Nečas Center for Mathematical Modeling

Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids

Eduard Feireisl, Antonín Novotný and Yongzhong Sun

Preprint no. 2010-019

Research Team 2
Mathematical Institute of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1
http://ncmm.karlin.mff.cuni.cz/

Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids

Eduard Feireisl* Antonín Novotný † Yongzhong Sun ‡

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

IMATH, Université du Sud Toulon-Var, BP 132, 839 57 La Garde, France

Department of Mathematics, Nanjing University
Nanjing, Jiangsu 210093, China

Abstract

We introduce a class of *suitable weak solutions* to the Navier-Stokes system of equations governing the motion of a compressible viscous fluid. These solutions satisfy the relative entropy inequality introduced by several authors, and, in particular, they enjoy the weak-strong uniqueness property.

1 Introduction

In a very interesting recent paper, Germain [12] obtained several results concerning a so-called weak-strong uniqueness property of solutions to the Navier-Stokes system of equations of a compressible fluid. Motivated by Dafermos [3], and several recent results by Berthelin and Vasseur [1], and Mellet and Vasseur [22], he considered a class of weak solutions satisfying the relative entropy inequality and shows that the latter may be

*The work of E.F. was supported by Grant 201/08/0315 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

†The work of A.N. was partially supported by the Nečas Center for Mathematical Modeling LC06052 financed by MŠMT

‡The work of Y.S. was supported by NSFC Grants 10771097, 10931007 and by the Institutional Research Plan AV0Z10190503 of the Academy of Sciences of the Czech Republic

used to deduce various weak-strong uniqueness results in this class of solutions. In the present paper, we introduce an intrinsic definition of *suitable weak solution* in the spirit of Germain, and, in particular, we establish global-in-time *existence* within the class of suitable weak solutions for any finite energy initial data.

Let $\Omega \subset R^3$ be a bounded regular domain, and let $Q_T \equiv (0,T) \times \Omega$. We consider an initial-boundary value problem for the Navier-Stokes system in the form:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \text{ in } Q_T, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \text{ in } Q_T, \tag{1.2}$$

with

$$\mathbb{S}(\nabla_x \mathbf{u}) \equiv \mu \left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.3}$$

supplemented with the standard no-slip boundary condition

$$\mathbf{u} = 0 \text{ on } (0,T) \times \partial\Omega, \tag{1.4}$$

together with the initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ (\varrho \mathbf{u})(0,\cdot) = \varrho_0 \mathbf{u}_0 \text{ in } \Omega. \tag{1.5}$$

The symbol $\varrho = \varrho(t,x)$ denotes the mass density and $\mathbf{u} = \mathbf{u}(t,x)$ the velocity of the fluid. Furthermore, $\mathbb{S}$ is the viscous stress, and $p = p(\varrho)$ is the barotropic pressure. The viscosity coefficients μ and η satisfy

$$\mu > 0, \ \eta \geq 0.$$

Existence of weak solutions in the spirit of Leray [20] was established in a seminal work by P.-L.Lions [21] on condition that $p(\varrho) \approx \varrho^\gamma$, with $\gamma \geq 9/5$. This result was later extended in [8] to a class of physically relevant adiabatic coefficients $\gamma > 3/2$, including, in particular, the isentropic pressure law of a monoatomic gas $p(\varrho) = a\varrho^{5/3}$. Note that existence of global-in-time smooth solutions for any choice of (large) initial data is an outstanding open problem, even in the two-dimensional case; see, however, Vaigant and Kazhikhov [29].

Similarly, the problem of *uniqueness* within the class of weak solutions seems extremely difficult, see Hoff [14], [15] for the case of small initial data. Even for the apparently "simpler" incompressible system uniqueness in the class of weak solutions is largely open, at least in the physically relevant three-dimensional setting, see Fefferman [6], Ladyzhenkaya [18]. On the other hand, there is a significant amount of work concerning conditional regularity and the property of weak-strong uniqueness in both incompressible and compressible case. Following the pioneering results by Prodi [26] and Serrin [27] many authors established regularity of the weak solutions to the incompressible Navier-Stokes system under various complementary assumptions. We refer to Galdi [10], Germain [12], Lemarié-Rieusset [19], and the references quoted therein.

The problem of *weak-strong uniqueness* for the compressible system (1.1 - 1.5) was first studied by Desjardins [4] in the class of solutions with bounded density. Recently, Germain [12] obtained several weak-strong uniqueness results for problem (1.1 - 1.5) under the extra hypothesis that $p(\varrho) = a\varrho^{\gamma}$, and

$$\nabla_x \varrho \in L^{2\gamma}(0,T; L^{\left(\frac{1}{2\gamma}+\frac{1}{3}\right)^{-1}}(\Omega)), \tag{1.6}$$

where Ω is a torus T^3 or the whole space R^3. More specifically, he shows that a weak solution satisfying (1.6) coincides with a hypothetical strong solution of the same problem as long as the latter exists. The main tool of Germain's approach is the so-called *relative entropy inequality* involving both the weak and the hypothetical strong solution. Germain also realized that the rather restrictive hypothesis (1.6) is needed only to justify the formal calculation leading to the relative entropy inequality. Accordingly, he suggests an alternative approach, incorporating the entropy inequality directly in the proper definition of weak solution.

The main goal of the present paper may be stated as follows:

- We introduce an *intrinsic definition* of suitable weak solutions to the Navier-Stokes system (1.1 - 1.5) satisfying, in particular, the relative entropy inequality with respect to any hypothetical strong solution to the problem.

- We propose an explicit approximation scheme and show *existence* of global-in-time suitable weak solutions for any finite energy initial data.

- We discuss basic properties of suitable weak solutions, in particular, the problem of weak-strong uniqueness and conditional regularity issues.

Thus, in contrast with Germain's result [12], we provide an *intrinsic* definition of suitable weak solution and show its *existence* for any finite energy initial data. Moreover, our definition is independent of existence of hypothetical smooth solutions and easily adaptable to various classes of domains and boundary conditions. Last but not least, we provide an explicit *construction* of suitable weak solutions by means of a family of approximate problems, therefore showing stability of the approximate solutions that may be relevant in numerical implementations.

The paper is organized as follows. In Section 2, we recall the well-known concept of renormalized weak solutions to problem (1.1 - 1.5) and introduce the suitable weak solutions. An approximation scheme to construct the suitable weak solutions is proposed in Section 3. Basic properties of the suitable weak solutions and several applications of the the existence theory are discussed in Section 4.

2 Weak and suitable weak solutions

We start with the nowadays standard definition of renormalized weak solution to the Navier-Stokes system, cf. P.-L.Lions [21].

Definition 2.1 *We say that ϱ, $\mathbf{u}$ represent* renormalized weak solution *to problem (1.1 - 1.5) if:*

-
$$\varrho \geq 0,\ \ \varrho \in L^\infty(0,T;L^\gamma(\Omega)) \text{ for a certain } \gamma > 3/2,\ \ \mathbf{u} \in L^2(0,T;W^{1,2}_0(\Omega;R^3)),$$
$$p(\varrho) \in L^1((0,T)\times\Omega);$$

- *equation of continuity (1.1) is satisfied in the sense of renormalized solutions (see DiPerna and P.-L.Lions [5]),*
$$\int_0^T \int_\Omega \Big((b(\varrho)+\varrho)\,\partial_t\varphi + (b(\varrho)+\varrho)\,\mathbf{u}\cdot\nabla_x\varphi + \Big(b(\varrho)-b'(\varrho)\varrho\Big)\mathrm{div}_x\mathbf{u}\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.1}$$
$$= -\int_\Omega (b(\varrho_0)+\varrho_0))\,\varphi(0,\cdot)\ \mathrm{d}x$$
for any test function $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$ and any $b \in C^\infty[0,\infty)$, $b' \in C^\infty_c[0,\infty)$;

- *momentum equation (1.2), together with the no-slip boundary condition (1.3), is satisfied in the sense of distributions,*
$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.2}$$
$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$
for any test function $\varphi \in C^\infty_c([0,T)\times\Omega;R^3)$.

As a matter of fact, the weak solutions constructed in [8], [21] also satisfy the *entropy (sometimes called also energy) inequality*

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \leq \tag{2.3}$$

$$\int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H(\varrho_0) \right) \ \mathrm{d}x \text{ for a.a. } \tau \in (0,T),$$

where

$$H(\varrho) \equiv \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z.$$

Solutions satisfying (2.3) may be constructed on *any* bounded domain in R^3, without any assumption concerning regularity of $\partial\Omega$, see [9].

Following Germain [12] we introduce a function

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r).$$

Let $r = r(t,x)$, $\mathbf{U} = \mathbf{U}(t,x)$ be smooth functions defined on $[0,T] \times \overline{\Omega}$,

$$r > 0 \text{ on } [0,T] \times \overline{\Omega}, \ \mathbf{U}|_{\partial\Omega} = 0. \tag{2.4}$$

Suppose that ϱ, $\mathbf{u}$ is a smooth solution of the Navier-Stokes system (1.1 - 1.5). A rather tedious but straightforward computation, specified in detail in Section 3, yields the following integral inequality

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho, r) \right) (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \tag{2.5}$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{U}(0,\cdot)|^2 + E(\varrho_0, r(0,\cdot)) \right) \ \mathrm{d}x + \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) \ \mathrm{d}t \text{ for a.a. } \tau \in (0,T),$$

where

$$\mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) = \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})(\mathbf{u} - \mathbf{U}) \Big) \ \mathrm{d}x \tag{2.6}$$

$$+ \int_\Omega \Big((r - \varrho) \partial_t P(r) + \nabla_x P(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) - \mathrm{div}_x \mathbf{U} \Big(\varrho \Big(P(\varrho) - P(r) \Big) - E(\varrho, r) \Big) \Big) \ \mathrm{d}x,$$

and

$$P = H'.$$

For

$$r = \frac{1}{|\Omega|} \int_\Omega \varrho_0 \ \mathrm{d}x, \ \mathbf{U} \equiv 0$$

relation (2.5) reduces to (2.3). This motivates the following definition:

Definition 2.2 *We shall say that ϱ,* $\mathbf{u}$ *represent a* suitable weak solution *to the Navier-Stokes system (1.1 - 1.5) if:*

- *the functions ϱ,* $\mathbf{u}$ *are renormalized solution to (1.1 - 1.5) in the sense of Definition 2.1;*
- *the integral inequality (2.5) holds for any smooth functions r,* $\mathbf{U}$ *satisfying (2.4).*

3 Global existence of suitable weak solutions

The main result of this paper reads as follows.

Theorem 3.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Suppose that the pressure p is continuously differentiable on $[0, \infty)$, and*

$$p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = a > 0 \tag{3.1}$$

for a certain $\gamma > 3/2$. Let the initial data ϱ_0, $\mathbf{u}_0$ satisfy

$$\varrho_0 \geq 0,\ \varrho_0 \not\equiv 0,\ \varrho_0 \in L^\gamma(\Omega),\ \varrho_0 |\mathbf{u}_0|^2 \in L^1(\Omega).$$

Then the Navier-Stokes system (1.1 - 1.5) possesses a suitable weak solution ϱ, $\mathbf{u}$ *in the sense of Definition 2.2.*

Remark 3.1 *Regularity of the boundary is not essential. The same result can be shown for a general class of domains in the spirit of [9].*

Remark 3.2 *Hypothesis (3.1) is not optimal, the result can be extended to a fairly general class of pressure-density state relations as long as p is an increasing function of ϱ.*

The rest of this section is devoted to the proof of Theorem 3.1.

3.1 Approximation scheme

We construct the suitable weak solution by means of the three-level approximation scheme proposed in [7, Chapter 7], [25, Chapter 7]. Specifically, we consider a family of finite dimensional spaces X_n, $n = 1, 2, \dots$ consisting of smooth vector valued functions defined on $\overline{\Omega}$ satisfying the no-slip boundary condition (1.4).

Equation of continuity is regularized by means of vanishing viscosity,

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \varepsilon \Delta_x \varrho \text{ in } Q_T, \tag{3.2}$$

supplemented with the homogeneous Neumann boundary conditions

$$\nabla_x \varrho \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega, \tag{3.3}$$

and the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta} \text{ in } \Omega \tag{3.4}$$

where $\varrho_{0,\delta}$ is a smooth, strictly positive function, satisfying the appropriate compatibility conditions.

Momentum equation (1.2) is replaced by a family of Faedo-Galerkin approximations

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho \mathbf{u} \cdot \mathbf{w} \ \mathrm{d}x = \int_\Omega \Big(\varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \mathbf{w} + p(\varrho) \mathrm{div}_x \mathbf{w} + \delta \varrho^\beta \mathrm{div}_x \mathbf{w} \Big) \ \mathrm{d}x \tag{3.5}$$

$$- \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u} \cdot \mathbf{w} \Big) \ \mathrm{d}x,$$

with the initial condition

$$\int_\Omega \varrho \mathbf{u}(0,\cdot) \cdot \mathbf{w} \ \mathrm{d}x = \int_\Omega \varrho_{0,\delta} \mathbf{u}_{0,\delta} \cdot \mathbf{w} \ \mathrm{d}x \tag{3.6}$$

for any $\mathbf{w} \in X_n$. Here, exactly as in [7, Chapter 7], $\varepsilon > 0$, $\delta > 0$ are small parameters and $\beta > 4$ is a suitable constant.

As shown in [7, Chapter 7], [25, Chapter 7], problem (3.2 - 3.6) possesses a (unique) regular solution ϱ_n, $\mathbf{u}_n$ for any fixed n. Specifically,

$$\mathbf{u}_n \in C^1([0,T]; X_n), \ \varrho_n, \partial_t \varrho_n, \nabla_x \varrho_n, \nabla_x \nabla_x \varrho_n \text{ Hölder continuous on } [0,T] \times \overline{\Omega},$$

and

$$0 < \min_{x \in \overline{\Omega}} \varrho_n(t,x) \leq \max_{x \in \overline{\Omega}} \varrho_n(t,x) < \infty \text{ for all } t \in [0,T].$$

3.2 Approximate relative entropy inequality

Our goal is to derive an approximate variant of relation (2.5). To this end, consider smooth functions r_m, $\mathbf{U}_m$ satisfying (2.4), and, in addition,

$$\mathbf{U}_m \in C^1([0,T]; X_m), \ m > 0 \text{ fixed.}$$

Furthermore, we set

$$g_m = \partial_t r_m + \mathrm{div}_x (r_m \mathbf{U}_m), \tag{3.7}$$

and

$$\mathbf{f}_m = \partial_t \mathbf{U}_m + \mathbf{U}_m \cdot \nabla_x \mathbf{U}_m + \nabla_x P(r_m) - \frac{1}{r_m} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m), \tag{3.8}$$

where

$$P(\varrho) = H'(\varrho), \ P'(\varrho) = \frac{p'(\varrho)}{\varrho} \text{ for all } \varrho > 0. \tag{3.9}$$

Subtracting (3.7) from (3.2), we get

$$\partial_t [\varrho_n - r_m] + \mathrm{div}_x \Big([\varrho_n - r_m] \mathbf{U}_m \Big) + \mathrm{div}_x \Big(\varrho_n [\mathbf{u}_n - \mathbf{U}_m] \Big) = \varepsilon \Delta_x \varrho_n - g_m. \tag{3.10}$$

Moreover, the quantity $\mathbf{u}_n - \mathbf{U}_m$ may be taken as a test function in (3.5) to obtain

$$\int_\Omega \partial_t (\varrho_n \mathbf{u}_n) \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x \tag{3.11}$$

$$= \int_\Omega \Big(\varrho_n [\mathbf{u}_n \otimes \mathbf{u}_n] : \nabla_x [\mathbf{u}_n - \mathbf{U}_m] + p(\varrho_n) \mathrm{div}_x [\mathbf{u}_n - \mathbf{U}_m] + \delta \varrho_n^\beta \mathrm{div}_x [\mathbf{u}_n - \mathbf{U}_m] \Big) \ \mathrm{d}x$$

$$- \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x [\mathbf{u}_n - \mathbf{U}_m] + \varepsilon \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot [\mathbf{u}_n - \mathbf{U}_m] \Big) \ \mathrm{d}x$$

as soon as $n \geq m$. Consequently, after a bit tedious but straightforward manipulation, relations (3.7 - 3.11) give rise to

$$\int_\Omega \varrho_n \Big(\partial_t [\mathbf{u}_n - \mathbf{U}_m] + \mathbf{u}_n \cdot \nabla_x [\mathbf{u}_n - \mathbf{U}_m] \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] + \nabla_x \left(p(\varrho_n) + \delta \varrho_n^\beta - p(r_m) \right) \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x \tag{3.12}$$

$$= - \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}_n) - \mathbb{S}(\mathbf{U}_m)] : \nabla_x [\mathbf{u}_n - \mathbf{U}_m] - r_m \mathbf{f}_m \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$+ \int_\Omega (r_m - \varrho_n) \Big(\partial_t \mathbf{U}_m + \mathbf{U}_m \cdot \nabla_x \mathbf{U}_m \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$+ \int_\Omega \varrho_n [\mathbf{U}_m - \mathbf{u}_n] \cdot \nabla_x \mathbf{U}_m \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$-\varepsilon \int_\Omega \Big(\Delta_x \varrho_n \mathbf{u}_n \cdot [\mathbf{u}_n - \mathbf{U}_m] + \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot [\mathbf{u}_n - \mathbf{U}_m] \Big) \ \mathrm{d}x,$$

where, in accordance with (3.2), (3.7),

$$\varrho_n \Big(\partial_t [\mathbf{u}_n - \mathbf{U}_m] + \mathbf{u}_n \cdot \nabla_x [\mathbf{u}_n - \mathbf{U}_m] \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] \tag{3.13}$$

$$= \partial_t \left(\frac{1}{2} \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 \right) + \mathrm{div}_x \left(\frac{1}{2} \varrho_n \mathbf{u}_n |\mathbf{u}_n - \mathbf{U}_m|^2 \right) - \frac{\varepsilon}{2} \Delta_x \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2.$$

Next, since

$$(r_m - \varrho_n) \Big(\partial_t \mathbf{U}_m + \mathbf{U}_m \cdot \nabla_x \mathbf{U}_m \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] - r_m \mathbf{f}_m \cdot [\mathbf{u}_n - \mathbf{U}_m]$$

$$= (r_m - \varrho_n) \left(\frac{1}{r_m} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m) - \frac{1}{r_m} \nabla_x p(r_m) \right) \cdot [\mathbf{u}_n - \mathbf{U}_m] - \varrho_n \mathbf{f}_m \cdot [\mathbf{u}_n - \mathbf{U}_m],$$

we may rewrite (3.12) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 \ \mathrm{d}x + \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}_n) - \mathbb{S}(\nabla_x \mathbf{U}_m)] : \nabla_x [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x \tag{3.14}$$

$$+ \int_\Omega \varrho_n \Big(\nabla_x P(\varrho_n) - \nabla_x P(r_m) \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x + \delta \int_\Omega \nabla_x \varrho_n^\beta \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$= \int_\Omega \varrho_n [\mathbf{U}_m - \mathbf{u}_n] \cdot \nabla_x \mathbf{U}_m \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$+ \int_\Omega \left(\left(\frac{r_m - \varrho_n}{r_m} \right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) - \varrho_n \mathbf{f}_m \cdot (\mathbf{u}_n - \mathbf{U}_m) \right) \ \mathrm{d}x$$

$$+ \varepsilon \int_\Omega \nabla_x \varrho_n \cdot \mathbf{U}_m \cdot \nabla_x [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x.$$

Furthermore, by virtue of (3.10), we have

$$\int_\Omega \varrho_n \Big(\nabla_x P(\varrho_n) - \nabla_x P(r_m) \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x = - \int_\Omega (P(\varrho_n) - P(r_m)) \, \mathrm{div}_x \left(\varrho_n [\mathbf{u}_n - \mathbf{U}_m] \right) \ \mathrm{d}x$$

$$= \int_\Omega (P(\varrho_n) - P(r_m)) \left(g_m - \varepsilon \Delta_x \varrho_n \right) \ \mathrm{d}x + \int_\Omega (P(\varrho_n) - P(r_m)) \, \mathrm{div}_x \left([\varrho_n - r_m] \mathbf{U}_m \right) \ \mathrm{d}x$$

$$+ \int_\Omega (P(\varrho_n) - P(r_m)) \, \partial_t \left(\varrho_n - r_m \right) \ \mathrm{d}x.$$

Following Germain [12] we write

$$E(\varrho, r) = E(s, r) = H(s + r) - H'(r) s - H(r), \ \text{where } s \equiv \varrho - r;$$

whence

$$\frac{\partial E(\varrho,r)}{\partial s} = P(\varrho) - P(r), \ \frac{\partial E(\varrho,r)}{\partial r} = P(\varrho) - P(r) - P'(r)(\varrho - r),$$

and, consequently,

$$\int_\Omega (P(\varrho_n) - P(r_m))\, g_m \ \mathrm{d}x = \int_\Omega \frac{\partial E}{\partial s}(\varrho_n, r_m) g_m \ \mathrm{d}x, \tag{3.15}$$

$$\int_\Omega (P(\varrho_n) - P(r_m))\, \mathrm{div}_x \left([\varrho_n - r_m] \mathbf{U}_m \right) \ \mathrm{d}x \tag{3.16}$$

$$= \int_\Omega \left(\frac{\partial E}{\partial s}(\varrho_n, r_m) \nabla_x [\varrho_n - r_m] \cdot \mathbf{U}_m + [\varrho_n - r_m] \frac{\partial E}{\partial s}(\varrho_n, r_m) \mathrm{div}_x \mathbf{U}_m \right) \ \mathrm{d}x$$

$$\int_\Omega \nabla_x E(\varrho_n, r_m) \cdot \mathbf{U}_m \ \mathrm{d}x - \int_\Omega \frac{\partial E}{\partial r}(\varrho_n, r_m) \nabla_x r_m \cdot \mathbf{U}_m \ \mathrm{d}x$$

$$+ \int_\Omega (\varrho_n - r_m) \frac{\partial E}{\partial s}(\varrho_n, r_m) \mathrm{div}_x \mathbf{U}_m \ \mathrm{d}x = \int_\Omega \frac{\partial E}{\partial r}(\varrho_n, r_m) \mathrm{div}_x (r_m \mathbf{U}_m) \ \mathrm{d}x$$

$$+ \int_\Omega \mathrm{div}_x \mathbf{U}_m \left(\frac{\partial E}{\partial s}(\varrho_n, r_m)(\varrho_n - r_m) + \frac{\partial E}{\partial r}(\varrho_n, r_m) r_m - E(\varrho_n, r_m) \right) \ \mathrm{d}x,$$

and, finally,

$$\int_\Omega (P(\varrho_n) - P(r_m))\, \partial_t (\varrho_n - r_m) \ \mathrm{d}x \tag{3.17}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E(\varrho_n, r_m) \ \mathrm{d}x - \int_\Omega \frac{\partial E}{\partial r}(\varrho_n, r_m) \partial_t r_m \ \mathrm{d}x.$$

Summing up (3.15 - 3.17) we conclude that

$$\int_\Omega \varrho_n \Big(\nabla_x P(\varrho_n) - \nabla_x P(r_m) \Big) \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x \tag{3.18}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E(\varrho_n, r_m) \ \mathrm{d}x + \int_\Omega \left(\frac{\partial E}{\partial s}(\varrho_n, r_m) g_m - \frac{\partial E}{\partial r} \left(\partial_t r_m + \mathrm{div}_x (r_m \mathbf{U}_m) \right) \right) \ \mathrm{d}x$$

$$+ \int_\Omega \mathrm{div}_x \mathbf{U}_m \left(\frac{\partial E}{\partial s}(\varrho_n, r_m)(\varrho_n - r_m) + \frac{\partial E}{\partial r}(\varrho_n, r_m) r_m - E(\varrho_n, r_m) \right) \ \mathrm{d}x$$

$$- \varepsilon \int_\Omega (P(\varrho_n) - P(r_m))\, \Delta_x \varrho_n \ \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E(\varrho_n, r_m) \ \mathrm{d}x + \int_\Omega P'(r_m)(\varrho_n - r_m) g_m \ \mathrm{d}x$$

$$+\int_\Omega \mathrm{div}_x \mathbf{U}_m \left(\frac{\partial E}{\partial s}(\varrho_n, r_m)(\varrho_n - r_m) + \frac{\partial E}{\partial r}(\varrho_n, r_m) r_m - E(\varrho_n, r_m) \right) \ \mathrm{d}x$$

$$-\varepsilon \int_\Omega \left(P(\varrho_n) - P(r_m)\right) \Delta_x \varrho_n \ \mathrm{d}x.$$

Relations (3.14), (3.18) give rise to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 + E(\varrho_n, r_m) \right) \ \mathrm{d}x + \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}_n) - \mathbb{S}(\nabla_x \mathbf{U}_m)] : \nabla_x [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x \tag{3.19}$$

$$+\delta \int_\Omega \nabla_x \varrho_n^\beta \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$= \int_\Omega \varrho_n [\mathbf{U}_m - \mathbf{u}_n] \cdot \nabla_x \mathbf{U}_m \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x - \int_\Omega P'(r_m)(\varrho_n - r_m) g_m \ \mathrm{d}x$$

$$+ \int_\Omega \left(\left(\frac{r_m - \varrho_n}{r_m} \right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) - \varrho_n \mathbf{f}_m \cdot (\mathbf{u}_n - \mathbf{U}_m) \right) \ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}_m \left(\frac{\partial E}{\partial s}(\varrho_n, r_m)(\varrho_n - r_m) + \frac{\partial E}{\partial r}(\varrho_n, r_m) r_m - E(\varrho_n, r_m) \right) \ \mathrm{d}x$$

$$+\varepsilon \int_\Omega \Big(\nabla_x \varrho_n \cdot \mathbf{U}_m \cdot \nabla_x [\mathbf{u}_n - \mathbf{U}_m] + \left(P(\varrho_n) - P(r_m)\right) \Delta_x \varrho_n \Big) \ \mathrm{d}x,$$

where

$$\int_\Omega \nabla_x \varrho_n^\beta \cdot [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x = \int_\Omega \varrho_n^\beta \mathrm{div}_x \mathbf{U}_m \ \mathrm{d}x - \int_\Omega \varrho_n^\beta \mathrm{div}_x \mathbf{u}_n \ \mathrm{d}x \tag{3.20}$$

$$= \int_\Omega \varrho_n^\beta \mathrm{div}_x \mathbf{U}_m \ \mathrm{d}x + \frac{1}{\beta - 1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_n^\beta \ \mathrm{d}x - \frac{\varepsilon \beta}{\beta - 1} \int_\Omega \Delta_x \varrho_n \varrho_n^{\beta - 1} \ \mathrm{d}x$$

$$= \int_\Omega \varrho_n^\beta \mathrm{div}_x \mathbf{U}_m \ \mathrm{d}x + \frac{1}{\beta - 1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_n^\beta \ \mathrm{d}x + \varepsilon \beta \int_\Omega \varrho_n^{\beta - 2} |\nabla_x \varrho_n|^2 \ \mathrm{d}x,$$

$$\int_\Omega \left(\left(\frac{r_m - \varrho_n}{r_m} \right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) - \varrho_n \mathbf{f}_m \cdot (\mathbf{u}_n - \mathbf{U}_m) \right) \ \mathrm{d}x \tag{3.21}$$

$$= \int_\Omega \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x$$

$$- \int_\Omega \varrho_n \Big(\partial_t \mathbf{U}_m + \mathbf{U}_m \cdot \nabla_x \mathbf{U}_m + \nabla_x P(r_m) \Big) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x,$$

and

$$- \int_\Omega P'(r_m)(\varrho_n - r_m) g_m \ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}_m \left(\frac{\partial E}{\partial s}(\varrho_n, r_m)(\varrho_n - r_m) + \frac{\partial E}{\partial r}(\varrho_n, r_m) r_m - E(\varrho_n, r_m) \right) \ \mathrm{d}x$$

$$= - \int_\Omega P'(r_m)(\varrho_n - r_m) g_m \ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}_m \Big(\varrho_n \left(P(\varrho_n) - P(r_m) \right) (\varrho_n, r_m) - r_m P'(r_m)(\varrho_n - r_m) - E(\varrho_n, r_m) \Big) \ \mathrm{d}x$$

$$= - \int_\Omega \Big(\partial_t P(r_m) + \mathbf{U}_m \cdot \nabla_x P(r_m) \Big) [\varrho_n - r_m] \ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}_m \Big(\varrho_n \left(P(\varrho_n) - P(r_m) \right) (\varrho_n, r_m) - E(\varrho_n, r_m) \Big) \ \mathrm{d}x.$$

Summarizing the previous estimates we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 + E(\varrho_n, r_m) \right) \ \mathrm{d}x + \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}_n) - \mathbb{S}(\nabla_x \mathbf{U}_m)] : \nabla_x [\mathbf{u}_n - \mathbf{U}_m] \ \mathrm{d}x$$

$$+ \frac{\delta}{\beta - 1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_n^\beta \ \mathrm{d}x \leq \int_\Omega \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x \tag{3.22}$$

$$- \int_\Omega \varrho_n \Big(\partial_t \mathbf{U}_m + \mathbf{u}_n \cdot \nabla_x \mathbf{U}_m \Big) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x,$$

$$- \int_\Omega \Big(\partial_t P(r_m) [\varrho_n - r_m] + \nabla_x P(r_m) \cdot [r_m \mathbf{U}_m - \varrho_n \mathbf{u}_n] \ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}_m \Big(\varrho_n \left(P(\varrho_n) - P(r_m) \right) - E(\varrho_n, r_m) \Big) \ \mathrm{d}x$$

$$+ \varepsilon \int_\Omega \Big(\nabla_x \varrho_n \cdot \mathbf{U}_m \cdot \nabla_x [\mathbf{u}_n - \mathbf{U}_m] + \nabla_x P(r_m) \cdot \nabla_x \varrho_n \Big) \ \mathrm{d}x - \delta \int_\Omega \varrho_n^\beta \mathrm{div}_x \mathbf{U}_m \ \mathrm{d}x.$$

Thus, at least formally, relation (3.22) coincides with (2.5), (2.6) modulo the extra terms proportional to ε, δ, respectively.

3.3 Convergence

The limit in the approximate relative entropy inequality follows step by step the existence proof in [7, Chapter 7], [25, Chapter 7]. More specifically:

- We perform the limit for $\varrho_n \to \varrho_\varepsilon$, $\mathbf{u}_n \to \mathbf{u}_\varepsilon$ for $n \to \infty$ in the Faedo-Galerkin approximations.
- The functions r_m and $\mathbf{U}_m$ are replaced by any smooth r and $\mathbf{U}$ by means of a density argument.
- The limits $\varrho_\varepsilon \to \varrho_\delta$, $\mathbf{u}_\varepsilon \to \mathbf{u}_\delta$ for $\varepsilon \to 0$, and $\varrho_\delta \to \varrho$, $\mathbf{u}_\delta \to \mathbf{u}$ for $\delta \to 0$ are performed, successively.

Note that the above programme can be carried over as relation (3.22) for $\mathbf{U}_m \equiv 0$, $r_m = \frac{1}{|\Omega|}\int_\Omega \varrho_n \ \mathrm{d}x$ reduces to the standard entropy inequality (2.3) yielding the uniform estimates:

$$\{\sqrt{\varrho}\mathbf{u}\}_{n,\varepsilon,\delta} \text{ bounded in } L^\infty(0,T;L^2(\Omega;R^3)), \tag{3.23}$$

$$\{\varrho\}_{n,\varepsilon,\delta} \text{ bounded in } L^\infty(0,T;L^\gamma(\Omega)), \tag{3.24}$$

$$\left\{\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right\}_{n,\varepsilon,\delta} \text{ bounded in } L^2((0,T)\times\Omega;R^{3\times 3}), \tag{3.25}$$

independent of n, ε, and δ. Moreover, there are uniform bounds in terms of ε and δ necessary to get rid of all ε, δ dependent quantities in (3.22) in the limit ε, $\delta \to 0$, specifically,

$$\varepsilon\int_0^T\int_\Omega |\nabla_x \varrho_n|^2 \ \mathrm{d}x \leq c, \ \varepsilon\int_0^T\int_\Omega |\nabla_x \varrho_\varepsilon|^2 \ \mathrm{d}x \leq c,$$

and the refined pressure estimates of the form

$$\int_0^T\int_\Omega \left(\delta\varrho_\varepsilon^{\beta+1} + \varrho_\varepsilon^{\gamma+1} \ \mathrm{d}x\right)\mathrm{d}t \leq c, \int_0^T\int_\Omega \left(\delta\varrho_\delta^{\beta+\nu} + \varrho_\delta^{\gamma+\nu} \ \mathrm{d}x\right)\mathrm{d}t \leq c \text{ for a certain } \nu > 0. \tag{3.26}$$

Moreover, the key idea of the existence theory for the compressible Navier-Stokes system asserts that

$$\varrho_n \to \varrho_\varepsilon, \ \varrho_\varepsilon \to \varrho_\delta, \ \varrho_\delta \to \varrho \text{ a.a. on } Q_T,$$

which, together with the uniform bounds established, enables us to perform the corresponding limits in the approximate entropy inequality (3.22), see [7, Chapter 7] for details.

Theorem 3.1 has been proved. As a matter of fact, the above procedure yields the relative entropy inequality (2.5) in a "stronger" differential form, namely

$$\partial_t \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho,r)\right)(\tau,\cdot) \ \mathrm{d}x + \int_\Omega [\mathbb{S}(\nabla_x\mathbf{u}) - \mathbb{S}(\nabla_x\mathbf{U})] : \nabla_x(\mathbf{u}-\mathbf{U}) \ \mathrm{d}x \tag{3.27}$$

$$\leq \mathcal{R}(\varrho,\mathbf{u},r,\mathbf{U}).$$

Unlike (2.5), however, validity of (3.27) is conditioned by certain regularity of the boundary $\partial\Omega$, cf. Kukučka [17].

4 Applications

To begin we identify the lowest level of admissible smoothness of the test functions r, $\mathbf{U}$ in the relative entropy inequality (2.5), (2.6). To this end, we suppose that the pressure

p satisfies hypothesis (3.1); whence any suitable weak solution in the sense of Definition 2.2 belongs to the regularity class (3.23 - 3.26).

Thus, for the left hand side (2.5) to be well defined, the functions r, $\mathbf{U}$ must belong at least to the class

$$r \in C_{\rm weak}([0,T]; L^\gamma(\Omega)),\ \mathbf{U} \in C_{\rm weak}([0,T]; L^{2\gamma/\gamma-1}(\Omega; R^3)), \tag{4.1}$$

$$\nabla_x \mathbf{U} \text{ in } L^2((0,T)\times\Omega; R^{3\times 3}),\ \mathbf{U}|_{\partial\Omega} = 0. \tag{4.2}$$

Similarly, a short inspection of the integrals appearing in (2.6) yields

$$\partial_t \mathbf{U} \in L^1(0,T; L^{2\gamma/(\gamma-1)}(\Omega, R^3)) \oplus L^2(0,T; L^{6\gamma/(5\gamma-6)}(\Omega, R^3)), \tag{4.3}$$

$$\nabla_x^2 \mathbf{U} \in L^1(0,T; L^{2\gamma/(\gamma-1)}(\Omega, R^{3\times3\times3})) \oplus L^2(0,T; L^{6\gamma/(5\gamma-6)}(\Omega, R^{3\times3\times3})), \tag{4.4}$$

and

$$\nabla_x \mathbf{U} \in L^1(0,T; L^\infty(\Omega, R^3)). \tag{4.5}$$

Moreover, the function r must be bounded below away from zero, and, similarly to (4.3), (4.4),

$$\partial_t P(r) \in L^1(0,T; L^{\gamma/(\gamma-1)}(\Omega)) \oplus L^2(0,T; L^{6\gamma/(5\gamma-6)}(\Omega)), \tag{4.6}$$

$$\nabla_x P(r) \in L^1(0,T; L^{2\gamma/(\gamma-1)}(\Omega; R^3)) \oplus L^2(0,T; L^{6\gamma/(5\gamma-6)}(\Omega; R^3)). \tag{4.7}$$

Note that

$$\partial_t P(r) = P'(r)\partial_t r, \text{ where } P'(r) = \frac{p'(r)}{r} \approx r^{\gamma-2} \text{ for } r >> 1.$$

It is easy to see that the relative entropy inequality (2.5), (2.6) can be extended to $\mathbf{U}$, r satisfying (4.1 - 4.7) by means of a simple density argument.

4.1 Weak-strong uniqueness

Our aim is to show that a suitable weak solution coincides with a strong solution of the Navier-Stokes system emanating form the same initial data provided the latter exists. More specifically, we assume that $\tilde{\varrho}$, $\tilde{\mathbf{u}}$ is a weak solution of the Navier-Stokes system in the sense of Definition 2.1 enjoying extra regularity properties, namely

$$0 < \underline{\varrho} \leq \tilde{\varrho}(t,x) \leq \overline{\varrho},\ |\tilde{\mathbf{u}}(t,x)| \leq \overline{u} \text{ for a.a. } (t,x) \in Q_T, \tag{4.8}$$

$$\nabla_x \tilde{\varrho} \in L^2(0,T; L^q(\Omega; R^3)),\ \nabla_x^2 \tilde{\mathbf{u}} \in L^2(0,T; L^q(\Omega; R^{3\times3\times3})),\ q > \max\{3, \frac{3}{\gamma-1}\}. \tag{4.9}$$

To begin, we observe that, by virtue of the standard embedding relations,

$$\tilde{\mathbf{u}} \in L^2(0,T;W^{1,\infty}(\Omega;R^3)), \tag{4.10}$$

and that

$$q > \frac{6\gamma}{5\gamma-6} \text{ for } \gamma > \frac{3}{2}.$$

Moreover, equations (2.1), (2.2) are satisfied in a strong sense, specifically,

$$\partial_t \tilde{\varrho} \in L^2(0,T;L^q(\Omega)),$$

and

$$\partial_t b(\tilde{\varrho}) + \nabla_x b(\tilde{\varrho}) \cdot \tilde{\mathbf{u}} + b'(\tilde{\varrho})\tilde{\varrho}\mathrm{div}_x \tilde{\mathbf{u}} = 0 \text{ a.a. in } Q_T$$

for any Lipschitz b. Similarly,

$$\partial_t \tilde{\mathbf{u}} \in L^2(0,T;L^q(\Omega;R^3)),$$

and

$$\partial_t \tilde{\varrho} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\mathbf{u}} + \nabla_x P(\tilde{\varrho}) = \frac{1}{\tilde{\varrho}} \mathrm{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \text{ a.a. in } Q_T.$$

As a matter of fact, hypothesis (4.9) implies (4.8) provided the initial distribution of the density $\tilde{\varrho}_0$ is bounded below away from zero on Ω. Indeed, by virtue of (4.9), there exists at least one point $\tau \in (0,T)$ such that $\varrho(\tau,\cdot) \in W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$; whence the uniform upper bound on $\tilde{\varrho}$ follows by integrating (1.1) along characteristics. Then equation (1.2) can be used to deduce the uniform bound on $\tilde{\mathbf{u}}$.

In view of (4.9), (4.10), the functions $r = \tilde{\varrho}$, $\mathbf{U} = \tilde{\mathbf{u}}$ can be used in the relative entropy inequality (2.5). After a straightforward manipulation, we deduce that

$$\mathcal{R}(\varrho, \mathbf{u}, \tilde{\varrho}, \tilde{\mathbf{u}}) = \int_\Omega \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \mathrm{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \ \mathrm{d}x + \int_\Omega \varrho(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \ \mathrm{d}x \tag{4.11}$$

$$+ \int_\Omega \Big(E(\varrho, \tilde{\varrho}) - \varrho \Big(P(\varrho) - P(\tilde{\varrho}) \Big) + \tilde{\varrho} P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \Big) \Big) \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x,$$

which coincides with the formula obtained by Germain [12]. Thus, similarly to [12], we have

$$\left| \int_\Omega \varrho(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \ \mathrm{d}x(\tau, \cdot) \right| \tag{4.12}$$

$$+ \left| \int_\Omega \Big(E(\varrho, \tilde{\varrho}) - \varrho \Big(P(\varrho) - P(\tilde{\varrho}) \Big) + \tilde{\varrho} P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \Big) \Big) \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \right|$$

$$\leq h(\tau) \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + E(\varrho, \tilde{\varrho}) \right) (\tau, \cdot) \ {\rm d}x \ \mbox{for a.a.} \ \tau \in (0,T), \ h \in L^2(0,T),$$

provided p is twice continuously differentiable in $(0, \infty)$ and (4.10) holds. Indeed we have

$$E(\varrho, \tilde{\varrho}) - \varrho \Big(P(\varrho) - P(\tilde{\varrho}) \Big) + \tilde{\varrho} P'(\tilde{\varrho})(\varrho - \tilde{\varrho})$$

$$= H(\varrho) - \varrho H'(\varrho) - \Big(H(\tilde{\varrho}) - \tilde{\varrho} H'(\tilde{\varrho}) \Big) + \tilde{\varrho} H''(\tilde{\varrho})(\varrho - \tilde{\varrho});$$

whence

$$\left| E(\varrho, \tilde{\varrho}) - \varrho \Big(P(\varrho) - P(\tilde{\varrho}) \Big) + \tilde{\varrho} P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \right| \leq c(\delta, \tilde{\varrho}) E(\varrho, \tilde{\varrho})$$

provided $\varrho \geq \delta > 0$, while, as $H \in C[0, \infty)$,

$$\left| E(\varrho, \tilde{\varrho}) - \varrho \Big(P(\varrho) - P(\tilde{\varrho}) \Big) + \tilde{\varrho} P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \right| \leq c_1(\delta, \tilde{\varrho}) |\varrho - \tilde{\varrho}|^2 \leq c_2(\delta, \tilde{\varrho}) E(\varrho, \tilde{\varrho})$$

as soon as $0 \leq \varrho \leq \delta$.

Finally, as a direct consequence of Korn's inequality, we have

$$\int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}})] : \nabla_x (\mathbf{u} - \tilde{\mathbf{u}}) \ {\rm d}x \geq \Lambda \|\mathbf{u} - \tilde{\mathbf{u}}\|^2_{W^{1,2}_0(\Omega; R^3))};$$

whence, by virtue of the standard embedding $W^{1,2} \hookrightarrow L^6(\Omega)$,

$$\int_{\{\varrho \leq M\}} \left| \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) {\rm div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \right| \ {\rm d}x \tag{4.13}$$

$$\leq \frac{\Lambda}{2} \|\mathbf{u} - \tilde{\mathbf{u}}\|^2_{W^{1,2}(\Omega;R^3)} + c(\Lambda, M) \|\tilde{\mathbf{u}}\|^2_{W^{2,3}(\Omega;R^3)} \int_\Omega E(\varrho, \tilde{\varrho}) \ {\rm d}x$$

for any $M >> 1$. On the other hand, in accordance with hypothesis (4.9),

$$\int_{\{\varrho > M\}} \left| \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) {\rm div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \right| \ {\rm d}x \tag{4.14}$$

$$\leq c(M, \tilde{\varrho}) \|\mathbf{u} - \tilde{\mathbf{u}}\|_{W^{1,2}_0(\Omega;R^3))} \|{\rm div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}})\|_{L^q(\Omega;R^3)} E(\varrho, \tilde{\varrho}).$$

Consequently, combining the relative entropy inequality (2.5) with relations (4.11 - 4.14), we can use Gronwall's lemma to conclude that

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + E(\varrho, \tilde{\varrho}) \right) (\tau, \cdot) \ {\rm d}x + \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \tilde{\mathbf{u}})] : \nabla_x (\mathbf{u} - \tilde{\mathbf{u}}) \ {\rm d}x \ {\rm d}t \tag{4.15}$$

$$\leq c(T) \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \tilde{\mathbf{u}}_0|^2 + E(\varrho_0, \tilde{\varrho}_0) \right) (\tau, \cdot) \ {\rm d}x \ \mbox{for a.a} \ \tau \in (0,T),$$

in particular, $\varrho \equiv \tilde{\varrho}$, $\mathbf{u} \equiv \tilde{\mathbf{u}}$ provided $\varrho_0 = \tilde{\varrho}_0$, $\mathbf{u}_0 = \tilde{\mathbf{u}}_0$.

Thus, we have shown the following weak-strong uniqueness result.

Theorem 4.1 *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\nu}$*,* $\nu > 0$*. In addition to hypotheses of Theorem 3.1, suppose that p is twice continuously differentiable on the open interval* $(0,\infty)$*. Assume that the Navier-Stokes system admits a weak solution* $\tilde{\varrho}$*,* $\tilde{\mathbf{u}}$ *in* Q_T *in the sense of Definition 2.1, belonging to the regularity class specified through (4.8), (4.9).*

Then $\tilde{\varrho} \equiv \varrho$*,* $\tilde{\mathbf{u}} \equiv \mathbf{u}$*, where* ϱ*,* $\mathbf{u}$ *is the suitable weak solution of the Navier-Stokes system emanating from the same initial data, the existence of which is guaranteed by Theorem 3.1.*

Remark 4.1 *Similar result can be shown for the Navier-Stokes system driven by an external force* $\mathbf{f}$*.*

Remark 4.2 *As observed above, hypothesis (4.8) can be omitted provided* $\tilde{\varrho}_0$ *is bounded from below, cf. similar hypotheses in Germain's paper [12].*

Remark 4.3 *Smoothness of the boundary* $\partial\Omega$ *is not necessary, see [9].*

4.2 Conditional regularity criteria

There are many results, mostly devoted to the more popular *incompressible* Navier-Stokes system, concerning conditional regularity of the weak solutions. Very roughly indeed, the weak solutions are regular as soon as they belong to a "critical" regularity class. The best known examples can be found in the work by Prodi [26], Serrin [27], or more recently, Neustupa et al. [23], [24]. Comparable results for the *compressible* Navier-Stokes system are in short supply and subject to various restrictions on the geometry of domains and/or viscosity coefficients (see e.g. [2], [13], [16]). This is mostly due to the fact that the viscosity provides only a partial smoothing effect on some but not all quantities in question. Recently, the authors of [28, Theorem 1.3] showed the following regularity criterion for the compressible Navier-Stokes system:

Proposition 4.1 *Let* $\Omega \subset R^3$ *be a bounded smooth domain and* $3 < q \leq 6$*. Moreover, let* $\varrho_0 \geq 0$*,* $\varrho_0 \in W^{1,q}(\Omega)$*,* $\mathbf{u}_0 \in W_0^{1,2}(\Omega; R^3) \cap W^{2,2}(\Omega; R^3)$*, and let the following compatibility condition holds:*

$$-\mathbb{S}(\nabla_x \mathbf{u}_0) + \nabla_x p(\varrho_0) = \sqrt{\varrho_0}\mathbf{g} \text{ for a certain } \mathbf{g} \in L^2(\Omega, R^3).$$

Then there exists a positive time $T > 0$ *such that the Navier-Stokes system (1.1 - 1.5) admits a unique strong solution* ϱ*,* $\mathbf{u}$ *belonging to the class*

$$\varrho \in C([0,T); W^{1,q}(\Omega)), \ \mathbf{u} \in C([0,T); W^{2,2}(\Omega; R^3)) \cap L^2(0,T; W^{2,6}(\Omega; R^3)).$$

Moreover, if the maximal existence time $T = T_{\max}$ *is finite, and*

$$0 \leq 3\eta < 23\mu,$$

then

$$\limsup_{t \to T_{\max}-} [\sup_{x \in \overline{\Omega}} \varrho(t,x)] = \infty. \tag{4.16}$$

Combining the previous result with Theorem 4.1 we obtain the following conditional regularity criterion for the Navier-Stokes system.

Theorem 4.2 *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\nu}$, $\nu > 0$. *Let* ϱ_0, $\mathbf{u}_0$ *be given such that*

$$\varrho_0 \in W^{1,6}(\Omega),\ 0 < \underline{\varrho} \leq \varrho_0(x) \leq \overline{\varrho} \text{ for all } x \in \Omega,$$

$$\mathbf{u}_0 \in W^{2,2}(\Omega; R^3) \cap W_0^{1,2}(\Omega; R^3).$$

Suppose that the pressure p satisfies the hypotheses of Theorem 3.1, and that

$$\mu > 0,\ \eta = 0.$$

Let ϱ, $\mathbf{u}$ *be a suitable weak solution of the Navier-Stokes system in* Q_T, *the existence of which is guaranteed by Theorem 3.1.*

If, in addition,

$$\operatorname*{ess\,sup}_{Q_T} \varrho < \infty,$$

then ϱ, $\mathbf{u}$ *is the unique (strong) solution of the Navier-Stokes system belonging to the regularity class specified in Proposition 4.1.*

Corollary 4.1 *Let* Ω *and the initial data* ϱ_0, $\mathbf{u}_0$ *be the same as in Theorem 4.2. Assume that* ϱ, $\mathbf{u}$ *is a suitable weak solution of the Navier-Stokes system in* Q_T *such that*

$$\operatorname*{ess\,inf}_{x \in \Omega} \varrho(\tau, x) = 0 \text{ for a certain } \tau \in (0, T).$$

The there exists $0 < \tau_0 \leq \tau$ *such that*

$$\limsup_{t \to \tau_0} [\operatorname*{ess\,sup}_{x \in \Omega} \varrho(t,x)] = \infty.$$

Corollary 4.1 may be interpreted in the way that the density must "blow-up" before developing a vacuum.

4.3 Stability issues

As a straightforward consequence of (4.15) we obtain the following stability result.

Theorem 4.3 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. In addition to hypotheses of Theorem 3.1, suppose that p is twice continuously differentiable on the open interval $(0, \infty)$. Assume that the Navier-Stokes system admits a weak solution $\tilde{\varrho}$, $\tilde{\mathbf{u}}$ in Q_T in the sense of Definition 2.1, belonging to the regularity class specified through (4.8), (4.9), with the initial data $\tilde{\varrho}_0$, $\tilde{\mathbf{u}}_0$. Moreover, suppose that*

$$\varrho_{0,\varepsilon} \to \tilde{\varrho}_0 \textit{ in } L^\gamma(\Omega),\ \varrho_{0,\varepsilon} \geq 0,\ \int_\Omega \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \tilde{\mathbf{u}}_0|^2 \ \mathrm{d}x \to 0 \textit{ as } \varepsilon \to 0.$$

Then

$$\sup_{\tau \in [0,T]} \|\varrho_\varepsilon(\tau, \cdot) - \tilde{\varrho}(\tau, \cdot)\|_{L^\gamma(\Omega)} + \sup_{\tau \in [0,T]} \|\varrho_\varepsilon \mathbf{u}_\varepsilon(\tau, \cdot) - \tilde{\varrho}\tilde{\mathbf{u}}(\tau, \cdot)\|_{L^1(\Omega;R^3))} \to 0,$$

and

$$\mathbf{u}_\varepsilon \to \tilde{\mathbf{u}} \textit{ in } L^2(0,T; W_0^{1,2}(\Omega; R^3)),$$

where ϱ_ε, $\mathbf{u}_\varepsilon$ is a suitable weak solution of the Navier-Stokes system in Q_T, emanating from the initial data $\varrho_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$.

The convergence obtained in the course of the proof of Theorem 3.1 may be interpreted in terms of *stability* of the approximation scheme proposed in (3.2 - 3.6) that may be of some interest in possible numerical implementations. The first observation - a direct consequence of Theorem 4.2 - asserts that the approximate solutions converge for $n \to \infty$, $\varepsilon \to 0$, and $\delta \to 0$ to a (unique) strong solution of the Navier-Stokes system provided the latter exists in Q_T. Secondly, the same is true if *a posteriori* bounds were available in the $L^\infty-$norm for the density at each level of approximations. We refer to Gallouët et al. [11] for other interesting problems related to numerical analysis of the compressible Navier-Stokes system.

References

[1] F. Berthelin and A. Vasseur. From kinetic equations to multidimensional isentropic gas dynamics before shocks. *SIAM J. Math. Anal.*, **36**:1807–1835, 2005.

[2] H.J. Choe and B.J. Jin. Regularity of weak solutions of the compressible Navier-Stokes equations. *J. Korean Math. Soc.*, **40**:1031–1050, 2003.

[3] C.M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, **70**:167–179, 1979.

[4] B. Desjardins. Regularity of weak solutions of the compressible isentropic Navier-Stokes equations. *Commun. Partial Differential Equations*, **22**:977–1008, 1997.

[5] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[6] C. L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The millennium prize problems*, pages 57–67. Clay Math. Inst., Cambridge, MA, 2006.

[7] E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford, 2004.

[8] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. *J. Math. Fluid Mech.*, **3**:358–392, 2001.

[9] E. Feireisl, A. Novotný, and H. Petzeltová. On the domain dependence of solutions to the compressible Navier-Stokes equations of a barotropic fluid. *Math. Meth. Appl. Sci.*, **25**:1045–1073, 2002.

[10] G. P. Galdi. *An introduction to the mathematical theory of the Navier - Stokes equations, I.* Springer-Verlag, New York, 1994.

[11] T. Gallouët, L. Gastaldo, R. Herbin, and J.-C. Latché. An unconditionally stable pressure correction scheme for the compressible barotropic Navier-Stokes equations. *ESAIM: M2AN*, **42**:303–331, 2008.

[12] P. Germain. Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. *J. Math. Fluid Mech.*, 2010. Published online.

[13] B. Haspot. Regularity of weak solutions of the compressible isentropic Navier-Stokes Navier-Stokes equations. *arXiv:1001.158 1v1*, 2010.

[14] D. Hoff. Compressible flow in a half-space with Navier boundary conditions. *J. Math. Fluid Mech.*, 7(3):315–338, 2005.

[15] D. Hoff. Uniqueness of weak solutions of the Navier-Stokes equations of multidimensional, compressible flow. *SIAM J. Math. Anal.*, 37(6):1742–1760 (electronic), 2006.

[16] L. Jiang and Y. Wang. On the blow up criterion for the 2-D compressible Navier-Stokes equations. *Czechoslovak Math. J.*, **60**:195–209, 2010.

[17] P. Kukučka. On the existence of finite energy weak solutions to the Navier-Stokes equations in irregular domains. *Math. Methods Appl. Sci.*, 32(11):1428–1451, 2009.

[18] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow.* Gordon and Breach, New York, 1969.

[19] P.-G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem.* Chapman and Hall, London, 2003.

[20] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, **63**:193–248, 1934.

[21] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, 1998.

[22] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.

[23] J. Neustupa, A. Novotný, and P. Penel. An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity. In *Topics in mathematical fluid mechanics*, volume 10 of *Quad. Mat.*, pages 163–183. Dept. Math., Seconda Univ. Napoli, Caserta, 2002.

[24] J. Neustupa and M. Pokorný. An interior regularity criterion for an axially symmetric suitable weak solution to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 2(4):381–399, 2000.

[25] A. Novotný and I. Straškraba. *Introduction to the mathematical theory of compressible flow.* Oxford University Press, Oxford, 2004.

[26] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl.*, **48**:173–182, 1959.

[27] J. Serrin. The initial value problem for the Navier-Stokes equations. *University of Wisconsin Press*, **9**:69–98, 1963.

[28] Y. Sun, C. Wang, and Z. Zhang. A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navier-Stokes equations. 2010. Preprint.

[29] V. A. Vaigant and A. V. Kazhikhov. On the existence of global solutions to two-dimensional Navier-Stokes equations of a compressible viscous fluid (in Russian). *Sibirskij Mat. Z.*, **36**(6):1283–1316, 1995.