Invent math (2013) 193:149–158
DOI 10.1007/s00222-012-0423-2

Suita conjecture and the Ohsawa-Takegoshi extension theorem

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Received: 3 April 2012 / Accepted: 26 August 2012 / Published online: 8 September 2012

Abstract We prove a conjecture of N. Suita which says that for any bounded domain D in $\mathbb{C}$ one has $c_D^2 \le \pi K_D$, where $c_D(z)$ is the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to $z \in D$ and K_D the Bergman kernel on the diagonal. We also obtain optimal constant in the Ohsawa-Takegoshi extension theorem.

1 Introduction

Suita [16] (see also [15, p. 179]) conjectured that for a bounded domain D in $\mathbb{C}$ one has

$$c_D^2 \le \pi K_D. \tag{1}$$

Here

$$c_D(z) = \exp\Big(\lim_{\zeta \to z}\big(G_D(\zeta, z) - \log|\zeta - z|\big)\Big)$$

is the logarithmic capacity of the complement of D with respect to $z \in D$, where $G_D(\cdot, z)$ the (negative) Green function for D with pole at z, and

$$K_D(z) = \sup\left\{\big|f(z)\big|^2 : f \text{ holomorphic in } D,\ \int_D |f|^2 d\lambda \le 1\right\}$$

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denotes the Bergman kernel on the diagonal. One can easily show that we have equality in (1) if D is simply connected. Suita in [16], using the theory of elliptic functions, proved strict inequality in (1) when D is an annulus, and hence also any regular doubly connected domain.

This conjecture has geometric interpretation and consequences. Suita also observed that

$$K_D = \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D)$$

and thus that (1) is equivalent to the fact that the curvature of the invariant metric $c_D|dz|$ is bounded above by -4. Therefore (1) means precisely that maximal curvature is attained on the boundary (if D is smooth then one can show that this curvature is equal to -4 on ∂D, or equivalently that the ratio $c_D^2/\pi K_D$ is equal to 1 there). This is not the case for other invariant metrics related to the Bergman kernel, e.g. $K_D|dz|^2$ and the Bergman metric $(\log K_D)_{z\bar{z}}|dz|^2$ (see [9] and [19]). On the other hand, this property is satisfied for the Carathéodory metric (see [17] and [18]) but this is much simpler to prove.

The relation of the Suita conjecture to the extension theorem was first observed and explored by Ohsawa [12]: to prove (1) for a given $z \in D$ we have to find a holomorphic f in D with $f(z) = 1$ and

$$\int_D |f|^2 d\lambda \le \frac{\pi}{(c_D(z))^2}.$$

Using methods developed in [14] he proved the estimate

$$c_D^2 \le C\pi K_D$$

with $C = 750$. This was later improved to $C = 2$ in [5], where the main tool was an estimate from [1]. A slightly better constant $C = 1.954$ was recently obtained in [10], see also [6] for a much simpler proof.

Our goal is to show the following version of the Ohsawa-Takegoshi theorem with optimal constant which in particular implies the Suita conjecture:

Theorem 1 *Assume that $\Omega \subset \mathbb{C}^{n-1} \times D$ is pseudoconvex, where D is a bounded domain in $\mathbb{C}$ containing the origin. Then for any holomorphic f in $\Omega' := \Omega \cap \{z_n = 0\}$ and φ plurisubharmonic in Ω one can find a holomorphic extension F of f to Ω with*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \le \frac{\pi}{(c_D(0))^2}\int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

This kind of result, with a worse constant though, was first formulated in [8] (it used the proof of the Ohsawa-Takegoshi theorem from [2]), see also [13].

Our proof is in big part motivated by the recent work of Chen [7] who showed that the Ohsawa-Takegoshi theorem can deduced directly from the classical Hörmander estimate for the $\bar\partial$-equation [11]. We use Chen's idea in the proof of Theorem 2 below. It improves a related $\bar\partial$-estimate from [6] which gave Theorem 1 but only with the same constant as in [10]. This $\bar\partial$-estimate was one of the crucial improvements compared to [6]. Another one was the solution of the following ODE problem: find $g \in C^{0,1}(\mathbb{R}_+)$, $h \in C^{1,1}(\mathbb{R}_+)$ such that h is convex and decreasing, $g + \log t$, $h + \log t$ vanish at ∞, and

$$\left(1 - \frac{(g')^2}{h''}\right)e^{2g-h+t} \geq 1.$$

The author would like to express his gratitude to Takeo Ohsawa for his constant interest and encouragement throughout the years and to Peter Pflug for essentially introducing him to the theory of Bergman kernel back in 1998.

2 The proof

Similarly as in [7] our main tool will be the Hörmander estimate: if Ω is pseudoconvex in $\mathbb{C}^n$ and

$$\alpha = \sum_j \alpha_j d\bar z_j \in L^2_{loc,(0,1)}(\Omega)$$

is such that $\bar\partial\alpha = 0$ then for any smooth, strongly plurisubharmonic φ in Ω one can find $u \in L^2_{loc}(\Omega)$ such that

$$\bar\partial u = \alpha$$

and

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \leq \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda. \tag{2}$$

Here

$$|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{j\bar k} \bar\alpha_j \alpha_k$$

(where $(\varphi^{j\bar k})$ is the inverse transposed of the complex Hessian $(\partial^2\varphi/\partial z_j \partial\bar z_k)$) denotes the length of α with respect to the Kähler metric $i\partial\bar\partial\varphi$.

It was observed in [4] that (2) makes also sense (and indeed remains true) for arbitrary plurisubharmonic φ: instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ one should take any $H \in L^\infty_{loc}(\Omega)$ with

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi.$$

In this convention we formulate our new estimate for the $\bar\partial$-equation:

Theorem 2 *Let $\alpha \in L^2_{loc,(0,1)}(\Omega)$ be a $\bar\partial$-closed form in a pseudoconvex domain Ω in $\mathbb{C}^n$. Assume that φ is plurisubharmonic in Ω, $\psi \in W^{1,2}_{loc}(\Omega)$ is locally bounded from above and they satisfy $|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \le H < 1$ in Ω for some $H \in L^\infty_{loc}(\Omega)$. Then there exists $u \in L^2_{loc}(\Omega)$ solving $\bar\partial u = \alpha$ and such that for every $b > 0$*

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi}d\lambda \le \left(1+\frac{1}{b}\right)\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}\frac{1+bH}{1-H}e^{2\psi-\varphi}d\lambda.$$

If in addition $H \le \delta < 1$ on $\operatorname{supp}\alpha$ *then*

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi}d\lambda \le \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda. \tag{3}$$

Proof By standard approximation we may assume that φ is smooth up to the boundary, strongly plurisubharmonic, and that ψ is bounded in Ω. Note that then $L^2(\Omega, e^{\psi-\varphi})$, $L^2(\Omega, e^{-\varphi})$ and $L^2(\Omega)$ are the same sets. We use a trick from [3]: define u to be the minimal solution to $\bar\partial u = \alpha$ in $L^2(\Omega, e^{\psi-\varphi})$ which means that u is perpendicular to $\ker\bar\partial$ there. Then $v := ue^\psi$ is perpendicular to $\ker\bar\partial$ in $L^2(\Omega, e^{-\varphi})$ and thus it is the minimal solution to $\bar\partial v = \beta$, where

$$\beta := e^\psi(\alpha + u\,\bar\partial\psi) \in L^2_{loc,(0,1)}(\Omega).$$

By Hörmander's estimate (2)

$$\int_\Omega |v|^2 e^{-\varphi}d\lambda \le \int_\Omega |\beta|^2_{i\partial\bar\partial\varphi}e^{-\varphi}d\lambda$$

which means that

$$\begin{aligned}\int_\Omega |u|^2 e^{2\psi-\varphi}d\lambda &\le \int_\Omega |\alpha + u\,\bar\partial\psi|^2_{i\partial\bar\partial\varphi}e^{2\psi-\varphi}d\lambda\\ &\le \int_\Omega \big(|\alpha|^2_{i\partial\bar\partial\varphi} + 2|u|\sqrt{H}|\alpha|_{i\partial\bar\partial\varphi} + |u|^2H\big)e^{2\psi-\varphi}d\lambda.\end{aligned}$$

For $t > 0$ we will get

$$\int_\Omega |u|^2(1-H)e^{2\psi-\varphi}d\lambda$$
$$\leq \int_\Omega \left[|\alpha|^2_{i\partial\bar\partial\varphi}\left(1+t^{-1}\frac{H}{1-H}\right)+t|u|^2(1-H)\right]e^{2\psi-\varphi}d\lambda.$$

We obtain the first estimate if we take $t := 1/(b+1)$, whereas the second one follows if we let $b := \delta^{-1/2}$. □

Proof of Theorem 1 Using appropriate approximation we may assume that Ω is bounded, smooth and strongly pseudoconvex, φ is smooth up to the boundary and that f is holomorphic in a neighborhood of $\overline{\Omega'}$. Fix $\varepsilon > 0$ and define

$$\alpha := \bar\partial\big(f(z')\chi(-2\log|z_n|)\big) = -\frac{f(z')\chi'(-2\log|z_n|)}{\bar z_n}d\bar z_n,$$

where we write $z = (z', z_n)$ and $\chi \in C^{0,1}(\mathbb{R})$ is non-decreasing and such that $\chi = 0$ on $\{t \leq -2\log\varepsilon\}$, $\chi(\infty) = 1$ (it will be precisely determined later). Note that α is defined in Ω for sufficiently small ε, $\operatorname{supp}\alpha \subset \{|z_n| \leq \varepsilon\}$, and for any solution of $\bar\partial u = \alpha$ the function

$$F := f(z')\chi(-2\log|z_n|) - u \tag{4}$$

is a holomorphic extension of f provided that $u = 0$ on Ω'.

In order to use Theorem 2 and define appropriate weights $\widetilde\varphi$, ψ the choice of the following two functions on $\mathbb{R}_+$ is absolutely crucial:

$$h(t) := -\log\big(t + e^{-t} - 1\big),$$
$$g(t) := -\log\big(t + e^{-t} - 1\big) + \log\big(1 - e^{-t}\big).$$

They solve the ODE problem formulated in the introduction: one can check that they are decreasing, convex,

$$\left(1 - \frac{(g')^2}{h''}\right)e^{2g-h+t} = 1 \tag{5}$$

and

$$\lim_{t\to\infty}\big(h(t) - 2g(t) + \log(-h'(t))\big) = 0. \tag{6}$$

(In fact, as noticed by the referee, we have $g = \log(-h')$. This substitution reduces (5) to

$$\frac{d^2}{dt^2}\big(e^{-h}\big) = e^{-t}$$

and enables to solve the desired ODE problem immediately.)

For $G := G_D(\cdot, 0)$ we can write $G = \log|\zeta| + v$, where v is a bounded harmonic function in D. There are positive constants C_0, C_1 such that

$$\big|2G - 2\log|\zeta|\big| \le C_0 \quad \text{in } D$$

and

$$\left|2G_\zeta - \frac{1}{\zeta}\right| \le C_1 \text{ near the origin.}$$

For $M := -2\log\varepsilon - C_0$ we define

$$\eta(t) := \begin{cases} h(t), & t < M, \\ -\delta\log(t - M + a) + b, & t \ge M \end{cases}$$

and

$$\gamma(t) := \begin{cases} g(t), & t < M, \\ -\delta\log(t - M + a) + \widetilde{b}, & t \ge M, \end{cases}$$

where $\delta = M^{-1/2}$ and $a, b, \widetilde{b}$ are chosen in such a way that $\eta \in C^{1,1}(\mathbb{R}_+)$ and $\gamma \in C^{0,1}(\mathbb{R}_+)$. Then

$$\begin{aligned} a &= a(\varepsilon) = -\frac{\delta}{h'(M)}, \\ b &= b(\varepsilon) = h(M) + \delta\log a, \\ \widetilde{b} &= \widetilde{b}(\varepsilon) = g(M) + \delta\log a. \end{aligned}$$

Note that δ is selected so that

$$\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0, \qquad \lim_{\varepsilon\to 0} a(\varepsilon) = \infty. \tag{7}$$

Now we are ready to define

$$\widetilde{\varphi} := \varphi + 2G + \eta(-2G), \qquad \psi := \gamma(-2G).$$

Then $\widetilde{\varphi}$ is plurisubharmonic in Ω and $\psi \in W^{1,2}_{loc}(D)$ is locally bounded from above. We have

$$|\bar\partial\psi|^2_{i\partial\bar\partial\widetilde{\varphi}} \le |\bar\partial\psi|^2_{i\partial\bar\partial(\eta(-2G))} = \frac{(\gamma'(-2G))^2}{\eta''(-2G)} \begin{cases} < 1 & \text{in } \Omega, \\ = \delta & \text{on } \operatorname{supp}\alpha \end{cases}$$

(note that $\operatorname{supp}\alpha \subset \{-2G \geq M\}$) and

$$|\alpha|^2_{i\partial\bar\partial\widetilde{\varphi}} \leq |\alpha|^2_{i\partial\bar\partial(\eta(-2G))} = \frac{|f(z')|^2(\chi'(-2\log|z_n|))^2}{4|G_{z_n}|^2|z_n|^2\eta''(-2G)},$$

where we follow the convention discussed before Theorem 2. Since the function $-\delta\log(t-M+a)+t$ is increasing in t by (5) we have

$$\left(1-\frac{(\gamma')^2}{\eta''}\right)e^{2\gamma-\eta+t}\begin{cases}=1 & \text{on } \{t<M\},\\ \geq (1-\delta)e^{2g(M)-h(M)+M} & \text{on } \{t\geq M\}\end{cases}$$

and thus for sufficiently small ε

$$\left(1-\frac{(\gamma')^2}{\eta''}\right)e^{2\gamma-\eta+t}\geq 1$$

on $\mathbb{R}_+$. Theorem 2 now gives a solution $u=u_\varepsilon$ of $\bar\partial u=\alpha$ such that

$$\int_\Omega |u|^2e^{-\varphi}d\lambda \leq \int_\Omega |u|^2\big(1-|\bar\partial\psi|^2_{i\partial\bar\partial\widetilde{\varphi}}\big)e^{2\psi-\widetilde{\varphi}}d\lambda \leq \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\mathcal{A}(\varepsilon), \tag{8}$$

where

$$\mathcal{A}(\varepsilon):=\int_\Omega \frac{|f(z')|^2(\chi'(-2\log|z_n|))^2}{4|G_{z_n}|^2|z_n|^2\eta''(-2G)}e^{2\psi-\widetilde{\varphi}}d\lambda.$$

We have

$$\varlimsup_{\varepsilon\to 0}\mathcal{A}(\varepsilon)\leq \varlimsup_{\varepsilon\to 0}\mathcal{B}(\varepsilon)\int_{\Omega'}|f|^2e^{-\varphi}d\lambda'$$

where

$$\mathcal{B}(\varepsilon):=\int_{\{|\zeta|\leq\varepsilon\}}\frac{(\chi'(-2\log|\zeta|))^2}{4|G_\zeta|^2|\zeta|^2\eta''(-2G)}e^{2\psi-\eta(-2G)-2G}d\lambda(\zeta).$$

We now want to replace G by $\log|\zeta|$ in the above integral. Note that

$$\lim_{\varepsilon\to 0}\sup_{\{|\zeta|\leq\varepsilon\}}|\zeta|^2e^{-2G}=\frac{1}{(c_D(0))^2}$$

and near the origin

$$4|G_\zeta|^2|\zeta|^2\geq\big(1-C_1|\zeta|\big)^2.$$

Moreover on $\{|\zeta|\leq\varepsilon\}$ we have

$$2\psi-\eta(-2G)\leq 2\gamma\big(-2\log|\zeta|-C_0\big)-\eta\big(-2\log|\zeta|-C_0\big)$$

$$\le 2\gamma\big(-2\log|\zeta|\big)-\eta\big(-2\log|\zeta|\big)+\delta\log\left(1+\frac{C_0}{a}\right)$$

and

$$\eta''(-2G)\ge\eta''\big(-2\log|\zeta|+C_0\big)\ge\left(\frac{a+C_0}{a+2C_0}\right)^2\eta''\big(-2\log|\zeta|\big).$$

Hence, from (7) it follows that

$$\overline{\lim_{\varepsilon\to 0}}\,\mathcal{B}(\varepsilon)\le\frac{1}{(c_D(0))^2}\,\overline{\lim_{\varepsilon\to 0}}\,\widetilde{\mathcal{B}}(\varepsilon),$$

where

$$\begin{aligned}\widetilde{\mathcal{B}}(\varepsilon)&=\int_{\{|\zeta|\le\varepsilon\}}\frac{(\chi'(-2\log|\zeta|))^2}{|\zeta|^2\eta''(-2\log|\zeta|)}e^{2\gamma(-2\log|\zeta|)-\eta(-2\log|\zeta|)}d\lambda(\zeta)\\&=\pi\int_{M+C_0}^{\infty}\frac{(\chi')^2e^{2\gamma-\eta}}{\eta''}\,dt.\end{aligned}$$

By the Schwarz inequality the optimal choice of increasing χ with

$$\chi(\infty)=\int_{M+C_0}^{\infty}\chi'\,dt=1$$

is

$$\chi(t):=\begin{cases}0, & t<M+C_0,\\ \frac{1}{c}\int_{M+C_0}^{t}w\,ds, & t\ge M+C_0,\end{cases}$$

where $w=\eta''e^{\eta-2\gamma}$ and $c=\int_{M+C_0}^{\infty}w\,dt$. Then

$$\widetilde{\mathcal{B}}(\varepsilon)=\frac{\pi}{\int_{M+C_0}^{\infty}\eta''e^{\eta-2\gamma}\,dt}$$

and

$$\begin{aligned}\int_{M+C_0}^{\infty}\eta''e^{\eta-2\gamma}\,dt&=\delta e^{b-2\widetilde{b}}\int_{C_0}^{\infty}(t+a)^{\delta-2}dt\\&=\frac{1}{1-\delta}\left(1+\frac{C_0}{a}\right)^{\delta-1}e^{h(M)-2g(M)+\log(-h'(M))}.\end{aligned}$$

By (6) and (7)

$$\lim_{\varepsilon\to 0}\widetilde{\mathcal{B}}(\varepsilon)=\pi$$

and we have thus obtained that

$$\overline{\lim_{\varepsilon \to 0}} \int_{\Omega} |u_\varepsilon|^2 e^{-\varphi} d\lambda \le \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

Finally, we note that if $0 < \widetilde{\varepsilon} \le \varepsilon$ then

$$\begin{aligned}
&\int_{\{|\zeta| \le \widetilde{\varepsilon}\}} \left(1 - \frac{(\gamma'(-2G))^2}{\eta''(-2G)}\right) e^{2\psi - \eta(-2G) - 2G} d\lambda(\zeta) \\
&\quad \ge \frac{1}{C(\varepsilon)} \int_{\{|\zeta| \le \widetilde{\varepsilon}\}} \frac{e^{2\gamma(-2\log|\zeta|) - \eta(-2\log|\zeta|)}}{|\zeta|^2} d\lambda(\zeta) \\
&\quad = \frac{\pi e^{2\widetilde{b} - b}}{C(\varepsilon)} \int_{-2\log\widetilde{\varepsilon}}^{\infty} (t - M + a)^{-\delta} dt \\
&\quad = \infty
\end{aligned}$$

and (8) implies that $u = 0$ almost everywhere on Ω'. The weak limit of $F = F_\varepsilon$ given by (4) is the required extension. □

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