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**SUMMABILITY AND FOURIER ANALYSIS**

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An integration on  $\beta N$ , the Stone-Cech compactification of the natural numbers  $N$ , is defined such that if  $s$  is a bounded sequence and  $\phi$  is a summation method evaluating  $s$  to  $\sigma$ ,  $\int s d\phi = \sigma$ . The Fourier transform  $\hat{\phi}$  of a summation method  $\phi$  is defined as a linear functional on a space of test functions analytic in the unit disc: if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, \quad |z| < 1, \quad \text{then } \phi(f) = \int \hat{f}(n)d\phi.$$

A functional which agrees with the Fourier transform of a regular summation method must annihilate the Hardy space  $H_1$ . Our space of test functions is often the space  $M_p$  of functions  $f = \sum \hat{f}(n)z^n$ , analytic in the unit disc, such that

$$\|f\|_{M_p} = \limsup [(1-r) \int_0^{2\pi} |f(re^{i\theta})|^p d\theta / 2\pi]^{1/p}$$

is finite for some  $p > 1$ . A functional  $L$  which is well defined on a space  $M_p$  for some  $p \geq 2$  such that  $L(1/(1-z)) = 1$  agrees with the Fourier transform of a summation method which is slightly stronger than convergence.

Let  $s = \{s_n\}$  be an infinite sequence of complex numbers, that is, a continuous function on the discrete additive semigroup of natural numbers  $N$ . The sequence  $s$  has a continuous extension  $s^\beta$  to  $\beta N$ , the Stone-Cech compactification of  $N$  ( $s^\beta$  takes the value  $\infty$  if  $s$  is unbounded). Throughout the paper, the symbol  $\beta Z$  denotes the Stone-Cech compactification of the space  $Z$ , and the continuous extension of a function  $f$  defined on  $Z$  to  $\beta Z$  will be denoted by  $f^\beta$ ; for a description of the Stone-Cech compactification we refer the reader to [2, pp. 82-93]. We impose the norm

$$\begin{aligned} \|s\| &= \limsup |s_n| \\ &= LUB |s^\beta(\gamma)|, \quad \gamma \in \beta N - N \end{aligned}$$

on the space  $m_0$  of bounded sequences. Thus  $m_0$  is isometric to  $C(\beta N - N)$ , the ring of continuous complex functions on  $\beta N - N$ ; sequences differing by a null sequence are identified in  $m_0$ .

Let  $\phi$  denote a summation method—that is, a linear functional on a subspace of  $m_0$ . We assume that the  $\phi$ -transform of every sequence  $s$  to which  $\phi$  is applicable is either a continuous function on  $N$  or else a continuous function on the half open unit interval  $I: [0, 1)$ .

For example, if  $\phi$  is representable by a summation matrix  $A = (a_{nk})$ , then the  $\phi$ -transform of a sequence  $s$  is the sequence  $t$  given by

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k \quad n = 0, 1, \dots,$$

which is continuous function on  $N$ ; if  $\phi$  is the Abel method  $\mathcal{A}$ , then the  $\phi$  transform of  $s$  is the continuous function on  $I$  given by

$$t(r) = (1 - r) \sum_{n=0}^{\infty} s_n r^n \quad 0 \leq r < 1.$$

If  $\phi$  is a regular and nonnegative summation method, then  $\underline{\phi}$  is a functional of norm one on a closed subspace of  $m_0$ . Moreover if we denote the  $\underline{\phi}$ -transform of  $s$  by  $t$  then  $\limsup |t|$  is a semi-norm on  $m_0$ . Thus by the Hahn Banach theorem, the linear functional  $\underline{\phi}$  may be extended to a nonnegative linear functional on  $m_0$  which satisfies

$$(1) \quad |\underline{\phi}(s)| \leq \limsup |t|,$$

for each bounded sequence  $s$ ; we shall denote this extension of  $\phi$  also by  $\underline{\phi}$ ; *throughout the paper we will assume that  $\phi$  has been extended to  $m_0$  in such a way that (1) is fulfilled.* Such an extension is never unique, and the results to be described hold for each such extension  $\underline{\phi}$ :

As a linear functional on  $m_0$ ,  $\underline{\phi}$  gives rise to a nonnegative measure on  $\beta N$  which we also denote by  $\underline{\phi}$ . Since  $\underline{\phi}$  is a regular summation method, the measure  $\underline{\phi}$  is concentrated on  $\beta N - N$  - we have  $\int_{\beta N} d\underline{\phi} = 1$ . We shall write  $\int s d\underline{\phi}$  for  $\int s^{[\beta]} d\underline{\phi}$ .

Using (1) we can show

**REMARK.** *If  $s$  is a bounded sequence and  $\underline{\phi}$  is a regular non-negative summation method which is representable by either a summation matrix or a sequence-to-function transformation, then*

$$\liminf t \leq \int_{\beta N} s d\underline{\phi} \leq \limsup t,$$

where  $t$  denotes the  $\underline{\phi}$ -transform of  $s$ .

The Abel summation method  $\mathcal{A}$  induces translation-invariant measures on  $\beta N$ . This summation method will play a vital role in our discussion of Fourier transforms of sequences.

1.  $L^p$  Spaces. If  $p \geq 1$  and  $\phi$  is a regular summation method which is representable either by a summation matrix or by a sequence-

to-function transformation, we define  $L^p(\phi)$  as the space of sequences  $s$  with the property that for each  $\varepsilon > 0$  there is a bounded sequence  $s^{(\varepsilon)}$  such that the sequence  $|s - s^{(\varepsilon)}|^p$  has a  $\phi$  transform whose limit superior is bounded in absolute value by  $\varepsilon$ ; this definition is more restrictive than the usual definition of  $L^p$  spaces. If  $s$  is a sequence in an  $L^p$  space we define

$$\int_{\beta.N} s d\phi = \lim_{\varepsilon \rightarrow 0} \int_{\beta.N} s^{(\varepsilon)} d\phi ,$$

where  $\{s^{(\varepsilon)}\}$  is a set of bounded sequences which approximate  $s$  in the sense that for each  $\varepsilon > 0$ , there is a bounded sequence  $s^{(\varepsilon)}$  such that the limit superior of the  $\phi$ -transform of  $|s - s^{(\varepsilon)}|^p$  is less than  $\varepsilon$  in absolute value. We norm  $L^p$  by:

$$\|s\|_p = \left( \int |s|^p d\phi \right)^{1/p} = \lim_{\varepsilon \rightarrow 0} \left[ \int |s|^{(\varepsilon)p} d\phi \right]^{1/p} .$$

(Clearly the limit is independent of the choice of  $\{s^{(\varepsilon)}\}$ ).

By Holder's inequality we have that for  $1 \leq q \leq p$ ,  $L^p(\phi) \subseteq L^q(\phi)$ , and moreover  $\|s\|_q \leq \|s\|_p$ .

As usual we identify two sequences  $s$  and  $t$  in  $L^p(\phi)$  if

$$\|s - t\|_p = 0 .$$

**THEOREM.** *Let  $\phi$  be a regular nonnegative summation method and let  $s$  be a sequence in  $L^p(\phi)$ ,  $p \geq 1$ . Let  $t$  denote the  $\phi$ -transform of  $|s|^p$ . Then*

$$\liminf t \leq \int |s|^p d\phi \leq \limsup t < \infty .$$

In particular if  $\phi$  evaluates the sequence  $|s_n|^p$  to  $\sigma$ , then

$$\int |s|^p d\phi = \sigma .$$

*Proof.* We deal only with the case where  $\phi$  is represented by a summation matrix  $A = (a_{nk})$  — the case where  $\phi$  is representable by a sequence-to-function may be dealt with in a similar fashion. Let  $s^{(\varepsilon)}$  be a set of bounded sequences approximating  $s$ , that is, for each  $\varepsilon > 0$  there is a bounded sequence  $s^{(\varepsilon)}$  such that

$$\limsup \sum_{k=0}^{\infty} a_{nk} |s_k - s_k^{(\varepsilon)}|^p \leq \varepsilon .$$

If we take  $\varepsilon = 1$ ,

$$\begin{aligned}
& \limsup \sum_{k=0}^{\infty} a_{nk} |s_k|^p \\
& \leq 2^p \left[ \limsup \sum_{k=0}^{\infty} a_{nk} |s_k^{(\varepsilon)}|^p \right. \\
& \quad \left. + \limsup \sum_{k=0}^{\infty} a_{nk} |s_k - s_k^{(s)}|^p \right] \\
& \leq 2^p [\limsup \sum a_{nk} |s_k^{(\varepsilon)}|^p + 1] .
\end{aligned}$$

Hence  $\limsup |t|$  is finite.

Also

$$\int |s|^p dA = \lim_{\varepsilon \rightarrow 0} \int |s^{(\varepsilon)}|^p dA .$$

Since each  $s^{(\varepsilon)}$  is a bounded sequence,

$$\begin{aligned}
\liminf t_n & \leq \liminf \sum a_{nk} |s_k^{(\varepsilon)}|^p + C_1 \varepsilon^{1/p} \\
& \leq \int |s^{(\varepsilon)}|^p dA + C_1 \varepsilon^{1/p} \\
& \leq \limsup \sum_{k=0}^{\infty} a_{nk} |s_k^{(\varepsilon)}|^p + C_1 \varepsilon^{1/p} \\
& \leq \limsup t_p + C_2 \varepsilon^{1/p} ,
\end{aligned}$$

where  $C_1$  and  $C_2$  are numbers not depending on  $\varepsilon$ . If we let  $\varepsilon$  tend to zero we have the theorem.

Holder's inequality together with the technique of the above proof may be used to yield:

**THEOREM.** *Let  $\phi$  be a regular nonnegative summation method and let  $s$  be a sequence in  $L^p(\phi)$   $p \geq 1$ . If  $t$  denotes the  $\phi$ -transform of  $s$ , then*

$$\liminf t \leq \int s d\phi \leq \limsup t .$$

*In particular if  $\phi$  evaluates  $s$  to  $\sigma$ , then  $\int_{\hat{\rho}, \lambda} s d\phi = \sigma$ .*

**2. Fourier transforms.** The Fourier transform  $\hat{\phi}$  of a summation method  $\phi$  is defined as a functional on a space  $M$  of test functions  $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$  analytic in the unit disc  $D: |z| < 1$ , given by

$$\begin{aligned}
\hat{\phi}(f) & = \int_{\hat{\rho}, \lambda} (\hat{f}(n))_{\hat{\rho}} d\hat{\phi} \\
& = \int_{\hat{\rho}, \lambda} \hat{f}(n) d\hat{\phi} ;
\end{aligned}$$

the Fourier transform  $\hat{s}$  of a sequence  $s = \{s_n\}$  is defined as the linear

functional on  $M$  given by

$$\begin{aligned} \hat{s}(f) &= \int_{\beta N} s_n^{\beta} (\hat{f}(n))^{\beta} d.\underline{\mathcal{A}} , \\ &= \int s_n \hat{f}(n) d.\underline{\mathcal{A}} , \quad f \in M , \end{aligned}$$

where  $\underline{\mathcal{A}}$  is any measure on  $\beta N - N$  induced by the Abel method.

The more customary definition of the Fourier transform, namely as the function of  $[0, 2\pi]$  given by

$$\int_N \exp(-i n \alpha) s_n d.\underline{\mathcal{A}} , \quad 0 \leq \alpha < 2\pi ,$$

is insufficient; S. P. Lloyd has given examples of sequences  $s$  such that  $|s_n| = 1$  for all  $\alpha$  and such that  $\int_N \exp(-i n \alpha) s_n d.\underline{\mathcal{A}}$  vanishes for all  $\alpha$  cf [6]. Later we shall make some remarks about sequences  $s$  which may be written

$$s_k = \sum_n a_n \exp(i \alpha_n k) ,$$

where the Fourier coefficients  $a_n$  are given by the formulas

$$a_n = \int_{\beta N} s_k \exp(-i \alpha_n k) d.\underline{\mathcal{A}} ,$$

(that is, the sequence  $s_k \exp(i \alpha k)$  is Abel summable for all  $\alpha$ ), where each  $\alpha_n$  is a number in  $[0, 2\pi)$ .

By  $H_p$ ,  $p \geq 1$  we understand the Hardy space of functions  $f$  analytic in  $D: |z| < 1$  such that  $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$  is bounded for  $0 \leq r < 1$  [cf. 5 pp. 39].

**THEOREM.** *If  $L$  is a linear functional on a space of functions analytic in  $D$  which agrees with the Fourier transform  $\hat{\phi}$  of a regular summation method  $\phi$ , then*

$$(1) \quad L(f) = 0$$

for each  $f \in M$  which is also in  $H_1$ ; also

$$(3) \quad L(1/(1 - z)) = 1 .$$

*Proof.* If  $f \in H_1$  then  $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ ,  $|z| < 1$ , and  $\{\hat{f}(n)\}$  is a null sequence [cf. 5 pp. 70]. Since  $\phi$  is a regular method,  $\phi$  must evaluate  $\{\hat{f}(n)\}$  to zero. Hence  $\hat{\phi}(f) = 0$  for each  $f \in H_1 \cap M$ . To establish (3) we simply note that since  $\phi$  is regular, it must evaluate the sequence  $\{1, 1, \dots\}$  to one, that is  $\hat{\phi}(1/(1 - z)) = 1$ .

Our spaces of test functions will be

(a) the space  $M_p$ ,  $p > 1$ , of functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

analytic in  $D$ , such that

$$\|f\|_{M_p} = \limsup_{r \rightarrow 1^-} (1-r)^{1/p'} \left[ \int_0^{2\pi} |f(r^{1/p'} \exp i\theta)|^p d\theta / 2\pi \right]^{1/p}$$

is finite through the paper the symbol  $p'$  denotes the number  $p/(p-1)$ :  
Two functions  $f, g$  are identified in  $M_p$  in case

$$(1-r)^{p/p'} \int_0^{2\pi} |f(r^{1/p'} \exp i\theta) - g(r^{1/p'} \exp i\theta)|^p d\theta$$

tends to zero as  $r$  tends to one. We norm each space  $M_p$  by  $\|\cdot\|_{M_p}$ ,

(b) the space of functions

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$$

such that

$$\|f\|_{M_\infty} = \limsup_{r \rightarrow 1^-} (1-r) |f(r \exp i\theta)|$$

is finite. We identify two functions  $f$  and  $g$  in  $M_\infty$  in case

$$(1-r) |f(r \exp i\theta) - g(r \exp i\theta)|$$

tends to zero as  $r$  tends to 1. We norm  $M_\infty$  by  $\|\cdot\|_{M_\infty}$ . For  $1 < p < q < \infty$  we have  $M_p \subseteq M_q$  of [3 pp. 623-625].

A linear functional  $L$  on a normed space  $M$  will be said to be well-defined if  $L(f) = L(g)$  whenever  $\|f - g\| = 0$ ,  $f, g \in M$ .

For  $p > 0$  a sequence  $s$  will be said to be strongly Abel- $p$ -summable to  $\sigma$  if

$$\lim_{r \rightarrow 1} (1-r) \sum_{n=0}^{\infty} |s_n - \sigma| r^n = 0.$$

The method of strong Abel- $p$ -summability is regular for  $p > 0$ .

**THEOREM.** *If  $2 \leq p < \infty$ , and  $L$  is a well-defined linear functional on  $M_p$  such that*

$$(4) \quad L(1/1-z) = 1,$$

*then there is a summation method  $\phi$  which includes strong Abel- $p$ -summability such that*

$$\hat{\phi}(f) = L(f) \quad f \in M_p.$$

*Proof.* We define a summation method  $\phi$  by  $\int_{\beta_N} sd\phi = L(S)$ , where  $S(z) = \sum_{n=0}^{\infty} s_n z^n$ , whenever the right hand is defined. If  $f \in M_p$ , then  $L(f)$  is defined and  $\hat{\phi}(f) = \int_N \hat{f}(n)d\phi = L(f)$ . Now let  $\{s_n\}$  be strongly Abel- $p'$ -summable to  $\sigma$ . Then  $(1-r) \sum |s_n - \sigma|^{p'} r^n \rightarrow 0$ . Since  $\sum (s_n - \sigma)z^n = S(z) - \sigma/(1-z)$  we have, by the Hausdorff-Young theorem cf [7, pp. 145],  $(1-r) \int_0^{2\pi} |S(r^{1/p'} e^{i\theta}) - \sigma/(1-r^{1/p'} e^{i\theta})|^p d\theta \rightarrow 0$ ; thus  $\|S - \sigma/(1-z)\|_{M_p} = 0$ . Since  $L$  is well defined,

$$L(S) = \sigma L(1/(1-z)) = \sigma$$

by (4). Hence  $\int_N sd\phi = \sigma$ , that is, the method  $\phi$  includes strong-Abel- $p'$ -summability.

Similarly

**THEOREM.** *If  $L$  is a well defined linear functional on  $M_{\infty}$  which satisfies (4), then there is a summation  $\phi$  which includes strong-Abel-1-summability such that  $\hat{\phi}(f) = L(f)$ ,  $f \in M_{\infty}$ .*

If a summation matrix  $A = (a_{n,k})$  has a sizable convergence field, then  $\lim_{n \rightarrow \infty} \max_k |a_{n,k}| = 0$ ; for example this condition must be satisfied if  $A$  has the Borel property (cf [3]).

We denote by  $\hat{A}$  the the Fourier transform of the summation method represented by the matrix  $A$ .

**THEOREM.** *If  $A = (a_{n,k})$  is a non-negative regular row-finite summation matrix such that  $\lim_{n \rightarrow \infty} \text{l.u.b.}_k |a_{n,k}| = 0$ ,  $a_{n0} \geq a_{n1} \geq a_{n2} \dots$ , then  $\hat{A}(1/(1-ze^{i\alpha})) = 1$  or  $0$  according as  $\alpha$  is or is not congruent to zero modulo  $2\pi$ .*

*Proof.* We have  $1/(1-ze^{i\alpha}) = \sum_{n=0}^{\infty} e^{in\alpha} z^n$ . If  $\alpha \equiv 0 \pmod{2\pi}$ , then  $\hat{A}(1/(1-ze^{i\alpha})) = 1$  by the regularity of  $A$ . If  $\alpha \not\equiv 0 \pmod{2\pi}$ , then since the sequence  $\{a_{n,k}\}$  is nonincreasing in  $k$ ,

$$\left| \sum_{k=0}^{\infty} a_{n,k} e^{ik\alpha} \right| \leq 8a_{n0}/\eta$$

where  $\eta$  is the distance of the point  $\alpha$  from the multiples of  $2\pi$ . Thus  $A$  evaluates to zero each sequence  $\{e^{in\alpha}\}$  such that  $\alpha$  is not a multiple of  $2\pi$ , that is,  $\hat{A}(1/(1-ze^{i\alpha})) = 0$  if  $\alpha \not\equiv 0 \pmod{2\pi}$ .



**THEOREM.** Let  $P$  denote the Norlund summation method, so that the  $P$ -transform of a sequence  $s$  is the sequence  $\{\sum_{k=0}^{\infty} p_{n-k}s_k/P_n\}$ , where the numbers  $p_n, P_n$  satisfy the conditions

$$P_n = \sum_{k=0}^n p_k, \quad p_k = o(1), \quad P_n \rightarrow \infty.$$

Then for almost all  $\alpha$  in  $[0, 2\pi)$

$$\hat{P}(1/1 - z \exp i\alpha) = 0.$$

This result is proved in [1, pp. 325-326].

**THEOREM.** If  $s$  is a sequence in  $L^p(\mathcal{A})$ ,  $1 < p \leq 2$ , then  $\hat{s}$  is a bounded functional on  $M_p$ , and

$$\|\hat{s}\|^p \leq \limsup (1-r) \sum_{n=0}^{\infty} |s_n|^p r^n,$$

*Proof.* If  $p \leq 2$ , then by the Hausdorff-Young theorem

$$\begin{aligned} & \left( \sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^n \right)^{1/p'} \\ & \leq \left[ \int_0^{2\pi} |f(r^{1/p'} \exp(i\theta))|^p d\theta/2\pi \right]^{1/p}, \quad f \in M_p. \end{aligned}$$

Hence, if  $s \in L^p(\mathcal{A})$ , we have by Holder's inequality

$$\begin{aligned} |\hat{s}(f)| & \leq \left| \int_{\beta_N} \{s_n \hat{f}(n)\} d\mathcal{A} \right| \\ & \leq \limsup_{r \rightarrow 1-} (1-r) \left( \sum_{n=0}^{\infty} |s_n|^p r^n \right)^{1/p} \left( \sum_{n=0}^{\infty} |\hat{f}(n)|^{p'} r^n \right)^{1/p'} \\ & \leq \|f\|_{M_p} \limsup [(1-r) \sum_{n=0}^{\infty} |s_n|^p r^n]^{1/p}. \end{aligned}$$

Since the last member is bounded,  $\hat{s}$  is a bounded functional on  $M_p$ . If  $s$  is a bounded sequence such that the sequence  $\{|s_n|^p\}$  is Abel summable, then  $\|\hat{s}\| \leq \|s\|_p$  — when  $\hat{s}$  is considered a linear functional on  $M_p$ .

**THEOREM.** If  $s$  is a sequence in  $L^p(\mathcal{A})$ ,  $2 \leq p < \infty$ , then

$$\|\hat{s}\| \geq \|s\| / \limsup (1-r) \sum |s_n|^p r^n,$$

when  $\hat{s}$  is considered a functional on  $M_p$ , provided that the sequence  $\{|s_n|^p\}$  is not Abel summable to zero. If the sequence  $\{|s_n|^p\}$  is Abel summable, then  $\|\hat{s}\| \geq \|s\|$ . If  $\hat{s}(f) = 0$  for all  $f \in M_p$ , then  $\|s\|_p = 0$ .

*Proof.* We let

$$\begin{aligned} \widehat{f}(n) &= |s_n|^{p-2} \overline{s_n} & \text{if } s_n \neq 0, \\ &= 0 & \text{if } s_n = 0. \end{aligned}$$

If follows from the Hausdorff Young theorem that  $f(z) = \sum \widehat{f}(n)z^n \in M_p$ , and

$$\begin{aligned} \|f\|_{M_p} &\leq \limsup[(1-r) \sum |\widehat{f}(n)|^{p'} r^n]^{1/p'} \\ &= \limsup \left[ (1-r) \sum_{n=0}^{\infty} |s_n|^{p'} r^n \right]^{1/p'}. \end{aligned}$$

Hence if  $\|f\|_{M_p} \neq 0$ ,

$$\begin{aligned} \|\widehat{s}\| &\geq |\widehat{s}(f)| / \|f\|_{M_p} \\ &\geq \|s\|_p^p / \limsup[(1-r) \sum |s_n|^{p'} r^n]^{1/p'}. \end{aligned}$$

If the sequence  $\{|s_n|^p\}$  is Abel summable to a nonzero value,

$$\|\widehat{s}\| \geq \|s\|_p^p / \|s\|_p^{p/p'} = \|s\|_p.$$

If  $\widehat{s}$  annihilates  $M_p$  it must annihilate the function  $f$  defined above, and thus  $\|s\|_p = 0$ .

We make a few remarks about the sequence  $s$  which may be written as exponential series

$$s_k = \sum_{n=0}^{\infty} a_n \exp(i\alpha_n k) \qquad k = 0, 1, \dots,$$

where the numbers  $\alpha_n$  lie in the interval  $[0, 2\pi)$  and the numbers  $a_n$  are given by the formulas

$$\begin{aligned} a_n &= \int_{\beta_N} s_k \exp(-i\alpha_n k) d\underline{\mathcal{N}} \\ &= \lim_{r \rightarrow 1-} (1-r) \sum_{k=0}^{\infty} s_k \exp(-i\alpha_n k) r^k \qquad n = 0, 1, \dots, \end{aligned}$$

(we assume that the sequence  $\{s_k \exp(i\alpha k)\}$  is Abel summable for each  $\alpha$  in  $[0, 2\pi)$ ). We also have

$$a_n = \widehat{s}(1/1 - z \exp(-i\alpha_n)).$$

We have the following version of the Riesz Fisher theorem:

**THEOREM.** *If  $\sum |a_p|^2 < \infty$ , then the Fourier transforms of the exponential polynomials*

$$s_k^{(j)} = \sum_{n=0}^j a_n \exp(i\alpha_n k), \qquad j = 1, 2, \dots,$$

*converge to a bounded linear functional  $\sigma$  on  $M_2$ , in the sense that*

$$\lim_{j \rightarrow \infty} \|\sigma - \hat{s}^{(j)}\| = 0,$$

and

$$\|\sigma\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \lim_{j \rightarrow \infty} \|\hat{s}^{(j)}\|_2^2,$$

when each  $\hat{s}^{(j)}$  is considered a functional on  $M_2$ .

*Proof.* Let  $f(z) = \sum \hat{f}(n)z^n$  be a function in  $M_2$ . Then

$$\begin{aligned} & |\hat{s}^{(j')} (f) - \hat{s}^{(j'')} (f)| \\ &= \int_{\beta_N} \left( \sum_j^{j''} a_n \exp(i\alpha_n k) \right) \hat{f}(k) d_{\mathcal{A}} \\ &\leq \left( \int_{\beta_N} \left| \sum_j^{j''} a_n \exp(i\alpha_n k) \right|^2 d_{\mathcal{A}} \right)^{1/2} \|f\|_{M_2} \\ &\leq \left( \sum_{n=j'}^{j''} |a_n|^2 \right)^{1/2} \|f\|_{M_2}, \end{aligned}$$

which tends to zero as  $j'$  and  $j''$  tend to infinity, where the above integration is carried out with respect to  $k$ . Therefore, for each  $f \in M_2$  the sequence  $\{\hat{s}^{(j)}(f)\}$  is a Cauchy sequence of numbers and hence converges. Let  $\sigma(f) = \lim \hat{s}^{(j)}(f)$ . It is readily verified that  $\sigma(f)$  depends linearly on  $f$ . Also

$$\begin{aligned} |\sigma(f)| &= |\lim \hat{s}^{(j)}(f)| \\ &\leq \left( \sum_{n=0}^j |a_n|^2 \right)^{1/2} \|f\|_{M_2}; \end{aligned}$$

hence if we regard  $\sigma$  as a functional on  $M_2$ ,  $\|\sigma\| < (\sum |a_j|^2)^{1/2}$ . If we take

$$f(z) = \sum \hat{f}(k)z^k,$$

where

$$\hat{f}(k) = \sum_{n=0}^j a_n \exp(-i\alpha_n k),$$

then the sequence  $\{|\hat{f}(k)|\}^2$  is Abel summable to  $\sum_{n=1}^j |a_n|^2$ ; thus

$$\int_{\beta_N} |\hat{f}(k)|^2 d_{\mathcal{A}} = \|f\|_{M_2}^2 = \sum_{n=1}^j |a_n|^2.$$

Since  $s^{(j)}(f) = \sum |a_n|^2$ ,  $\|\hat{s}^{(j)}\|^2 = \sum_{n=1}^j |a_n|^2$ . Since  $\|\sigma\| = \lim_{j \rightarrow \infty} \|\hat{s}^{(j)}\|$ ,  $\|\sigma\|^2 = \sum_{n=1}^{\infty} |a_n|^2$ .

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