Sum Capacity of the Vector Gaussian Broadcast Channel and Uplink–Downlink Duality

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Abstract—We characterize the sum capacity of the vector Gaussian broadcast channel by showing that the existing inner bound of Marton and the existing upper bound of Sato are tight for this channel. We exploit an intimate four-way connection between the vector broadcast channel, the corresponding point-to-point channel (where the receivers can cooperate), the multiple-access channel (MAC) (where the role of transmitters and receivers are reversed), and the corresponding point-to-point channel (where the transmitters can cooperate).

Index Terms—Broadcast channels, downlink, multiple antennas, wireless system.

I. INTRODUCTION

WE consider a memoryless vector Gaussian broadcast channel to model the downlink of a wireless system with N antennas at the base station and K users with a single antenna at each receiver. Focusing on one particular time instant, denote the received symbol at receiver j by y_j and $\boldsymbol{y}_{\mathrm{dl}} \stackrel{\mathrm{def}}{=} (y_1, \ldots, y_K)^t$. They are related by

$$\boldsymbol{y}_{\mathrm{dl}} = H^\dagger \boldsymbol{x}_{\mathrm{dl}} + \boldsymbol{z}. \qquad (1)$$

Here H is a fixed $N \times K$ matrix with H_{ij}^* entry denoting the (flat-fading) channel gain from the ith antenna to the jth user, assumed to be known at both the transmitter and receivers. $\boldsymbol{x}_{\mathrm{dl}}$ is the vector input to the antenna array with an average total power constraint of P. The additive noise z_j is zero mean, unit variance, complex circular symmetric Gaussian. Each user j has access only to the received symbol y_j. For $N > 1$, this is a vector Gaussian broadcast channel, and unlike the scalar Gaussian broadcast channel ($N = 1$), it is in general not degraded and the capacity region is unknown. The main result of the paper is the following characterization of the *sum capacity* of this channel.

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Theorem: The sum capacity of the vector Gaussian broadcast channel is

$$C_{\mathrm{sum}} = \sup_{D \in \mathcal{A}_1} \log \det \left(I + HDH^\dagger \right)$$

where $\mathcal{A}_1$ is the set of K by K nonnegative diagonal matrices D with $\mathrm{Tr}[D] \leq P$.

Recent work by Caire and Shamai [2] obtains the sum capacity for the special case of $K = 2$ users. They propose a transmission scheme which uses Costa's "Writing on Dirty Paper" precoding technique [3]. This scheme can also be interpreted as Marton's broadcast coding technique [9] applied to the vector Gaussian channel. For the case of two users, Caire and Shamai showed that this scheme is optimal in achieving the sum capacity, by demonstrating that the achievable rate meets the Sato's upper bound [12], which is the capacity of a point-to-point channel where the receivers in the downlink can cooperate. The proof involves a direct calculation and seems difficult to be generalized to $K > 2$.

In this paper, we generalize Caire and Shamai's result to arbitrary number of users through a deeper investigation of the structure of the relevant optimization problems. We first analyze the maximum sum rate achievable by the Costa precoding technique, by exploiting a duality between the downlink (broadcast) and the uplink (multiple access). We develop this duality (as a change of variable) in Section II. In Section III, we show that this sum rate meets Sato's upper bound. The key step is to show that the optimization problems for the Sato bound and for the sum capacity of the multiple-access channels (MACs) are convex duals of each other.

To keep the notations simple, we will confine ourselves in most of the paper to the case when each user has a single antenna element. Our techniques can be naturally generalized to the case when the users have multiple antennas; this is discussed in Section IV.

Independent proofs of the same result were reported in [20] and [22].

Notations: we use lower case letters to denote scalars, upper case letters to denote matrices, and boldface to denote vectors. $\mathcal{CN}(\mu, K)$ denotes a complex circular symmetric Gaussian distribution with mean μ and covariance matrix K.

II. UPLINK–DOWNLINK DUALITY AND COSTA PRECODING

In this section, we analyze the performance of the Costa precoding based broadcast strategy. The key step is to observe an equivalence between the performance of a class of receive and
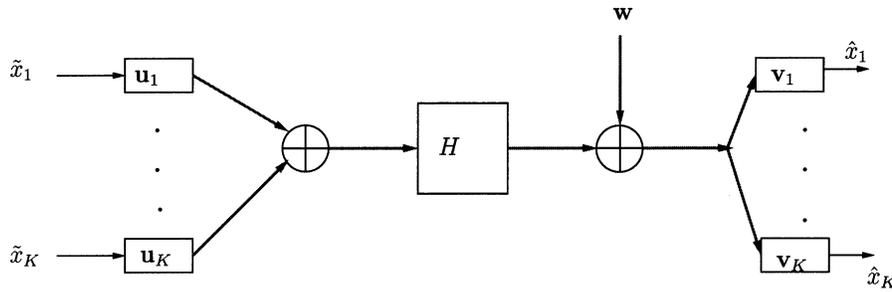
Fig. 1. Linear transmit–receive strategy.

transmit strategies when the role of transmitters and receivers are reversed for vector Gaussian channels. This equivalence has been observed in seemingly different contexts in the literature.

1) In the context of the capacity of a point-to-point multiple-transmit, multiple-receive antenna channel, [15] shows that the capacity is unchanged when the role of the transmitters and receivers is interchanged. The author calls this a *reciprocity* result.

2) In the context of a downlink of a multiple antenna system employing simple linear beamforming strategies followed by single-user receivers by the users, [19] and [10] show that the optimal choice of transmit and receive beamforming vectors is closely related to a *virtual uplink problem*.

3) In the context of the degraded Gaussian broadcast channel, [8] shows that the capacity region is the same as the capacity region of the corresponding MAC with the transmit power constraint of the broadcast channel translated to the sum of powers in the MAC. The authors name their result a *duality* connection.

We focus on a certain class of transmission and receive strategies which have equivalence of performance when the role of transmitters and receivers are reversed with a natural conservation of total power transmitted. This allowed us to succinctly generalize the underlying common phenomenon of the observations above and gives a simple characterization of the maximum achievable sum rate of the Costa precoding strategy. An independent and similar derivation of the duality in the context of *linear* beamforming strategies is presented in [13].

A. Point-to-Point Reciprocity Revisited

Let us start with a point-to-point vector Gaussian channel

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{w} \tag{2}$$

with H being a fixed matrix of dimension N by K. The additive noise $\boldsymbol{w}$ is $\mathcal{CN}(0, I)$. We consider a linear transmission and reception strategy as shown in Fig. 1. The transmitted signal is

$$\boldsymbol{x} = \sum_{k=1}^{K} \tilde{x}_k \boldsymbol{u}_k$$

where $\tilde{x}_k$ and $\boldsymbol{u}_k$ can be interpreted as the information symbol and the transmit vector for the kth data stream respectively. The kth information stream is demodulated by the receive vector $\boldsymbol{v}_k$

$$\hat{x}_k = \boldsymbol{v}_k^\dagger \boldsymbol{y}.$$

The vectors $\boldsymbol{u}_k$'s and $\boldsymbol{v}_k$'s are normalized so that their l_2 norm is unity.

The output of the receive filter contains in part the signal of interest and in part the interference from the other streams plus the background noise. A key performance measure is the signal-to-interference ratio (SIR)

$$\mathrm{SIR}_k \stackrel{\text{def}}{=} \frac{p_k \left| \boldsymbol{v}_k^\dagger H \boldsymbol{u}_k \right|^2}{1 + \sum_{j \neq k} p_j \left| \boldsymbol{v}_k^\dagger H \boldsymbol{u}_j \right|^2} \tag{3}$$

where $p_k \stackrel{\text{def}}{=} E[\tilde{x}_k^2]$ is the power allocated to stream k.

Denoting $\boldsymbol{a} \stackrel{\text{def}}{=} (a_1, \ldots, a_K)^t$ where

$$a_k \stackrel{\text{def}}{=} \frac{\mathrm{SIR}_k}{(1 + \mathrm{SIR}_k) \left| \boldsymbol{v}_k^\dagger H \boldsymbol{u}_k \right|^2},$$

we can rewrite (3) in matrix notation as

$$(I - \mathrm{diag}\{a_1, \ldots, a_K\} A) \boldsymbol{p} = \boldsymbol{a} \tag{4}$$

where the $K \times K$ matrix A has (k, j)th component equal to $|\boldsymbol{v}_k^\dagger H \boldsymbol{u}_j|^2$. As in [7, Sec. II-E], a positive solution to $\boldsymbol{p}$ that satisfies (4) exists if and only if the Perron–Frobenius eigenvalue of $\mathrm{diag}\{a_1, \ldots, a_K\} A$ is less than 1. This characterizes the *performance region* of the bank of linear transmit and receive filters, i.e., the set of vectors $\boldsymbol{a}$'s or, equivalently, the set of SIR requirements that can be met. For a given set of SIR requirements, the corresponding component-wise minimum transmit power required is

$$\boldsymbol{p}_{\min} \stackrel{\text{def}}{=} (I - \mathrm{diag}\{a_1, \ldots, a_K\} A)^{-1} \boldsymbol{a} = (D_a - A)^{-1} \mathbf{1} \tag{5}$$

where $D_a \stackrel{\text{def}}{=} \mathrm{diag}(1/a_1, \ldots, 1/a_K)$ and $\mathbf{1}$ is the vector of all 1's [14, Theorem 2.1].

Now we turn to the reciprocal channel of (2).

$$\boldsymbol{y} = H^\dagger \boldsymbol{x} + \boldsymbol{z} \tag{6}$$

with input of dimension N and output of dimension K (i.e., the transmitter and receiver reversed). Let us consider a linear
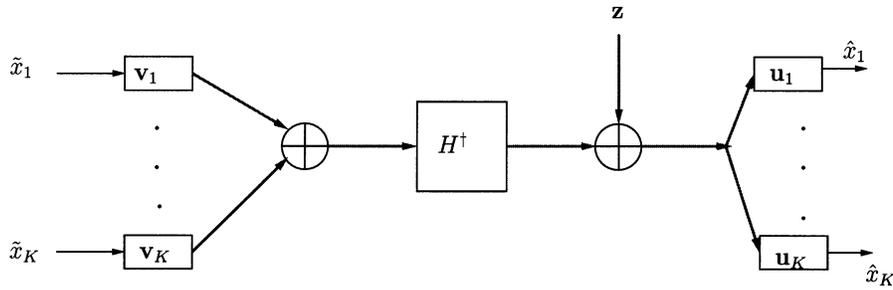
Fig. 2. The reciprocal channel.

transmission strategy in which the transmit and receive filters are the reverse of those used for the original channel

$$\boldsymbol{x} = \sum_{k=1}^{K} \tilde{x}_k \boldsymbol{v}_k \quad \text{and} \quad \hat{x}_k = \boldsymbol{u}_k^{\dagger} \boldsymbol{y}$$

(see Fig. 2). The SIR of user k with this transmission–reception strategy is

$$\text{SIR}_k \stackrel{\text{def}}{=} \frac{q_k \left| \boldsymbol{u}_k^{\dagger} H^{\dagger} \boldsymbol{v}_k \right|^2}{1 + \sum_{j \neq k} q_j \left| \boldsymbol{u}_k^{\dagger} H^{\dagger} \boldsymbol{v}_j \right|^2} \tag{7}$$

where q_k is the power allocated to stream k.

Denoting $\boldsymbol{b} \stackrel{\text{def}}{=} (b_1, \ldots, b_K)^t$ where

$$b_k \stackrel{\text{def}}{=} \frac{\text{SIR}_k}{(1 + \text{SIR}_k) \left| \boldsymbol{u}_k^{\dagger} H^{\dagger} \boldsymbol{v}_k \right|^2}$$

we can rewrite (3) in matrix notation as

$$(I - \text{diag}\{b_1, \ldots, b_K\} B) \boldsymbol{q} = \boldsymbol{b} \tag{8}$$

where the $K \times K$ matrix B has (k, j)th component equal to $|\boldsymbol{u}_k^{\dagger} H^{\dagger} \boldsymbol{v}_j|^2$. A positive solution to $\boldsymbol{q}$ that satisfies (8) exists if and only if the Perron–Frobenius eigenvalue of $\text{diag}\{b_1, \ldots, b_K\} B$ is less than 1. This characterizes the *performance region* of the transmission strategy, and for given SIR requirements, the corresponding component-wise minimum transmit power required is

$$\boldsymbol{q}_{\min} \stackrel{\text{def}}{=} (I - \text{diag}\{b_1, \ldots, b_K\} B)^{-1} \boldsymbol{b} = (D_b - B)^{-1} \mathbf{1} \tag{9}$$

where $D_b \stackrel{\text{def}}{=} \text{diag}(1/b_1, \ldots, 1/b_K)$.

We now state the equivalence between the two reciprocal systems. The performance of the two systems are fully characterized by the pairs $(\boldsymbol{a}, A)$ and $(\boldsymbol{b}, B)$, respectively. Note that $A = B^t$ and for the same SIR requirements, $\boldsymbol{a} = \boldsymbol{b}$. From this we can make two observations. First, the achievable SIR performance region in both these cases is the *same*, i.e., given SIR requirements can be met in one system if and only if they can be met in the other system. This is seen by noting that the Perron–Frobenius eigenvalues of $\text{diag}\{a_1, \ldots, a_K\} A$ and $\text{diag}\{a_1, \ldots, a_K\} A^t$ are the same. Second, we observe that for any given achievable performance $\boldsymbol{a} = \boldsymbol{b}$, the sum of the min-

imal transmit powers required to achieve this performance is also identical in the two systems

$$\sum_k p_{\min, k} = \mathbf{1}^t (D_a - A)^{-1} \mathbf{1} = \mathbf{1}^t \left[(D_a - A)^{-1} \right]^t \mathbf{1}$$

$$= \mathbf{1}^t (D_a - A^t)^{-1} \mathbf{1} = \sum_k q_{\min, k}.$$

An immediate consequence of this equivalence is that the capacities of the original point-to-point channel (2) and its reciprocal (6) under the same total power constraint are equal. This follows from the fact that the capacity-achieving transmit–receive strategies are linear, with independent Gaussian signaling on parallel, noninterfering links (the $\boldsymbol{u}_k$'s and the $\boldsymbol{v}_k$'s are then the left and right eigenvectors of H, respectively.) One can also see this by observing that H and $H^{\dagger}$ have the same nonzero singular values and the capacity of a point-to-point vector Gaussian channel depends only on the nonzero singular values of the channel matrices [15]. However, the preceding equivalence is stronger as it applies to *every* choice of linear transmit–receive filters. It also reveals the underlying structure that can be generalized to other settings, as we shall now see.

B. Linear Beamforming in Uplink and Downlink

The point-to-point reciprocity developed in the preceding subsection can be directly applied to establish a connection between the uplink and downlink linear beamforming problems. Consider an uplink (multiple-access) channel with K users and N antenna elements at the receiver

$$\boldsymbol{y}_{\text{ul}} = H \boldsymbol{x}_{\text{ul}} + \boldsymbol{w}. \tag{10}$$

The difference with the point-to-point channel is that the transmit antennas cannot cooperate, i.e., we are constrained to using transmit vector $\boldsymbol{u}_k = (0, \ldots, 1, \ldots, 0)^t$ (with 1 in the kth position) for the data stream for the kth user. The receive vector $\boldsymbol{v}_k$ can now be interpreted as a linear receive beamforming vector for demodulating the signal for user k. Direct application of the equivalence in Section II-A yields the following performance equivalence between this uplink and the downlink:

$$\boldsymbol{y}_{\text{dl}} = H^{\dagger} \boldsymbol{x}_{\text{dl}} + \boldsymbol{z} \tag{11}$$

where $\boldsymbol{v}_k$ now acts as the *transmit* beamforming vector for the kth user. The receive vector $\boldsymbol{u}_k = (0, \ldots, 1, \ldots, 0)^t$, signifying that the users cannot cooperate in demodulating the signals (see Fig. 3). More precisely
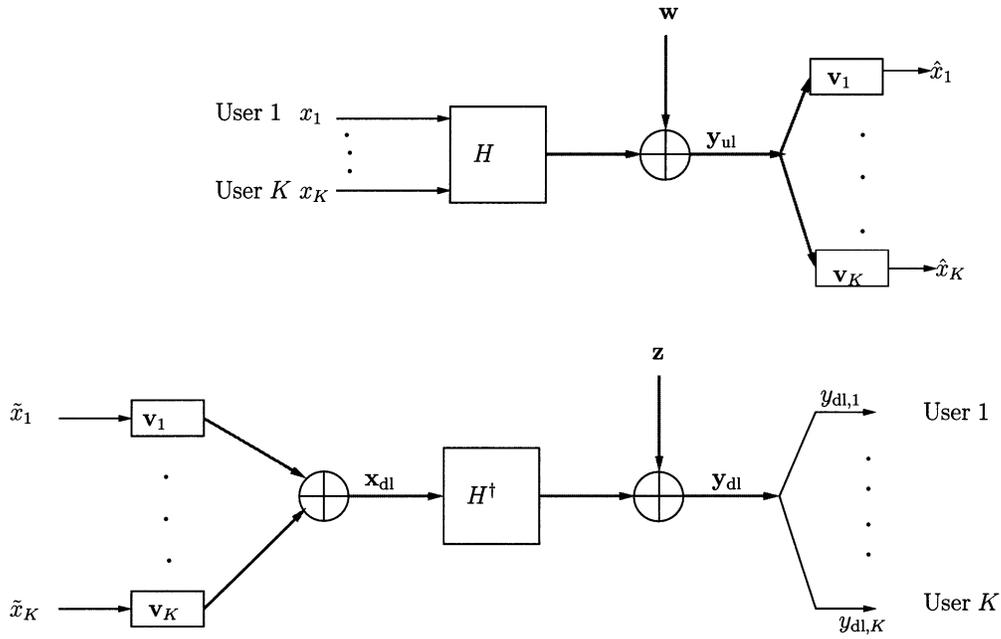
Fig. 3.   Uplink with linear receive beamforming and downlink with linear transmit beamforming.

- the sets of achievable SIRs are the same in both cases;

- to meet given SIR requirements, the *sum* of the powers of the users in the uplink is equal to the total transmit power in the downlink. This holds for all choice of the filters $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_K$.

In the uplink, it is easy to compute the optimal receive vectors and power allocation that minimizes the total power consumed. Given a set of powers of the users, the minimum mean-square error (MMSE) filter $\boldsymbol{v}_k$ should be used, since it maximizes the SIR for user k. The optimal allocation of powers can be obtained by a simple iterative algorithm that exploits the monotonicity of the problem [17]. A direct solution to the downlink is not as obvious. However, the equivalence derived above shows that the downlink can be solved by converting it to an uplink problem. The optimal transmit filters in the downlink are exactly the MMSE receive filters used in the uplink. This fact was first discovered in [10] and [19], but by showing the equivalence between the two optimal solutions rather than the equivalence between the performance for all choice of transmit/receive vectors, as is done here.

C. Costa Precoding and Successive Cancellation

There is an important difference between the point-to-point and multiuser scenarios: whereas linear transmit–receive strategies are capacity achieving for the point-to-point scenario, they are not for the multiuser uplink and downlink scenarios. Nevertheless, it turns out that a very similar equivalence holds even for nonlinear strategies that *are* capacity achieving for the multiuser scenarios.

We first focus on the uplink and order the users as $1, \ldots, K$. We retain the bank of linear receive filters $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_K$ but demodulate user symbols using successive cancellation (this is a multiuser receiver in contrast to the earlier single-user receiver

structure), canceling in the order $1, \ldots, K$. The SIR of user k with this receiver structure is

$$\text{SIR}_k = \frac{p_k \left| \boldsymbol{h}_k^\dagger \boldsymbol{v}_k \right|^2}{1 + \sum_{j>k} p_j \left| \boldsymbol{h}_j^\dagger \boldsymbol{v}_k \right|^2}$$

with the signals from users $1, \ldots, j-1$ are decoded and perfectly canceled. In the matrix notation of (4), we have

$$(I - \text{diag}\{a_1, \ldots, a_K\}\mathcal{U}(A))\boldsymbol{p} = \boldsymbol{a}. \qquad (12)$$

Here we have introduced the notation $\mathcal{U}(A)$ to indicate the upper triangular part of A, i.e., the (k, j)th entry of $\mathcal{U}(A)$ is A_{kj} if $j \geq k$ and 0 otherwise. For the specific choice of $\boldsymbol{v}_k$ as the MMSE receiver of user k treating signals from users $1, \ldots k-1$ to be nonexistent and treating signals from users $k+1, \ldots, K$ as noise, for every k, i.e., the unnormalized vector $\boldsymbol{v}_k$ has the expression

$$\boldsymbol{v}_k = \left(I + \sum_{j=k+1}^{K} p_j \boldsymbol{h}_j \boldsymbol{h}_j^\dagger \right)^{-1} \boldsymbol{h}_k \qquad (13)$$

we know that the sum capacity of the MAC is achieved [18] and

$$\sum_{k=1}^{K} \log\left(1 + \text{SIR}_k\right) = \log\det\left(I + H\text{diag}\{p_1, \ldots, p_K\}H^\dagger\right). \qquad (14)$$

In the broadcast channel, we retain the bank of linear filters but use a transmission strategy that codes for the users based on *known interference at the transmitter*. This strategy was proposed for the vector Gaussian broadcast channel in [2] and is built on a result of Costa [3]. Consider the scalar point-to-point channel

$$y = x + s + z$$

where s, z are independent Gaussian noise with s known to the transmitter but not to the receiver and z known to no one. Costa showed that the capacity of this channel is the same as that of the additive white Gaussian noise (AWGN) channel $y = x + z$, i.e., having the side information on s at the transmitter is as powerful as knowing s both at the transmitter *and* the receiver. This idea can be applied to the broadcast channel to improve the performance of linear transmit beamforming strategies. As before, we transmit $\boldsymbol{x}_{\mathrm{dl}} = \sum_{k=1}^{K} \tilde{x}_k \boldsymbol{v}_k$. The received signal at user k is

$$y_k = \tilde{x}_k \boldsymbol{h}_k^\dagger \boldsymbol{v_k} + \sum_{j>k} \tilde{x}_j \boldsymbol{h}_k^\dagger \boldsymbol{v_j} + \sum_{j<k} \tilde{x}_j \boldsymbol{h}_k^\dagger \boldsymbol{v_j} + \boldsymbol{h}_k^\dagger \boldsymbol{z}.$$

We use Gaussian independent inputs for $\tilde{x}_1, \ldots \tilde{x}_K$ with variances $q_1, \ldots, q_K$ and perform Costa precoding for each user k, encoding the information in $\tilde{x}_k$ treating the interference $\sum_{j>k} \tilde{x}_j \boldsymbol{h}_k^\dagger \boldsymbol{v_j}$ from users $k+1, \ldots, K$ as side information s known at the transmitter and the interference from the other users as Gaussian noise. Hence, we obtain the rates

$$R_k = \log\left(1 + \mathrm{SIR}_k\right) \tag{15}$$

where

$$\mathrm{SIR}_k = \frac{q_k \left|\boldsymbol{h}_k^\dagger \boldsymbol{v}_k\right|^2}{1 + \sum_{j=1}^{k-1} q_j \left|\boldsymbol{h}_k^\dagger \boldsymbol{v}_j\right|^2}$$

for each $k = 1, \ldots, K$. Note that user k now only sees interference from users $1, \ldots, k-1$, in contrast to the linear beamforming strategy where it sees interference from all other users.

In the matrix notation of (8), we can write this as

$$\left(I - \mathrm{diag}\{b_1, \ldots, b_K\}\mathcal{L}(A^t)\right)\boldsymbol{q} = \boldsymbol{b}. \tag{16}$$

Here we defined $\mathcal{L}(A)$ as a matrix whose (k, j)th component is equal to A_{kj} if $k \geq j$ and 0 otherwise. Observing that $\mathcal{L}(A^t) = (\mathcal{U}(A))^t$, we see that the performance of the uplink and downlink channels are equivalent (in terms of the SIRs achievable and the minimum transmit power required to achieve it) even with this extended set of transmission–reception strategies. Thus, for every transmit power vector $\boldsymbol{p}$ in the uplink and the choice of $\boldsymbol{v}$ as in (13), we have that there exists a downlink transmit power vector $\boldsymbol{q}$ with $\sum_k p_k = \sum_k q_k$ such that the sum of achievable rates can be written as, (from (15) and (14))

$$\sum_{k=1}^{K} R_i = \log \det\left(I + H \mathrm{diag}\{p_1, \ldots, p_K\}H^\dagger\right).$$

Since the nonnegative power vector $\boldsymbol{p}$ is only constrained by

$$\sum_k p_k = \sum_k q_k = \mathrm{tr}\left[\boldsymbol{x}\boldsymbol{x}^\dagger\right] \leq P$$

we have shown that the Costa achievable sum rate of the broadcast channel is equal to that of the MAC in (10) with a constraint

on the sum of the transmit powers of the users. This yields a lower bound to C_{sum}. Summarizing, we have the following.

Lemma 2: The maximum achievable sum rate R_{costa} is equal to the sum capacity C_{mac} of the uplink MAC under a total power constraint P on the users. More explicitly

$$R_{\mathrm{costa}} = C_{\mathrm{mac}} = \sup_{D \in \mathcal{A}_1} \log \det\left(I + HDH^\dagger\right) \tag{17}$$

where $\mathcal{A}_1$ is the set of K by K nonnegative diagonal matrices D with $\mathrm{Tr}[D] \leq P$.

Here the elements of D are the powers allocated to the users in the uplink MAC. We stated the correspondence between the uplink and the downlink in terms of the sum rate, but it is not too difficult to see that the *entire* Costa achievable rate region (i.e., all the rate vectors achievable by arbitrary choice of beamforming vectors, power allocation across users, and precoding orders) is, in fact, the same as the MAC capacity region. This is because of the one-to-one correspondence between the Costa precoding strategies in the downlink and the successive decoding strategies in the uplink, and the latter being sufficient to achieve any rate point in the MAC capacity region. (This result is independently arrived at in [20].)

Although we introduced the broadcast transmission scheme in terms of Costa precoding, it should be noted that the achievable rates are, in fact, a subset of the achievable region for general broadcast channel first proposed by Marton [9] (this strategy provides the largest known achievable region for a general broadcast channel).

Theorem 3 (Marton): Fix the joint distribution $p(u_1, \ldots, u_K, \boldsymbol{x})$ for some auxiliary random variables $u_1, \ldots, u_K$ (with no constraints on the cardinality of their alphabets) and $\boldsymbol{x}$ is a random variable on the input alphabets. The following is an achievable rate region:

$$C \stackrel{\mathrm{def}}{=} \left\{ \boldsymbol{R} : \begin{matrix} R_i \leq I(u_i; y_i) \\ \sum_{s \in S} R_s \leq \sum_{s \in S}(I(u_s; y_s) - H(u_s)) + H(u_s, s \in S) \end{matrix} \right\}. \tag{18}$$

Let $\tilde{x}_k \sim CN(0, q_k)$ and independent across k. If we let $\boldsymbol{x} = \sum_{k=1}^{K} \tilde{x}_k \boldsymbol{v}_k$ and

$$u_k \stackrel{\mathrm{def}}{=} \tilde{x}_k \boldsymbol{h}_k^\dagger \boldsymbol{v}_k + \frac{q_k \left|\boldsymbol{h}_k^\dagger \boldsymbol{v}_k\right|^2}{1 + \sum_{j=k}^{K} q_j \left|\boldsymbol{h}_k^\dagger \boldsymbol{v}_j\right|^2} \sum_{j=1}^{k-1} \tilde{x}_j \boldsymbol{h}_k^\dagger \boldsymbol{v}_j$$

then it can be seen by a direct calculation that the resulting rate point in the Marton's region is exactly the same as that achieved by Costa's precoding strategy defined above. The conceptual connection between Marton's and Costa's result is that they are both based on a random binning encoding technique. In fact, the connection between the broadcast channel problem and the problem of channel coding with side information at the transmitter has been known for some time [6].

III. CONVERSE

A. Sato Upper Bound

To show that the sum rate R_{costa} is the best that can be achieved by *any* strategy, we begin with an upper bound on the sum capacity of an arbitrary broadcast channel by Sato [12]. A cooperative upper bound to the sum capacity of the broadcast channel is the capacity of the point-to-point channel with all the receivers cooperating. Observe that while the capacity region of the broadcast channel depends only on the marginal distribution of the z_j's and not on the joint distribution, this is not the case for the capacity of the point-to-point channel. Hence the capacity $C_{\text{pp-dl}}(\Sigma_z)$ of the point-to-point channel generated from the downlink by cooperating receivers

$$\boldsymbol{y}_{\text{dl}} = H^\dagger \boldsymbol{x}_{\text{dl}} + \boldsymbol{z}, \qquad \boldsymbol{z} \sim \mathcal{CN}(0, \Sigma_z) \tag{19}$$

is an upper bound to the sum capacity C_{sum} of the broadcast channel (11) for any choice of Σ_z with the diagonal elements less than or equal to 1. Here

$$C_{\text{pp-dl}}(\Sigma_z) = \sup_{\mathbb{E}[\boldsymbol{x}^\dagger \boldsymbol{x}] \le P} I\left(\boldsymbol{x}; H^\dagger \boldsymbol{x} + \boldsymbol{z}\right).$$

We can now minimize over all such possible noise covariance matrices to obtain (and as derived explicitly in [2] for the case of $K = 2$)

$$C_{\text{sum}} \le \inf_{\Sigma_z \in \mathcal{A}_2} C_{\text{pp-dl}}\left(\Sigma_z\right) \tag{20}$$

where $\mathcal{A}_2$ is the set of positive semidefinite matrices with diagonal values less than or equal to 1.

In the next two subsections, we show the existence of Σ_z such that R_{costa} equals the upper bound in (20).

B. Point-to-Point Reciprocity

So far, we have considered three channels: 1) the original downlink broadcast channel; 2) the point-to-point channel in Sato bound, by having the receivers in the downlink cooperate; 3) the uplink MAC which is dual to the broadcast channel under Costa strategies. A natural way to connect the Costa lower bound with the Sato upper bound is to introduce a fourth channel, the reciprocal of the point-to-point channel in Sato bound, with the roles of the transmitters and receivers again reversed. This is a point-to-point channel with K transmitting antennas and N receiving antennas (generated from the uplink by a cooperation among the transmitters)

$$\boldsymbol{y}_{\text{ul}} = H \boldsymbol{x}_{\text{ul}} + \boldsymbol{w}, \qquad \boldsymbol{w} \sim \mathcal{CN}(0, I_N). \tag{21}$$

There is a quadratic cost function on the input: $\rho(\boldsymbol{x}_{\text{ul}}) := \boldsymbol{x}_{\text{ul}}^\dagger Q \boldsymbol{x}_{\text{ul}}$, where Q is Hermitian nonnegative definite, and a constraint that the average cost per unit time must be no greater than P. The capacity of this channel is

$$C_{\text{pp-ul}}(Q) = \sup_{\mathbb{E}[\boldsymbol{x}_{\text{ul}}^\dagger Q \boldsymbol{x}_{\text{ul}}] \le P} I(\boldsymbol{x}_{\text{ul}}; H \boldsymbol{x}_{\text{ul}} + \boldsymbol{z}). \tag{22}$$

The following result relates the capacities of the downlink point-to-point channel (in Sato bound) and this uplink point-to-point channel.

Lemma 4:

$$C_{\text{pp-dl}}(Q) = C_{\text{pp-ul}}(Q)$$

for all nonnegative-definite Hermitian matrix Q.

Proof: This result is a slight generalization of the point-to-point reciprocity described in Section II-A. If Q is invertible, then we can factorize $Q = Q^{1/2}(Q^{1/2})^\dagger$ and whiten the noise in the channel (19) and obtain the equivalent channel

$$\tilde{\boldsymbol{y}}_{\text{dl}} := Q^{-1/2} \boldsymbol{y}_{\text{dl}} = Q^{-1/2} H^\dagger \boldsymbol{x}_{\text{dl}} + \boldsymbol{w}$$

where $\boldsymbol{w} \sim \mathcal{CN}(0, I_K)$. From point-to-point reciprocity, the capacity $C_{\text{pp-dl}}(Q)$ of the channel (19) is the same as that of the channel

$$\boldsymbol{y} = H(Q^{-1/2})^\dagger \tilde{\boldsymbol{x}} + \boldsymbol{w}$$

with the constraint that $\mathbb{E}[\|\tilde{\boldsymbol{x}}\|^2] \le P$. With a change of variable $\boldsymbol{x} := (Q^{-1/2})^\dagger \tilde{\boldsymbol{x}}$, we get the equivalent channel

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{w}$$

with the constraint that $\mathbb{E}[\boldsymbol{x}^\dagger Q \boldsymbol{x}] \le P$. This is precisely the reciprocal channel (21). Hence, $C_{\text{pp-dl}}(Q) = C_{\text{pp-ul}}(Q)$.

Now suppose Q is not invertible, Consider first the case when $\text{Ker}\, Q$ is not perpendicular to $\text{Im}\, H^\dagger$. Let $\boldsymbol{v} \in \text{Im}\, H^\dagger$ which has a nonzero projection in $\text{Ker}\, Q$. Then by signaling along the direction $\boldsymbol{u}$, where $\boldsymbol{v} = H^\dagger \boldsymbol{u}$, then one can get infinite rate in the point-to-point channel (19), since one can just project the received signal onto $\text{Ker}\, Q$ where there would be no noise and nonzero signal power. Hence, $C_{\text{pp-dl}}(Q) = \infty$. Next pick a $\boldsymbol{v}'$ in $\text{Ker}\, Q$ which has a nonzero projection in $\text{Im}\, H^\dagger$. Since $\text{Im}\, H^\dagger$ is perpendicular to $\text{Ker}\, H$, $\boldsymbol{v}'$ is not in $\text{Ker}\, H$. Hence, by signaling along the direction $\boldsymbol{v}'$, we can use arbitrary high power and get arbitrarily high rate without incurring any cost in the reciprocal channel (21). Hence, $C_{\text{pp-u}}(Q) = \infty$ as well. $C_{\text{pp-dl}}(Q) = C_{\text{pp-ul}}(Q)$.

Now let us consider the case when $\text{Ker}\, Q$ is perpendicular to $\text{Im}\, H^\dagger$. Let

$$Q = \sum_{i=1}^r \sigma_i^2 \boldsymbol{v}_i \boldsymbol{v}_i^\dagger$$

where r is the rank of Q and the $\boldsymbol{v}_i$'s are the orthonormal eigenvectors corresponding to the nonzero eigenvalues σ_i^2 of Q. Let

$$V := [\boldsymbol{v}_1, \ldots \boldsymbol{v}_r] \cdot \text{diag}(1/\sigma_1, \ldots 1/\sigma_r).$$

Since $\text{Im}\, H^\dagger \subset \text{span}(\boldsymbol{v}_1, \ldots \boldsymbol{v}_r)$

$$\tilde{\boldsymbol{y}} = V^\dagger H^\dagger \boldsymbol{x} + \boldsymbol{w}'$$

is a sufficient statistic for the channel (19), where $\boldsymbol{w}' \sim \mathcal{CN}(0, I_r)$. By reciprocity, this channel has the same capacity as

$$\boldsymbol{y} = HV\tilde{\boldsymbol{x}} + \boldsymbol{w}$$

where $\mathbb{E}[\|\tilde{\boldsymbol{x}}\|^2] \le P$. Now if we define $\boldsymbol{x} = V\tilde{\boldsymbol{x}}$, then

$$\boldsymbol{x}^\dagger Q \boldsymbol{x} = \tilde{\boldsymbol{x}}^\dagger V^\dagger Q V \tilde{\boldsymbol{x}} = \|\tilde{\boldsymbol{x}}\|^2.$$

Hence in this case $C_{\text{pp-dl}}(Q) = C_{\text{pp-ul}}(Q)$ as well.
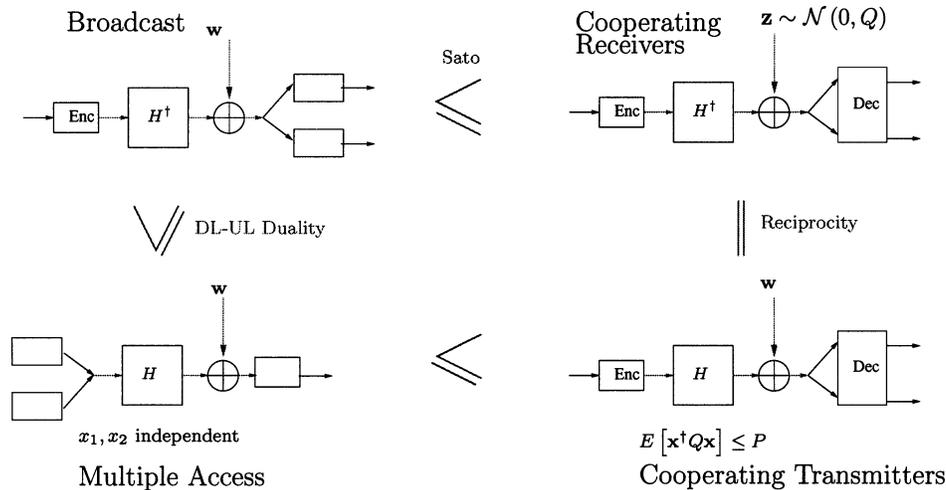
Fig. 4. The four channels, multiple access, broadcast, and their corresponding point-to-point channels, depicted along with the relationship between their capacities.

C. Finding the Right Cost Matrix

The relationship between the four channels are shown in Fig. 4. The uplink–downlink duality allows a correspondence between Costa strategies (a subset of all possible broadcast strategies) and the reciprocal uplink MAC. We need to show that the optimal Costa precoding strategy achieves the Sato upper bound. Using the uplink–downlink duality and the point-to-point reciprocity developed above, this is equivalent to showing that there exists a cost matrix Q such that the capacity of the reciprocal point-to-point channel equals the sum capacity of the MAC . There is actually a close connection between the two channels. We can rewrite the capacity (22) of the uplink point-to-point channel as

$$C_{\text{pp-ul}}(Q) = \sup_{\Sigma \in \mathcal{A}_3} \log \det \left(I + H\Sigma H^{\dagger} \right) \qquad (23)$$

where $\mathcal{A}_3$ is the set of K by K positive semidefinite matrices Σ with $\text{Tr}[Q\Sigma] \leq P$. Comparing this to the expression (17) for the sum capacity of the MAC , we see that in the reciprocal point-to-point channel, the users are allowed to cooperate (i.e., D can be nondiagonal) but a price has to be paid (as dictated by the cost matrix Q). If $Q_{ii} \leq 1$ for all i, we can see that any feasible (noncooperating) input D for the MAC is also a valid input for the point-to-point channel. Hence, $C_{\text{mac}} \leq C_{\text{pp-ul}}(Q)$. We now find a Q^* with diagonal elements ≤ 1 such that $C_{\text{mac}} = C_{\text{pp-ul}}(Q^*)$, i.e., a Q^* such that the input D^* that is optimal for the MAC is also optimal for the uplink point-to-point channel.

To this end, define the Lagrangian for the constrained optimization problem (23)

$$L(\Sigma, \lambda) \overset{\text{def}}{=} \log \det \left(I + H\Sigma H^{\dagger} \right) - \lambda \left(\text{Tr}[Q\Sigma] - P \right)$$

where λ is the Lagrange multiplier for the cost constraint $\text{Tr}[Q\Sigma] \leq P$. A *sufficient* Kuhn–Tucker condition for $\Sigma = D^*$ to be optimal for problem (23) is that

$$\text{Tr}[QD^*] = P \qquad (24)$$

$$\nabla_{\Sigma} L(\Sigma, \lambda)|_{\Sigma = D^*} = 0, \qquad \text{for some } \lambda > 0. \qquad (25)$$

By directly computing the gradient, the second condition (25) can be rewritten as

$$H^{\dagger}(I + HD^*H^{\dagger})^{-1}H - \lambda Q = 0, \qquad \text{for some } \lambda > 0. \quad (26)$$

Hence, if we set

$$Q^* = \frac{1}{\lambda} H^{\dagger} \left(I + HD^*H^{\dagger} \right)^{-1} H \qquad (27)$$

condition (26) is satisfied. To show that there is a λ such that condition (24) is satisfied as well, we observe that since $D^* = \text{diag}(d_1^*, \ldots, d_K^*)$ solves the optimization problem (17) for the MAC channel, it satisfies the Kuhn–Tucker conditions, which can be easily derived: for all $i = 1, \ldots, K$

$$h_i^{\dagger}(I + HD^*H^{\dagger})^{-1}h_i = \lambda^*, \qquad \text{if } d_i^* > 0$$

$$h_i^{\dagger}(I + HD^*H^{\dagger})^{-1}h_i \leq \lambda^*, \qquad \text{if } d_i^* = 0$$

for some $\lambda^* > 0$. Thus, if we now set $\lambda = \lambda^*$ in (27), then the diagonal entries of Q^* are equal to or less than 1 and

$$\text{Tr}[Q^*D^*] = \sum_{i=1}^{K} Q_{ii}^* d_i^* = \sum_{i:d_i^* > 0} Q_{ii}^* d_i^* = \sum_{i:d_i^* > 0} d_i^* = P$$

i.e., condition (24) is satisfied as well. Hence,

$$C_{\text{pp-ul}}(Q^*) = C_{\text{mac}} \quad \text{and} \quad Q_{ii} \leq 1, \qquad \forall i = 1 \cdots K.$$

The choice of the cost matrix $Q = Q^*$ gives no incentive for the users to cooperate in the uplink, even if they could.

Returning to Fig. 4, this establishes the existence of a cost matrix Q such that the sum capacity of the MAC equals the capacity of the reciprocal point-to-point channel. Through the uplink–downlink duality and the point-to-point reciprocity, this, in turn, implies that the optimal Costa precoding strategy achieves the Sato bound. The proof of Theorem 1 is now complete.

Essentially, what we constructed is a cost function such that the optimal input for the point-to-point channel is the desired one (the optimal noncooperating input for the MAC.) Interestingly, a similar line of thinking is useful in the seemingly unrelated problem of optimality of uncoded transmission [5].

D. Convex Duality Interpretation

Should one be surprised by the existence of such a Q^* which leads to the desirable state of affairs? To get more insight, let us prove the result in a slightly different and more abstract way. Define

$$f(\Sigma) := \log \det(I + H\Sigma H^\dagger).$$

The MAC sum capacity optimization problem can be written as

$$C_{\text{mac}} = \max_{\Sigma} f(\Sigma)$$

subject to

$$\text{Tr}\,[\Sigma] \le P,\ \Sigma_{ii} \ge 0 \text{ for all } i,\ \Sigma_{ij} = 0 \text{ for all } i \ne j. \quad (28)$$

Introducing Lagrange multipliers λ, λ_{ij}, the convex dual of this problem is

$$C_{\text{mac}} = \min_{\lambda > 0, \lambda_{ii} > 0, \lambda_{ij}} \max_{\Sigma}$$

$$\left[f(\Sigma) - \lambda(\text{Tr}\,[\Sigma] - P) + \sum_i \lambda_{ii} D_{ii} + \sum_{i \ne j} \lambda_{ij}\Sigma_{ij} \right].$$

If we define a K by K matrix Q with $Q_{ii} := 1 - \lambda_{ii}/\lambda$ and $Q_{ij} = \lambda_{ij}/\lambda$, then we can rewrite the above problem as

$$C_{\text{mac}} = \min_{Q: Q_{ii} \le 1} \min_{\lambda > 0} \max_{\Sigma} \{f(\Sigma) - \lambda[\text{Tr}\,[Q\Sigma] - P]\}. \quad (29)$$

We can introduce an additional positive-semidefinite matrix $\Lambda \succeq 0$ and it is easy to see that the following is an equivalent form:

$$C_{\text{mac}} = \min_{Q: Q_{ii} \le 1} \min_{\lambda > 0, \Lambda \succeq 0} \max_{\Sigma}$$

$$\cdot \{f(\Sigma) - \lambda[\text{Tr}\,[Q\Sigma] - P] + \text{Tr}\,[\Lambda\Sigma]\}. \quad (30)$$

By convex duality theory with positive-semidefinite constraints ([1, Sec. 4.8] is the appropriate modern reference while [11, Theorem 28.4] is the classical reference on the topic)

$$\min_{\lambda > 0, \Lambda \succeq 0} \max_{\Sigma} \{f(\Sigma) - \lambda[\text{Tr}\,[Q\Sigma] - P] + \text{Tr}\,[\Lambda\Sigma]\}$$

$$= \max_{\Sigma \succeq 0, \text{Tr}[Q\Sigma] \le P} f(\Sigma).$$

Substituting into (30)

$$C_{\text{mac}} = \min_{Q: Q_{ii} \le 1} \max_{\Sigma \succeq 0, \text{Tr}[Q\Sigma] \le P} f(\Sigma) = \min_{Q: Q_{ii} \le 1} C_{\text{pp-ul}}(Q).$$

We conclude that the Sato bound (in the form of the reciprocal channel) is essentially the convex dual of the MAC sum capacity maximization problem, the only difference being that in the Sato's bound, the minimization is over only positive-semidefinite matrices Q while in the convex dual the minimization is over *all* Q. However, since in the latter problem the saddle point (Q^*, Σ^*) must satisfy

$$\nabla_\Sigma f(\Sigma)|_{\Sigma=\Sigma^*} = \lambda Q^*$$

for some $\lambda > 0$, where $\nabla_\Sigma f(\Sigma) = H^\dagger(I + H\Sigma H^\dagger)^{-1}H$, it follows that Q^* must also be positive semidefinite. Thus, constraining the minimization to positive-semidefinite matrices in the convex dual problem does not affect its value.

This identification gives a Lagrangian interpretation to the matrix Q^*: it forms the (scaled) Kuhn–Tucker coefficients associated with the constraints of the MAC (namely, independent inputs and an overall power constraint).

IV. GENERALIZATION TO MULTIPLE RECEIVE ANTENNAS

So far we have considered only single receive antenna at each user. Consider now M_k multiple receive antennas at user k. The received signal at user k is now a vector

$$\boldsymbol{y}_{\text{dl},\,k} = H_k^\dagger \boldsymbol{x}_{\text{dl}} + \boldsymbol{z}_k \quad (31)$$

with the entries of the $N \times M_k$ matrix H_k representing the channel from the N transmit antennas to the M_k receive antennas. Without loss of generality let $\boldsymbol{z}_k \sim \mathcal{CN}(0, I_{M_k})$. So far we have discussed the situation of $M_k = 1$ for all users k. Following this discussion, a natural extension of our main result, Theorem 1, is the following.

Theorem 5: The sum capacity of the broadcast channel in (31) with an overall transmit power constraint of P is

$$C_{\text{sum}} = \max_{\Sigma_k \succeq 0,\, k=1,\dots,K,\, \sum_{k=1}^K \text{Tr}[\Sigma_k] \le P}$$

$$\cdot \log \det \left(I + \sum_{k=1}^K H_k \Sigma_k H_k^\dagger \right). \quad (32)$$

In complete analogy with the single receive antenna discussion, the expression in (32) is the sum capacity of the reciprocal uplink MAC with multiple *transmit* antennas M_k at user k

$$\boldsymbol{y}_{\text{ul}} = \sum_{k=1}^K H_k \boldsymbol{x}_{\text{ul},\,k} + \boldsymbol{w}. \quad (33)$$

We first consider the forward part of Theorem 5, i.e., we want to show that the largest sum rate with Costa strategies is equal to the sum capacity of (33). We will use the uplink–downlink duality developed in Section II-B. First, each user k now transmits its data on M_k separate streams. Each data stream is communicated using linear transmit and receive filters (Fig. 1). The filters $\boldsymbol{u}_{k,1}, \dots \boldsymbol{u}_{k,M_k}$ corresponding to the M_k data streams of user k all have the property that they have nonzero components only in the positions $\sum_{i=1}^{k-1} M_i + 1$ to $\sum_{i=1}^k M_i$. This is analogous to the only nonzero entry in the kth position in our earlier discussion (Section II-B). These filters serve as transmit filters in the reciprocal MAC (and the specification of the nonzero entries indicates which of the antennas can be used jointly to send the stream) and as receive filters in the broadcast channel (and here the specification of the nonzero entries indicates which of the antennas can be used jointly to receive the stream). Since these linear strategies combined with successive cancellation achieve the capacity region of the MAC (in (33)) we can conclude by the uplink–downlink duality that the maximum sum rate using Costa coding strategies in the broadcast channel (of (31)) is equal to the sum capacity of the MAC. This shows the forward part of Theorem 5.

The converse is similar to the approach in Section III. The Sato bound lets all the users cooperate thus creating a giant point-to-point MAC (with $\sum_{k=1}^K M_k$ number of receive antennas). We want to correlate the noises across the antennas of the users so that the benefit of this cooperation is minimized. An explicit construction of such a correlation structure with which there is no benefit to cooperation, analogous to the Q^* in Section III-D, is done through a convex duality argument: the optimal correlation structure forms the Kuhn–Tucker coefficients
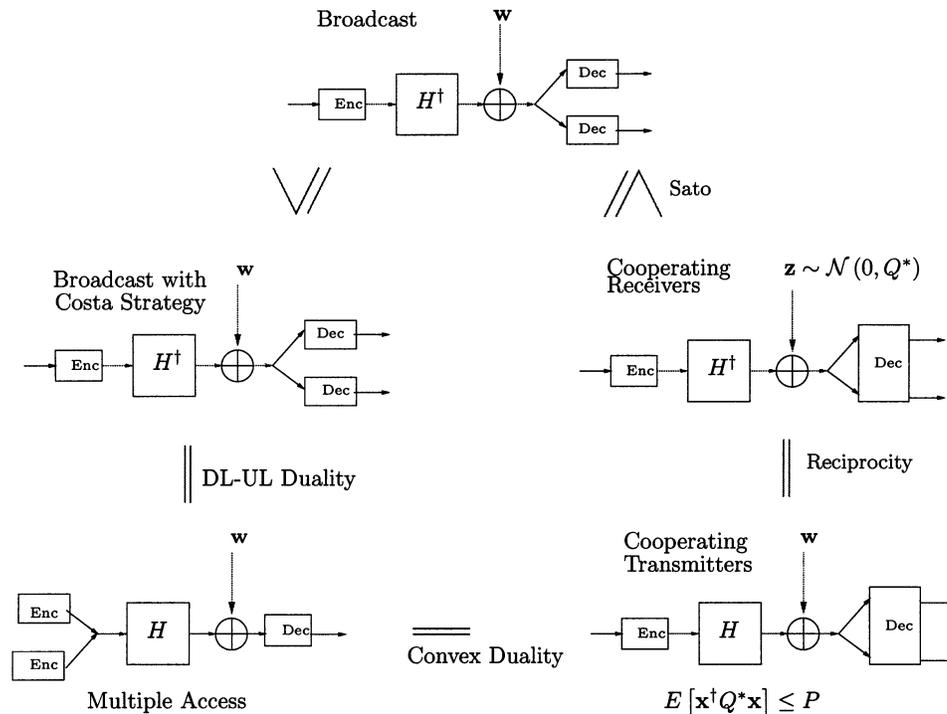
Fig. 5. The four channels, multiple access, broadcast with Costa inputs, and their corresponding point-to-point channels. Costa strategies are a subset of general strategies on the broadcast channel. The capacity of the point-to-point channels under the minimizing Q^* equals the sum capacities of the multiple access and broadcast channels.

associated with the constraints in the reciprocal MAC (in (33)) (namely, independent vector Gaussian inputs and a total power constraint).

Both the forward part and the converse for multiple-receive antennas are carefully carried out in [16], which also derives other results that shed insight into the entire capacity region of the vector Gaussian broadcast channel.

V. CONCLUSION

In this paper, we computed the sum capacity of the vector Gaussian broadcast channel. The central problem is to show that the maximum achievable rate by Costa's strategy is equal to the capacity of the point-to-point channel where receivers cooperate, for a suitable choice of the noise covariance Q. We solve this problem in three steps.

1) We showed that the achievable region by Costa's strategy equals the multiple-access capacity region with transmitters and receivers reversed.

2) We showed that the capacity of the downlink point-to-point channel with noise covariance Q and receivers cooperating equals the capacity of the uplink point-to-point channel with transmitters cooperating and a quadratic cost function $\boldsymbol{x}^\dagger Q \boldsymbol{x}$.

3) We showed that there exists a cost matrix Q such that the uplink point-to-point capacity equals the sum capacity of the MAC. The cost matrix Q has the interpretation of a Lagrangian price to force users not to cooperate in the uplink.

The proof is summarized in Fig. 5.

An independent and different proof is given in [22], where the authors directly worked with the downlink channel and showed the existence of a noise covariance Σ_z such that cooperating among the receivers did not provide any additional benefit.

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