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Edison Tse (M'70), for a photograph and biography see page 16 of the February 1973 issue of this TRANSACTIONS.

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# Sufficiently Informative Functions and the Minimax Feedback Control of Uncertain Dynamic Systems

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**Abstract**—The problem of optimal feedback control of uncertain discrete-time dynamic systems is considered where the uncertain quantities do not have a stochastic description but instead are known to belong to given sets. The problem is converted to a sequential minimax problem and dynamic programming is suggested as a general method for its solution. The notion of a sufficiently informative function, which parallels the notion of a sufficient statistic of stochastic optimal control, is introduced, and conditions under which the optimal controller decomposes into an estimator and an actuator are identified. A limited class of problems for which this decomposition simplifies the computation and implementation of the optimal controller is delineated.

## I. INTRODUCTION

THIS PAPER is concerned with the optimal feedback control of a discrete-time dynamic system in the presence of uncertainty. The traditional treatment of this problem has been to assign probability distributions

to the uncertain quantities and to formulate the optimization problem as one of minimizing the expected value of a suitable cost functional. In this paper, a nonprobabilistic description of the uncertainty is adopted, where, instead of being modeled as random vectors with given probability distributions, the uncertainties are considered to be unknown except for the fact that they belong to given subsets of appropriate vector spaces. The optimization problem is then cast as one of finding the feedback controller within a prescribed admissible class that minimizes the maximum value (over all possible values of the uncertain quantities) of a suitable cost functional. This worst case approach to the optimal control of uncertain dynamic systems is applicable to problems where a set-membership description of the uncertain quantities is more natural or more readily available than a probabilistic one, or when specified tolerances must be met with certainty.

The modeling of uncertainties as quantities that are unknown except that they belong to prescribed sets and the adoption of a worst case viewpoint in the context of the problem of feedback control of a dynamic system was first considered by Witsenhausen [1], [2] and received further attention in [4]–[10]. In this paper a general minimax feedback control problem which involves a

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discrete-time dynamic system defined on a Euclidean space is formulated in Section II, and, in Section III, a dynamic programming algorithm is given for its solution. This algorithm is similar to one given earlier by Witsenhausen [1], but is somewhat more detailed and explicit than and somewhat different in its form from that in [1], although the same basic ideas are involved. Subsequently, in Section IV, an effort is made to identify conditions under which this algorithm can be simplified, and to deduce structural properties of the optimal controller. This is accomplished by introducing the notion of a sufficiently informative function, in analogy with the familiar notion of a sufficient statistic of stochastic optimal control. It is proved in Section IV that the decomposition of the optimal controller into an estimator and an actuator is possible. Finally, in Section V some special cases for which this decomposition is profitable are delineated.

## II. PROBLEM FORMULATION

*Problem 1:* Given is the discrete-time dynamic system

$$x_{k+1} = f_k(x_k, u_k, w_k), \quad k = 0, 1, \dots, N-1 \quad (1)$$

where  $x_k \in R^n$ ,  $k = 0, 1, \dots, N$ , is the state vector;  $u_k \in R^m$ ,  $k = 0, 1, \dots, N-1$ , is the control vector;  $w_k \in R^r$ ,  $k = 0, 1, \dots, N-1$ , is the input disturbance vector; and  $f_k: R^n \times R^m \times R^r \rightarrow R^n$  is a known function for each  $k = 0, 1, \dots, N-1$ .

Available to the controller are measurements of the form

$$z_k = h_k(x_k, v_k), \quad k = 1, 2, \dots, N-1 \quad (2)$$

where  $z_k \in R^s$ ,  $k = 1, 2, \dots, N-1$ , is the measurement vector;  $v_k \in R^p$ ,  $k = 1, 2, \dots, N-1$ , is the measurement noise vector; and  $h_k: R^n \times R^p \rightarrow R^s$  is a known function for each  $k = 0, 1, \dots, N-1$ .

The uncertain quantities lumped in a vector  $q \in R^{n+Nr+(N-1)p}$

$$q = (x_0', w_0', u_1', \dots, w_{N-1}', v_1', v_2', \dots, v_{N-1}')' \quad (3)$$

are known to belong to a given subset  $Q$  of  $R^{n+Nr+(N-1)p}$ .

Attention is restricted to control laws of the form

$$\mu_k: R^{k(s+m)} \rightarrow R^m, \quad k = 0, 1, \dots, N-1 \quad (4)$$

taking values

$$u^k = \mu_k(z_1, z_2, \dots, z_k, u_0, u_1, \dots, u_{k-1}), \quad k = 0, 1, \dots, N-1. \quad (5)$$

Because the control at time 0 depends only on *a priori* data,  $\mu_0$  may be interpreted as a constant vector.

It is required to find (if it exists) the control law in this class for which the cost functional

$$J(\mu_0, \mu_1, \dots, \mu_{N-1}) = \sup_{q \in Q} F[x_1, x_2, \dots, x_N, \mu_0, \mu_1(z_1, u_0), \dots, \mu_{N-1}(z_1, \dots, u_{N-1})] \quad (6)$$

is minimized subject to the system equation constraint, where  $F: R^{N(n+m)} \rightarrow (-\infty, +\infty]$  is a given function.

It should be noted that this problem formulation implicitly includes the possible presence of state and control constraints, since we allow the function  $F$  to take the value  $+\infty$ . We need simply specify that  $F$  take the value  $+\infty$  whenever some constraint is violated. Thus, for example, state and control constraints of the form  $x_k \in X_k$  or  $u_{k-1} \in U_{k-1}$ , where  $X_k$  and  $U_{k-1}$ ,  $k = 1, 2, \dots, N$ , are given sets, may be accounted for by additively including in  $F$  the function

$$\sum_{i=1}^N \{ \delta(x_i | X_i) + \delta[\mu_{i-1}(z_1, \dots, u_{i-2}) | U_{i-1}] \}$$

where  $(y|Y)$  denotes the indicator function of a set  $Y$ , viz.,

$$\delta(y|Y) = \begin{cases} 0, & y \in Y \\ \infty, & y \notin Y. \end{cases} \quad (7)$$

In the next section we present a dynamic programming algorithm for the solution of Problem 1. Using this algorithm we will then be able to reach some conclusions concerning the structure of the optimal control law.

## III. SOLUTION BY DYNAMIC PROGRAMMING

Consider the optimal value of the cost function (6)

$$\bar{J} = \inf_{\mu_k} \sup_{q \in Q} F(x_1, x_2, \dots, x_N, u_0, u_1, \dots, u_{N-1}), \quad k = 0, 1, \dots, N-1. \quad (8)$$

The purpose of the dynamic programming algorithm is to convert the minimization problem indicated in the above equation to a sequence of simpler minimization problems by taking advantage of the sequential evolution of the system state and the information available to the controller according to (1) and (2). However, matters are somewhat complicated in the above problem by the presence of uncertainty, since in the process of generating the state and measurement vectors the disturbances are immediately selected by, say, Nature with the objective of maximizing the value of the cost. For this reason the development of the dynamic programming algorithm requires a somewhat elaborate construction.

In order to simplify the notation we will make use of the vector  $\zeta_k \in R^{k(s+m)}$ ,  $k = 1, 2, \dots, N-1$ , which consists of all the information available to the controller at time  $k$ , viz.,

$$\zeta_k = (z_1', z_2', \dots, z_k', u_0', u_1', \dots, u_{k-1}')'. \quad (9)$$

With this notation we write for the control law

$$\mu_k(z_1, z_2, \dots, z_k, u_0, u_1, \dots, u_{k-1}) = \mu_k(\zeta_k) = u_k. \quad (10)$$

Now consider the following definitions. Let  $P(R^s)$  be the power set (the set of all subsets) of  $R^s$  and consider the following function:

$$\hat{Z}_k: R^{(k-1)s+km} \rightarrow P(R^s), \quad (11)$$

which assigns to the vectors  $\zeta_{k-1}, u_{k-1}$  the set  $\hat{Z}_k(\zeta_{k-1}, u_{k-1}) \subset R^s$  of all measurement vectors  $z_k$  given by (2)

which are consistent with the constraint set  $Q$ , the previous measurement vectors  $z_1, z_2, \dots, z_{k-1}$ , and the previous control vectors  $u_0, u_1, \dots, u_{k-1}$ . In other words,  $z_k \in \hat{Z}_k(\zeta_{k-1}, u_{k-1})$  if and only if there exists a vector  $q = (x_0', w_0', w_1', \dots, w_{N-1}', v_1', v_2', \dots, v_{N-1}')' \in Q$  such that the vectors  $x_0, w_0, \dots, w_{k-1}, v_1, \dots, v_k, z_1, \dots, z_k, u_0, \dots, u_{k-1}$  together satisfy the system and measurement equations (1) and (2) for times  $0, 1, \dots, k$ .

We also define the function

$$\hat{Q}: R^{(N-1)s + Nm} \rightarrow P(R^{n + Nr + (N-1)p}), \quad (12)$$

which assigns to the vectors  $\zeta_{N-1}, u_{N-1}$  the set  $\hat{Q}(\zeta_{N-1}, u_{N-1}) \subset R^{n + Nr + (N-1)p}$  of all vectors  $q \in Q$  [recall (3)] which are consistent with the measurements  $z_1, z_2, \dots, z_{N-1}$  and the control vectors  $u_0, u_1, \dots, u_{N-1}$ . In other words a vector  $q$  belongs to the set  $\hat{Q}(\zeta_{N-1}, u_{N-1})$  if and only if  $q \in Q$  and the vectors  $x_0, w_0, \dots, w_{N-1}, v_1, \dots, v_{N-1}, z_1, \dots, z_{N-1}, u_0, \dots, u_{N-1}$  together satisfy the system and measurement equations (1) and (2) for all  $k$ .

It should be noted that for some vectors  $\zeta_{k-1}$  it is possible that the set  $\hat{Z}_k(\zeta_{k-1}, u_{k-1})$  or the set  $\hat{Q}(\zeta_{N-1}, u_{N-1})$  is empty for all  $u_{k-1} \in R^m$ , implying that the vector  $\zeta_{k-1}$  is inconsistent with the constraint set  $Q$  and the system and measurement equations. Notice also that whether the set  $\hat{Z}_k(\zeta_{k-1}, u_{k-1})$  is empty or nonempty depends on the vector  $\zeta_{k-1}$  alone and is entirely independent of  $u_{k-1}$ . From (2) the consistency of a vector  $\zeta_k$  is equivalent to the existence of vectors  $x_k, v_k$  consistent with  $\zeta_{k-1}$  and the constraint  $q \in Q$  (for all  $u_{k-1}$ ) and therefore is equivalent to the existence of a vector  $z_k \in \hat{Z}_k(\zeta_{k-1}, u_{k-1})$ . In subsequent equations in which empty sets appear we will adopt the convention that the supremum of the empty set is  $-\infty$  ( $\sup \phi = -\infty$ ). Another possible approach would be to restrict the domain of definition of the functions  $\hat{Z}_k, \hat{Q}$  to include only those vectors  $\zeta_{k-1}$  for which the sets  $\hat{Z}_k(\zeta_{k-1}, u_{k-1})$ ,  $\hat{Q}(\zeta_{N-1}, u_{N-1})$  are nonempty. Since in any actual operation of the system these sets will always be nonempty, this restriction results in no loss of generality.

We are now ready to state and prove the following dynamic programming algorithm for the solution of Problem 1.

**Proposition 1:** Assume that for the functions  $H_k$  defined below we have  $-\infty < H_k(\zeta_k)$ ,  $k = 1, 2, \dots, N-2$ , for all vectors  $\zeta_k$  such that the set  $\hat{Z}_{k+1}(\zeta_k, u_k)$  is nonempty (for all  $u_k \in R^m$ ), and  $-\infty < H_{N-1}(\zeta_{N-1})$  for all vectors  $\zeta_{N-1}$  such that the set  $\hat{Q}(\zeta_{N-1}, u_{N-1})$  is nonempty. Then the optimal value  $\bar{J}$  of the cost functional (6) is given by

$$\bar{J} = \inf_{u_0} E_1(u_0) \quad (13)$$

where the function  $E_1: R^m \rightarrow (-\infty, +\infty]$  is given by the last step of the recursive algorithm

$$E_N(\zeta_{N-1}, u_{N-1}) = \sup_{q \in \hat{Q}(\zeta_{N-1}, u_{N-1})} F(x_1, x_2, \dots, x_N, u_0, u_1, \dots, u_{N-1}) \quad (14)$$

$$H_k(\zeta_k) = \inf_{u_k} E_{k+1}(\zeta_k, u_k), \quad k = 1, 2, \dots, N-1 \quad (15)$$

$$\begin{aligned} E_{k+1}(\zeta_k, u_k) &= \sup_{z_{k+1} \in \hat{Z}_{k+1}(\zeta_k, u_k)} H_{k+1}(\zeta_k, u_k, z_{k+1}) \\ &= \sup_{z_{k+1} \in \hat{Z}_{k+1}(\zeta_k, u_k)} \bar{H}_{k+1}(\zeta_{k+1}), \quad k = 0, 1, \dots, N-2. \end{aligned} \quad (16)$$

The above proposition will not be proved here. Its proof involves standard but lengthy dynamic programming arguments and can be found in [4].

The dynamic programming algorithm of Proposition 1 can be profitably interpreted in terms of game theory, and in particular in terms of multistage games of perfect information [13]. The optimal value of the cost  $\bar{J}$  can be viewed as the upper value (or min-max) of a game played by two opponents, the Controller selecting the control law  $(\mu_0, \mu_1, \dots, \mu_{N-1})$ , and Nature selecting the uncertain quantities  $q$  from the set  $Q$ . The information, based on which the decision of the Controller is made, is fixed by the form of the functions  $\mu_k$ , i.e., by the information vectors  $\zeta_k$ . Since, however, only the upper value of the game is of interest here, a variety of equivalent methods of selections of the vector  $q$  and corresponding information patterns can be assigned to Nature. One such information pattern and method for selection of the components of the vector  $q$  corresponds to the following sequence of events: 1) Controller selects  $u_0$ ; 2) Nature selects  $z_1$  from the set  $\hat{Z}_1(u_0)$ ; 3) Controller selects  $u_1$ ; 4) Nature selects  $z_2$  from the set  $\hat{Z}_2(z_1, u_0, u_1), \dots, 2N-1u_{N-1}$ ; 2N) Nature selects all the uncertain quantities  $q = (x_0', w_0', w_1', \dots, w_{N-1}', v_1', v_2', \dots, v_{N-1}')'$  from the set  $\hat{Q}(\zeta_{N-1}, u_{N-1})$ . Each selection by either Controller or Nature is made with full knowledge of the outcomes of previous selections.

This sequence of events is fictitious; however, it accurately reflects the sequence of events as viewed by the Controller whose only information concerning the course of the game at time  $k$  is the information  $\zeta_k$ , i.e., all measurements and all control selections up to that time.

A moment's reflection shows that in fact the dynamic programming algorithm determines the (pure) value  $\bar{J}$  of the game of perfect information described above. This value is the same as the optimal cost  $\bar{J}$  of the Problem 1.

Finding the optimal cost  $\bar{J}$  and the optimal control law from the dynamic programming algorithm of Proposition 1 is in general a very difficult task. Part of the difficulty stems from the fact that, loosely speaking, the objective of the Controller is dual in nature: first, to actuate the system in a favorable fashion, and, second, to try to improve the quality of his estimate of the uncertainty in the system. This is a familiar situation from stochastic optimal control, known as dual control problem [11], the formidable complexities of which have been widely discussed in the literature. In stochastic optimal control, insight into the structure of the optimal controller, and its dual function, can be obtained through the notion of a sufficient statistic [12]–[14]. Similar insight will be obtained for the minimax controller of this chapter by introducing in the next section the analogous concept of a sufficiently informative function.

#### IV. SUFFICIENTLY INFORMATIVE FUNCTIONS

Let us consider the following definition.

**Definition 1:** A function  $S_k: R^{k(s+m)} \rightarrow \Sigma_k$ , where  $\Sigma_k$  is some space, will be called *sufficiently informative* with respect to Problem 1 if there exists a function  $\bar{E}_{k+1}: \Sigma_k \times R^m \rightarrow (-\infty, +\infty]$  for all  $k = 0, 1, \dots, N-1$  such that

$$\bar{E}_{k+1}[S_k(\zeta_k), u_k] = E_{k+1}(\zeta_k, u_k) \quad (17)$$

where  $E_{k+1}$  is the function defined in (14) and (16) for  $k = 0, 1, \dots, N-1$ . The value of a sufficiently informative function at any point will be called *sufficient information*.

The clear consequence of the above definition is that, if  $S_k$  is a sufficiently informative function, then (15) may be rewritten as

$$\bar{H}_k(S_k(\zeta_k)) = H_k(\zeta_k) = \inf_{u_k} \bar{E}_{k+1}(S_k(\zeta_k), u_k);$$

so that we now seek the infimum over  $u_k$  of a function of  $u_k$  and the sufficient information  $S_k(\zeta_k)$ . If this infimum is attained for all  $\zeta_k$ , then there exists an optimal control law  $\bar{\mu}_k$  that can also be written as

$$\bar{\mu}_k(\zeta_k) = \bar{\mu}_k^* \cdot S_k(\zeta_k) \quad (18)$$

where  $\bar{\mu}_k^*$  is a suitable function which can be determined by minimizing the function  $\bar{E}_{k+1}$  of (17) with respect to  $u_k$ . As a result the control at any time need only depend on the sufficient information  $S_k(\zeta_k)$ . If this sufficient information can be more easily generated or stored than the information vector  $\zeta_k$ , and, furthermore, if it is easier to minimize the function  $\bar{E}_{k+1}$  over  $u_k$  rather than the function  $E_{k+1}$ , then it is advantageous to compute and implement the control law in the form of (18).

Factorizations of the optimal control law into the composition of two functions, as in (18), have been widely considered in stochastic optimal control theory, and are commonly referred to as separation theorems whenever the function  $S_k$  can be interpreted as an estimator. In such problems the function  $S_k$  or its value is usually called a sufficient statistic. Particularly simple sufficient statistics have been found for problems involving a linear system, linear measurements, and Gaussian white input and measurement noises [12]. In other problems sufficient statistics of interest take the form of conditional probability distributions conditioned on the information available [12]. Such sufficient statistics imply the factorization of the optimal control law into an estimator  $S_k$  computing the conditional probability distribution of some quantities, which may differ depending on the problem given, and an actuator  $\bar{\mu}_k^*$  applying a control input to the system. It has been demonstrated [3]–[5] that, in estimation problems which involve a set-membership description of the uncertainty, the set of possible states consistent with the measurements received plays a role analogous to that of conditional probability distributions in stochastic estimation problems. Thus it should not come as a surprise that for Problem 1 we shall be able to derive sufficiently informative functions that involve

sets of possible system states (or other quantities) consistent with the measurements received. In what follows we obtain such sufficiently informative functions and further discuss the well-behaved case of a linear system and an energy constraint on the uncertain quantities for which, as was demonstrated in [4] and [7], the set of possible states can be characterized by a finite set of numbers. We first introduce the following notation.

We denote for all  $k$  by

$$S_k(x_1, \dots, x_k, w_k, \dots, w_{N-1}, v_{k+1}, \dots, v_{N-1} | \zeta_k)$$

the subset of  $R^{kn + (N-k)r + (N-k-1)p}$  which consists of all vectors  $(x_1, \dots, x_k, w_k, \dots, w_{N-1}, v_{k+1}, \dots, v_{N-1})$  that are consistent with the measurements  $z_1, z_2, \dots, z_k$ , the control vectors  $u_0, u_1, \dots, u_{k-1}$ , the system and measurement equations (1) and (2) and the constraint  $q \in Q$ . Similarly, we denote by  $S_k(\cdot, \dots, \cdot | \zeta_k)$  the respective sets of all possible quantities within the parentheses that are consistent with the information vector  $\zeta_k$ , the system and measurement equations, and the constraint  $q \in Q$ .

With the above notation we have the following proposition.

**Proposition 2:** A sufficiently informative function with respect to Problem 1 is the function

$$S_k: R^{k(s+m)} \rightarrow P(R^{kn + (N-k)r + (N-k-1)p}) \times R^{km}$$

given for all  $\zeta_k$  and  $k$  by

$$S_k(\zeta_k) = [S_k(x_1, \dots, x_k, w_k, \dots, w_{N-1}, v_{k+1}, \dots, v_{N-1} | \zeta_k), u_0, u_1, \dots, u_{k-1}]. \quad (19)$$

Again, the proof of the above proposition is straightforward but tedious, and will not be presented here. It can be found in [4].

Proposition 2 shows that sufficient information in the case of Problem 1 is provided at each time by the set of past inputs together with the set of past and present states and future uncertainties that are consistent with the observed output sequence up to that time. In addition, Proposition 2 clearly illustrates the dual function of the optimal controller. By (18) the optimal control law is of the form

$$\bar{\mu}_k = \bar{\mu}_k^* \cdot S_k, \quad (20)$$

i.e., it is the composition of the sufficiently informative function  $S_k$  and the function  $\bar{\mu}_k^*$ . The function  $S_k$  may be interpreted as an estimator, and the function  $\bar{\mu}_k^*$  as an actuator. Alternatively, the optimal controller can be viewed as being composed of two cascaded parts. The first part produces an estimate set and the second part accepts as input this estimate set and produces a control vector. This control vector is stored and recalled in the future by the controller.

By adding additional structure to Problem 1, various important simplifications can be achieved in the sufficiently informative function (19). The additional structure takes the form of further assumptions on the form of the set  $Q$  in which the uncertainties lie, and the cost functional

$F$ . The simplifications that these additional assumptions induce in the sufficiently informative function are reductions in the number of entities whose consistency with the past output measurements needs to be considered. For example, under certain conditions there is no need to include the past states  $x_1, \dots, x_{k-1}$  in the set of entities whose consistency with the output data is part of the sufficient information (19). Under other conditions there is no need to retain the future uncertainties in  $S(\cdot|\zeta_k)$ . In other cases, the controller need not recall past control inputs. Because the verification of each of these simplifications requires only a straightforward specialization of the proof of Proposition 2, we state them as a sequence of Corollaries to Proposition 2.

*Corollary 1:* If the cost functional  $F$  in (6) has the form

$$F(x_1, \dots, x_N, u_0, \dots, u_{N-1}) = f(x_N) + \sum_{i=0}^{N-1} g_i(u_i), \quad (21)$$

then the function

$$S_k(\zeta_k) = S_k(x_k, w_k, w_{k+1}, \dots, w_{N-1}, v_{k+1}, v_{k+2}, \dots, v_{N-1} | \zeta_k) \quad (22)$$

is sufficiently informative.

Thus when  $F$  has the form (21) the past states  $x_1, \dots, x_{k-1}$  and controls  $u_0, \dots, u_{k-1}$  no longer appear in the sufficiently informative function. The dependence of the sufficient information on the future uncertainties can be removed if the constraint set  $Q$  for the uncertain quantities has a property implying that the set of values that any particular uncertain quantity can take is independent of the values of the other uncertain quantities.

*Corollary 2:* If the set  $Q$  has the form

$$Q = \{x_0, w_0, w_1, \dots, w_{N-1}, v_1, \dots, v_{N-1} | x_0 \in X_0, \\ w_i \in W_i, \quad i = 0, 1, \dots, N-1, \\ v_k \in V_k, \quad k = 1, 2, \dots, N-1\} \quad (23)$$

where  $X_0, W_i, V_k$  are given subsets of the corresponding Euclidean spaces, then the function  $S_k$  given for all  $k$  by

$$S_k(\zeta_k) = [S_k(x_1, x_2, \dots, x_k | \zeta_k), u_0, u_1, \dots, u_{N-1}]$$

is sufficiently informative.

The case where the constraint  $Q$  is of the form (23) should be considered analogous to the case of uncorrelated white input and measurement noises in the corresponding stochastic problem.

The natural combination of Corollaries 1 and 2 yields the following.

*Corollary 3:* If  $F$  has the form (21) and  $Q$  has the form (23), then the function

$$S_k(\zeta_k) = S_k(x_k | \zeta_k)$$

is sufficiently informative.

In the stochastic problem analogous to Problem 1, important simplifications in the sufficient statistic result when the function  $F$  in (6) is additively separable [12], i.e., when

$$F(x_1, x_2, \dots, x_N, u_0, u_1, \dots, u_{N-1}) = \sum_{i=0}^{N-1} g_i(x_i, u_i) \quad (24)$$

where  $g_i(\cdot, \cdot)$  are given real valued functions. In the minimax framework of interest here, simplification of the sufficiently informative function can be achieved only at the expense of adjoining to the system equation (1) an additional state  $x_k^{(n+1)}$  defined by

$$x_{k+1}^{(n+1)} = x_k^{(n+1)} + g_k(x_k, u_k), \quad x_0^{(n+1)} = 0$$

so that the function  $F$  given by (24) becomes simply  $x_N^{(n+1)}$  and the simplification of the sufficient information afforded by Corollary 1 is applicable to the augmented system. The sufficiently informative function is reduced to

$$S_k(\zeta_k) = S_k(\tilde{x}_k, w_k, \dots, w_{N-1}, v_{k+1}, \dots, v_{N-1} | \zeta_k)$$

where  $\tilde{x}_k \triangleq (x_k, x_k^{(n+1)})$  is the augmented state. If, in addition,  $Q$  has the form of (23), we have

$$S_k(\zeta_k) = S_k(\tilde{x}_k | \zeta_k).$$

The difference between the simplifications available for additively separable cost functionals in the stochastic setting and those available under the minimax formulation may be attributed to the fact that, whereas the expectation operation is linear and distributes over addition, the maximization operation is not.

Equation (20) demonstrates the structure of the optimal control law, provides an alternative conceptual framework for considering Problem 1, and can give insight concerning the complexity of the optimal control law. Furthermore, it can form the basis for the development and the analysis of suboptimal control schemes [15]. However, it appears that only for a limited class of problems is it profitable to implement the optimal control law in the form given by (20) since the estimator  $S_k$  is infinite dimensional in most cases. Some typical examples of this limited class of problems are discussed in the next section.

## V. SOME SPECIAL CASES

It is advantageous to use a specific sufficiently informative function  $S_k$  only if the sufficient information  $S(\zeta_k)$  has smaller dimension than  $\zeta_k$ , i.e.,  $S_k$  maps into a finite-dimensional space with dimension less than  $k(s+m)$ . Under these circumstances by using  $S_k$  the solution of the problem by dynamic programming will be implemented over a space of smaller dimension. Furthermore, the computation of the optimal control law  $\mu^*[S_k(\zeta_k)]$ ,  $k = 0, 1, \dots, N-1$ , must be at least as easy as the computation of the optimal control law  $\mu(\zeta_k)$ . Such situations occur in problems involving finite-state systems, one-dimensional problems with terminal cost where the set of possible states is an interval, and, of course, the case of perfect state information where system state is measured exactly and constitutes a sufficient information. Another situation in which the sufficient information has smaller dimension than  $\zeta_k$  and leads to simplified computations is the case of a linear system

with linear measurements and an energy constraint on the uncertain quantities. This case bears great similarity to the linear quadratic Gaussian case of stochastic control. We have

$$x_{k+1} = A_k x_k + B_k u_k + G_k w_k, \quad k = 0, 1, \dots, N-1 \quad (25)$$

$$z_k = C_k x_k + v_k, \quad k = 1, 2, \dots, N-1 \quad (26)$$

$$x_0' P^{-1} x_0 + \sum_{i=0}^{N-1} w_i' Q_i^{-1} w_i + \sum_{i=1}^{N-1} v_i' R_i^{-1} v_i \leq 1 \quad (27)$$

where  $P$ ,  $Q_i$ ,  $R_i$  are positive-definite symmetric matrices. It has been shown in [7] that under these circumstances the set  $S(x_k, w_k, \dots, w_{N-1}, v_{k+1}, \dots, v_{N-1}, \xi_k)$  is the ellipsoid

$$\left\{ x_k, w_k, \dots, w_{N-1}, v_{k+1}, \dots, v_{N-1} \mid (x - \hat{x}_k)' \Sigma_{k|k}^{-1} (x - \hat{x}_k) + \sum_{i=k}^{N-1} w_i' Q_i^{-1} w_i + \sum_{i=k+1}^{N-1} v_i' R_i^{-1} v_i \leq 1 - \delta_k^2 \right\} \quad (28)$$

where the  $n$ -vector  $\hat{x}_k$ , the  $n \times n$  matrix  $\Sigma_{k|k}$ , and the real number  $\delta_k^2$  are generated recursively by

$$\hat{x}_{i+1} = A_i \hat{x}_i + B_i u_i + \Sigma_{i+1|i} C_{i+1}' R_{i+1}^{-1} \cdot (z_{i+1} - C_{i+1} A_i \hat{x}_i - C_{i+1} B_i u_i) \quad (29)$$

$$\Sigma_{i+1|i} = [\Sigma_{i|i}^{-1} + C_i' R_i^{-1} C_i]^{-1} \quad (30)$$

$$\Sigma_{i+1|i} = A_i \Sigma_{i|i} A_i' + G_i Q_i G_i' \quad (31)$$

$$\delta_{i+1}^2 = \delta_i^2 + (z_{i+1} - C_{i+1} A_i \hat{x}_i - C_{i+1} B_i u_i)' (C_{i+1} \Sigma_{i+1|i} \cdot C_{i+1}' + R_{i+1})^{-1} (z_{i+1} - C_{i+1} A_i \hat{x}_i - C_{i+1} B_i u_i) \quad (32)$$

with initial conditions  $\hat{x}_0 = 0$ ,  $\Sigma_{0|0} = P$ ,  $\delta_0^2 = 0$ . Since the matrix  $\Sigma_{k|k}$  is independent of the output  $z$  and precomputable from the problem data, the ellipsoid (28) is completely specified by the  $n$ -vector  $\hat{x}_k$  and the scalar  $\delta_k^2$ . If the function  $F$  has the form (21), combination of this result with Corollary 1 immediately yields the following result.

**Corollary 4:** For the system (25) and (26), the constraint set  $Q$  specified by (27), and the function  $F$  given by (21), the function  $S_k: R^{k(s+m)} \rightarrow R^n \times [0, 1]$  defined by

$$S_k(\xi) = \{\hat{x}_k, \delta_k^2\}$$

is sufficiently informative, where  $\hat{x}_k$  and  $\delta_k^2$  are given above.

Thus, for the problem involving a linear system, linear measurements, an energy constraint on the uncertain quantities, and a cost functional involving a function  $F$  of the form (21), the estimator part of the optimal controller can be completely and efficiently characterized. Furthermore, the computational requirements of the dynamic programming algorithm leading to the calculation of the optimal controller are greatly reduced. This is due to the fact that the algorithm can be redefined over the space of the sufficient information as follows:

$$\bar{H}_{N-1}(\hat{x}_{N-1}, \delta_{N-1}^2) = \inf_{u_{N-1}} \sup_{x \in X_{N|N-1}} \{f(x + B_{N-1} u_{N-1}) + g_{N-1}(u_{N-1})\} \quad (33)$$

where  $X_{N|N-1}$  is the ellipsoid

$$X_{N|N-1} = \left\{ x \mid (x - A_{N-1} \hat{x}_{N-1})' \cdot \Sigma_{N|N-1}^{-1} (x - A_{N-1} \hat{x}_{N-1}) \leq 1 - \delta_{N-1}^2 \right\} \quad (34)$$

$$\bar{H}_k(\hat{x}_k, \delta_k^2) = \inf_{u_k} \sup_{z_{k-1} \in \hat{Z}_{k-1}} \{ \bar{H}_{k+1}(\hat{x}_{k+1}, \delta_{k+1}^2) + g_k(u_k) \}, \quad k = 0, 1, \dots, N-2 \quad (35)$$

where  $\hat{x}_{k+1}$ ,  $\delta_{k+1}^2$  are given in terms of  $\hat{x}_k$ ,  $\delta_k^2$ ,  $u_k$ , and  $z_{k+1}$  by the estimator equations (29)–(32) and  $\hat{Z}_{k+1}$  is the ellipsoid

$$\hat{Z}_{k+1} = \{ z_{k+1} \mid (z_{k+1} - C_{k+1} A_k \hat{x}_k - C_{k+1} B_k u_k)' \cdot (C_{k+1} \Sigma_{k+1|k} C_{k+1}' + R_{k+1})^{-1} \cdot (z_{k+1} - C_{k+1} A_k \hat{x}_k - C_{k+1} B_k u_k) \leq 1 - \delta_k^2 \}. \quad (36)$$

The optimal controller is of the form

$$u_k = \bar{\mu}_k^*(\hat{x}_k, \delta_k^2), \quad k = 0, 1, \dots, N-1$$

where  $\hat{x}_k, \delta_k^2$  are generated recursively by the estimator of (29)–(32). The function  $\bar{\mu}_k^*$  is computed from the dynamic programming algorithm of (33)–(36). This algorithm is carried over a space of dimension  $(n+1)$ —a substantial improvement over the case where the optimal control law is calculated as a function of  $\xi_k$  by means of the algorithm of Proposition 1.

A special case of the function (21) occurs in the target set reachability problem in which

$$F(x_1, \dots, x_N, u_0, \dots, u_{N-1}) = \delta(x_N | X_N) + \sum_{i=0}^{N-1} \delta(u_i | U_i)$$

where  $\delta(y | Y)$  denotes the indicator function of a set  $Y$  given by (7). This problem has been examined in detail in [4].

As another representative example of the limited class of problems where the use of a suitable sufficiently informative function results in substantial reduction of computational and storage requirements, consider the following scalar system:

$$x_{k+1} = ax_k + u_k + w_k, \quad k = 0, 1, \dots, N-1$$

where  $a \geq 0$ , with measurement equation

$$z_k = x_k + v_k,$$

and where the initial state and the input and measurement noises are known to belong to the following intervals:

$$x_0 \in [s_0^1, s_0^2], \quad u_k \in [p^1, p^2], \quad v_k \in [r^1, r^2].$$

It is required to find a control law  $\mu_k$ ,  $k = 0, 1, \dots, N-1$  with  $\mu_k(\xi_k) \in U \subset R$ , that minimizes a cost functional involving a function  $F$  of the form

$$F(x_1, \dots, x_N, u_0, \dots, u_{N-1}) = f(x_N) + \sum_{i=0}^{N-1} g(u_i)$$

where  $f, g$  are given real valued functions.

From Corollary 3 we have that the function

$$S_k(\xi_k) = S(x_k | \xi_k), \quad k = 0, 1, \dots, N-1$$

is sufficiently informative where  $S(x_k | \xi_k)$  is the set of all states  $x_k$  consistent with the information vector  $\xi_k = (z_1, z_2, \dots, z_k, u_0, u_1, \dots, u_{k-1})$ . The set  $S(x_k | \xi_k)$  is a closed interval for this problem and can be computed by using the recursive set algorithm given in [5]. We have

$$S(x_k | \xi_k) = [s_k^1, s_k^2]$$

where  $s_k^1, s_k^2$  are generated by the following estimator equations:

$$s_{k+1}^1 = \max \{as_k^1 + u_k + p^1, z_{k+1} - d^2\} \quad (37)$$

$$s_{k+1}^2 = \min \{as_k^2 + u_k + p^2, z_{k+1} - d^1\} \quad (38)$$

with initial conditions the endpoints  $s_0^1, s_0^2$  of the interval of uncertainty for the initial state.

The dynamic programming algorithm in terms of the sufficiently informative function takes the form of

$$\begin{aligned} \bar{H}_{N-1}(s_{N-1}^1, s_{N-1}^2) = \inf_{u_{N-1} \in U} \sup_{\substack{w_{N-1} \in [p^1, p^2] \\ x_{N-1} \in [s_{N-1}^1, s_{N-1}^2]}} \{f(ax_{N-1} + u_{N-1} \\ + w_{N-1}) + g(u_{N-1})\} \quad (39) \end{aligned}$$

$$\begin{aligned} \bar{H}_{k-1}(s_{k-1}^1, s_{k-1}^2) = \inf_{u_{k-1} \in U} \sup_{z_k \in \hat{Z}_k(s_{k-1}^1, s_{k-1}^2, u_{k-1})} \{\bar{H}_k[\max \{as_{k-1}^1 \\ + u_{k-1} + p^1, z_k - d^2\}, \min \{as_{k-1}^2 + u_{k-1} \\ + p^2, z_k - d^1\}] + g(u_{k-1})\}, \quad k = 1, 2, \dots, N \quad (40) \end{aligned}$$

where the interval  $\hat{Z}_k(s_{k-1}^1, s_{k-1}^2, u_{k-1})$  is given by

$$\begin{aligned} \hat{Z}_k(s_{k-1}^1, s_{k-1}^2, u_{k-1}) = a[s_{k-1}^1, s_{k-1}^2] + u_{k-1} \\ + [p^1, p^2] + [r^1, r^2]. \end{aligned}$$

Thus for the above problem the optimal controller is of the form

$$u_k = \bar{\mu}_k^*(s_k^1, s_k^2), \quad k = 0, 1, \dots, N-1$$

where  $s_k^1, s_k^2$  are generated by the estimator of (37) and (38). The function  $\bar{\mu}_k^*$ ,  $k = 0, 1, \dots, N-1$ , is computed from the dynamic programming algorithm (39), (40). This algorithm is carried over a two-dimensional half-space (recall that  $s_k^1 \leq s_k^2$ ). It can be seen that the reduction in computational and storage requirements is substantial over the case where the optimal controller would be computed in the form  $\mu_k(\xi_k)$  by means of the algorithm of Proposition 1, since the dimension of  $\xi_k$  is  $2k$  for this problem.

## VI. CONCLUSIONS

In this paper we considered some general aspects of a minimax feedback control problem with imperfect state

information. A dynamic programming algorithm was given for the solution of this problem, which in general must be implemented over the space of the information available to the controller. The notion of a sufficiently informative function, which parallels the notion of a sufficient statistic of stochastic control, was introduced with a twofold purpose. First, to provide an alternative conceptual framework for viewing the problem and to demonstrate the separation of the optimal controller into an estimator and an actuator. Second, to demonstrate the possibility of redefining and implementing the dynamic programming algorithm so that it is carried over the space of the sufficient information. For a limited class of problems it was shown that this alternate implementation is profitable. The results in this paper should not come as a surprise to anyone familiar with dynamic programming, sequential games, and stochastic control since they represent a formalization and extension of well-known concepts within the framework of the minimax problem. It is to be noted that, similar to the stochastic control case, the notion of a sufficiently informative function is useful for only a limited class of problems; however, this class does not include any special case with a solution as elegant as the case of a linear system with a quadratic cost in stochastic control.

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# Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relation to Deterministic Differential Games

DAVID H. JACOBSON

**Abstract**—Two stochastic optimal control problems are solved whose performance criteria are the expected values of exponential functions of quadratic forms. The optimal controller is linear in both cases but depends upon the covariance matrix of the additive process noise so that the certainty equivalence principle does not hold. The controllers are shown to be equivalent to those obtained by solving a cooperative and a noncooperative quadratic (differential) game, and this leads to some interesting interpretations and observations.

Finally, some stability properties of the asymptotic controllers are discussed.

## I. INTRODUCTION

THE SO-CALLED linear-quadratic-Gaussian (LQG) problem<sup>1</sup> of optimal stochastic control [1] possesses a number of interesting features. First, the optimal feedback controller is a linear (time-varying) function of the state

variables. Second, this linear controller is identical to that which is obtained by neglecting the additive Gaussian noise and solving the resultant deterministic linear-quadratic problem (LQP)<sup>2</sup> (certainty equivalence principle). Thus the controller for the stochastic system is independent of the statistics of the additive noise. This is appealing for small noise intensity, but for large noise (large covariance) one has the intuitive feeling that perhaps a different controller would be more appropriate.

In this paper we consider optimal control of linear systems disturbed by additive Gaussian noise, whose associated performance criteria are the expected values of exponential functions of negative-semidefinite and positive-semidefinite quadratic forms. We shall refer to the former case as the LE-G problem, and the latter as the LE+G problem, and to their deterministic counterparts as LE-P and LE+P, respectively. In the deterministic cases LE+P, the solutions are identical to that for the LQP (the natural logarithm of the exponential performance criteria yields quadratic forms). However, when noise is present, LE±G problems, the optimal controllers are different from that of the LQG problem. In particular, though as in the case of the LQG problem, these are *linear functions* of the state

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<sup>1</sup> This is a problem with linear dynamics disturbed by additive Gaussian noise, together with a performance criterion which is the expected value of a positive-semidefinite quadratic form.

<sup>2</sup> This is the same as the LQG problem, but with noise set to zero.