 1$, we obtain the integral operator

$$\text{(6)} \qquad I(f_1,\ \ldots,\ f_m)(z) = \int_0^z \left(\frac{D^n f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{D^n f_m(t)}{t}\right)^{\alpha_n} dt,$$

   where $D^n f$ is the Ruscheweyh derivative of f. The operator $I(f_1,\ldots,f_m)$ was studied by G. I. Oros, G. Oros, and D. Breaz in [9].

Apart from the above, the integral operator $F_\alpha(a,c;z)$ generalizes the well known operators like Alexander transforms and Libera integral operator.

We now state the following lemma which we need to establish our results in the sequel.

Lemma 1.2 ([1]). *If $f \in \mathcal{A}$ satisfies the inequality*

$$(1-\mid z\mid^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \quad \textit{for all } z \in \mathcal{U},$$

then the function f is univalent in $\mathcal{U}$.

2. Main results

We begin with the following:

Theorem 2.1. *Let $f_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$, $i \in \{1, 2, \dots n\}$. If*

$$\left|\frac{z(L(a,c)f_i(z))'}{L(a,c)f_i(z)} - 1\right| \le 1, \quad \mid \alpha_1 \mid + \mid \alpha_2 \mid + \cdots + \mid \alpha_n \mid \le 1, \quad z \in \mathcal{U},$$

then $F_\alpha(a, c; z)$ given by (3) *is univalent.*

Proof. From the definition of the operator $L(a, c)f(z)$, it can be easily seen that

$$\frac{L(a,c)f(z)}{z} \neq 0 \quad (z \in \mathcal{U})$$

and moreover for $z = 0$, we have

$$\left(\frac{L(a,c)f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{L(a,c)f_n(z)}{z}\right)^{\alpha_n} = 1.$$

From (3), we obtain

$$F_\alpha'(a,c;z) = \left(\frac{L(a,c)f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{L(a,c)f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F_\alpha''(a,c;z) = \sum_{i=1}^{n} \alpha_i \left(\frac{z(L(a,c)f_i(z))' - L(a,c)f_i(z)}{zL(a,c)f_i(z)}\right) F_\alpha'(a,c;z).$$

From the above equalities, we have

$$\frac{F_\alpha''(a,c;z)}{F_\alpha'(a,c;z)} = \alpha_1 \left(\frac{z(L(a,c)f_1(z))' - L(a,c)f_1(z)}{zL(a,c)f_1(z)}\right) + \cdots + \alpha_n \left(\frac{z(L(a,c)f_n(z))' - L(a,c)f_n(z)}{zL(a,c)f_n(z)}\right)$$

and

$$(7) \quad \frac{F_\alpha''(a,c;z)}{F_\alpha'(a,c;z)} = \alpha_1 \left(\frac{(L(a,c)f_1(z))'}{L(a,c)f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n \left(\frac{(L(a,c)f_n(z))'}{L(a,c)f_n(z)} - \frac{1}{z}\right).$$

By multiplying the relation (7) with z, we obtain

$$(8) \quad \frac{zF_\alpha''(a,c;z)}{F_\alpha'(a,c;z)} = \alpha_1 \left(\frac{z(L(a,c)f_1(z))'}{L(a,c)f_1(z)} - 1\right) + \ldots + \alpha_n \left(\frac{z(L(a,c)f_n(z))'}{L(a,c)f_n(z)} - 1\right).$$

On multiplying the modulus of equation (8) by $(1-\mid z\mid^2)$, we obtain

$$
\begin{aligned}
(1-\mid z\mid^2)\left|\frac{zF''_\alpha(a,c;z)}{F'_\alpha(a,c;z)}\right| &\le (1-\mid z\mid^2)\left[\mid\alpha_1\mid\left|\frac{z(L(a,c)f_1(z))'}{L(a,c)f_1(z)}-1\right|+\cdots\right.\\
&\qquad\left.+\mid\alpha_n\mid\left|\frac{z(L(a,c)f_n(z))'}{L(a,c)f_n(z)}-1\right|\right]\\
&\le (1-\mid z\mid^2)\big[\mid\alpha_1\mid+\mid\alpha_2\mid+\cdots+\mid\alpha_n\mid\big]\\
&\le \mid\alpha_1\mid+\mid\alpha_2\mid+\cdots+\mid\alpha_n\mid\le 1.
\end{aligned}
$$

From Lemma 1.2, we have that $F_\alpha(a,c;z)\in\mathcal{S}$. □

Putting $a=c$ in Theorem 2.1, we have:

Corollary 2.2 ([2]). *Let $f_i\in\mathcal{A}$, $\alpha_i\in\mathbb{C}$, $i\in\{1,2,\ldots,n\}$. If*

$$\left|\frac{zf'_i(z)}{f_i(z)}-1\right|\le 1,\quad \mid\alpha_1\mid+\mid\alpha_2\mid+\cdots+\mid\alpha_n\mid\le 1,\quad z\in\mathcal{U},$$

then F_α given by (4) *is univalent.*

Putting $a=n+1$ and $c=1$ in Theorem 2.1, we have:

Corollary 2.3 ([9]). *Let $f_i\in\mathcal{A}$, $\alpha_i\in\mathbb{C}$, $i\in\{1,2,\ldots,m\}$. If*

$$\left|\frac{z(D^nf_i(z))'}{D^nf_i(z)}-1\right|\le 1,\quad \mid\alpha_1\mid+\mid\alpha_2\mid+\cdots+\mid\alpha_m\mid\le 1,\quad z\in\mathcal{U},$$

then $I(f_1,\ldots,f_m)$ given by (6) *is univalent.*

Theorem 2.4. *Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be real numbers with the properties $\alpha_i>0$ for $i\in\{1,2,\ldots,n\}$ and*

$$0\le(\gamma-1)\sum_{i=1}^{n}\alpha_i+1<1. \tag{9}$$

If $f_i\in\mathcal{S}^*_\gamma(a,c;b)$ for $i=\{1,\ldots,n\}$, then $F_\alpha(a,c;z)$ given by (3) *belongs to $\mathcal{C}_\delta(b)$, where $\delta=(\gamma-1)\sum_{i=1}^{n}\alpha_i+1$.*

Proof. Using (8), we obtain

$$
\begin{aligned}
\frac{1}{b}\frac{zF''_\alpha(a,c;z)}{F'_\alpha(a,c;z)} &= \frac{1}{b}\sum_{i=1}^{n}\alpha_i\left(\frac{z(L(a,c)f_i(z))'}{L(a,c)f_i(z)}-1\right)\\
&=\sum_{i=1}^{n}\alpha_i\left[1+\frac{1}{b}\left(\frac{z(L(a,c)f_i(z))'}{L(a,c)f_i(z)}-1\right)\right]-\sum_{i=1}^{n}\alpha_i.
\end{aligned}\tag{10}
$$

The relation (10) is equivalent to

$$1+\frac{1}{b}\frac{zF''_\alpha(a,c;z)}{F'_\alpha(a,c;z)}=\sum_{i=1}^{n}\alpha_i\left[1+\frac{1}{b}\left(\frac{z(L(a,c)f_i(z))'}{L(a,c)f_i(z)}-1\right)\right]-\sum_{i=1}^{n}\alpha_i+1. \tag{11}$$

We now calculate the real part of both terms of (11) and obtain
(12)
$$\operatorname{Re}\left\{1+\frac{1}{b}\frac{zF_\alpha''(a,c;z)}{F_\alpha'(a,c;z)}\right\} = \sum_{i=1}^{n}\alpha_i \operatorname{Re}\left[1+\frac{1}{b}\left(\frac{z(L(a,c)f_i(z))'}{L(a,c)f_i(z)}-1\right)\right] - \sum_{i=1}^{n}\alpha_i + 1.$$

Since each $f_i \in \mathcal{S}^*_\gamma(a,c;b)$ for $i = \{1, \ldots, n\}$, by using (2) in (12) we have

$$\operatorname{Re}\left\{1+\frac{1}{b}\frac{zF_\alpha''(a,c;z)}{F_\alpha'(a,c;z)}\right\} > (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1. \tag{13}$$

Since by hypothesis $0 \le (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1 < 1$, we obtain $F_\alpha(a,c;z) \in \mathcal{C}_\delta(b)$, where $\delta = (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1$. □

It is interesting to observe that all the results in [3] follows for the choice of the parameters a and c in Theorem 2.4, which we state below:

If we let $a = c$ in Theorem 2.4, we have:

Corollary 2.5 ([3])**.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, 2, \ldots, n\}$ and*

$$0 \le (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1 < 1.$$

If $f_i \in \mathcal{S}^*_\gamma(b)$ for $i = \{1, \ldots, n\}$, then F_α given by (4) belongs to $\mathcal{C}_\delta(b)$, where $\delta = (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1$.

Putting $a = 2$ and $c = 1$ in Theorem 2.4, we have:

Corollary 2.6 ([3])**.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, 2, \ldots, n\}$ and*

$$0 \le (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1 < 1.$$

If $f_i \in \mathcal{C}_\gamma(b)$ for $i = \{1, \ldots, n\}$, then G_α given by (5) belongs to $\mathcal{C}_\delta(b)$, where $\delta = (\gamma-1)\sum_{i=1}^{n}\alpha_i + 1$.

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