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GAINER and HOFFMAN

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Thomas G. Gainer and Sherwood Hoffman

Langley Research Center
Hampton, Virginia

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PREFACE

A summary of equations often used in free-flight and wind-tunnel data reduction and analysis is presented. Included are transfer equations for accelerometer, rate-gyro, and angle-of-attack instrumentation; axes-system transfers of aerodynamic derivatives; and methods for measuring moments of inertia. In general, the equations are in a complete form; for example, those terms are retained that are missing when planar symmetry is assumed for airplanes.

TABLE OF CONTENTS

INTRODUCTION

The equations in this report are the coordinate transformation and motion equations used in the various tasks associated with free-flight and wind-tunnel data reduction and analysis. These tasks range from reducing flight data to calculating the motions on a digital or analog computer and to applying various techniques for analyzing the data, such as in references 1 and 2.

While many publications contain a number of these equations, no one contains all that are usually needed in a complicated aerodynamic analysis; even the more nearly complete reports (refs. 3 and 4, for example) omit the equations for transferring aerodynamic stability derivatives from one moment reference to another. Moreover, in most cases the equations are simplified, when they are presented, by assumptions such as small angles of attack, zero sideslip, and small perturbation motions. Expanded forms of many of the equations, on the other hand, are needed in special problems that may arise. For example, parawing vehicles, which have their center of gravity located well below the wing surface, require the expanded forms of the axes transformations when data measured about a point on the wing are to be transferred to the center of gravity; reentry motion studies sometimes involve large-amplitude motions so the complete forms of the transformations, without the assumptions of small angles of attack or sideslip, are needed. The engineer working on any of these special problems usually has to derive these equations himself, and this can be time consuming.

The purpose of this report is to provide the basic equations from which many of the equations needed in a particular analysis can be generated. A comprehensive summary of the basic axes transformation and motion equations is included, with most of these given in their expanded, most general forms. Once these expanded forms are available, the simpler forms can be written out fairly easily, and yet the general forms are here when needed for special cases.

The general forms presented include axes transformations that enable transfer back and forth between any of the five axes systems that are encountered in aerodynamic analysis. Equations of motion are presented that enable calculation of motions anywhere in the vicinity of the earth. Special problems are also considered; since flight instruments, such as accelerometers or rate gyros, are not always alined along mutually perpendicular axes, the procedure for correcting instrument readings for nonorthogonal alinements is outlined.

In addition to these general forms, many of the simplified forms used frequently in practical applications are summarized in appendix A.

Other relationships are presented in appendixes B to F. For example, appendix C summarizes methods used to measure moments of inertia of models and full-scale vehicles; appendix E discusses the use of the direction-cosine and the quaternion methods, often used in place of Euler angles in specifying vehicle alinement; appendix F discusses the scaling parameters used in model testing. However, throughout this paper, the emphasis is on providing the basic equations. For discussions of their development and of the procedures used in their application, the user should turn to general published works on flight-motion analysis. A comprehensive bibliography of these works is provided and includes textbooks and reports dealing with stability, control, and performance as well as reports discussing various techniques for extracting stability derivatives from flight data.

SYMBOLS

Throughout this paper, symbols are defined in terms of SI Units with equivalent U.S. Customary Units given parenthetically. Factors for converting from U.S. Customary to SI Units are given in table I.

A — generalized angle of attack, defined for various axes systems in tables III and IV, rad (deg); cross-sectional area in eq. (F-11) of appendix F, m^2 (ft^2)

A' — angle between surface wind vector and plane of local horizontal, measured perpendicular to plane of local horizontal (fig. 8), rad (deg)

A_D — acceleration along flight path, g units ($1g = 9.807\ m/sec^2$)

A_L — acceleration in lift direction, g units ($1g = 9.807\ m/sec^2$)

$A_{X,cg}, A_{Y,cg}, A_{Z,cg}$ — components of acceleration along X,Y,Z vehicle reference axes at c.g., respectively, g units ($1g = 9.807\ m/sec^2$)

$A_{X,i}, A_{Y,i}, A_{Z,i}$ — components of acceleration indicated by accelerometers along X_i, Y_i, Z_i instrument axes, respectively, g units ($1g = 9.807\ m/sec^2$)

a — speed of sound, m/sec (ft/sec)

a' — damping constant defined by eq. (D-4) of appendix D, dimensionless

$\bar{a}$ — diameter of circle around which wires or rods are attached in bifilar or trifilar methods of measuring moments of inertia in eq. (C-9) of appendix C (see, also, fig. 14(b)), m (in.)

a_e equatorial radius of earth's reference ellipsoid, m (ft)

a_s distance from knife edge to spring (fig. 13), m (ft)

B generalized angle of sideslip, defined for various axes systems in tables III and IV, rad (deg)

B' angle between projection of wind vector onto plane of local horizontal and component of wind velocity tangent to circle of local horizontal (fig. 8), rad (deg)

b span, m (ft)

C constant in eq. (D-3) of appendix D, rad (deg)

C_A axial-force coefficient

C_C crosswind coefficient

C_D drag coefficient

C'_D drag coefficient for wind-tunnel stability axes

C_L lift coefficient

$C_{L,o}$ lift coefficient in steady-state level flight

C_l, C_m, C_n $\left\{ \begin{array}{l} \text{generalized aerodynamic moment coefficients about } X, Y, Z \text{ vehicle reference axes, respectively, } \dfrac{\text{Moment}}{q_\infty S \ell} \\ \text{body-axes moment coefficients} \end{array} \right.$

$C_{m,o}$ pitching-moment coefficient in steady-state level flight

C_N normal-force coefficient

C_X, C_Y, C_Z $\left\{ \begin{array}{l} \text{generalized aerodynamic force coefficients about } X, Y, Z \text{ vehicle reference axes, respectively, } \dfrac{\text{Force}}{q_\infty S} \\ \text{body-axes force coefficients} \end{array} \right.$

$\bar{c}$ reference chord, m (ft)

c_p specific heat at constant pressure, J/kg-$^{\circ}$C (Btu(thermochemical)/lbm-$^{\circ}$F)

c_V specific heat at constant volume, J/kg-$^{\circ}$C (Btu(thermochemical)/lbm-$^{\circ}$F)

D $\begin{cases}\text{aerodynamic drag in eqs. (I-25) of Sec. I, N (lbf)} \\ \text{direction-cosine matrix defined in eq. (E-2) of appendix E}\end{cases}$

d_X,d_Y,d_Z displacements of centroid of jet interface, measured with respect to X,Y,Z vehicle reference axes, respectively, m (ft)

E modulus of elasticity, N/m^2 (lbf/ft^2)

e_0,e_1,e_2,e_3 Euler parameters in eq. (E-9) of appendix E

$F_{r,X},F_{r,Y},F_{r,Z}$ forces due to jet control (not main rockets) along X,Y,Z vehicle reference axes, respectively, N (lbf)

F_X,F_Y,F_Z forces along X,Y,Z vehicle reference axes, respectively, N (lbf)

G transformation matrix composed of Euler parameters

GE geocentric gravitational constant, m^3/sec^2 (ft^3/sec^2)

g acceleration due to gravity, m/sec^2 (ft/sec^2)

g_X,g_Y,g_Z components of gravitational acceleration along X_g,Y_g,Z_g gravity axes, respectively, for an oblate earth, m/sec^2 (ft/sec^2)

H angular direction vehicle is traveling (fig. 6); angle measured positive clockwise from north, i.e., for H = 90°, vehicle is traveling toward east, rad (deg)

h altitude, m (ft)

I moment of inertia, kg-m^2 (slug-ft^2)

$I' = \dfrac{I_Y}{q_\infty S\bar{c}}$

I_c mounting-cradle moment of inertia about vehicle roll axis, $kg\text{-}m^2$ ($slug\text{-}ft^2$)

I_{rm} moment of inertia of engine rotating mass, $kg\text{-}m^2$ ($slug\text{-}ft^2$)

I_X, I_Y, I_Z moments of inertia about X,Y,Z vehicle reference axes, respectively, $kg\text{-}m^2$ ($slug\text{-}ft^2$)

I_{XY}, I_{XZ}, I_{YZ} products of inertia, $I_{XY} = \int xy\, dm$, $I_{XZ} = \int xz\, dm$, $I_{YZ} = \int yz\, dm$, $kg\text{-}m^2$ ($slug\text{-}ft^2$)

i,j,k unit vectors having properties given in eqs. (E-10) of appendix E

J_2, J_3, J_4 second, third, fourth zonal harmonics, respectively, of earth's reference ellipsoid (values given following eq. (V-45) of Sec. V)

j unit vector (see i,j,k)

K spring constant, N/m (lbf/in.)

$K_1, K_2, \ldots, K_6$ constants defined in eqs. (I-13) of Sec. I

K_o boom misalinement angle at zero g, measured with respect to X vehicle reference axis; positive with boom inclined above X-axis, rad (deg)

K_s constant for springs shown in fig. 13, N/m (lbf/ft)

$K_{s,1}, K_{s,2}$ values of K_s for springs of different strengths, N/m (lbf/ft)

$K_{t,1}, K_{t,2}$ constants, determined from $K_{s,1}$ and $K_{s,2}$, used in measuring moments of inertia in eqs. (C-6) and (C-7) of appendix C, N/m (lbf/ft)

k
- unit vector (see i,j,k)
- torsion-spring constant in eq. (C-8) of appendix C, N-m/rad (ft-lbf/rad)
- coefficient of heat conduction in eq. (F-9) of appendix F, J/m-sec-K (Btu(thermochemical)-in./ft^2-sec-$^{\circ}F$)

L — aerodynamic lift in eqs. (I-25) of Sec. I, N (lbf); geocentric latitude of vehicle, positive in northern hemisphere, in Sec. V and appendix E, rad (deg)

$\bar{L},\bar{M},\bar{N}$ — aerodynamic moments about X,Y,Z vehicle reference axes, respectively, N-m (ft-lbf)

L_{rm},M_{rm},N_{rm} — moments due to gyroscopic action of engine rotating mass about X,Y,Z vehicle reference axes, respectively, N-m (ft-lbf)

ℓ — length used in nondimensionalizing moments, m (ft); characteristic length, m (ft)

ℓ' — length of wire or rod on which model is suspended in measuring moment of inertia by multifilar-pendulum method in appendix C, m (ft)

$\bar{\ell}$ — distance between fore and aft weighing scales in eq. (C-1) of appendix C, m (ft); distance from knife edge to supporting spring in eq. (C-5) of appendix C, m (ft)

$\ell_{XX},\ell_{XY},\ldots,\ell_{ZZ}$ — direction cosines defining orientation of rotated (primed) axes system with respect to initial (unprimed) axes system; ℓ_{XX} is cosine of angle between X-axes of two systems; ℓ_{XY}, cosine of angle between X-axis of rotated system and Y-axis of initial system; . . .; and ℓ_{ZZ}, cosine of angle between Z-axes of two systems (eqs. (E-3) of appendix E)

M — Mach number

$\bar{M}$ — aerodynamic moment (see $\bar{L},\bar{M},\bar{N}$)

M_{rm} — moment due to engine rotating mass (see L_{rm},M_{rm},N_{rm})

$M_{r,X},M_{r,Y},M_{r,Z}$ — moments due to jet controls (not main rockets) about X,Y,Z vehicle reference axes, respectively, N-m (ft-lbf)

M_X,M_Y,M_Z — moments about X,Y,Z vehicle reference axes, respectively, N-m (ft-lbf)

m instantaneous mass of vehicle, kg (slugs)

$m' = \dfrac{mV_\infty}{q_\infty S}$

$\dot{m}$ time rate of change of vehicle mass (such as due to fuel consumption); negative when vehicle losing mass, kg/sec (slugs/sec)

m_c mass of weighing cradle used in measuring moments of inertia, kg (slugs)

$\overline{N}$ aerodynamic moment (see $\overline{L},\overline{M},\overline{N}$)

N_{rm} moment due to engine rotating mass (see L_{rm},M_{rm},N_{rm})

P period of oscillation, sec

P_1 period of oscillation with spring constant $K_{t,1}$, sec

P_2 period of oscillation with spring constant $K_{t,2}$, sec

P_M period of oscillation for model alone, sec

P_{M+S} period of oscillation for model plus supporting hardware, sec

P_S period of oscillation for supporting hardware, sec

P_o,Q_o,R_o initial, steady-state values of angular-velocity components along X,Y,Z vehicle reference axes, respectively, rad/sec

$\bar{p}$ static pressure, N/m^2 (lbf/ft^2)

p,q,r components of angular velocity about X,Y,Z vehicle reference axes, respectively, rad/sec (deg/sec)

p_i,q_i,r_i components of angular velocity indicated by rate gyros about X_i,Y_i,Z_i instrument axes, respectively, rad/sec

Q_o angular-velocity component (see P_o,Q_o,R_o)

q	angular-velocity component (see p,q,r) quaternion in eq. (E-9) of appendix E
$\bar{q}_c$	impact pressure, N/m^2 (lbf/ft^2)
q_i	angular-velocity component $\left(\text{see } p_i,q_i,r_i\right)$
q_∞	free-stream dynamic pressure, N/m^2 (lbf/ft^2)
q^*	conjugate of quaternion in eq. (E-12) of appendix E
R	distance of c.g. of vehicle from center of earth (fig. 4), m (ft) scale factors, with subscripts, as defined in table VI and discussed in appendix F
R_e	radius of assumed spherical earth, 6378.123 km (20 925 631 ft)
R_o	angular-velocity component $\left(\text{see } P_o,Q_o,R_o\right)$
R_1	vertical reaction force (fig. 10) in eq. (C-2) of appendix C, N (lbf)
R_1,R_2,R_3	forces at supports in setups for determining c.g. location in eq. (C-1) of appendix C, N (lbf)
r	angular-velocity component (see p,q,r)
$\bar{r}$	vector expressing distance from c.g. to origin of instrument-axes system, m (ft)
r_g	distance from origin of gravity-axes system to vehicle reference center, m (ft)
r_i	angular-velocity component $\left(\text{see } p_i,q_i,r_i\right)$
S	vehicle reference area, m^2 (ft^2)
T	total vehicle thrust in eq. (V-36) of Sec. V and in eq. (A-134) of appendix A, N (lbf) temperature in eq. (F-9) of appendix F, K (^oF)

T_D component of thrust along flight path, N (lbf)

T_O $\begin{cases}\text{initial thrust in eq. (V-36) of Sec. V} \\ \text{representative temperature of flow in eq. (F-10) of appendix F, K } (^{o}\text{F})\end{cases}$

T_X,T_Y,T_Z components of thrust along X,Y,Z vehicle reference axes, respectively, N (lbf)

t time, sec

$t_{1/2}$ time to damp to one-half amplitude, sec

t_1 arbitrary time defined in fig. 15, sec

t_2 arbitrary time defined in fig. 15, sec

U_O,V_O,W_O initial, steady-state values of linear-velocity components along X,Y,Z vehicle reference axes, respectively, m/sec (ft/sec)

u,v,w components of vehicle absolute (inertial) velocity along X,Y,Z vehicle reference axes, respectively, m/sec (ft/sec)

$\bar{u},\bar{v},\bar{w}$ components of relative velocity (velocity of vehicle with respect to air), along X,Y,Z vehicle reference axes, respectively, m/sec (ft/sec)

$\bar{u}_i,\bar{v}_i,\bar{w}_i$ indicated components of relative velocity along X_i,Y_i,Z_i instrument axes, respectively, m/sec (ft/sec)

V resultant velocity, m/sec (ft/sec)

ΔV increment in resultant free-stream velocity, m/sec (ft/sec)

V_O velocity component (see U_O,V_O,W_O)

V_∞ free-stream reference velocity, m/sec (ft/sec)

V_X,V_Y,V_Z components of geostrophic-wind velocity, due to earth rotation (atmosphere rotates with earth), relative to X,Y,Z vehicle reference axes, respectively, m/sec (ft/sec)

v inertial-velocity component (see u,v,w)

$\bar{v}$ relative-velocity component (see $\bar{u},\bar{v},\bar{w}$)

$\bar{v}_i$	relative-velocity component (see $\bar{u}_i,\bar{v}_i,\bar{w}_i$)
W	vehicle weight, N (lbf)
$\overline{W}$	resultant velocity of earth surface wind, m/sec (ft/sec)
W_c	weight of weighing cradle, N (lbf)
W_M	weight of model alone, N (lbf)
W_{M+S}	weight of model plus supporting hardware, N (lbf)
W_S	weight of model supporting hardware, N (lbf)
W_o	velocity component (see U_o,V_o,W_o)
W_X,W_Y,W_Z	components of wind velocity, due to surface winds, relative to X,Y,Z vehicle reference axes, respectively, m/sec (ft/sec)
w	inertial-velocity component (see u,v,w)
$\bar{w}$	relative-velocity component (see $\bar{u},\bar{v},\bar{w}$)
$\bar{w}_i$	relative-velocity component (see $\bar{u}_i,\bar{v}_i,\bar{w}_i$)
X,Y,Z	vehicle reference axes
$\overline{X},\overline{Y},\overline{Z}$	aerodynamic forces along X,Y,Z vehicle reference axes, respectively, N (lbf)
X_e,Y_e,Z_e	right-handed inertial axes with origin at center of earth (fig. 4)
X_g,Y_g,Z_g	gravity axes with origin at surface of earth (fig. 4)
X_i,Y_i,Z_i	axes in orthogonal coordinate system with origin at an instrument at point $\bar{x},\bar{y},\bar{z}$ and with system alined at angles ψ,θ,ϕ with X,Y,Z vehicle reference axes at c.g., respectively, as shown in fig. 1
x,y,z	distances measured from X,Y,Z vehicle reference axes, respectively, m (ft)

$\bar{x},\bar{y},\bar{z}$ — distances from c.g. to another point on vehicle, m (ft)

x_g,y_g,z_g — displacements of vehicle with respect to X_g,Y_g,Z_g gravity axes, respectively (figs. 4 and 5), m (ft)

$\bar{x}_X,\bar{y}_X,\bar{z}_X$ — coordinates of X-axis accelerometer or rate gyro, measured from X,Y,Z vehicle reference axes at c.g., respectively, m (ft)

$\bar{x}_Y,\bar{y}_Y,\bar{z}_Y$ — coordinates of Y-axis accelerometer or rate gyro, measured from X,Y,Z vehicle reference axes at c.g., respectively, m (ft)

$\bar{x}_Z,\bar{y}_Z,\bar{z}_Z$ — coordinates of Z-axis accelerometer or rate gyro, measured from X,Y,Z vehicle reference axes at c.g., respectively, m (ft)

y_1 — distance from knife edge to vertical reaction force R_1 (fig. 10), m (ft)

$\bar{z}_a$ — distance from knife edge to vehicle c.g. (fig. 13), m (ft)

$\bar{z}_c$ — vertical distance to c.g. of weighing cradle (figs. 10 and 13), m (ft)

z_M — distance from knife edge to c.g. of model, m (ft)

z_{M+S} — distance from knife edge to c.g. of model plus supporting hardware, m (ft)

z_S — distance from knife edge to c.g. of supporting hardware, m (ft)

α — $\begin{cases} \text{angle of attack, rad (deg)} \\ \tan^{-1} \dfrac{\bar{w}_b}{\bar{u}_b} = \tan^{-1} \dfrac{w_b - V_{Z,b} - W_{Z,b}}{u_b - V_{X,b} - W_{X,b}} \end{cases}$

$\Delta\alpha$ — position error, error due to location of angle-of-attack sensor in flow of vehicle, equal to free-stream angle of attack minus true or corrected local angle of attack; positive for a free-stream angle of attack greater than local angle of attack, rad

$\Delta\alpha_1$ — difference between angle of attack and trim angle of attack at time t_1, rad

$\Delta\alpha_2$ — difference between angle of attack and trim angle of attack at time t_2, rad

α_a increment in angle of attack caused by boom bending under static air load, measured with respect to boom center line; positive for upward deflection of boom, rad

α_b change in angle of attack caused by bending of mounting boom under inertial load, measured with respect to boom center line; positive for upward deflection of boom, rad

α_d change in angle of attack caused by vehicle rotation, measured with respect to vehicle reference axis, rad

α_i angle of attack indicated by vane or other sensor, measured with respect to center line of mounting boom; positive with nose of vane pointed down, rad

α_0 angle between X body axis and X flight stability axis (fig. 3), rad (deg)

α_t trim angle of attack, rad (deg)

α_u change in angle of attack caused by upwash from mounting boom, measured with respect to free-stream velocity; positive for upwash (upward flow normal to free stream), rad

α_v vane floating angle, angle caused by slight mass unbalance or by warp in vane surface due to imperfections in manufacture, measured with respect to local velocity vector; positive for nose-up deflections of vane, rad

β $\begin{cases} \text{angle of sideslip, rad (deg)} \\ \sin^{-1} \dfrac{\bar{v}_b}{V} = \sin^{-1} \dfrac{v_b - V_{Y,b} - W_{Y,b}}{V} \end{cases}$

$\Delta\beta$ sideslip position error, rad

β_a increment in sideslip angle caused by boom bending under static air load, measured with respect to boom center line; positive for boom deflected to right, rad

β_b change in sideslip angle caused by bending of mounting boom under inertial loads, measured with respect to boom center line; positive for boom deflections to right, rad

β_d change in sideslip angle caused by vehicle rotation, measured with respect to vehicle reference axis, rad

β_i sideslip angle indicated by vane or other sensor, measured with respect to center line of mounting boom; positive with nose of vane pointed to right of boom, rad

β_u change in sideslip angle caused by sidewash from mounting boom, measured with respect to free-stream velocity; positive for positive sidewash, rad

β_v sideslip-vane floating angle, angle caused by warp in sideslip vane due to imperfections in manufacture, measured with respect to local velocity vector; positive for vane deflected to right, rad

Γ transformation matrix for orthogonal axes system, defined in eq. (A-2) of appendix A

$\Gamma_{nonorthogonal}$ transformation matrix for nonorthogonal axes system, defined in eq. (I-14) of Sec. I

γ $\begin{cases} \text{ratio of specific heats in eqs. (I-19) and (I-20) of Sec. I, } c_p/c_v \\ \text{flight-path angle in eq. (V-17) of Sec. V (fig. 6), rad (deg)} \end{cases}$

δ control deflection, rad (deg)

$\delta_a, \delta_e, \delta_r$ control deflections (aileron, elevator, rudder, respectively) rad (deg)

δ_{RPM} change in engine rpm

ϵ angle between X_P principal and X body axes (fig. 3), rad (deg)

ϵ_m strain (elongation per unit length) measured on model, cm/cm (in./in.)

ϵ_p strain (elongation per unit length) measured on prototype, cm/cm (in./in.)

ζ angle between X_S flight stability and X_{wt} wind-tunnel stability axes (fig. 3), rad (deg)

η angle between X_P principal and X_S flight stability axes (fig. 3), rad (deg)

λ	geocentric longitude of vehicle, positive counterclockwise looking in direction of positive Z_e inertial axis (fig. 4), rad
μ	coefficient of viscosity, $N\text{-sec/m}^2$ (slugs/ft-sec)
ν	kinematic viscosity in eq. (F-9) of appendix F, μ/ρ, m^2/sec (ft^2/sec)
ξ	phase angle, rad (deg)
ρ	atmospheric density, kg/m^3 ($slugs/ft^3$)
σ	range angle in eq. (V-21) of Sec. V, rad (deg) surface tension in eq. (F-7) of appendix F, N/m (lbf/ft)
ϕ	roll angle defined in fig. 10, rad (deg)
ψ,θ,ϕ	Euler angles defining angular alinement of one axes system with respect to another axes system, rad (deg)
ψ_g,θ_g,ϕ_g	Euler angles defining alinement of X,Y,Z vehicle reference axes, respectively, with respect to gravity-axes system (fig. 5), rad (deg)
ψ_{rm},θ_{rm}	Euler angles describing alinement of engine-thrust axes (fig. 9), rad (deg)
ψ_X,θ_X,ϕ_X	Euler angles of X-axis accelerometer or rate gyro, measured with respect to X,Y,Z vehicle reference axes at c.g., respectively, rad (deg)
ψ_Y,θ_Y,ϕ_Y	Euler angles of Y-axis accelerometer or rate gyro, measured with respect to X,Y,Z vehicle reference axes at c.g., respectively, rad (deg)
ψ_Z,θ_Z,ϕ_Z	Euler angles of Z-axis accelerometer or rate gyro, measured with respect to X,Y,Z vehicle reference axes at c.g., respectively, rad (deg)
Ω_e	rate of rotation of earth, rad/sec
Ω_{rm}	angular velocity of engine rotating mass, rad/sec
ω	resultant angular velocity in Sec. I, rad/sec frequency of oscillation in appendix D, rad/sec

$\omega_{X'}, \omega_{Y'}, \omega_{Z'}$ angular-velocity components along X', Y', Z' vehicle reference axes, respectively, usually measured with respect to gravity-axes or inertial-axes system, rad/sec

Subscripts:

b body axes

o initial conditions

P principal axes

s flight stability axes

w wind axes

wt wind-tunnel stability axes

Notation:

. (Dot) first derivative with respect to time

' (Prime) unless otherwise specified, a primed quantity is one referred to axes system located at point $\bar{x}, \bar{y}, \bar{z}$ and/or alined at angles ψ, θ, ϕ with respect to initial reference axes system

Δ perturbation quantity (unless specified otherwise)

Subscripts used with coefficient symbols denote derivatives as follows:

α with respect to α

$\dot{\alpha}$ with respect to $\dot{\alpha}\ell/2V_\infty$

β with respect to β

$\dot{\beta}$ with respect to $\dot{\beta}\ell/2V_\infty$

u with respect to u/V_∞

v	with respect to v/V_∞
w	with respect to w/V_∞
p	with respect to $p\ell/2V_\infty$
q	with respect to $q\ell/2V_\infty$
r	with respect to $r\ell/2V_\infty$
δ	with respect to control deflection
V	with respect to V/V_∞. This derivative, for a force such as the X-force, is equal to $2C_X + \frac{\partial C_X}{\partial(V/V_\infty)}$, where the term $2C_X$ represents the change in X-force due to changes in free-stream velocity and the term $\frac{\partial C_X}{\partial(V/V_\infty)}$ represents the change in C_X due to effects such as Mach number or aeroelastic effects. If these effects are negligible, then $C_{X_V} = 2C_X$, $C_{m_V} = 2C_m$, etc.

SECTION I

EQUATIONS INVOLVING BASIC FLIGHT MEASUREMENTS

This section summarizes some of the relationships used in working with basic flight measurements. Included are general equations for transferring accelerations, linear velocities, and angular velocities between any two axes systems on a flight vehicle that are related by a ψ,θ,ϕ Euler rotation sequence; the simplified forms that are frequently used to correct accelerometer and rate-gyro readings for instrument displacement and misalinement are given as equations (A-5) to (A-16) in appendix A. Also included are the correction equations for angles of attack and sideslip, equations for determining flight Mach number from measurements of static and impact pressures, and equations for determining vehicle forces and moments from accelerometer and rate-gyro readings.

GENERAL AXES TRANSFORMATIONS FOR COMPONENTS OF ACCELERATION, LINEAR VELOCITY, AND ANGULAR VELOCITY

The general equations given here transfer between the two axes systems shown in figure 1, where the "instrument-axes" system represents any system displaced and/or misalined with respect to the vehicle reference axes at the c.g. It should be noted that the X_i,Y_i,Z_i components of the instrument-axes system need not be mutually perpendicular (orthogonal) nor referred to the same origin. This would be the case, for example, if accelerometers were used to measure the separate components of acceleration but the accelerometers were not orthogonally alined and were located at different points on the vehicle. Hence, three separate instrument axes have to be considered. For the equations developed here, one instrument is assumed to be alined along the X-axis of a system located at point $\bar{x}_X,\bar{y}_X,\bar{z}_X$ and alined at Euler angles ψ_X,θ_X,ϕ_X with the vehicle reference axes; a second instrument is assumed to be alined along the Y-axis of a system located at point $\bar{x}_Y,\bar{y}_Y,\bar{z}_Y$ and alined at Euler angles ψ_Y,θ_Y,ϕ_Y with the reference axes, and so on. Nine displacement coordinates and nine Euler angles in all are needed to define the locations and alinements of the three instruments.

Both axes rotation and axes translation can be performed by using these equations $\Bigg($in the acceleration equations, axes-translation terms are of the form $\vec{a}_{inertial} = \frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{V}$; in the velocity equations, they are of the form $\vec{V}_{inertial} = \frac{d\vec{r}}{dt} + \vec{\omega} \times \vec{r}\Bigg)$. The transformations from the vehicle reference axes to the instrument axes are given in equation form; however, those from the instrument axes to the vehicle reference axes are given only in matrix form, since they involve a matrix inversion which becomes long and complicated unless certain simplifying assumptions are made. Two simplified cases, in which (1) instrument axes are orthogonal and (2) misalinement angles are small, are given as instrument-correction equations in equations (A-5) to (A-16) of appendix A.

Transfer From Vehicle Reference Axes to Instrument Axes

Acceleration.-

$$\begin{aligned} gA_{X,i} = &\left[gA_{X,cg} - \left(r^2 + q^2\right)\bar{x}_X + (pq - \dot{r})\bar{y}_X + (rp + \dot{q})\bar{z}_X\right]\cos\theta_X \cos\psi_X \\ &+ \left[gA_{Y,cg} + (pq + \dot{r})\bar{x}_Y - \left(p^2 + r^2\right)\bar{y}_Y + (qr - \dot{p})\bar{z}_Y\right]\cos\theta_X \sin\psi_X \\ &- \left[gA_{Z,cg} + (pr - \dot{q})\bar{x}_Z + (qr + \dot{p})\bar{y}_Z - \left(q^2 + p^2\right)\bar{z}_Z\right]\sin\theta_X \end{aligned} \qquad \text{(I-1)}$$

$$\begin{aligned} gA_{Y,i} = &\left[gA_{X,cg} - \left(r^2 + q^2\right)\bar{x}_X + (pq - \dot{r})\bar{y}_X + (rp + \dot{q})\bar{z}_X\right]\left(\cos\psi_Y \sin\theta_Y \sin\phi_Y - \sin\psi_Y \cos\phi_Y\right) \\ &+ \left[gA_{Y,cg} + (pq + \dot{r})\bar{x}_Y - \left(p^2 + r^2\right)\bar{y}_Y + (qr - \dot{p})\bar{z}_Y\right]\left(\sin\psi_Y \sin\theta_Y \sin\phi_Y + \cos\psi_Y \cos\phi_Y\right) \\ &+ \left[gA_{Z,cg} + (pr - \dot{q})\bar{x}_Z + (qr + \dot{p})\bar{y}_Z - \left(q^2 + p^2\right)\bar{z}_Z\right]\cos\theta_Y \sin\phi_Y \end{aligned} \qquad \text{(I-2)}$$

$$\begin{aligned} gA_{Z,i} = &\left[gA_{X,cg} - \left(r^2 + q^2\right)\bar{x}_X + (pq - \dot{r})\bar{y}_X + (rp + \dot{q})\bar{z}_X\right]\left(\cos\psi_Z \sin\theta_Z \cos\phi_Z + \sin\psi_Z \sin\phi_Z\right) \\ &+ \left[gA_{Y,cg} + (pq + \dot{r})\bar{x}_Y - \left(p^2 + r^2\right)\bar{y}_Y + (qr - \dot{p})\bar{z}_Y\right]\left(\sin\psi_Z \sin\theta_Z \cos\phi_Z - \cos\psi_Z \sin\phi_Z\right) \\ &+ \left[gA_{Z,cg} + (pr - \dot{q})\bar{x}_Z + (qr + \dot{p})\bar{y}_Z - \left(q^2 + p^2\right)\bar{z}_Z\right]\cos\theta_Z \cos\phi_Z \end{aligned} \qquad \text{(I-3)}$$

Linear velocity.-

$$\begin{aligned} \bar{u}_i = &\left(\bar{u} + q\bar{z}_X - r\bar{y}_X\right)\cos\theta_X \cos\psi_X + \left(\bar{v} + r\bar{x}_Y - p\bar{z}_Y\right)\cos\theta_X \sin\psi_X \\ &- \left(\bar{w} + p\bar{y}_Z - q\bar{x}_Z\right)\sin\theta_X \end{aligned} \qquad \text{(I-4)}$$

$$\bar{v}_i = \left(\bar{u} + q\bar{z}_X - r\bar{y}_X\right)\left(\sin\phi_Y \sin\theta_Y \cos\psi_Y - \sin\psi_Y \cos\phi_Y\right)$$
$$+ \left(\bar{v} + r\bar{x}_Y - p\bar{z}_Y\right)\left(\sin\psi_Y \sin\theta_Y \sin\phi_Y + \cos\psi_Y \cos\phi_Y\right)$$
$$+ \left(\bar{w} + p\bar{y}_Z - q\bar{x}_Z\right)\sin\phi_Y \cos\theta_Y \tag{I-5}$$

$$\bar{w}_i = \left(\bar{u} + q\bar{z}_X - r\bar{y}_X\right)\left(\cos\psi_Z \cos\phi_Z \sin\theta_Z + \sin\psi_Z \sin\phi_Z\right)$$
$$+ \left(\bar{v} + r\bar{x}_Y - p\bar{z}_Y\right)\left(\sin\psi_Z \cos\phi_Z \sin\theta_Z - \cos\psi_Z \sin\phi_Z\right)$$
$$+ \left(\bar{w} + p\bar{y}_Z - q\bar{x}_Z\right)\cos\phi_Z \cos\theta_Z \tag{I-6}$$

Angular velocity.-

$$p_i = p \cos\theta_X \cos\psi_X + q \cos\theta_X \sin\psi_X - r \sin\theta_X \tag{I-7}$$

$$q_i = p\left(\cos\psi_Y \sin\theta_Y \sin\phi_Y - \sin\psi_Y \cos\phi_Y\right)$$
$$+ q\left(\sin\phi_Y \sin\psi_Y \sin\theta_Y + \cos\psi_Y \cos\phi_Y\right) + r \sin\phi_Y \cos\theta_Y \tag{I-8}$$

$$r_i = p\left(\cos\psi_Z \cos\phi_Z \sin\theta_Z + \sin\psi_Z \sin\phi_Z\right)$$
$$+ q\left(\sin\psi_Z \cos\phi_Z \sin\theta_Z - \cos\psi_Z \sin\phi_Z\right) + r \cos\theta_Z \cos\phi_Z \tag{I-9}$$

Transfer From Instrument Axes to Vehicle Reference Axes

If the axes along which data are measured are not orthogonally alined, the following procedure must be used to transfer instrument readings to the vehicle reference axes at the c.g.:

Accelerations are given by

$$\begin{bmatrix} gA_{X,cg} \\ gA_{Y,cg} \\ gA_{Z,cg} \end{bmatrix} = \left[\Gamma_{nonorthogonal}\right]^{-1} \begin{bmatrix} gA_{X,i} \\ gA_{Y,i} \\ gA_{Z,i} \end{bmatrix} + \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} \tag{I-10}$$

angular velocities, by

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \left[\Gamma_{nonorthogonal}\right]^{-1} \begin{bmatrix} p_i \\ q_i \\ r_i \end{bmatrix} \tag{I-11}$$

and linear velocities by

$$\begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix} = \left[\Gamma_{nonorthogonal}\right]^{-1} \begin{bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \end{bmatrix} + \begin{bmatrix} K_4 \\ K_5 \\ K_6 \end{bmatrix} \tag{I-12}$$

where

$$\left.\begin{aligned} K_1 &= \bar{x}_X\left(r^2 + q^2\right) - \bar{y}_X(pq - \dot{r}) - \bar{z}_X(rp + \dot{q}) \\ K_2 &= -\bar{x}_Y(pq + \dot{r}) + \bar{y}_Y\left(p^2 + r^2\right) - \bar{z}_Y(qr - \dot{p}) \\ K_3 &= -\bar{x}_Z(pr - \dot{q}) - \bar{y}_Z(qr + \dot{p}) + \bar{z}_Z\left(q^2 + p^2\right) \\ K_4 &= r\bar{y}_X - q\bar{z}_X \\ K_5 &= p\bar{z}_Y - r\bar{x}_Y \\ K_6 &= q\bar{x}_Z - p\bar{y}_Z \end{aligned}\right\} \tag{I-13}$$

In equations (I-10) to (I-12) $\left[\Gamma_{nonorthogonal}\right]^{-1}$ is the inverse of a transformation matrix made up of the sines and cosines of nine Euler angles, one set ψ,θ,ϕ for each of the three instruments. This transformation matrix is defined as

$$\left[\Gamma_{nonorthogonal}\right] = \begin{bmatrix} \cos\theta_X \cos\psi_X & \cos\theta_X \sin\psi_X & -\sin\theta_X \\ \begin{aligned}&\cos\psi_Y \sin\phi_Y \sin\theta_Y \\ &- \sin\psi_Y \cos\phi_Y\end{aligned} & \begin{aligned}&\sin\psi_Y \sin\phi_Y \sin\theta_Y \\ &+ \cos\psi_Y \cos\phi_Y\end{aligned} & \sin\phi_Y \cos\theta_Y \\ \begin{aligned}&\cos\psi_Z \cos\phi_Z \sin\theta_Z \\ &+ \sin\psi_Z \sin\phi_Z\end{aligned} & \begin{aligned}&\sin\psi_Z \cos\phi_Z \sin\theta_Z \\ &- \cos\psi_Z \sin\phi_Z\end{aligned} & \cos\theta_Z \cos\phi_Z \end{bmatrix} \tag{I-14}$$

Except under certain simplifying assumptions, the inverted matrix $\left[\Gamma_{nonorthogonal}\right]^{-1}$ needed to solve these equations is too long and involved to write out; however, the matrix inversion can be performed fairly easily on a digital computer. Equations for two cases in which the matrix can be inverted are given in appendix A as equations (A-5) to (A-16).

CORRECTIONS TO ANGLES OF ATTACK AND SIDESLIP

Angle of Attack

For a vane-type sensor, the true or corrected angle of attack (referred to the vehicle reference axes at the c.g.) is given by

$$\alpha = \alpha_i - \alpha_a - \alpha_u + \alpha_v - K_o - \alpha_b + \alpha_d + \Delta\alpha \tag{I-15}$$

where $\Delta\alpha$ is the position error, the error due to the location of the sensor in the flow field of the body, and the other corrections are those due to boom bending, misalinement, and flow-field effects (see the list of symbols).

Equation (I-15) is for an angle-of-attack vane mounted on an instrument boom extending from the body or a wing tip but generally applies to any type of sensor. Other types of sensor and methods of calibration are discussed in reference 5. The importance of the different types of error is discussed in reference 3.

The angle α_d is the correction for vehicle rotation and is given by

$$\alpha_d = \frac{2\bar{x}}{\ell}\frac{q\ell}{2V_\infty} - \frac{2\bar{y}}{\ell}\frac{p\ell}{2V_\infty} \tag{I-16}$$

The other errors are determined by calibration as discussed in reference 5.

Angle of Sideslip

The equation for corrected sideslip angle is

$$\beta = \beta_i + \beta_a + \beta_u - \beta_v + \beta_b + \beta_d + \Delta\beta \tag{I-17}$$

where the correction for vehicle rotation is given by

$$\beta_d = -\frac{2\bar{x}}{\ell}\frac{r\ell}{2V_\infty} + \frac{2\bar{z}}{\ell}\frac{p\ell}{2V_\infty} \tag{I-18}$$

DETERMINATION OF FREE-STREAM MACH NUMBER

For flight tests of high-speed aircraft or missiles, the following relationships are used to determine Mach number from onboard measurements of impact pressure $\bar{q}_c$ and static pressure $\bar{p}$: For subsonic conditions (M < 1.0),

$$\frac{\bar{q}_c}{\bar{p}} = \left[\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right] = (1 + 0.2M)^{7/2} - 1 \tag{I-19}$$

For supersonic conditions (M > 1.0), the equation is modified to include the loss in total pressure behind the shock wave and becomes

$$\frac{\bar{q}_c}{\bar{p}} = \frac{\gamma + 1}{2} M^2 \left(\frac{\frac{\gamma + 1}{2} M^2}{\frac{2\gamma}{\gamma + 1} M^2 - \frac{\gamma - 1}{\gamma + 1}} \right)^{\frac{1}{\gamma - 1}} - 1 = 1.2M^2 \left(\frac{5.76M^2}{5.6M^2 - 0.8} \right)^{5/2} - 1 \qquad \text{(I-20)}$$

The indicated Mach number M is determined from tables or plots, based on equations (I-19) and (I-20), of $\bar{q}_c/\bar{p}$ as functions of Mach number.

DETERMINATION OF AERODYNAMIC FORCES AND MOMENTS FROM ACCELEROMETER AND RATE-GYRO READINGS

The components of the total force acting on the flight vehicle can be determined by multiplying the corrected accelerometer reading (i.e., corrected for displacement and misalinement) by vehicle weight. In coefficient form

$$\left.\begin{aligned} C_X &= \frac{WA_{X,cg}}{q_\infty S} \\ C_Y &= \frac{WA_{Y,cg}}{q_\infty S} \\ C_Z &= \frac{WA_{Z,cg}}{q_\infty S} \end{aligned}\right\} \qquad \text{(I-21)}$$

The total force thus determined includes the static and the dynamic aerodynamic forces, the engine thrust, the jet damping force, and the reaction and other control forces. It does not include the components of vehicle weight, however, since gravitational effects appear as accelerations that are measured directly by the accelerometer; that is, the products of the weight and the corrected accelerometer readings are

$$\left.\begin{aligned} WA_{X,cg} &= W\left(\dot{u} + wq - vr + g \sin \theta_g\right) = \sum F_X \\ WA_{Y,cg} &= W\left(\dot{v} + ur - wp - g \cos \theta_g \sin \phi_g\right) = \sum F_Y \\ WA_{Z,cg} &= W\left(\dot{w} + vp - uq - g \cos \theta_g \cos \phi_g\right) = \sum F_Z \end{aligned}\right\} \qquad \text{(I-22)}$$

where the sine and cosine terms are components of vehicle weight.

The total moments are determined from the rate-gyro readings according to the equations

$$\left.\begin{aligned}
\sum M_X &= \dot{p}I_X - \dot{q}I_{XY} - \dot{r}I_{XZ} + q\left(rI_Z - pI_{XZ} - qI_{YZ}\right) - r\left(qI_Y - rI_{YZ} - pI_{XY}\right) \\
\sum M_Y &= \dot{q}I_Y - \dot{r}I_{YZ} - \dot{p}I_{XY} + r\left(pI_X - qI_{XY} - rI_{XZ}\right) - p\left(rI_Z - pI_{XZ} - qI_{YZ}\right) \\
\sum M_Z &= \dot{r}I_Z - \dot{p}I_{XZ} - \dot{q}I_{YZ} + p\left(qI_Y - rI_{YZ} - pI_{XY}\right) - q\left(pI_X - qI_{XY} - rI_{XZ}\right)
\end{aligned}\right\} \qquad \text{(I-23)}$$

The moment coefficients are

$$\left.\begin{aligned}
C_l &= \sum \frac{M_X}{q_\infty Sb} \\
C_m &= \sum \frac{M_Y}{q_\infty S\bar{c}} \\
C_n &= \sum \frac{M_Z}{q_\infty Sb}
\end{aligned}\right\} \qquad \text{(I-24)}$$

(Although b and $\bar{c}$ are commonly used to nondimensionalize the moments, these lengths are arbitrary and any convenient length can be used.)

Methods by which the total forces and moments are broken down into trim, static, and cross-coupling components so that longitudinal and lateral coefficients can be determined are quite complicated and beyond the scope of this paper. The simplified method for determining basic static- and dynamic-stability derivatives from oscillatory flight motions is developed in appendix D. In this and in most methods, linear systems are used to represent transient flight data; these linear systems are obtained by assuming small angles, constant aerodynamic coefficients, constant free-stream conditions, rigid-body mass and inertial characteristics, and separation of the longitudinal and lateral modes. For data that do not fit these limitations, exact methods of simulation based on the equations of motion for six degrees of freedom are used. Other methods that are now becoming popular involve parameter identification in which linear and quasilinear estimation techniques are used. (See, for example, refs. 2 and 6.)

DIRECT FLIGHT MEASUREMENTS OF LIFT AND DRAG

Lift and drag components for an airplane can be measured directly by using flight-path accelerometers which differ from body-fixed accelerometers in that they are operated by a vane or pressure sensor that rotates the sensor into or normal to the stream direction. Accelerometers measuring parallel and normal to the flight path measure vehicle thrust minus drag and lift, respectively, if the component of thrust in the lift direction can be assumed negligible. In equation form the lift and drag are

$$\left.\begin{aligned} T_D - D &= -WA_D \\ L &= WA_L \end{aligned}\right\} \qquad \text{(I-25)}$$

where T_D is the component of thrust along the flight path. The accelerations should be corrected for the usual boom-type errors as described in correction with equation (I-15) for the angle of attack measured from a boom. Details of this flight-testing technique are given in reference 7.

SECTION II

TRANSFER OF AERODYNAMIC FORCE AND MOMENT COEFFICIENTS AND DERIVATIVES TO ANOTHER REFERENCE CENTER

The equations in this section are the general forms for transferring aerodynamic force and moment coefficients and stability derivatives from a coordinate axes system with origin at the c.g. to a parallel axes system with origin at a point $\bar{x},\bar{y},\bar{z}$ away from the vehicle c.g. (Simplified forms obtained by assuming zero angles of attack and sideslip and neglecting aerodynamic cross derivatives are given as equations (A-17) to (A-43) in appendix A.) In these general equations, unprimed coefficients are referred to the c.g.; primed coefficients are referred to the point at $\bar{x},\bar{y},\bar{z}$ shown in figure 2.

The transformations are given for both systems of variables used in aerodynamics. Equations for the α,β,V,p,q,r system are given by equations (II-1) to (II-40); those for the u,v,w,p,q,r system are given by equations (II-41) to (II-76). (The relationships between the two sets of variables are given in table II.) The equations are derived in appendix B. They are general in that no assumptions are made as to angle of attack or sideslip and in that all transfer distances, lateral as well as vertical, are included; however, they are still not complete in that, for the purposes of these transformations, second-order derivatives with respect to time are assumed negligible and are omitted. Also, the transformations given for the static forces and moments (derivatives with respect to α or β) are the simplified forms that apply only when the body is not undergoing any significant rotation and p, q, and r are essentially zero. The transformations for static-stability derivatives that apply when there is significant vehicle rotation can be derived as indicated in appendix B.

TRANSFORMATIONS FOR α,β,V DERIVATIVES

X-Axis Force Coefficients and Derivatives

$$C'_X = C_X \tag{II-1}$$

$$C'_{X_\alpha} = C_{X_\alpha} \tag{II-2}$$

$$C'_{X_\beta} = C_{X_\beta} \tag{II-3}$$

$$C'_{X_{\dot{\alpha}}} = C_{X_{\dot{\alpha}}} \tag{II-4}$$

$$C'_{X_{\dot{\beta}}} = C_{X_{\dot{\beta}}} \tag{II-5}$$

$$C'_{X_p} = C_{X_p} - 2C_{X_\alpha} \frac{\bar{y}}{\ell} \frac{\cos\alpha}{\cos\beta} + 2C_{X_\beta}\left(\frac{\bar{z}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\sin\alpha\sin\beta\right)$$
$$+ 4C_X\left(\frac{\bar{z}}{\ell}\sin\beta - \frac{\bar{y}}{\ell}\sin\alpha\cos\beta\right) \qquad \text{(II-6)}$$

$$C'_{X_q} = C_{X_q} + \frac{2}{\cos\beta} C_{X_\alpha}\left(\frac{\bar{x}}{\ell}\cos\alpha + \frac{\bar{z}}{\ell}\sin\alpha\right) + 2C_{X_\beta}\left(\frac{\bar{z}}{\ell}\cos\alpha - \frac{\bar{x}}{\ell}\sin\alpha\right)\sin\beta$$
$$+ 4C_X\left(\frac{\bar{x}}{\ell}\sin\alpha - \frac{\bar{z}}{\ell}\cos\alpha\right)\cos\beta \qquad \text{(II-7)}$$

$$C'_{X_r} = C_{X_r} - 2C_{X_\alpha} \frac{\bar{y}}{\ell} \frac{\sin\alpha}{\cos\beta} - 2C_{X_\beta}\left(\frac{\bar{x}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\cos\alpha\sin\beta\right)$$
$$+ 4C_X\left(\frac{\bar{y}}{\ell}\cos\alpha\cos\beta - \frac{\bar{x}}{\ell}\sin\beta\right) \qquad \text{(II-8)}$$

Y-Axis Force Coefficients and Derivatives

$$\left.\begin{aligned} C'_Y &= C_Y \\ C'_{Y_\alpha} &= C_{Y_\alpha} \\ C'_{Y_\beta} &= C_{Y_\beta} \\ C'_{Y_{\dot\alpha}} &= C_{Y_{\dot\alpha}} \\ C'_{Y_{\dot\beta}} &= C_{Y_{\dot\beta}} \end{aligned}\right\} \qquad \text{(II-9)}$$

$$C'_{Y_p} = C_{Y_p} - 2C_{Y_\alpha} \frac{\bar{y}}{\ell} \frac{\cos\alpha}{\cos\beta} + 2C_{Y_\beta}\left(\frac{\bar{z}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\sin\alpha\sin\beta\right)$$
$$+ 4C_Y\left(\frac{\bar{z}}{\ell}\sin\beta - \frac{\bar{y}}{\ell}\sin\alpha\cos\beta\right) \qquad \text{(II-10)}$$

$$C'_{Y_q} = C_{Y_q} + \frac{2C_{Y_\alpha}}{\cos\beta}\left(\frac{\bar{x}}{\ell}\cos\alpha + \frac{\bar{z}}{\ell}\sin\alpha\right) + 2C_{Y_\beta}\left(\frac{\bar{z}}{\ell}\cos\alpha - \frac{\bar{x}}{\ell}\sin\alpha\right)\sin\beta$$
$$+ 4C_Y\left(\frac{\bar{x}}{\ell}\sin\alpha - \frac{\bar{z}}{\ell}\cos\alpha\right)\cos\beta \qquad \text{(II-11)}$$

$$C'_{Y_r} = C_{Y_r} - 2C_{Y_\alpha} \frac{\bar{y}}{\ell} \frac{\sin\alpha}{\cos\beta} - 2C_{Y_\beta}\left(\frac{\bar{x}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\cos\alpha\sin\beta\right)$$

$$+ 4C_Y\left(\frac{\bar{y}}{\ell}\cos\alpha\cos\beta - \frac{\bar{x}}{\ell}\sin\beta\right) \qquad \text{(II-12)}$$

Z-Axis Force Coefficients and Derivatives

$$\left.\begin{aligned} C'_Z &= C_Z \\ C'_{Z_\alpha} &= C_{Z_\alpha} \\ C'_{Z_\beta} &= C_{Z_\beta} \\ C'_{Z_{\dot{\alpha}}} &= C_{Z_{\dot{\alpha}}} \\ C'_{Z_{\dot{\beta}}} &= C_{Z_{\dot{\beta}}} \end{aligned}\right\} \qquad \text{(II-13)}$$

$$C'_{Z_p} = C_{Z_p} - 2C_{Z_\alpha} \frac{\bar{y}}{\ell} \frac{\cos\alpha}{\cos\beta} + 2C_{Z_\beta}\left(\frac{\bar{z}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\sin\alpha\sin\beta\right)$$

$$+ 4C_Z\left(\frac{\bar{z}}{\ell}\sin\beta - \frac{\bar{y}}{\ell}\sin\alpha\cos\beta\right) \qquad \text{(II-14)}$$

$$C'_{Z_q} = C_{Z_q} + \frac{2C_{Z_\alpha}}{\cos\beta}\left(\frac{\bar{x}}{\ell}\cos\alpha + \frac{\bar{z}}{\ell}\sin\alpha\right) + 2C_{Z_\beta}\left(\frac{\bar{z}}{\ell}\cos\alpha - \frac{\bar{x}}{\ell}\sin\alpha\right)\sin\beta$$

$$+ 4C_Z\left(\frac{\bar{x}}{\ell}\sin\alpha - \frac{\bar{z}}{\ell}\cos\alpha\right)\cos\beta \qquad \text{(II-15)}$$

$$C'_{Z_r} = C_{Z_r} - 2C_{Z_\alpha} \frac{\bar{y}}{\ell} \frac{\sin\alpha}{\cos\beta} - 2C_{Z_\beta}\left(\frac{\bar{x}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\cos\alpha\sin\beta\right)$$

$$+ 4C_Z\left(\frac{\bar{y}}{\ell}\cos\alpha\cos\beta - \frac{\bar{x}}{\ell}\sin\beta\right) \qquad \text{(II-16)}$$

X-Axis Moment (Roll) Coefficients and Derivatives

$$C'_l = C_l + \frac{\bar{z}}{\ell} C_Y - \frac{\bar{y}}{\ell} C_Z \qquad \text{(II-17)}$$

$$C'_{l_\alpha} = C_{l_\alpha} + \frac{\bar{z}}{\ell} C_{Y_\alpha} - \frac{\bar{y}}{\ell} C_{Z_\alpha} \qquad \text{(II-18)}$$

$$C'_{l_\beta} = C_{l_\beta} + \frac{\bar{z}}{\ell} C_{Y_\beta} - \frac{\bar{y}}{\ell} C_{Z_\beta} \tag{II-19}$$

$$C'_{l_{\dot{\alpha}}} = C_{l_{\dot{\alpha}}} + \frac{\bar{z}}{\ell} C_{Y_{\dot{\alpha}}} - \frac{\bar{y}}{\ell} C_{Z_{\dot{\alpha}}} \tag{II-20}$$

$$C'_{l_{\dot{\beta}}} = C_{l_{\dot{\beta}}} + \frac{\bar{z}}{\ell} C_{Y_{\dot{\beta}}} - \frac{\bar{y}}{\ell} C_{Z_{\dot{\beta}}} \tag{II-21}$$

$$\begin{aligned} C'_{l_p} = {} & C_{l_p} + \frac{\bar{z}}{\ell} C_{Y_p} - \frac{\bar{y}}{\ell} C_{Z_p} - \frac{2\bar{y}}{\ell} \frac{\cos\alpha}{\cos\beta}\left(C_{l_\alpha} + \frac{\bar{z}}{\ell} C_{Y_\alpha} - \frac{\bar{y}}{\ell} C_{Z_\alpha}\right) \\ & + 2\left(\frac{\bar{z}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\sin\alpha\sin\beta\right)\left(C_{l_\beta} + \frac{\bar{z}}{\ell} C_{Y_\beta} - \frac{\bar{y}}{\ell} C_{Z_\beta}\right) \\ & + 4\left(\frac{\bar{z}}{\ell}\sin\beta - \frac{\bar{y}}{\ell}\sin\alpha\cos\beta\right)\left(C_l + \frac{\bar{z}}{\ell} C_Y - \frac{\bar{y}}{\ell} C_Z\right) \end{aligned} \tag{II-22}$$

$$\begin{aligned} C'_{l_q} = {} & C_{l_q} + \frac{\bar{z}}{\ell} C_{Y_q} - \frac{\bar{y}}{\ell} C_{Z_q} + \frac{2}{\cos\beta}\left(\frac{\bar{x}}{\ell}\cos\alpha + \frac{\bar{z}}{\ell}\sin\alpha\right)\left(C_{l_\alpha} + \frac{\bar{z}}{\ell} C_{Y_\alpha} - \frac{\bar{y}}{\ell} C_{Z_\alpha}\right) \\ & + 2\sin\beta\left(\frac{\bar{z}}{\ell}\cos\alpha - \frac{\bar{x}}{\ell}\sin\alpha\right)\left(C_{l_\beta} + \frac{\bar{z}}{\ell} C_{Y_\beta} - \frac{\bar{y}}{\ell} C_{Z_\beta}\right) \\ & + 4\cos\beta\left(\frac{\bar{x}}{\ell}\sin\alpha - \frac{\bar{z}}{\ell}\cos\alpha\right)\left(C_l + \frac{\bar{z}}{\ell} C_Y - \frac{\bar{y}}{\ell} C_Z\right) \end{aligned} \tag{II-23}$$

$$\begin{aligned} C'_{l_r} = {} & C_{l_r} + \frac{\bar{z}}{\ell} C_{Y_r} - \frac{\bar{y}}{\ell} C_{Z_r} - 2\frac{\bar{y}}{\ell} \frac{\sin\alpha}{\cos\beta}\left(C_{l_\alpha} + \frac{\bar{z}}{\ell} C_{Y_\alpha} - \frac{\bar{y}}{\ell} C_{Z_\alpha}\right) \\ & - 2\left(\frac{\bar{x}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\cos\alpha\sin\beta\right)\left(C_{l_\beta} + \frac{\bar{z}}{\ell} C_{Y_\beta} - \frac{\bar{y}}{\ell} C_{Z_\beta}\right) \\ & + 4\left(\frac{\bar{y}}{\ell}\cos\alpha\cos\beta - \frac{\bar{x}}{\ell}\sin\beta\right)\left(C_l + \frac{\bar{z}}{\ell} C_Y - \frac{\bar{y}}{\ell} \cdot C_Z\right) \end{aligned} \tag{II-24}$$

Y-Axis Moment (Pitch) Coefficients and Derivatives

$$C'_m = C_m + \frac{\bar{x}}{\ell} C_Z - \frac{\bar{z}}{\ell} C_X \tag{II-25}$$

$$C'_{m_\alpha} = C_{m_\alpha} + \frac{\bar{x}}{\ell} C_{Z_\alpha} - \frac{\bar{z}}{\ell} C_{X_\alpha} \tag{II-26}$$

$$C'_{m_\beta} = C_{m_\beta} + \frac{\bar{x}}{\ell} C_{Z_\beta} - \frac{\bar{z}}{\ell} C_{X_\beta} \tag{II-27}$$

$$C'_{m_{\dot{\alpha}}} = C_{m_{\dot{\alpha}}} + \frac{\bar{x}}{\ell} C_{Z_{\dot{\alpha}}} - \frac{\bar{z}}{\ell} C_{X_{\dot{\alpha}}} \tag{II-28}$$

$$C'_{m_{\dot{\beta}}} = C_{m_{\dot{\beta}}} + \frac{\bar{x}}{\ell} C_{Z_{\dot{\beta}}} - \frac{\bar{z}}{\ell} C_{X_{\dot{\beta}}} \tag{II-29}$$

$$\begin{aligned} C'_{m_p} = {} & C_{m_p} + \frac{\bar{x}}{\ell} C_{Z_p} - \frac{\bar{z}}{\ell} C_{X_p} - 2 \frac{\bar{y}}{\ell} \frac{\cos\alpha}{\cos\beta} \left(C_{m_\alpha} + \frac{\bar{x}}{\ell} C_{Z_\alpha} - \frac{\bar{z}}{\ell} C_{X_\alpha}\right) \\ & + 2\left(\frac{\bar{z}}{\ell} \cos\beta + \frac{\bar{y}}{\ell} \sin\alpha \sin\beta\right)\left(C_{m_\beta} + \frac{\bar{x}}{\ell} C_{Z_\beta} - \frac{\bar{z}}{\ell} C_{X_\beta}\right) \\ & + 4\left(\frac{\bar{z}}{\ell} \sin\beta - \frac{\bar{y}}{\ell} \sin\alpha \cos\beta\right)\left(C_m + \frac{\bar{x}}{\ell} C_Z - \frac{\bar{z}}{\ell} C_X\right) \end{aligned} \tag{II-30}$$

$$\begin{aligned} C'_{m_q} = {} & C_{m_q} + \frac{\bar{x}}{\ell} C_{Z_q} - \frac{\bar{z}}{\ell} C_{X_q} + \frac{2}{\cos\beta}\left(\frac{\bar{x}}{\ell} \cos\alpha + \frac{\bar{z}}{\ell} \sin\alpha\right)\left(C_{m_\alpha} + \frac{\bar{x}}{\ell} C_{Z_\alpha} - \frac{\bar{z}}{\ell} C_{X_\alpha}\right) \\ & + 2 \sin\beta\left(\frac{\bar{z}}{\ell} \cos\alpha - \frac{\bar{x}}{\ell} \sin\alpha\right)\left(C_{m_\beta} + \frac{\bar{x}}{\ell} C_{Z_\beta} - \frac{\bar{x}}{\ell} C_{X_\beta}\right) \\ & + 4 \cos\beta\left(\frac{\bar{x}}{\ell} \sin\alpha - \frac{\bar{z}}{\ell} \cos\alpha\right)\left(C_m + \frac{\bar{x}}{\ell} C_Z - \frac{\bar{z}}{\ell} C_X\right) \end{aligned} \tag{II-31}$$

$$\begin{aligned} C'_{m_r} = {} & C_{m_r} + \frac{\bar{x}}{\ell} C_{Z_r} - \frac{\bar{z}}{\ell} C_{X_r} - 2 \frac{\bar{y}}{\ell} \frac{\sin\alpha}{\cos\beta}\left(C_{m_\alpha} + \frac{\bar{x}}{\ell} C_{Z_\alpha} - \frac{\bar{z}}{\ell} C_{X_\alpha}\right) \\ & - 2\left(\frac{\bar{x}}{\ell} \cos\beta + \frac{\bar{y}}{\ell} \cos\alpha \sin\beta\right)\left(C_{m_\beta} + \frac{\bar{x}}{\ell} C_{Z_\beta} - \frac{\bar{z}}{\ell} C_{X_\beta}\right) \\ & + 4\left(\frac{\bar{y}}{\ell} \cos\alpha \cos\beta - \frac{\bar{x}}{\ell} \sin\beta\right)\left(C_m + \frac{\bar{x}}{\ell} C_Z - \frac{\bar{z}}{\ell} C_X\right) \end{aligned} \tag{II-32}$$

Z-Axis Moment (Yaw) Coefficients and Derivatives

$$C'_n = C_n + \frac{\bar{y}}{\ell} C_X - \frac{\bar{x}}{\ell} C_Y \tag{II-33}$$

$$C'_{n_\alpha} = C_{n_\alpha} + \frac{\bar{y}}{\ell} C_{X_\alpha} - \frac{\bar{x}}{\ell} C_{Y_\alpha} \tag{II-34}$$

$$C'_{n_\beta} = C_{n_\beta} + \frac{\bar{y}}{\ell} C_{X_\beta} - \frac{\bar{x}}{\ell} C_{Y_\beta} \tag{II-35}$$

$$C'_{n_{\dot\alpha}} = C_{n_{\dot\alpha}} + \frac{\bar{y}}{\ell} C_{X_{\dot\alpha}} - \frac{\bar{x}}{\ell} C_{Y_{\dot\alpha}} \tag{II-36}$$

$$C'_{n_{\dot\beta}} = C_{n_{\dot\beta}} + \frac{\bar{y}}{\ell} C_{X_{\dot\beta}} - \frac{\bar{x}}{\ell} C_{Y_{\dot\beta}} \tag{II-37}$$

$$\begin{aligned} C'_{n_p} = {} & C_{n_p} + \frac{\bar{y}}{\ell} C_{X_p} - \frac{\bar{x}}{\ell} C_{Y_p} - 2\frac{\bar{y}}{\ell}\frac{\cos\alpha}{\cos\beta}\left(C_{n_\alpha} + \frac{\bar{y}}{\ell} C_{X_\alpha} - \frac{\bar{x}}{\ell} C_{Y_\alpha}\right) \\ & + 2\left(\frac{\bar{z}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\sin\alpha\sin\beta\right)\left(C_{n_\beta} + \frac{\bar{y}}{\ell} C_{X_\beta} - \frac{\bar{x}}{\ell} C_{Y_\beta}\right) \\ & + 4\left(\frac{\bar{z}}{\ell}\sin\beta - \frac{\bar{y}}{\ell}\sin\alpha\cos\beta\right)\left(C_n + \frac{\bar{y}}{\ell} C_X - \frac{\bar{x}}{\ell} C_Y\right) \end{aligned} \tag{II-38}$$

$$\begin{aligned} C'_{n_q} = {} & C_{n_q} + \frac{\bar{y}}{\ell} C_{X_q} - \frac{\bar{x}}{\ell} C_{Y_q} + \frac{2}{\cos\beta}\left(\frac{\bar{x}}{\ell}\cos\alpha + \frac{\bar{z}}{\ell}\sin\alpha\right)\left(C_{n_\alpha} + \frac{\bar{y}}{\ell} C_{X_\alpha} - \frac{\bar{x}}{\ell} C_{Y_\alpha}\right) \\ & + 2\sin\beta\left(\frac{\bar{z}}{\ell}\cos\alpha - \frac{\bar{x}}{\ell}\sin\alpha\right)\left(C_{n_\beta} + \frac{\bar{y}}{\ell} C_{X_\beta} - \frac{\bar{x}}{\ell} C_{Y_\beta}\right) \\ & + 4\cos\beta\left(\frac{\bar{x}}{\ell}\sin\alpha - \frac{\bar{z}}{\ell}\cos\alpha\right)\left(C_n + \frac{\bar{y}}{\ell} C_X - \frac{\bar{x}}{\ell} C_Y\right) \end{aligned} \tag{II-39}$$

$$\begin{aligned} C'_{n_r} = {} & C_{n_r} + \frac{\bar{y}}{\ell} C_{X_r} - \frac{\bar{x}}{\ell} C_{Y_r} - 2\frac{\bar{y}}{\ell}\frac{\sin\alpha}{\cos\beta}\left(C_{n_\alpha} + \frac{\bar{y}}{\ell} C_{X_\alpha} - \frac{\bar{x}}{\ell} C_{Y_\alpha}\right) \\ & - 2\left(\frac{\bar{x}}{\ell}\cos\beta + \frac{\bar{y}}{\ell}\cos\alpha\sin\beta\right)\left(C_{n_\beta} + \frac{\bar{y}}{\ell} C_{X_\beta} - \frac{\bar{x}}{\ell} C_{Y_\beta}\right) \\ & + 4\left(\frac{\bar{y}}{\ell}\cos\alpha\cos\beta - \frac{\bar{x}}{\ell}\sin\beta\right)\left(C_n + \frac{\bar{y}}{\ell} C_X - \frac{\bar{x}}{\ell} C_Y\right) \end{aligned} \tag{II-40}$$

TRANSFORMATIONS FOR u,v,w DERIVATIVES

X-Axis Force Coefficients and Derivatives

$$C'_{X_u} = C_{X_u} \tag{II-41}$$

$$C'_{X_v} = C_{X_v} \tag{II-42}$$

$$C'_{X_w} = C_{X_w} \tag{II-43}$$

$$C'_{X_p} = C_{X_p} + \frac{\bar{z}}{\ell} C_{X_v} - \frac{\bar{y}}{\ell} C_{X_w} \tag{II-44}$$

$$C'_{X_q} = C_{X_q} - \frac{\bar{z}}{\ell} C_{X_u} + \frac{\bar{x}}{\ell} C_{X_w} \tag{II-45}$$

$$C'_{X_r} = C_{X_r} + \frac{\bar{y}}{\ell} C_{X_u} - \frac{\bar{x}}{\ell} C_{X_v} \tag{II-46}$$

Y-Axis Force Coefficients and Derivatives

$$C'_{Y_u} = C_{Y_u} \tag{II-47}$$

$$C'_{Y_v} = C_{Y_v} \tag{II-48}$$

$$C'_{Y_w} = C_{Y_w} \tag{II-49}$$

$$C'_{Y_p} = C_{Y_p} + \frac{\bar{z}}{\ell} C_{Y_v} - \frac{\bar{y}}{\ell} C_{Y_w} \tag{II-50}$$

$$C'_{Y_q} = C_{Y_q} - \frac{\bar{z}}{\ell} C_{Y_u} + \frac{\bar{x}}{\ell} C_{Y_w} \tag{II-51}$$

$$C'_{Y_r} = C_{Y_r} + \frac{\bar{y}}{\ell} C_{Y_u} - \frac{\bar{x}}{\ell} C_{Y_v} \tag{II-52}$$

Z-Axis Force Coefficients and Derivatives

$$C'_{Z_u} = C_{Z_u} \tag{II-53}$$

$$C'_{Z_v} = C_{Z_v} \tag{II-54}$$

$$C'_{Z_w} = C_{Z_w} \tag{II-55}$$

$$C'_{Z_p} = C_{Z_p} + \frac{\bar{z}}{\ell} C_{Z_v} - \frac{\bar{y}}{\ell} C_{Z_w} \tag{II-56}$$

$$C'_{Z_q} = C_{Z_q} - \frac{\bar{z}}{\ell} C_{Z_u} + \frac{\bar{x}}{\ell} C_{Z_w} \tag{II-57}$$

$$C'_{Z_r} = C_{Z_r} + \frac{\bar{y}}{\ell} C_{Z_u} - \frac{\bar{x}}{\ell} C_{Z_v} \tag{II-58}$$

X-Axis Moment (Roll) Coefficients and Derivatives

$$C'_{l_u} = C_{l_u} + \frac{\bar{z}}{\ell} C_{Y_u} - \frac{\bar{y}}{\ell} C_{Z_u} \qquad \text{(II-59)}$$

$$C'_{l_v} = C_{l_v} + \frac{\bar{z}}{\ell} C_{Y_v} - \frac{\bar{y}}{\ell} C_{Z_v} \qquad \text{(II-60)}$$

$$C'_{l_w} = C_{l_w} + \frac{\bar{z}}{\ell} C_{Y_w} - \frac{\bar{y}}{\ell} C_{Z_w} \qquad \text{(II-61)}$$

$$C'_{l_p} = C_{l_p} + \frac{\bar{z}}{\ell} C_{Y_p} - \frac{\bar{y}}{\ell} C_{Z_p} + \frac{\bar{z}}{\ell}\left(C_{l_v} + \frac{\bar{z}}{\ell} C_{Y_v} - \frac{\bar{y}}{\ell} C_{Z_v}\right) - \frac{\bar{y}}{\ell}\left(C_{l_w} + \frac{\bar{z}}{\ell} C_{Y_w} - \frac{\bar{y}}{\ell} C_{Z_w}\right) \qquad \text{(II-62)}$$

$$C'_{l_q} = C_{l_q} + \frac{\bar{z}}{\ell} C_{Y_q} - \frac{\bar{y}}{\ell} C_{Z_q} - \frac{\bar{z}}{\ell}\left(C_{l_u} + \frac{\bar{z}}{\ell} C_{Y_u} - \frac{\bar{y}}{\ell} C_{Z_u}\right) + \frac{\bar{x}}{\ell}\left(C_{l_w} + \frac{\bar{z}}{\ell} C_{Y_w} - \frac{\bar{y}}{\ell} C_{Z_w}\right) \qquad \text{(II-63)}$$

$$C'_{l_r} = C_{l_r} + \frac{\bar{z}}{\ell} C_{Y_r} - \frac{\bar{y}}{\ell} C_{Z_r} + \frac{\bar{y}}{\ell}\left(C_{l_u} + \frac{\bar{z}}{\ell} C_{Y_u} - \frac{\bar{y}}{\ell} C_{Z_u}\right) - \frac{\bar{x}}{\ell}\left(C_{l_v} + \frac{\bar{z}}{\ell} C_{Y_v} - \frac{\bar{y}}{\ell} C_{Z_v}\right) \qquad \text{(II-64)}$$

Y-Axis Moment (Pitch) Coefficients and Derivatives

$$C'_{m_u} = C_{m_u} + \frac{\bar{x}}{\ell} C_{Z_u} - \frac{\bar{z}}{\ell} C_{X_u} \qquad \text{(II-65)}$$

$$C'_{m_v} = C_{m_v} + \frac{\bar{x}}{\ell} C_{Z_v} - \frac{\bar{z}}{\ell} C_{X_v} \qquad \text{(II-66)}$$

$$C'_{m_w} = C_{m_w} + \frac{\bar{x}}{\ell} C_{Z_w} - \frac{\bar{z}}{\ell} C_{X_w} \qquad \text{(II-67)}$$

$$C'_{m_p} = C_{m_p} + \frac{\bar{x}}{\ell} C_{Z_p} - \frac{\bar{z}}{\ell} C_{X_p} + \frac{\bar{z}}{\ell}\left(C_{m_v} + \frac{\bar{x}}{\ell} C_{Z_v} - \frac{\bar{z}}{\ell} C_{X_v}\right) - \frac{\bar{y}}{\ell}\left(C_{m_w} + \frac{\bar{x}}{\ell} C_{Z_w} - \frac{\bar{z}}{\ell} C_{X_w}\right) \qquad \text{(II-68)}$$

$$C'_{m_q} = C_{m_q} + \frac{\bar{x}}{\ell} C_{Z_q} - \frac{\bar{z}}{\ell} C_{X_q} - \frac{\bar{z}}{\ell}\left(C_{m_u} + \frac{\bar{x}}{\ell} C_{Z_u} - \frac{\bar{z}}{\ell} C_{X_u}\right) + \frac{\bar{x}}{\ell}\left(C_{m_w} + \frac{\bar{x}}{\ell} C_{Z_w} - \frac{\bar{z}}{\ell} C_{X_w}\right) \qquad \text{(II-69)}$$

$$C'_{m_r} = C_{m_r} + \frac{\bar{x}}{\ell} C_{Z_r} - \frac{\bar{z}}{\ell} C_{X_r} + \frac{\bar{y}}{\ell}\left(C_{m_u} + \frac{\bar{x}}{\ell} C_{Z_u} - \frac{\bar{z}}{\ell} C_{X_u}\right)$$

$$- \frac{\bar{x}}{\ell}\left(C_{m_v} + \frac{\bar{x}}{\ell} C_{Z_v} - \frac{\bar{z}}{\ell} C_{X_v}\right) \qquad \text{(II-70)}$$

Z-Axis Moment (Yaw) Coefficients and Derivatives

$$C'_{n_u} = C_{n_u} + \frac{\bar{y}}{\ell} C_{X_u} - \frac{\bar{x}}{\ell} C_{Y_u} \qquad \text{(II-71)}$$

$$C'_{n_v} = C_{n_v} + \frac{\bar{y}}{\ell} C_{X_v} - \frac{\bar{x}}{\ell} C_{Y_v} \qquad \text{(II-72)}$$

$$C'_{n_w} = C_{n_w} + \frac{\bar{y}}{\ell} C_{X_w} - \frac{\bar{x}}{\ell} C_{Y_w} \qquad \text{(II-73)}$$

$$C'_{n_p} = C_{n_p} + \frac{\bar{y}}{\ell} C_{X_p} - \frac{\bar{x}}{\ell} C_{Y_p} + \frac{\bar{z}}{\ell}\left(C_{n_v} + \frac{\bar{y}}{\ell} C_{X_v} - \frac{\bar{x}}{\ell} C_{Y_v}\right)$$

$$- \frac{\bar{y}}{\ell}\left(C_{n_w} + \frac{\bar{y}}{\ell} C_{X_w} - \frac{\bar{x}}{\ell} C_{Y_w}\right) \qquad \text{(II-74)}$$

$$C'_{n_q} = C_{n_q} + \frac{\bar{y}}{\ell} C_{X_q} - \frac{\bar{x}}{\ell} C_{Y_q} - \frac{\bar{z}}{\ell}\left(C_{n_u} + \frac{\bar{y}}{\ell} C_{X_u} - \frac{\bar{x}}{\ell} C_{Y_u}\right)$$

$$+ \frac{\bar{x}}{\ell}\left(C_{n_w} + \frac{\bar{y}}{\ell} C_{X_w} - \frac{\bar{x}}{\ell} C_{Y_w}\right) \qquad \text{(II-75)}$$

$$C'_{n_r} = C_{n_r} + \frac{\bar{y}}{\ell} C_{X_r} - \frac{\bar{x}}{\ell} C_{Y_r} + \frac{\bar{y}}{\ell}\left(C_{n_u} + \frac{\bar{y}}{\ell} C_{X_u} - \frac{\bar{x}}{\ell} C_{Y_u}\right)$$

$$- \frac{\bar{x}}{\ell}\left(C_{n_v} + \frac{\bar{y}}{\ell} C_{X_v} - \frac{\bar{x}}{\ell} C_{Y_v}\right) \qquad \text{(II-76)}$$

SECTION III

TRANSFER OF AERODYNAMIC FORCE AND MOMENT COEFFICIENTS AND DERIVATIVES TO A ROTATED AXES SYSTEM

The equations in this section are the general forms of the axes transformations for aerodynamic force and moment coefficients and stability derivatives that can be used to transfer between any two of the five axes systems used in aerodynamic analysis. (Particular forms for transfer between body and wind-tunnel stability axes are given in appendix A.) Transformations are presented for both α,β,V derivatives and u,v,w derivatives. (For the relationships between the two sets of derivatives, see table II.) The equations are developed in terms of a generalized angle of attack A and generalized sideslip angle B; transformations between any two axes systems can be obtained by substituting specific angles for A and B in these equations. (See tables III and IV.)

DESCRIPTIONS OF AXES SYSTEMS

The five axes systems considered are those shown in figure 3. While three of these systems, body, principal, and wind axes, are clearly defined in the literature, there is some confusion concerning the definition of the stability axes. The stability axes described in some reports are vehicle or flight stability axes about which the equations of motion are written; in other reports they are wind-tunnel stability axes about which aerodynamic data are measured in the wind tunnel. The differences between the two are pointed out in these brief descriptions of the axes systems.

Body Axes

The orthogonal body-axes system is fixed within the vehicle with the X-axis along the longitudinal center line of the body, the Y-axis normal to the plane of symmetry, and the Z-axis in the plane of symmetry. This is the axes system about which aircraft instruments are usually mounted. Its main advantage in motion calculations is that vehicle moments of inertia about the axes are constant, so that the $\dot{I}$ terms can be omitted from the equations of motion. It is the logical system to which to refer velocities, accelerations, and stability and control parameters in the study of aircraft handling qualities because the pilot's orientation with respect to this frame is fixed.

Principal Axes

The principal axes are an orthogonal body-fixed system for which the products of inertia are zero. The X and Z principal axes lie in the plane of symmetry; the angle between the X body axis and the X principal axis is usually small so that in many cases the body axes can be assumed to coincide with the principal axes.

Flight Stability Axes

The flight stability axes (sometimes referred to as vehicle stability axes) are an orthogonal body-axes system fixed to the vehicle, the X-axis of which is alined with the relative wind vector when the vehicle is in a steady-state trim condition but then rotates with the vehicle after a disturbance as the vehicle changes angle of attack. This system is preferred in many stability studies because, as with other body-fixed axes, the moments of inertia about the axes remain constant and also because the motions defined are primarily those about the flight path rather than about body reference lines.

Wind-Tunnel Stability Axes

The wind-tunnel stability axes are the system about which most wind-tunnel data are obtained. For this system the X-axis is in the same horizontal plane as the relative wind at all times (fig. 3). The angle α between the X-axis of this system and the X body axes is variable. (It is a constant α_0 for the flight stability axes.) This means that vehicle moments of inertia about the X-axis change. It also means that additional terms are required in the transformation equations for static-stability derivatives and for u,v,w derivatives when data are transferred to or from the wind axes or the wind-tunnel stability axes. These additional terms are designated (1), as in equation (III-7), for example.

Wind Axes

The wind axes are the system generally used in calculating motions of the vehicle as a point mass. The X-axis for this system is alined with the relative wind at all times so that vehicle moments of inertia about this axis change. As with the wind-tunnel stability axes, additional terms, designated (1), are required in the transformations to or from the wind axes and either the body, principal, or flight stability axes, since the angle A between the X wind axis and the X-axis of either of these systems is variable. Also, since the lateral angle B between the X-axes is variable, there are additional terms, designated (2), as in equation (III-13), required in the transformations for some of the lateral derivatives between the wind axes and either of the other axes systems.

NOTES ON USE OF TRANSFORMATION EQUATIONS

In the transformations that follow, symbols such as C'_X, C'_Y, C'_Z and C_X, C_Y, C_Z are used in a general sense to designate coefficients and derivatives about corresponding

X',Y',Z' and X,Y,Z axes systems; specific designations to use with each axes system are given in table V. Two types of transformation are given: "Direct" which include those defined in table III and "inverse" which include those defined in table IV. Transformations between any two axes systems are obtained by selecting the proper angles for A and B from table III or IV and then the proper coefficient designations from table V. For example, in transferring from flight stability axes to body axes (a direct transformation, according to table III), angle A is replaced by α_0 and angle B equals zero; the transformation for C_X then becomes, by using equation (III-1) and table V,

$$C_X = C_{X,s} \cos \alpha_0 - C_{Z,s} \sin \alpha_0$$

As a second example, in determining the C_{l_p} about the wind-tunnel stability axes from derivatives given about the body axes (an inverse transformation according to table IV), equation (III-82) and tables IV and V are used to obtain

$$C_{l_{p,wt}} = C_{l_p} \cos^2 \alpha + C_{n_r} \sin^2 \alpha + \left(C_{l_r} + C_{n_p}\right) \sin \alpha \cos \alpha$$

In the transformations for static-stability derivatives and also in the transformations for u,v,w derivatives, the terms designated (1) are included only in transferring from either wind or wind-tunnel stability axes to either body, principal, or flight stability axes and not in other transformations. For example, terms (1) are included when the body-axes derivative C_{X_α} (eq. (III-7)) is determined from data given about wind-tunnel stability axes as in

$$C_{X_\alpha} = C_{X_{\alpha,wt}} \cos \alpha - C_{Z_{\alpha,wt}} \sin \alpha - C_{X,wt} \sin \alpha - C_{Z,wt} \cos \alpha$$

but are not included when C_{X_α} is determined from data about flight stability axes as in

$$C_{X_\alpha} = C_{X_{\alpha,s}} \cos \alpha_0 - C_{Z_{\alpha,s}} \sin \alpha_0$$

The terms designated (2) are included only in transferring to or from wind axes. For example, in transferring from wind axes, the body-axes derivative C_{l_β} (eq. (III-16)) is given by

$$C_{l_\beta} = C_{l_{\beta,w}} \cos \alpha \cos \beta - C_{m_{\beta,w}} \cos \alpha \sin \beta - C_{n_{\beta,w}} \sin \alpha$$

$$- C_{l,w} \cos \alpha \sin \beta - C_{m,w} \cos \alpha \cos \beta$$

but, in transferring from wind-tunnel stability axes, becomes

$$C_{l_\beta} = C_{l_{\beta,wt}} \cos \alpha - C_{n_{\beta,wt}} \sin \alpha$$

If the need arises, derivatives with respect to α,β,V can be converted to derivatives with respect to u,v,w by using the relationships given in table II.

DIRECT TRANSFORMATIONS

Static Force and Moment Coefficients[1] (Direct, Table III)

$$C'_X = C_X \cos A \cos B - C_Y \cos A \sin B - C_Z \sin A \tag{III-1}$$

$$C'_Y = C_X \sin B + C_Y \cos B \tag{III-2}$$

$$C'_Z = C_X \sin A \cos B - C_Y \sin A \sin B + C_Z \cos A \tag{III-3}$$

$$C'_l = C_l \cos A \cos B - C_m \cos A \sin B - C_n \sin A \tag{III-4}$$

$$C'_m = C_l \sin B + C_m \cos B \tag{III-5}$$

$$C'_n = C_l \sin A \cos B - C_m \sin A \sin B + C_n \cos A \tag{III-6}$$

Static-Stability Derivatives (Direct, Table III)

$$C'_{X_\alpha} = C_{X_\alpha} \cos A \cos B - C_{Y_\alpha} \cos A \sin B - C_{Z_\alpha} \sin A$$
$$- \underbrace{C_X \sin A \cos B + C_Y \sin A \sin B - C_Z \cos A}_{(1)} \tag{III-7}$$

where the terms designated (1) are included only in transferring from either wind or wind-tunnel stability axes to either body, principal, or flight stability axes and not in other transformations.

$$C'_{Y_\alpha} = C_{X_\alpha} \sin B + C_{Y_\alpha} \cos B \tag{III-8}$$

$$C'_{Z_\alpha} = C_{X_\alpha} \sin A \cos B - C_{Y_\alpha} \sin A \sin B + C_{Z_\alpha} \cos A$$
$$+ \underbrace{C_X \cos A \cos B - C_Y \cos A \sin B - C_Z \sin A}_{(1)} \tag{III-9}$$

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[1] Equations for control derivatives and velocity derivatives have this same form; for example,

$$C'_{X_{\delta_a}} = C_{X_{\delta_a}} \cos A \cos B - C_{Y_{\delta_a}} \cos A \sin B - C_{Z_{\delta_a}} \sin A$$

$$C'_{X_V} = C_{X_V} \cos A \cos B - C_{Y_V} \cos A \sin B - C_{Z_V} \sin A$$

$$C'_{l_\alpha} = C_{l_\alpha} \cos A \cos B - C_{m_\alpha} \cos A \sin B - C_{n_\alpha} \sin A$$

$$- \underbrace{C_l \sin A \cos B + C_m \sin A \sin B - C_n \cos A}_{(1)} \qquad \text{(III-10)}$$

$$C'_{m_\alpha} = C_{l_\alpha} \sin B + C_{m_\alpha} \cos B \qquad \text{(III-11)}$$

$$C'_{n_\alpha} = C_{l_\alpha} \sin A \cos B - C_{m_\alpha} \sin A \sin B + C_{n_\alpha} \cos A$$

$$+ \underbrace{C_l \cos A \cos B - C_m \cos A \sin B - C_n \sin A}_{(1)} \qquad \text{(III-12)}$$

$$C'_{X_\beta} = C_{X_\beta} \cos A \cos B - C_{Y_\beta} \cos A \sin B - C_{Z_\beta} \sin A$$

$$- \underbrace{C_X \cos A \sin B - C_Y \cos A \cos B}_{(2)} \qquad \text{(III-13)}$$

where the terms designated (2) are included only in transferring to or from wind axes.

$$C'_{Y_\beta} = C_{X_\beta} \sin B + C_{Y_\beta} \cos B + \underbrace{C_X \cos B - C_Y \sin B}_{(2)} \qquad \text{(III-14)}$$

$$C'_{Z_\beta} = C_{X_\beta} \sin A \cos B - C_{Y_\beta} \sin A \sin B + C_{Z_\beta} \cos A$$

$$- \underbrace{C_X \sin A \sin B - C_Y \sin A \cos B}_{(2)} \qquad \text{(III-15)}$$

$$C'_{l_\beta} = C_{l_\beta} \cos A \cos B - C_{m_\beta} \cos A \sin B - C_{n_\beta} \sin A$$

$$- \underbrace{C_l \cos A \sin B - C_m \cos A \cos B}_{(2)} \qquad \text{(III-16)}$$

$$C'_{m_\beta} = C_{l_\beta} \sin B + C_{m_\beta} \cos B + \underbrace{C_l \cos B - C_m \sin B}_{(2)} \qquad \text{(III-17)}$$

$$C'_{n_\beta} = C_{l_\beta} \sin A \cos B - C_{m_\beta} \sin A \sin B + C_{n_\beta} \cos A$$

$$- \underbrace{C_l \sin A \sin B - C_m \sin A \cos B}_{(2)} \qquad \text{(III-18)}$$

Dynamic-Stability Derivatives (Direct, Table III)

$$C'_{X_p} = \left[C_{X_p} \cos^2 B - \left(C_{X_q} + C_{Y_p}\right) \sin B \cos B + C_{Y_q} \sin^2 B\right] \cos^2 A$$

$$+ C_{Z_r} \sin^2 A - \left[\left(C_{X_r} + C_{Z_p}\right) \cos B - \left(C_{Y_r} + C_{Z_q}\right) \sin B\right] \sin A \cos A \qquad \text{(III-19)}$$

$$C'_{X_q} = \left[C_{X_q} \cos^2 B - C_{Y_p} \sin^2 B + \left(C_{X_p} - C_{Y_q}\right) \sin B \cos B\right] \cos A$$

$$- \left(C_{Z_q} \cos B + C_{Z_p} \sin B\right) \sin A \qquad \text{(III-20)}$$

$$C'_{X_r} = \left(C_{X_r} \cos B - C_{Y_r} \sin B\right) \cos^2 A - \left(C_{Z_p} \cos B - C_{Z_q} \sin B\right) \sin^2 A$$

$$+ \left[C_{X_p} \cos^2 B + C_{Y_q} \sin^2 B - \left(C_{X_q} + C_{Y_p}\right) \sin B \cos B - C_{Z_r}\right] \sin A \cos A \qquad \text{(III-21)}$$

$$C'_{Y_p} = \left[C_{Y_p} \cos^2 B - C_{X_q} \sin^2 B + \left(C_{X_p} - C_{Y_q}\right) \sin B \cos B\right] \cos A$$

$$- \left(C_{Y_r} \cos B + C_{X_r} \sin B\right) \sin A \qquad \text{(III-22)}$$

$$C'_{Y_q} = C_{Y_q} \cos^2 B + C_{X_p} \sin^2 B + \left(C_{X_q} + C_{Y_p}\right) \sin B \cos B \qquad \text{(III-23)}$$

$$C'_{Y_r} = \left(C_{Y_r} \cos B + C_{X_r} \sin B\right) \cos A$$

$$+ \left[C_{Y_p} \cos^2 B - C_{X_q} \sin^2 B + \left(C_{X_p} - C_{Y_q}\right) \sin B \cos B\right] \sin A \qquad \text{(III-24)}$$

$$C'_{Z_p} = \left(C_{Z_p} \cos B - C_{Z_q} \sin B\right) \cos^2 A - \left(C_{X_r} \cos B - C_{Y_r} \sin B\right) \sin^2 A$$

$$+ \left[C_{X_p} \cos^2 B + C_{Y_q} \sin^2 B - \left(C_{X_q} + C_{Y_p}\right) \sin B \cos B - C_{Z_r}\right] \sin A \cos A \qquad \text{(III-25)}$$

$$C'_{Z_q} = \left(C_{Z_q} \cos B + C_{Z_p} \sin B\right) \cos A$$
$$+ \left[C_{X_q} \cos^2 B - C_{Y_p} \sin^2 B + \left(C_{X_p} - C_{Y_q}\right) \sin B \cos B\right] \sin A \qquad \text{(III-26)}$$

$$C'_{Z_r} = C_{Z_r} \cos^2 A + \left[C_{X_p} \cos^2 B + C_{Y_q} \sin^2 B - \left(C_{X_q} + C_{Y_p}\right) \sin B \cos B\right] \sin^2 A$$
$$+ \left[\left(C_{X_r} + C_{Z_p}\right) \cos B - \left(C_{Y_r} + C_{Z_q}\right) \sin B\right] \sin A \cos A \qquad \text{(III-27)}$$

$$C'_{l_p} = \left[C_{l_p} \cos^2 B - \left(C_{l_q} + C_{m_p}\right) \sin B \cos B + C_{m_q} \sin^2 B\right] \cos^2 A + C_{n_r} \sin^2 A$$
$$+ \left[-\left(C_{l_r} + C_{n_p}\right) \cos B + \left(C_{m_r} + C_{n_q}\right) \sin B\right] \sin A \cos A \qquad \text{(III-28)}$$

$$C'_{l_q} = \left[C_{l_q} \cos^2 B - C_{m_p} \sin^2 B + \left(C_{l_p} - C_{m_q}\right) \sin B \cos B\right] \cos A$$
$$- \left(C_{n_q} \cos B + C_{n_p} \sin B\right) \sin A \qquad \text{(III-29)}$$

$$C'_{l_r} = \left(C_{l_r} \cos B - C_{m_r} \sin B\right) \cos^2 A - \left(C_{n_p} \cos B - C_{n_q} \sin B\right) \sin^2 A$$
$$+ \left[C_{l_p} \cos^2 B + C_{m_q} \sin^2 B - \left(C_{l_q} + C_{m_p}\right) \sin B \cos B - C_{n_r}\right] \sin A \cos A \qquad \text{(III-30)}$$

$$C'_{m_p} = \left[C_{m_p} \cos^2 B - C_{l_q} \sin^2 B + \left(C_{l_p} - C_{m_q}\right) \sin B \cos B\right] \cos A$$
$$- \left(C_{m_r} \cos B + C_{l_r} \sin B\right) \sin A \qquad \text{(III-31)}$$

$$C'_{m_q} = C_{m_q} \cos^2 B + C_{l_p} \sin^2 B + \left(C_{l_q} + C_{m_p}\right) \sin B \cos B \qquad \text{(III-32)}$$

$$C'_{m_r} = \left(C_{m_r} \cos B + C_{l_r} \sin B\right) \cos A$$
$$+ \left[C_{m_p} \cos^2 B - C_{l_q} \sin^2 B + \left(C_{l_p} - C_{m_q}\right) \sin B \cos B\right] \sin A \qquad \text{(III-33)}$$

$$C'_{n_p} = \left(C_{n_p} \cos B - C_{n_q} \sin B\right)\cos^2 A - \left(C_{l_r} \cos B - C_{m_r} \sin B\right)\sin^2 A$$

$$+ \left[C_{l_p} \cos^2 B + C_{m_q} \sin^2 B - \left(C_{l_q} + C_{m_p}\right)\sin B \cos B - C_{n_r}\right]\sin A \cos A \qquad \text{(III-34)}$$

$$C'_{n_q} = \left(C_{n_q} \cos B + C_{n_p} \sin B\right)\cos A$$

$$+ \left[C_{l_q} \cos^2 B - C_{m_p} \sin^2 B + \left(C_{l_p} - C_{m_q}\right)\sin B \cos B\right]\sin A \qquad \text{(III-35)}$$

$$C'_{n_r} = C_{n_r} \cos^2 A + \left[C_{l_p} \cos^2 B + C_{m_q} \sin^2 B - \left(C_{l_q} + C_{m_p}\right)\sin B \cos B\right]\sin^2 A$$

$$+ \left[\left(C_{l_r} + C_{n_p}\right)\cos B - \left(C_{m_r} + C_{n_q}\right)\sin B\right]\sin A \cos A \qquad \text{(III-36)}$$

u,v,w Derivatives (Direct, Table III)

$$C'_{X_u} = \left[C_{X_u} \cos^2 B - \left(C_{X_v} + C_{Y_u}\right)\sin B \cos B + C_{Y_v} \sin^2 B\right]\cos^2 A + C_{Z_w} \sin^2 A$$

$$+ \left[-\left(C_{X_w} + C_{Z_u}\right)\cos B + \left(C_{Y_w} + C_{Z_v}\right)\sin B\right]\sin A \cos A$$

$$+ \underbrace{\frac{1}{\cos B}\left(C_X \sin^2 A \cos B - C_Y \sin^2 A \sin B + C_Z \sin A \cos A\right)}_{(1)}$$

$$+ \underbrace{C_X \cos^2 A \sin^2 B + C_Y \cos^2 A \sin B \cos B}_{(2)} \qquad \text{(III-37)}$$

$$C'_{X_v} = \left[C_{X_v} \cos^2 B - C_{Y_u} \sin^2 B + \left(C_{X_u} - C_{Y_v}\right)\sin B \cos B\right]\cos A$$

$$- \left(C_{Z_v} \cos B + C_{Z_u} \sin B\right)\sin A - \underbrace{\left(C_X \cos A \sin B + C_Y \cos A \cos B\right)\cos B}_{(2)}$$

(III-38)

$$C'_{X_w} = \left(C_{X_w} \cos B - C_{Y_w} \sin B\right)\cos^2 A - \left(C_{Z_u} \cos B - C_{Z_v} \sin B\right)\sin^2 A$$

$$+ \left[C_{X_u} \cos^2 B + C_{Y_v} \sin^2 B - \left(C_{X_v} + C_{Y_u}\right)\sin B \cos B - C_{Z_w}\right] \sin A \cos A$$

$$- \underbrace{\frac{\cos A}{\cos B}\left(C_X \sin A \cos B - C_Y \sin A \sin B + C_Z \cos A\right)}_{(1)}$$

$$+ \underbrace{\sin A \sin B\left(C_X \cos A \sin B + C_Y \cos A \cos B\right)}_{(2)} \qquad \text{(III-39)}$$

$$C'_{Y_u} = \left[C_{Y_u} \cos^2 B - C_{X_v} \sin^2 B + \left(C_{X_u} - C_{Y_v}\right) \sin B \cos B\right] \cos A$$

$$- \left(C_{Y_w} \cos B + C_{X_w} \sin B\right) \sin A - \underbrace{\left(C_X \cos B - C_Y \sin B\right) \cos A \sin B}_{(2)} \qquad \text{(III-40)}$$

$$C'_{Y_v} = C_{Y_v} \cos^2 B + C_{X_u} \sin^2 B + \left(C_{X_v} + C_{Y_u}\right) \sin B \cos B$$

$$+ \underbrace{\left(C_X \cos B - C_Y \sin B\right) \cos B}_{(2)} \qquad \text{(III-41)}$$

$$C'_{Y_w} = \left(C_{Y_w} \cos B + C_{X_w} \sin B\right) \cos A$$

$$+ \left[C_{Y_u} \cos^2 B - C_{X_v} \sin^2 B + \left(C_{X_u} - C_{Y_v}\right) \sin B \cos B\right] \sin A$$

$$- \underbrace{\left(C_X \cos B - C_Y \sin B\right) \sin A \sin B}_{(2)} \qquad \text{(III-42)}$$

$$C'_{Z_u} = \left(C_{Z_u} \cos B - C_{Z_v} \sin B\right)\cos^2 A - \left(C_{X_w} \cos B - C_{Y_w} \sin B\right)\sin^2 A$$

$$+ \left[C_{X_u} \cos^2 B + C_{Y_v} \sin^2 B - \left(C_{X_v} + C_{Y_u}\right)\sin B \cos B - C_{Z_w}\right] \sin A \cos A$$

$$- \underbrace{\left(C_X \cos A \cos B - C_Y \cos A \sin B - C_Z \sin A\right)\frac{\sin A}{\cos B}}_{(1)}$$

$$+ \underbrace{\left(C_X \sin A \sin B + C_Y \sin A \cos B\right) \cos A \sin B}_{(2)} \qquad \text{(III-43)}$$

$$C'_{Z_v} = \left(C_{Z_v} \cos B + C_{Z_u} \sin B\right)\cos A$$

$$+ \left[C_{X_v} \cos^2 B - C_{Y_u} \sin^2 B + \left(C_{X_u} - C_{Y_v}\right)\sin B \cos B\right]\sin A$$

$$- \underbrace{\left(C_X \sin B + C_Y \cos B\right)\sin A \cos B}_{(2)} \tag{III-44}$$

$$C'_{Z_w} = C_{Z_w} \cos^2 A + \left[C_{X_u} \cos^2 B + C_{Y_v} \sin^2 B - \left(C_{X_v} + C_{Y_u}\right)\sin B \cos B\right]\sin^2 A$$

$$+ \left[\left(C_{X_w} + C_{Z_u}\right)\cos B - \left(C_{Y_w} + C_{Z_v}\right)\sin B\right]\sin A \cos A$$

$$+ \underbrace{\left(C_X \cos A \cos B - C_Y \cos A \sin B - C_Z \sin A\right)\frac{\cos A}{\cos B}}_{(1)}$$

$$+ \underbrace{\left(C_X \sin A \sin B + C_Y \sin A \cos B\right)\sin A \sin B}_{(2)} \tag{III-45}$$

$$C'_{l_u} = \left[C_{l_u} \cos^2 B - \left(C_{l_v} + C_{m_u}\right)\sin B \cos B + C_{m_v} \sin^2 B\right]\cos^2 A$$

$$+ C_{n_w} \sin^2 A + \left[-\left(C_{l_w} + C_{n_u}\right)\cos B + \left(C_{m_w} + C_{n_v}\right)\sin B\right]\sin A \cos A$$

$$+ \underbrace{\left(C_l \sin A \cos B - C_m \sin A \sin B + C_n \cos A\right)\frac{\sin A}{\cos B}}_{(1)}$$

$$+ \underbrace{\left(C_l \sin B + C_m \cos B\right)\cos^2 A \sin B}_{(2)} \tag{III-46}$$

$$C'_{l_v} = \left[C_{l_v} \cos^2 B - C_{m_u} \sin^2 B + \left(C_{l_u} - C_{m_v}\right)\sin B \cos B\right]\cos A$$

$$- \left(C_{n_v} \cos B + C_{n_u} \sin B\right)\sin A - \underbrace{\left(C_l \sin B + C_m \cos B\right)\cos A \cos B}_{(2)} \tag{III-47}$$

$$C'_{l_w} = \left(C_{l_w} \cos B - C_{m_v} \sin B\right)\cos^2 A - \left(C_{n_u} \cos B - C_{n_v} \sin B\right)\sin^2 A$$

$$+ \left[C_{l_u} \cos^2 B + C_{m_v} \sin^2 B - \left(C_{l_v} + C_{m_u}\right)\sin B \cos B - C_{n_w}\right]\sin A \cos A$$

$$- \underbrace{\left(C_l \sin A \cos B - C_m \sin A \sin B + C_n \cos A\right)\frac{\cos A}{\cos B}}_{(1)}$$

$$+ \underbrace{\left(C_l \sin B + C_m \cos B\right)\sin A \cos A \sin B}_{(2)} \qquad \text{(III-48)}$$

$$C'_{m_u} = \left[C_{m_u} \cos^2 B - C_{l_v} \sin^2 B + \left(C_{l_u} - C_{m_v}\right)\sin B \cos B\right]\cos A$$

$$- \left(C_{m_w} \cos B + C_{l_w} \sin B\right)\sin A - \underbrace{\left(C_l \cos B - C_m \sin B\right)\cos A \sin B}_{(2)} \qquad \text{(III-49)}$$

$$C'_{m_v} = C_{m_v} \cos^2 B + C_{l_u} \sin^2 B + \left(C_{l_v} + C_{m_u}\right)\sin B \cos B$$

$$+ \underbrace{\left(C_l \cos B - C_m \sin B\right)\cos B}_{(2)} \qquad \text{(III-50)}$$

$$C'_{m_w} = \left(C_{m_w} \cos B + C_{l_w} \sin B\right)\cos A$$

$$+ \left[C_{m_u} \cos^2 B - C_{l_v} \sin^2 B + \left(C_{l_u} - C_{m_v}\right)\sin B \cos B\right]\sin A$$

$$- \underbrace{\left(C_l \cos B - C_m \sin B\right)\sin A \sin B}_{(2)} \qquad \text{(III-51)}$$

$$C'_{n_u} = \left(C_{n_u} \cos B - C_{n_v} \sin B\right)\cos^2 A - \left(C_{l_w} \cos B - C_{m_w} \sin B\right)\sin^2 A$$

$$+ \left[C_{l_u} \cos^2 B + C_{m_v} \sin^2 B - \left(C_{l_v} + C_{m_u}\right)\sin B \cos B - C_{n_w}\right] \sin A \cos A$$

$$- \underbrace{\left(C_l \cos A \cos B - C_m \cos A \sin B - C_n \sin A\right)\frac{\sin A}{\cos B}}_{(1)}$$

$$+ \underbrace{\left(C_l \sin A \sin B + C_m \sin A \cos B\right)\cos A \sin B}_{(2)} \qquad \text{(III-52)}$$

$$C'_{n_v} = \left(C_{n_v} \cos B + C_{n_u} \sin B\right)\cos A$$

$$+ \left[C_{l_v} \cos^2 B - C_{m_u} \sin^2 B + \left(C_{l_u} - C_{m_v}\right)\sin B \cos B\right]\sin A$$

$$- \underbrace{\left(C_l \sin B + C_m \cos B\right)\sin A \cos B}_{(2)} \qquad \text{(III-53)}$$

$$C'_{n_w} = C_{n_w} \cos^2 A + \left[C_{l_u} \cos^2 B + C_{m_v} \sin^2 B - \left(C_{l_v} + C_{m_u}\right)\sin B \cos B\right]\sin^2 A$$

$$+ \left[\left(C_{l_w} + C_{n_u}\right)\cos B - \left(C_{m_w} + C_{n_v}\right)\sin B\right]\sin A \cos A$$

$$+ \underbrace{\left(C_l \cos A \cos B - C_m \cos A \sin B - C_n \sin A\right)\frac{\cos A}{\cos B}}_{(1)}$$

$$+ \underbrace{\left(C_l \sin A \sin B + C_m \sin A \cos B\right)\sin A \sin B}_{(2)} \qquad \text{(III-54)}$$

INVERSE TRANSFORMATIONS

Static Force and Moment Coefficients[1] (Inverse, Table IV)

$$C_X = \left(C'_X \cos A + C'_Z \sin A\right)\cos B + C'_Y \sin B \tag{III-55}$$

$$C_Y = -\left(C'_X \cos A + C'_Z \sin A\right)\sin B + C'_Y \cos B \tag{III-56}$$

$$C_Z = -C'_X \sin A + C'_Z \cos A \tag{III-57}$$

$$C_l = \left(C'_l \cos A + C'_n \sin A\right)\cos B + C'_m \sin B \tag{III-58}$$

$$C_m = -\left(C'_l \cos A + C'_n \sin A\right)\sin B + C'_m \cos B \tag{III-59}$$

$$C_n = -C'_l \sin A + C'_n \cos A \tag{III-60}$$

Static-Stability Derivatives (Inverse, Table IV)

$$C_{X_\alpha} = \left(C'_{X_\alpha} \cos A + C'_{Z_\alpha} \sin A\right)\cos B + C'_{Y_\alpha} \sin B - \underbrace{\left(C'_X \sin A - C'_Z \cos A\right)\cos B}_{(1)} \tag{III-61}$$

$$C_{Y_\alpha} = -\left(C'_{X_\alpha} \cos A + C'_{Z_\alpha} \sin A\right)\sin B + C'_{Y_\alpha} \cos B + \underbrace{\left(C'_X \sin A - C'_Z \cos A\right)\sin B}_{(1)} \tag{III-62}$$

$$C_{Z_\alpha} = -C'_{X_\alpha} \sin A + C'_{Z_\alpha} \cos A - \underbrace{C'_X \cos A - C'_Z \sin A}_{(1)} \tag{III-63}$$

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[1] Equations for control derivatives and velocity derivatives have this same form; for example,

$$C_{X_{\delta_a}} = \left(C'_{X_{\delta_a}} \cos A + C'_{Z_{\delta_a}} \sin A\right)\cos B + C'_{Y_{\delta_a}} \sin B$$

$$C_{X_V} = \left(C'_{X_V} \cos A + C'_{Z_V} \sin A\right)\cos B + C'_{Y_V} \sin B$$

$$C_{l_\alpha} = \left(C'_{l_\alpha} \cos A + C'_{n_\alpha} \sin A\right) \cos B + C'_{m_\alpha} \sin B$$

$$- \underbrace{\left(C'_l \sin A - C'_n \cos A\right) \cos B}_{(1)} \tag{III-64}$$

$$C_{m_\alpha} = -\left(C'_{l_\alpha} \cos A + C'_{n_\alpha} \sin A\right) \sin B + C'_{m_\alpha} \cos B$$

$$+ \underbrace{\left(C'_l \sin A - C'_n \cos A\right) \sin B}_{(1)} \tag{III-65}$$

$$C_{n_\alpha} = -C'_{l_\alpha} \sin A + C'_{n_\alpha} \cos A - \underbrace{C'_l \cos A - C'_n \sin A}_{(1)} \tag{III-66}$$

$$C_{X_\beta} = \left(C'_{X_\beta} \cos A + C'_{Z_\beta} \sin A\right) \cos B + C'_{Y_\beta} \sin B$$

$$- \underbrace{\left(C'_X \cos A + C'_Z \sin A\right) \sin B + C'_Y \cos B}_{(2)} \tag{III-67}$$

$$C_{Y_\beta} = -\left(C'_{X_\beta} \cos A + C'_{Z_\beta} \sin A\right) \sin B + C'_{Y_\beta} \cos B$$

$$- \underbrace{\left(C'_X \cos A + C'_Z \sin A\right) \cos B - C'_Y \sin B}_{(2)} \tag{III-68}$$

$$C_{Z_\beta} = -C'_{X_\beta} \sin A + C'_{Z_\beta} \cos A \tag{III-69}$$

$$C_{l_\beta} = \left(C'_{l_\beta} \cos A + C'_{n_\beta} \sin A\right) \cos B + C'_{m_\beta} \sin B$$

$$- \underbrace{\left(C'_l \cos A + C'_n \sin A\right) \sin B + C'_m \cos B}_{(2)} \tag{III-70}$$

$$C_{m_\beta} = -\left(C'_{l_\beta} \cos A + C'_{n_\beta} \sin A\right) \sin B + C'_{m_\beta} \cos B$$

$$- \underbrace{\left(C'_{l} \cos A + C'_{n} \sin A\right) \cos B - C'_{m} \sin B}_{(2)} \qquad \text{(III-71)}$$

$$C_{n_\beta} = -C'_{l_\beta} \sin A + C'_{n_\beta} \cos A \qquad \text{(III-72)}$$

Dynamic-Stability Derivatives (Inverse, Table IV)

$$C_{X_p} = \left[C'_{X_p} \cos^2 A + C'_{Z_r} \sin^2 A + \left(C'_{X_r} + C'_{Z_p}\right) \sin A \cos A\right] \cos^2 B$$

$$+ C'_{Y_q} \sin^2 B + \left[\left(C'_{X_q} + C'_{Y_p}\right) \cos A + \left(C'_{Y_r} + C'_{Z_q}\right) \sin A\right] \sin B \cos B \qquad \text{(III-73)}$$

$$C_{X_q} = \left(C'_{X_q} \cos A + C'_{Z_q} \sin A\right) \cos^2 B - \left(C'_{Y_p} \cos A + C'_{Y_r} \sin A\right) \sin^2 B$$

$$- \left[C'_{X_p} \cos^2 A + C'_{Z_r} \sin^2 A + \left(C'_{X_r} + C'_{Z_p}\right) \sin A \cos A - C'_{Y_q}\right] \sin B \cos B \qquad \text{(III-74)}$$

$$C_{X_r} = \left[C'_{X_r} \cos^2 A - C'_{Z_p} \sin^2 A - \left(C'_{X_p} - C'_{Z_r}\right) \sin A \cos A\right] \cos B$$

$$+ \left(C'_{Y_r} \cos A - C'_{Y_p} \sin A\right) \sin B \qquad \text{(III-75)}$$

$$C_{Y_p} = \left(C'_{Y_p} \cos A + C'_{Y_r} \sin A\right) \cos^2 B - \left(C'_{X_q} \cos A + C'_{Z_q} \sin A\right) \sin^2 B$$

$$- \left[C'_{X_p} \cos^2 A + C'_{Z_r} \sin^2 A + \left(C'_{X_r} + C'_{Z_p}\right) \sin A \cos A - C'_{Y_q}\right] \sin B \cos B \qquad \text{(III-76)}$$

$$C_{Y_q} = C'_{Y_q} \cos^2 B + \left[C'_{X_p} \cos^2 A + C'_{Z_r} \sin^2 A + \left(C'_{X_r} + C'_{Z_p}\right) \sin A \cos A\right] \sin^2 B$$

$$- \left[\left(C'_{X_q} + C'_{Y_p}\right) \cos A + \left(C'_{Y_r} + C'_{Z_q}\right) \sin A\right] \sin B \cos B \qquad \text{(III-77)}$$

$$C_{Y_r} = \left(C'_{Y_r} \cos A - C'_{Y_p} \sin A\right) \cos B$$

$$+ \left[-C'_{X_r} \cos^2 A + C'_{Z_p} \sin^2 A + \left(C'_{X_p} - C'_{Z_r}\right) \sin A \cos A\right] \sin B \qquad \text{(III-78)}$$

$$C_{Z_p} = \left[C'_{Z_p} \cos^2 A - C'_{X_r} \sin^2 A - \left(C'_{X_p} - C'_{Z_r}\right) \sin A \cos A\right] \cos B$$
$$+ \left(C'_{Z_q} \cos A - C'_{X_q} \sin A\right) \sin B \qquad \text{(III-79)}$$

$$C_{Z_q} = \left(C'_{Z_q} \cos A - C'_{X_q} \sin A\right) \cos B$$
$$+ \left[-C'_{Z_p} \cos^2 A + C'_{X_r} \sin^2 A + \left(C'_{X_p} - C'_{Z_r}\right) \sin A \cos A\right] \sin B \qquad \text{(III-80)}$$

$$C_{Z_r} = C'_{Z_r} \cos^2 A + C'_{X_p} \sin^2 A - \left(C'_{X_r} + C'_{Z_p}\right) \sin A \cos A \qquad \text{(III-81)}$$

$$C_{l_p} = \left[C'_{l_p} \cos^2 A + C'_{n_r} \sin^2 A + \left(C'_{l_r} + C'_{n_p}\right) \sin A \cos A\right] \cos^2 B$$
$$+ C'_{m_q} \sin^2 B + \left[\left(C'_{l_q} + C'_{m_p}\right) \cos A + \left(C'_{m_r} + C'_{n_q}\right) \sin A\right] \sin B \cos B \qquad \text{(III-82)}$$

$$C_{l_q} = \left(C'_{l_q} \cos A + C'_{n_q} \sin A\right) \cos^2 B - \left(C'_{m_p} \cos A + C'_{m_r} \sin A\right) \sin^2 B$$
$$- \left[C'_{l_p} \cos^2 A + C'_{n_r} \sin^2 A + \left(C'_{l_r} + C'_{n_p}\right) \sin A \cos A - C'_{m_q}\right] \sin B \cos B \qquad \text{(III-83)}$$

$$C_{l_r} = \left[C'_{l_r} \cos^2 A - C'_{n_p} \sin^2 A - \left(C'_{l_p} - C'_{n_r}\right) \sin A \cos A\right] \cos B$$
$$+ \left(C'_{m_r} \cos A - C'_{m_p} \sin A\right) \sin B \qquad \text{(III-84)}$$

$$C_{m_p} = \left(C'_{m_p} \cos A + C'_{m_r} \sin A\right) \cos^2 B - \left(C'_{l_q} \cos A + C'_{n_q} \sin A\right) \sin^2 B$$
$$- \left[C'_{l_p} \cos^2 A + C'_{n_r} \sin^2 A + \left(C'_{l_r} + C'_{n_p}\right) \sin A \cos A - C'_{m_q}\right] \sin B \cos B \qquad \text{(III-85)}$$

$$C_{m_q} = C'_{m_q} \cos^2 B + \left[C'_{l_p} \cos^2 A + C'_{n_r} \sin^2 A + \left(C'_{l_r} + C'_{n_p}\right) \sin A \cos A\right] \sin^2 B$$
$$- \left[\left(C'_{l_q} + C'_{m_p}\right) \cos A + \left(C'_{m_r} + C'_{n_q}\right) \sin A\right] \sin B \cos B \qquad \text{(III-86)}$$

$$C_{m_r} = \left(C'_{m_r} \cos A - C'_{m_p} \sin A\right) \cos B$$

$$+ \left[-C'_{l_r} \cos^2 A + C'_{n_p} \sin^2 A + \left(C'_{l_p} - C'_{n_r}\right) \sin A \cos A\right] \sin B \qquad \text{(III-87)}$$

$$C_{n_p} = \left[C'_{n_p} \cos^2 A - C'_{l_r} \sin^2 A - \left(C'_{l_p} - C'_{n_r}\right) \sin A \cos A\right] \cos B$$

$$+ \left(C'_{n_q} \cos A - C'_{l_q} \sin A\right) \sin B \qquad \text{(III-88)}$$

$$C_{n_q} = \left(C'_{n_q} \cos A - C'_{l_q} \sin A\right) \cos B$$

$$+ \left[-C'_{n_p} \cos^2 A + C'_{l_r} \sin^2 A + \left(C'_{l_p} - C'_{n_r}\right) \sin A \cos A\right] \sin B \qquad \text{(III-89)}$$

$$C_{n_r} = C'_{n_r} \cos^2 A + C'_{l_p} \sin^2 A - \left(C'_{l_r} + C'_{n_p}\right) \sin A \cos A \qquad \text{(III-90)}$$

u,v,w Derivatives (Inverse, Table IV)

$$C_{X_u} = \left[C'_{X_u} \cos^2 A + C'_{Z_w} \sin^2 A + \left(C'_{X_w} + C'_{Z_u}\right) \sin A \cos A\right] \cos^2 B$$

$$+ C'_{Y_v} \sin^2 B + \left[\left(C'_{X_v} + C'_{Y_u}\right) \cos A + \left(C'_{Y_w} + C'_{Z_v}\right) \sin A\right] \sin B \cos B$$

$$+ \underbrace{\left(C'_X \sin A - C'_Z \cos A\right) \sin A}_{(1)}$$

$$+ \underbrace{\left[\left(C'_X \cos A + C'_Z \sin A\right) \sin B - C'_Y \cos B\right] \cos A \sin B}_{(2)} \qquad \text{(III-91)}$$

$$C_{X_v} = \left(C'_{X_v} \cos A + C'_{Z_v} \sin A\right) \cos^2 B - \left(C'_{Y_u} \cos A + C'_{Y_w} \sin A\right) \sin^2 B$$

$$- \left[C'_{X_u} \cos^2 A + C'_{Z_w} \sin^2 A + \left(C'_{X_w} + C'_{Z_u}\right) \sin A \cos A - C'_{Y_v}\right] \sin B \cos B$$

$$+ \underbrace{\left[-\left(C'_X \cos A + C'_Z \sin A\right) \sin B + C'_Y \cos B\right] \cos B}_{(2)} \qquad \text{(III-92)}$$

$$C_{X_w} = \left[C'_{X_w} \cos^2 A - C'_{Z_u} \sin^2 A - \left(C'_{X_u} - C'_{Z_w}\right) \sin A \cos A\right] \cos B$$

$$+ \left(C'_{Y_w} \cos A - C'_{Y_u} \sin A\right) \sin B - \underbrace{\left(C'_X \sin A - C'_Z \cos A\right) \cos A}_{(1)}$$

$$+ \underbrace{\left[\left(C'_X \cos A + C'_Z \sin A\right) \sin B - C'_Y \cos B\right] \sin A \sin B}_{(2)} \qquad \text{(III-93)}$$

$$C_{Y_u} = \left(C'_{Y_u} \cos A + C'_{Y_w} \sin A\right) \cos^2 B - \left(C'_{X_v} \cos A + C'_{Z_v} \sin A\right) \sin^2 B$$

$$- \left[C'_{X_u} \cos^2 A + C'_{Z_w} \sin^2 A + \left(C'_{X_w} + C'_{Z_u}\right) \sin A \cos A - C'_{Y_v}\right] \sin B \cos B$$

$$- \underbrace{\left(C'_X \sin A - C'_Z \cos A\right) \frac{\sin A \sin B}{\cos B}}_{(1)}$$

$$+ \underbrace{\left[\left(C'_X \cos A + C'_Z \sin A\right) \cos B + C'_Y \sin B\right] \cos A \sin B}_{(2)} \qquad \text{(III-94)}$$

$$C_{Y_v} = C'_{Y_v} \cos^2 B + \left[C'_{X_u} \cos^2 A + C'_{Z_w} \sin^2 A + \left(C'_{X_w} + C'_{Z_u}\right) \sin A \cos A\right] \sin^2 B$$

$$- \left[\left(C'_{X_v} + C'_{Y_u}\right) \cos A + \left(C'_{Y_w} + C'_{Z_v}\right) \sin A\right] \sin B \cos B$$

$$- \underbrace{\left[\left(C'_X \cos A + C'_Z \sin A\right) \cos B + C'_Y \sin B\right] \cos B}_{(2)} \qquad \text{(III-95)}$$

$$C_{Y_w} = \left(C'_{Y_w} \cos A - C'_{Y_u} \sin A\right) \cos B$$

$$+ \left[-C'_{X_w} \cos^2 A + C'_{Z_u} \sin^2 A + \left(C'_{X_u} - C'_{Z_w}\right) \sin A \cos A\right] \sin B$$

$$+ \underbrace{\left(C'_X \sin A - C'_Z \cos A\right) \frac{\cos A \sin B}{\cos B}}_{(1)}$$

$$+ \underbrace{\left[\left(C'_X \cos A + C'_Z \sin A\right) \cos B + C'_Y \sin B\right] \sin A \sin B}_{(2)} \qquad \text{(III-96)}$$

$$C_{Z_u} = \left[C'_{Z_u} \cos^2 A - C'_{X_w} \sin^2 A - \left(C'_{X_u} - C'_{Z_w}\right)\sin A \cos A\right] \cos B$$

$$+ \left(C'_{Z_v} \cos A - C'_{X_v} \sin A\right)\sin B + \underbrace{\left(C'_X \cos A + C'_Z \sin A\right)\frac{\sin A}{\cos B}}_{(1)} \qquad \text{(III-97)}$$

$$C_{Z_v} = \left(C'_{Z_v} \cos A - C'_{X_v} \sin A\right)\cos B$$

$$+ \left[-C'_{Z_u} \cos^2 A + C'_{X_w} \sin^2 A + \left(C'_{X_u} - C'_{Z_w}\right)\sin A \cos A\right]\sin B \qquad \text{(III-98)}$$

$$C_{Z_w} = C'_{Z_w} \cos^2 A + C'_{X_u} \sin^2 A - \left(C'_{X_w} + C'_{Z_u}\right)\sin A \cos A$$

$$- \underbrace{\left(C'_X \cos A + C'_Z \sin A\right)\frac{\cos A}{\cos B}}_{(1)} \qquad \text{(III-99)}$$

$$C_{l_u} = \left[C'_{l_u} \cos^2 A + C'_{n_w} \sin^2 A + \left(C'_{l_w} + C'_{n_u}\right)\sin A \cos A\right]\cos^2 B$$

$$+ C'_{m_v} \sin^2 B + \left[\left(C'_{l_v} + C'_{m_u}\right)\cos A + \left(C'_{m_w} + C'_{n_v}\right)\sin A\right]\sin B \cos B$$

$$+ \underbrace{\left(C'_l \sin A - C'_n \cos A\right)\sin A}_{(1)}$$

$$+ \underbrace{\left[\left(C'_l \cos A + C'_n \sin A\right)\sin B - C'_m \cos B\right]\cos A \sin B}_{(2)} \qquad \text{(III-100)}$$

$$C_{l_v} = \left(C'_{l_v} \cos A + C'_{n_v} \sin A\right)\cos^2 B - \left(C'_{m_u} \cos A + C'_{m_w} \sin A\right)\sin^2 B$$

$$- \left[C'_{l_u} \cos^2 A + C'_{n_w} \sin^2 A + \left(C'_{l_w} + C'_{n_u}\right)\sin A \cos A - C'_{m_v}\right]\sin B \cos B$$

$$- \underbrace{\left[\left(C'_l \cos A + C'_n \sin A\right)\sin B - C'_m \cos B\right]\cos B}_{(2)} \qquad \text{(III-101)}$$

$$C_{l_w} = \left[C'_{l_w}\cos^2 A - C'_{n_u}\sin^2 A - \left(C'_{l_u} - C'_{n_w}\right)\sin A\cos A\right]\cos B$$

$$+ \left(C'_{m_w}\cos A - C'_{m_u}\sin A\right)\sin B - \underbrace{\left(C'_l \sin A - C'_n \cos A\right)\cos A}_{(1)}$$

$$+ \underbrace{\left[\left(C'_l \cos A + C'_n \sin A\right)\sin B - C'_m \cos B\right]\sin A \sin B}_{(2)} \qquad \text{(III-102)}$$

$$C_{m_u} = \left(C'_{m_u}\cos A + C'_{m_w}\sin A\right)\cos^2 B - \left(C'_{l_v}\cos A + C'_{n_v}\sin A\right)\sin^2 B$$

$$- \left[C'_{l_u}\cos^2 A + C'_{n_w}\sin^2 A + \left(C'_{l_w} + C'_{n_u}\right)\sin A\cos A - C'_{m_v}\right]\sin B\cos B$$

$$- \underbrace{\left(C'_l \sin A - C'_n \cos A\right)\frac{\sin A \sin B}{\cos B}}_{(1)}$$

$$+ \underbrace{\left[\left(C'_l \cos A + C'_n \sin A\right)\cos B + C'_m \sin B\right]\cos A \sin B}_{(2)} \qquad \text{(III-103)}$$

$$C_{m_v} = C'_{m_v}\cos^2 B + \left[C'_{l_u}\cos^2 A + C'_{n_w}\sin^2 A + \left(C'_{l_w} + C'_{n_u}\right)\sin A\cos A\right]\sin^2 B$$

$$- \left[\left(C'_{l_v} + C'_{m_u}\right)\cos A + \left(C'_{m_w} + C'_{n_v}\right)\sin A\right]\sin B\cos B$$

$$- \underbrace{\left[\left(C'_l \cos A + C'_Z \sin A\right)\cos B + C'_m \sin B\right]\cos B}_{(2)} \qquad \text{(III-104)}$$

$$C_{m_w} = \left(C'_{m_w} \cos A - C'_{m_u} \sin A\right)\cos B$$

$$+ \left[-C_{l_w} \cos^2 A + C'_{n_u} \sin^2 A + \left(C'_{l_u} - C'_{n_w}\right)\sin A \cos A\right]\sin B$$

$$+ \underbrace{\left(C'_l \sin A - C'_n \cos A\right)\frac{\cos A \sin B}{\cos B}}_{(1)}$$

$$+ \underbrace{\left[\left(C'_l \cos A + C'_n \sin A\right)\cos B + C'_m \sin B\right]\sin A \sin B}_{(2)} \qquad \text{(III-105)}$$

$$C_{n_u} = \left[C'_{n_u} \cos^2 A - C'_{l_w} \sin^2 A - \left(C'_{l_u} - C'_{n_w}\right)\sin A \cos A\right]\cos B$$

$$+ \left(C'_{n_v} \cos A - C'_{l_v} \sin A\right)\sin B + \underbrace{\left(C'_l \cos A + C'_n \sin A\right)\frac{\sin A}{\cos B}}_{(1)} \qquad \text{(III-106)}$$

$$C_{n_v} = \left(C'_{n_v} \cos A - C'_{l_v} \sin A\right)\cos B$$

$$+ \left[-C'_{n_u} \cos^2 A + C'_{l_w} \sin^2 A + \left(C'_{l_u} - C'_{n_w}\right)\sin A \cos A\right]\sin B \qquad \text{(III-107)}$$

$$C_{n_w} = C'_{n_w} \cos^2 A + C'_{l_u} \sin^2 A - \left(C'_{l_w} + C'_{n_u}\right)\sin A \cos A$$

$$- \underbrace{\left(C'_l \cos A + C'_n \sin A\right)\frac{\cos A}{\cos B}}_{(1)} \qquad \text{(III-108)}$$

SECTION IV

TRANSFORMATION EQUATIONS FOR MOMENTS OF INERTIA

This section gives equations for transferring moments of inertia to a rotated axes system in a general form, similar to that used in section III, so that transformations can be made between any two of the five axes systems used in aerodynamic analysis. As in section III, these transformations are defined as direct and inverse in tables III and IV, respectively, and can be performed between any two axes systems by selecting proper angles for A and B from these tables.

These equations are derived from a general transformation of the form

$$\begin{bmatrix} I'_X & -I'_{XY} & -I'_{XZ} \\ -I'_{XY} & I'_Y & -I'_{YZ} \\ -I'_{XZ} & -I'_{YZ} & I'_Z \end{bmatrix} = \left[\Gamma\right] \begin{bmatrix} I_X & -I_{XY} & -I_{XZ} \\ -I_{XY} & I_Y & -I_{YZ} \\ -I_{XZ} & -I_{YZ} & I_Z \end{bmatrix} \left[\Gamma\right]^{-1} \qquad \text{(IV-1)}$$

where Γ is the transformation matrix given in appendix A and is applied to the aerodynamic axes by setting $\psi = -B$, $\theta = A$, and $\phi = 0$.

The most commonly used forms of these transformations, those for transfers among body, principal, and flight stability axes, are given in appendix A.

DIRECT TRANSFORMATIONS (TABLE III)

$$I'_X = \left(I_X \cos^2 B + 2I_{XY} \sin B \cos B + I_Y \sin^2 B\right)\cos^2 A + I_Z \sin^2 A$$
$$+ \left(2I_{XZ} \cos B - 2I_{YZ} \sin B\right) \sin A \cos A \qquad \text{(IV-2)}$$

$$I'_Y = I_Y \cos^2 B + I_X \sin^2 B - 2I_{XY} \sin B \cos B \qquad \text{(IV-3)}$$

$$I'_Z = I_Z \cos^2 A + \left(I_X \cos^2 B + I_Y \sin^2 B + 2I_{XY} \sin B \cos B\right)\sin^2 A$$
$$- \left(2I_{XZ} \cos B - 2I_{YZ} \sin B\right) \sin A \cos A \qquad \text{(IV-4)}$$

$$I'_{XY} = \left[I_{XY}\left(\cos^2 B - \sin^2 B\right) - \left(I_X - I_Y\right) \sin B \cos B\right] \cos A$$
$$- \left(I_{YZ} \cos B + I_{XZ} \sin B\right) \sin A \qquad \text{(IV-5)}$$

$$I'_{XZ} = \left(I_{XZ} \cos B - I_{YZ} \sin B\right)\cos^2 A - \left(I_{XZ} \cos B - I_{YZ} \sin B\right)\sin^2 A$$
$$- \left(I_X \cos^2 B + I_Y \sin^2 B + 2I_{XY} \sin B \cos B - I_Z\right)\sin A \cos A \qquad \text{(IV-6)}$$

$$I'_{YZ} = \left(I_{YZ} \cos B + I_{XZ} \sin B\right)\cos A$$
$$+ \left[I_{XY}\left(\cos^2 B - \sin^2 B\right) - \left(I_X - I_Y\right)\sin B \cos B\right] \sin A \qquad \text{(IV-7)}$$

INVERSE TRANSFORMATIONS (TABLE IV)

$$I_X = \left(I'_X \cos^2 A + I'_Z \sin^2 A - 2I'_{XZ} \sin A \cos A\right)\cos^2 B + I'_Y \sin^2 B$$
$$- 2\left(I'_{XY} \cos A + I'_{YZ} \sin A\right) \sin B \cos B \qquad \text{(IV-8)}$$

$$I_Y = I'_Y \cos^2 B + \left(I'_X \cos^2 A + I'_Z \sin^2 A - 2I'_{XZ} \sin A \cos A\right)\sin^2 B$$
$$+ 2\left(I'_{XY} \cos A + I'_{YZ} \sin A\right) \sin B \cos B \qquad \text{(IV-9)}$$

$$I_Z = I'_Z \cos^2 A + I'_X \sin^2 A + 2I'_{XZ} \sin A \cos A \qquad \text{(IV-10)}$$

$$I_{XY} = \left(I'_{XY} \cos A + I'_{YZ} \sin A\right)\cos^2 B - \left(I'_{XY} \cos A + I'_{YZ} \sin A\right) \sin^2 B$$
$$+ \left(I'_X \cos^2 A + I'_Z \sin^2 A - 2I'_{XZ} \sin A \cos A - I'_Y\right) \sin B \cos B \qquad \text{(IV-11)}$$

$$I_{XZ} = \left[I'_{XZ}\left(\cos^2 A - \sin^2 A\right) + \left(I'_X - I'_Z\right) \sin A \cos A\right] \cos B$$
$$+ \left(I'_{YZ} \cos A - I'_{XY} \sin A\right) \sin B \qquad \text{(IV-12)}$$

$$I_{YZ} = \left(I'_{YZ} \cos A - I'_{XY} \sin A\right) \cos B + \left[I'_{XZ}\left(\sin^2 A - \cos^2 A\right)\right.$$
$$\left. - \left(I'_X - I'_Z\right) \sin A \cos A\right] \sin B \qquad \text{(IV-13)}$$

SECTION V

EQUATIONS OF MOTION FOR SIX DEGREES OF FREEDOM

The equations presented here are the general forms that include the variables likely to be of interest in computing motions in the vicinity of the earth (moon and sun perturbations are ignored). The terms of each equation are grouped so that various effects (for example, oblateness of the earth) can be accounted for by adding or omitting certain terms. Linearized equations and the wind-axes equations for a point mass are given in appendix A.

The general equations apply to any of the five systems of vehicle reference axes shown in figure 3. The Euler angles ψ_g, θ_g, ϕ_g are referred to the gravity-axes system, with origin located at the surface of the earth, which rotates with the earth as shown in figure 4. (The relationship of these gravity axes to the vehicle reference axes is shown in fig. 5.)

FORCE EQUATIONS

Equations for the forces along the X,Y,Z axes are given in general form as equations (V-1) to (V-3), respectively. Equations for specialized cases can be obtained from the general forms as follows:

(1) For an oblate earth, equations (V-43) to (V-45) of auxiliary equations are used in place of term (2) in X,Y,Z force equations, respectively.

(2) If mass of vehicle is constant (zero thrust is also implied), terms (3) to (5) are omitted.

(3) For flight outside the atmosphere, terms (6) to (8) are omitted.

X-Axis Forces

$$\underbrace{m(\dot{u} + qw - rv)}_{(1)} + \underbrace{mg \sin \theta_g}_{(2)} = \underbrace{T_X}_{(3)} - \underbrace{\dot{m}(q\, d_Z - r\, d_Y)}_{(4)} + \underbrace{F_{r,X}}_{(5)}$$

$$+ \frac{1}{2}\rho V_\infty^2 S \Bigg(\underbrace{C_{X,o} + C_{X_\alpha}\alpha + C_{X_V}\frac{V}{V_\infty} + C_{X_{\dot{\alpha}}}\frac{\dot{\alpha}\ell}{2V_\infty} + C_{X_q}\frac{q\ell}{2V_\infty}}_{(6)}$$

$$+ \underbrace{C_{X_\beta}\beta + C_{X_{\dot{\beta}}}\frac{\dot{\beta}\ell}{2V_\infty} + C_{X_p}\frac{p\ell}{2V_\infty} + C_{X_r}\frac{r\ell}{2V_\infty}}_{(7)} + \underbrace{C_{X_{\delta_a}}\delta_a + C_{X_{\delta_e}}\delta_e + C_{X_{\delta_r}}\delta_r}_{(8)}\Bigg) + \underbrace{0}_{(9)} \qquad \text{(V-1)}$$

where the different terms are defined as follows:

(1) Mass times acceleration

(2) Component of vehicle weight (for a more nearly complete weight component, one that includes earth-oblateness effects, see auxiliary eqs. (V-43) to (V-45))

(3) Primary rocket thrust (see auxiliary eqs. (V-36) to (V-39))

(4) Jet damping force

(5) Reaction control force

(6) Basic aerodynamic forces

(7) Aerodynamic cross-coupling terms

(8) Aerodynamic control forces

(9) Higher order terms

Expansions of the aerodynamic forces and moments (terms (6) to (9) in force equations and terms (11) to (14) in moment equations) neglect all aerodynamic partial derivatives with respect to rates of change of velocities and angles except those with respect to α and β. The forces are expanded in terms of the independent variables α,β,V,p,q,r but could as easily be expanded in terms of the variables u,v,w,p,q,r. The X-axis aerodynamic force, for example, could also be written as

$$X = \frac{1}{2}\rho V_\infty^2 S\left(C_{X,o} + C_{X_u}u + C_{X_v}v + C_{X_w}w + C_{X_p}\frac{p\ell}{2V_\infty} + C_{X_q}\frac{q\ell}{2V_\infty} + C_{X_r}\frac{r\ell}{2V_\infty} + \sum C_{X_\delta}\delta + \text{Higher order terms}\right)$$

Y-Axis Forces

$$\underbrace{m(\dot{v} + ru - pw)}_{(1)} \underbrace{- mg\cos\theta_g \sin\phi_g}_{(2)} = \underbrace{T_Y}_{(3)} \underbrace{- \dot{m}(r\,d_X - p\,d_Z)}_{(4)} + \underbrace{F_{r,Y}}_{(5)}$$

$$+ \frac{1}{2}\rho V_\infty^2 S\Bigg(\underbrace{C_{Y,o} + C_{Y_\beta}\beta + C_{Y_V}\frac{V}{V_\infty} + C_{Y_{\dot\beta}}\frac{\dot\beta\ell}{2V_\infty} + C_{Y_p}\frac{p\ell}{2V_\infty} + C_{Y_r}\frac{r\ell}{2V_\infty}}_{(6)}$$

$$+ \underbrace{C_{Y_\alpha}\alpha + C_{Y_{\dot\alpha}}\frac{\dot\alpha\ell}{2V_\infty} + C_{Y_q}\frac{q\ell}{2V_\infty}}_{(7)} + \underbrace{C_{Y_{\delta_r}}\delta_r}_{(8)}\Bigg) + \underbrace{0}_{(9)} \qquad \text{(V-2)}$$

where the terms (1) to (9) are defined as in equation (V-1).

Z-Axis Forces

$$\underbrace{m(\dot{w} + pv - qu)}_{(1)} \underbrace{- mg\cos\theta_g\cos\phi_g}_{(2)} = \underbrace{T_Z}_{(3)} \underbrace{- \dot{m}(p\,d_Y - q\,d_X)}_{(4)} + \underbrace{F_{r,Z}}_{(5)}$$

$$+ \frac{1}{2}\rho V_\infty^2 S\Bigg(\underbrace{C_{Z,o} + C_{Z_\alpha}\alpha + C_{Z_V}\frac{V}{V_\infty} + C_{Z_{\dot\alpha}}\frac{\dot\alpha\ell}{2V_\infty} + C_{Z_q}\frac{q\ell}{2V_\infty}}_{(6)}$$

$$+ \underbrace{C_{Z_\beta}\beta + C_{Z_{\dot\beta}}\frac{\dot\beta\ell}{2V_\infty} + C_{Z_p}\frac{p\ell}{2V_\infty} + C_{Z_r}\frac{r\ell}{2V_\infty}}_{(7)} + \underbrace{C_{Z_{\delta_a}}\delta_a + C_{Z_{\delta_e}}\delta_e}_{(8)}\Bigg) + \underbrace{0}_{(9)} \qquad \text{(V-3)}$$

where the terms (1) to (9) are defined as in equation (V-1).

MOMENT EQUATIONS

Equations of motion involving moments about the X,Y,Z axes are given in general form as equations (V-4) to (V-6), respectively. Equations for specialized cases can be obtained from the general forms as follows:

(1) If vehicle mass is constant, terms (2) to (4) and (7) to (9) are omitted

(2) If principal axes are used, terms (3) to (6) are omitted

(3) If vehicle has a plane of symmetry (X-Z plane) but principal axes are not used, terms (3) and (5) are omitted

X-Axis Moment (Roll)

$$\underbrace{\dot{p}I_X + qr(I_Z - I_Y)}_{(1)} + \underbrace{p\dot{I}_X}_{(2)} - \underbrace{q\dot{I}_{XY}}_{(3)} - \underbrace{r\dot{I}_{XZ}}_{(4)} + \underbrace{(r^2 - q^2)I_{YZ} + (pr - \dot{q})I_{XY}}_{(5)} - \underbrace{(pq + \dot{r})I_{XZ}}_{(6)}$$

$$= \dot{m}\Big[\underbrace{-p\left(d_Y^2 + d_Z^2\right) + d_X(q\,d_Y + r\,d_Z)}_{(7)}\Big] - \underbrace{(T_Y d_Z - T_Z d_Y)}_{(8)} + \underbrace{M_{r,X}}_{(9)} + \underbrace{L_{rm}}_{(10)}$$

$$+ \frac{1}{2}\rho V_\infty^2 S\ell\Bigg(\underbrace{C_{l,o} + C_{l_\beta}\beta + C_{l_V}\frac{V}{V_\infty} + C_{l_{\dot\beta}}\frac{\dot\beta\ell}{2V_\infty} + C_{l_p}\frac{p\ell}{2V_\infty} + C_{l_r}\frac{r\ell}{2V_\infty}}_{(11)} + \underbrace{C_{l_\alpha}\alpha + C_{l_{\dot\alpha}}\frac{\dot\alpha\ell}{2V_\infty} + C_{l_q}\frac{q\ell}{2V_\infty}}_{(12)}$$

$$\underbrace{+ C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r}_{(13)}\Bigg) + \underbrace{0}_{(14)} \qquad \text{(V-4)}$$

where the different terms are defined as follows:

(1), (5), and (6) moments of inertia times angular accelerations

(2) to (4) and (7) jet damping moments

(8) Moments due to main rocket thrust

(9) Moment due to reaction control

(10) Moment due to gyroscopic action of engine rotating mass (see auxiliary eqs. (V-40) to (V-42))

(11) Basic aerodynamic moments [1]

(12) Aerodynamic cross-coupling terms

(13) Moments due to aerodynamic controls

(14) Higher order terms

Y-Axis Moment (Pitch)

$$\underbrace{\dot{q}I_Y + pr(I_X - I_Z)}_{(1)} + \underbrace{q\dot{I}_Y}_{(2)} \underbrace{- p\dot{I}_{XY}}_{(3)} \underbrace{- r\dot{I}_{YZ}}_{(4)} + \underbrace{(pq - \dot{r})I_{YZ} - (qr + \dot{p})I_{XY}}_{(5)} + \underbrace{(p^2 - r^2)I_{XZ}}_{(6)}$$

$$= \dot{m}\left[\underbrace{-q\left(d_X^2 + d_Z^2\right) + d_Y(p\,d_X + r\,d_Z)}_{(7)}\right] + \underbrace{(T_X d_Z - T_Z d_X)}_{(8)} + \underbrace{M_{r,Y}}_{(9)} + \underbrace{M_{rm}}_{(10)}$$

$$+ \frac{1}{2}\rho V_\infty^2 S\ell\left(\underbrace{C_{m,o} + C_{m_\alpha}\alpha + C_{m_V}\frac{V}{V_\infty} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}\ell}{2V_\infty} + C_{m_q}\frac{q\ell}{2V_\infty}}_{(11)}\right.$$

$$\left. + \underbrace{C_{m_\beta}\beta + C_{m_{\dot{\beta}}}\frac{\dot{\beta}\ell}{2V_\infty} + C_{m_p}\frac{p\ell}{2V_\infty} + C_{m_r}\frac{r\ell}{2V_\infty}}_{(12)} + \underbrace{C_{m_{\delta_a}}\delta_a + C_{m_{\delta_e}}\delta_e}_{(13)}\right) + \underbrace{0}_{(14)} \qquad \text{(V-5)}$$

where terms (1) to (14) are defined as in equation (V-4).

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[1] See statement following equation (V-1) about terms (11) to (14) in expansion of aerodynamic forces and moments.

$$\underbrace{\dot{r}I_Z + pq(I_Y - I_X)}_{(1)} + \underbrace{r\dot{I}_Z}_{(2)} - \underbrace{q\dot{I}_{YZ}}_{(3)} - \underbrace{p\dot{I}_{XZ}}_{(4)} + \underbrace{(q^2 - p^2)I_{XY} - (pr + \dot{q})I_{YZ}}_{(5)} + \underbrace{(qr - \dot{p})I_{XZ}}_{(6)}$$

$$= \dot{m}\left[\underbrace{-r(d_X{}^2 + d_Y{}^2) + d_Z(p\, d_X + q\, d_Y)}_{(7)}\right] - \underbrace{(T_X\, d_Y - T_Y\, d_X)}_{(8)} + \underbrace{M_{r,Z}}_{(9)} + \underbrace{N_{rm}}_{(10)}$$

$$+ \frac{1}{2}\rho V_\infty^2 S\ell\Bigg(\underbrace{C_{n,o} + C_{n_\beta}\beta + C_{n_V}\frac{V}{V_\infty} + C_{n_{\dot{\beta}}}\frac{\dot{\beta}\ell}{2V_\infty} + C_{n_p}\frac{p\ell}{2V_\infty} + C_{n_r}\frac{r\ell}{2V_\infty}}_{(11)} + \underbrace{C_{n_\alpha}\alpha + C_{n_{\dot{\alpha}}}\frac{\dot{\alpha}\ell}{2V_\infty} + C_{n_q}\frac{q\ell}{2V_\infty}}_{(12)}$$

$$+ \underbrace{C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r}_{(13)}\Bigg) + \underbrace{0}_{(14)} \tag{V-6}$$

where terms (1) to (14) are defined as in equation (V-4).

AUXILIARY EQUATIONS

General equations that take into account various relationships and effects in developing the equations of motion are given as equations (V-7) to (V-45). Equations for specialized cases may be obtained as follows:

(1) For flat earth, terms containing $\dot{L}$ and $\dot{\lambda}$ are omitted

(2) For nonrotating earth, Ω_e, V_X, V_Y, and V_Z are omitted

(3) For no surface winds, terms containing W_X, W_Y, and W_Z are omitted

Relationship Between Euler Angles and Angular Velocities

The Euler angles specifying vehicle alinement with the gravity-axes system can be determined from the angular velocities p,q,r by the equations

$$\dot{\psi}_g = \frac{r\cos\phi_g}{\cos\theta_g} + \frac{q\sin\phi_g}{\cos\theta_g} + \dot{L}\cos\psi_g\tan\theta_g + (\Omega_e + \dot{\lambda})(\sin L$$
$$+ \cos L\sin\psi_g\tan\theta_g) \tag{V-7}$$

$$\dot{\theta}_g = q\cos\phi_g - r\sin\phi_g - \left[\dot{L}\sin\psi_g - (\Omega_e + \dot{\lambda})\cos L\cos\psi_g\right] \tag{V-8}$$

$$\dot{\phi}_g = p + q \tan \theta_g \sin \phi_g + r \tan \theta_g \cos \phi_g + \left[\frac{\dot{L} \cos \psi_g}{\cos \theta_g} + \left(\Omega_e + \dot{\lambda}\right) \cos L \frac{\sin \psi_g}{\cos \theta_g}\right] \qquad \text{(V-9)}$$

The inverse relationships are

$$p = \dot{\phi}_g - \dot{\psi}_g \sin \theta_g - \dot{L} \cos \theta_g \cos \psi_g - \left(\Omega_e + \dot{\lambda}\right)\left(\cos L \cos \theta_g \sin \psi_g - \sin L \sin \theta_g\right) \qquad \text{(V-10)}$$

$$q = \dot{\theta}_g \cos \phi_g + \dot{\psi}_g \sin \phi_g \cos \theta_g - \dot{L}\left(\sin \theta_g \cos \psi_g \sin \phi_g - \sin \psi_g \cos \phi_g\right) - \left(\Omega_e + \dot{\lambda}\right) \cos L\left(\sin \theta_g \sin \psi_g \sin \phi_g + \cos \psi_g \cos \phi_g\right) - \left(\Omega_e + \dot{\lambda}\right) \sin L \sin \phi_g \cos \theta_g \qquad \text{(V-11)}$$

$$r = -\dot{\theta}_g \sin \phi_g + \dot{\psi}_g \cos \phi_g \cos \theta_g - \dot{L}\left(\sin \theta_g \cos \psi_g \cos \phi_g + \sin \psi_g \sin \phi_g\right) - \left(\Omega_e + \dot{\lambda}\right) \cos L\left(\sin \theta_g \sin \psi_g \cos \phi_g - \cos \psi_g \sin \phi_g\right) - \left(\Omega_e + \dot{\lambda}\right) \sin L \cos \phi_g \cos \theta_g \qquad \text{(V-12)}$$

Vehicle Coordinates

Vehicle coordinates can be computed from the X,Y,Z axes velocity components and vehicle Euler angles by integrating the equations

$$\dot{x}_g = u \cos \psi_g \cos \theta_g + v\left(\cos \psi_g \sin \theta_g \sin \phi_g - \sin \psi_g \cos \phi_g\right) + w\left(\cos \psi_g \sin \theta_g \cos \phi_g + \sin \psi_g \sin \phi_g\right) \qquad \text{(V-13)}$$

$$\dot{y}_g = u \sin \psi_g \cos \theta_g + v\left(\sin \psi_g \sin \theta_g \sin \phi_g + \cos \psi_g \cos \phi_g\right) + w\left(\sin \psi_g \sin \theta_g \cos \phi_g - \cos \psi_g \sin \phi_g\right) \qquad \text{(V-14)}$$

$$\dot{z}_g = -u \sin \theta_g + v \cos \theta_g \sin \phi_g + w \cos \theta_g \cos \phi_g \qquad \text{(V-15)}$$

$$r_g = \sqrt{x_g^2 + y_g^2 + z_g^2} \qquad \text{(V-16)}$$

Trajectory Parameters (See Fig. 6)

Flight-path angle, longitude, and latitude can be determined from the equations

$$\gamma = \tan^{-1} \frac{\dot{h}}{\sqrt{\left(\dot{x} - R_e\Omega_e \cos L\right)^2 + \dot{y}^2}} \tag{V-17}$$

$$\dot{\lambda} = \frac{\dot{x}_g}{R_e \cos L} - \Omega_e \tag{V-18}$$

$$\dot{L} = \frac{-\dot{y}_g}{R_e} \tag{V-19}$$

The range for a spherical earth is

$$\text{Range} = R_e\sigma \tag{V-20}$$

where, for a total range of less than 50 000 feet,

$$\sigma = \sqrt{\left(L - L_o\right)^2 + \left[\left(\lambda - \lambda_o\right) \cos L_o\right]^2} \tag{V-21}$$

For a total range greater than 50 000 feet,

$$\sigma = \cos^{-1}\left[\sin L_o \sin L + \cos L_o \cos L \cos\left(\lambda - \lambda_o\right)\right] \tag{V-22}$$

range for a flat earth is

$$\text{Range} = \sqrt{x_g{}^2 + y_g{}^2} \tag{V-23}$$

Angle of Attack, Sideslip, and Relative Velocity (See Fig. 7)

Angle of attack, sideslip, and resultant relative velocity are related to components of velocity along the vehicle body axes by

$$\alpha = \tan^{-1} \frac{\bar{w}_b}{\bar{u}_b} \tag{V-24}$$

$$\beta = \sin^{-1} \frac{\bar{v}_b}{V} \tag{V-25}$$

$$V = \bar{u}_b{}^2 + \bar{v}_b{}^2 + \bar{w}_b{}^2 \tag{V-26}$$

$$\bar{u}_b = V \cos \alpha \cos \beta = u_b - V_{X,b} - W_{X,b} \tag{V-27}$$

$$\bar{v}_b = V \sin \beta = v_b - V_{Y,b} - W_{Y,b} \tag{V-28}$$

$$\overline{w}_b = V \sin \alpha \cos \beta = w_b - V_{Z,b} - W_{Z,b} \tag{V-29}$$

where subscripts b; X,b; Y,b; and Z,b denote body-axes components.

Wind Corrections (See Fig. 8)

Geostrophic (due to earth's rotation).- The components of the geostrophic wind along the X,Y,Z vehicle reference axes are

$$V_X = R_e \Omega_e \cos L \cos \theta_g \cos \psi_g \tag{V-30}$$

$$V_Y = -R_e \Omega_e \cos L \left(\cos \phi_g \sin \psi_g - \sin \theta_g \sin \phi_g \cos \psi_g\right) \tag{V-31}$$

$$V_Z = R_e \Omega_e \cos L \left(\sin \phi_g \sin \psi_g + \sin \theta_g \cos \phi_g \cos \psi_g\right) \tag{V-32}$$

Surface winds.- Components of the surface winds along the X,Y,Z vehicle reference axes are

$$W_X = \overline{W}\left(\cos A' \sin B' \cos \theta_g \cos \psi_g - \cos A' \cos B' \cos \theta_g \sin \psi_g + \sin A' \sin \theta_g\right) \tag{V-33}$$

$$W_Y = \overline{W}\left[\cos A' \sin B' \left(\sin \phi_g \sin \theta_g \cos \psi_g - \sin \psi_g \cos \phi_g\right) - \cos A' \cos B' \left(\sin \psi_g \sin \theta_g \sin \phi_g + \cos \psi_g \cos \phi_g\right) - \sin A' \sin \phi_g \cos \theta_g\right] \tag{V-34}$$

$$W_Z = \overline{W}\left[\cos A' \sin B' \left(\cos \psi_g \cos \phi_g \sin \theta_g + \sin \psi_g \sin \phi_g\right) - \cos A' \cos B' \left(\sin \psi_g \cos \phi_g \sin \theta_g - \cos \psi_g \sin \phi_g\right) - \sin A' \cos \phi_g \cos \theta_g\right] \tag{V-35}$$

Resolution of Engine Thrust and Torque Into Components Along Vehicle Reference Axes (See Fig. 9)

Vehicle thrust and moment can be resolved into components along the vehicle reference axes by the equations

$$T = T_O + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{RPM}} \delta_{RPM} \tag{V-36}$$

$$T_X = T \cos \theta_{rm} \cos \psi_{rm} \tag{V-37}$$

$$T_Y = T \cos \theta_{rm} \sin \psi_{rm} \tag{V-38}$$

$$T_Z = -T \sin \theta_{rm} \tag{V-39}$$

and, if it is assumed that $\dot{I}_{rm} = \dot{\Omega}_{rm} = \dot{\theta}_{rm} = \dot{\psi}_{rm} = 0$,

$$L_{rm} = I_{rm}\Omega_{rm}\left(-q \sin \theta_{rm} - r \cos \theta_{rm} \sin \psi_{rm}\right) \tag{V-40}$$

$$M_{rm} = I_{rm}\Omega_{rm}\left(r \cos \theta_{rm} \cos \psi_{rm} + p \sin \theta_{rm}\right) \tag{V-41}$$

$$N_{rm} = I_{rm}\Omega_{rm}\left(p \cos \theta_{rm} \sin \psi_{rm} - q \cos \theta_{rm} \cos \psi_{rm}\right) \tag{V-42}$$

Components of Gravitational Acceleration Along X,Y,Z Vehicle Reference Axes With Earth-Oblateness Effects Included

If the effect of earth oblateness on the acceleration due to gravity is considered important, terms (2) in the X,Y,Z axes force equations (V-1) to (V-3) should be replaced, respectively, by the following X,Y,Z axes weight components:

$$\begin{aligned} mg_X = m \frac{GE}{R^2}\Bigg\{ & \left[J_2\left(\frac{a_e}{R}\right)^2 (3 \sin L \cos L) + \frac{3}{2} J_3\left(\frac{a_e}{R}\right)^3\left(5 \sin^2 L - 1\right)\cos L\right. \\ & \left. - \frac{5}{2} J_4\left(\frac{a_e}{R}\right)^4\left(7 \sin^3 L - 3 \sin L\right)\cos L - \ldots\right] \cos \theta_g \sin \psi_g \\ & - \left[1 - \frac{3}{2} J_2\left(\frac{a_e}{R}\right)^2\left(3 \sin^2 L - 1\right) - 2J_3\left(\frac{a_e}{R}\right)^3\left(5 \sin^3 L - 3 \sin L\right)\right. \\ & \left. - \frac{5}{8} J_4\left(\frac{a_e}{R}\right)^4\left(35 \sin^4 L - 30 \sin^2 L + 3\right) - \ldots\right] \sin \theta_g \Bigg\} \end{aligned} \tag{V-43}$$

$$\begin{aligned} mg_Y = m \frac{GE}{R^2}\Bigg\{ & \left[J_2\left(\frac{a_e}{R}\right)^2 (3 \sin L \cos L) + \frac{3}{2} J_3\left(\frac{a_e}{R}\right)^3\left(5 \sin^2 L - 1\right)\cos L\right. \\ & \left. - \frac{5}{2} J_4\left(\frac{a_e}{R}\right)^4\left(7 \sin^3 L - 3 \sin L\right)\cos L - \ldots\right]\left(\sin \psi_g \sin \theta_g \sin \phi_g + \cos \psi_g \cos \phi_g\right) \\ & + \left[1 - \frac{3}{2} J_2\left(\frac{a_e}{R}\right)^2\left(3 \sin^2 L - 1\right) - 2J_3\left(\frac{a_e}{R}\right)^3\left(5 \sin^3 L - 3 \sin L\right)\right. \\ & \left. - \frac{5}{8} J_4\left(\frac{a_e}{R}\right)^4\left(35 \sin^4 L - 30 \sin^2 L + 3\right) - \ldots\right] \sin \phi_g \cos \theta_g \Bigg\} \end{aligned} \tag{V-44}$$

$$mg_Z = m\frac{GE}{R^2}\left\{\left[J_2\left(\frac{a_e}{R}\right)^2(3\sin L\cos L) + \frac{3}{2}J_3\left(\frac{a_e}{R}\right)^3\left(5\sin^2 L - 1\right)\cos L\right.\right.$$

$$\left. - \frac{5}{2}J_4\left(\frac{a_e}{R}\right)^4\left(7\sin^3 L - 3\sin L\right)\cos L - \ldots\right]\left(\sin\psi_g\cos\phi_g\sin\theta_g - \cos\psi_g\sin\phi_g\right)$$

$$+\left[1 - \frac{3}{2}J_2\left(\frac{a_e}{R}\right)^2\left(3\sin^2 L - 1\right) - 2J_3\left(\frac{a_e}{R}\right)^3\left(5\sin^3 L - 3\sin L\right)\right.$$

$$\left.\left. - \frac{5}{8}J_4\left(\frac{a_e}{R}\right)^4\left(35\sin^4 L - 30\sin^2 L + 3\right) - \ldots\right]\cos\phi_g\cos\theta_g\right\} \tag{V-45}$$

where

$GE = 398\ 601.2 \pm 0.4\ \text{km}^3/\text{sec}^2\ \left((14\ 074\ 901.1 \pm 14) \times 10^9\ \text{ft}^3/\text{sec}^2\right)$

$a_e = 6378.160 \pm 0.005$ km $(20\ 925\ 721.8 \pm 16$ ft$)$

and J_2, J_3, and J_4 are the second, third, and fourth zonal harmonics. The values from reference 8 are

$$J_2 = 1082.7(1 \pm 0.1) \times 10^{-6}$$

$$J_3 = -2.56(1 \pm 0.1) \times 10^{-6}$$

$$J_4 = -1.58(1 \pm 0.2) \times 10^{-6}$$

In equations (V-43) to (V-45), which were derived from the gravitational potential given in reference 8, oblateness terms through the 4th order are considered and the earth's longitudinal oblateness is neglected.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., February 16, 1972.

APPENDIX A

SUMMARY OF FREQUENTLY USED FORMS OF AXES TRANSFORMATIONS AND EQUATIONS OF MOTION

EULER ANGLE TRANSFORMATION BETWEEN TWO ORTHOGONAL AXES SYSTEMS

For two orthogonal axes systems, an initial X,Y,Z reference system and an X',Y',Z' system obtained by rotating the initial system through Euler angles ψ,θ,ϕ (in that order), the transformations between the two systems are given by these relationships.

Direct Transformation

The transformation from the initial X,Y,Z system to the X',Y',Z' system alined at Euler angles ψ,θ,ϕ with the initial system is

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \Gamma \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{A-1}$$

where

$$\Gamma = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\phi\cos\theta \\ \cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi & \sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{A-2}$$

Inverse Transformation

The transformation from the X',Y',Z' system back to the X,Y,Z system is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} \tag{A-3}$$

where

$$\Gamma^{-1} = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi & \cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \tag{A-4}$$

TRANSFORMATIONS FOR ACCELEROMETER AND RATE-GYRO MEASUREMENTS

These equations are simplified forms of the general inverse transformations for accelerations and angular velocities given in section I; however, they can be used in most practical applications to correct accelerometer and rate-gyro readings for displacement and misalinement. For cases that do not fit the assumptions made, the general forms in section I must be used.

Case I

Orthogonal Instrument Axes; No Restrictions on Misalinement Angles

If X,Y,Z axes accelerometers or rate gyros making flight measurements are orthogonally alined then, even though the misalinements of the instruments with respect to the vehicle reference axes are large, their readings can be corrected by using these equations.

Acceleration corrections.-

$$\begin{aligned} gA_{X,cg} = {} & gA_{X,i}\cos\theta\cos\psi + gA_{Y,i}(\sin\phi\sin\theta\cos\psi - \sin\psi\cos\phi) \\ & + gA_{Z,i}(\cos\psi\cos\phi\sin\theta + \sin\psi\sin\phi) + \left(r^2 + q^2\right)\bar{x}_X \\ & - (pq - \dot{r})\bar{y}_X - (pr + \dot{q})\bar{z}_X \end{aligned} \tag{A-5}$$

$$\begin{aligned} gA_{Y,cg} = {} & gA_{X,i}\cos\theta\sin\psi + gA_{Y,i}(\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi) \\ & + gA_{Z,i}(\sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi) - (pq + \dot{r})\bar{x}_Y \\ & + \left(p^2 + r^2\right)\bar{y}_Y - (qr - \dot{p})\bar{z}_Y \end{aligned} \tag{A-6}$$

$$gA_{Z,cg} = -gA_{X,i} \sin\theta + gA_{Y,i} \sin\phi \cos\theta + gA_{Z,i} \cos\phi \cos\theta$$

$$- (pr - \dot{q})\bar{x}_Z - (qr + \dot{p})\bar{y}_Z + \left(p^2 + q^2\right)\bar{z}_Z \tag{A-7}$$

Angular-velocity corrections.-

$$p = p_i \cos\theta \cos\psi + q_i(\cos\psi \sin\phi \sin\theta - \sin\psi \cos\phi)$$

$$+ r_i(\cos\psi \cos\phi \sin\theta + \sin\psi \sin\phi) \tag{A-8}$$

$$q = p_i \cos\theta \sin\psi + q_i(\sin\psi \sin\phi \sin\theta + \cos\psi \cos\phi)$$

$$+ r_i(\sin\psi \cos\phi \sin\theta - \cos\psi \sin\phi) \tag{A-9}$$

$$r = -p_i \sin\theta + q_i \sin\phi \cos\theta + r_i \cos\theta \cos\phi \tag{A-10}$$

Case II

Nonorthogonal Instrument Axes; Small Misalinement Angles

If X,Y,Z axes accelerometers or rate gyros are not orthogonally alined but the misalinements with the vehicle reference axes are small, then these correction equations can be used. It should be noted that, in equations (A-11) to (A-16), the angles ψ,θ,ϕ are in radians.

Acceleration corrections.-

$$gA_{X,cg} = g\left(A_{X,i} - \psi_X A_{Y,i} + \theta_X A_{Z,i}\right) + \left(q^2 + r^2\right)\bar{x}_X - (pq - \dot{r})\bar{y}_X - (pr + \dot{q})\bar{z}_X \tag{A-11}$$

$$gA_{Y,cg} = g\left(\psi_Y A_{X,i} + A_{Y,i} - \phi_Y A_{Z,i}\right) - (pq + \dot{r})\bar{x}_Y + \left(p^2 + r^2\right)\bar{y}_Y - (qr - \dot{p})\bar{z}_Y \tag{A-12}$$

$$gA_{Z,cg} = g\left(-\theta_Z A_{X,i} + \phi_Z A_{Y,i} + A_{Z,i}\right) - (pr - \dot{q})\bar{x}_Z - (qr + \dot{p})\bar{y}_Z + \left(q^2 + p^2\right)\bar{z}_Z \tag{A-13}$$

Angular-velocity corrections.-

$$p = p_i - \psi_X q_i + \theta_X r_i \tag{A-14}$$

$$q = \psi_Y p_i + q_i - \phi_Y r_i \tag{A-15}$$

$$r = -\theta_Z p_i + \phi_Z q_i + r_i \tag{A-16}$$

SPECIAL FORMS OF TRANSFORMATIONS FOR AERODYNAMIC FORCE AND MOMENT COEFFICIENTS AND STABILITY DERIVATIVES

Simplified Forms for Transferring Coefficients and Derivatives to Another Reference Center

These transformations are the simplified forms obtained from the general equations of section II by assuming zero angles of attack and sideslip and neglecting aerodynamic cross derivatives.

X-axis force coefficients and derivatives.-

$$C'_X = C_X \tag{A-17}$$

$$C'_{X_\alpha} = C_{X_\alpha} \tag{A-18}$$

$$C'_{X_{\dot{\alpha}}} = C_{X_{\dot{\alpha}}} \tag{A-19}$$

$$C'_{X_q} = C_{X_q} + \frac{2\bar{x}}{\ell} C_{X_\alpha} - \frac{4\bar{z}}{\ell} C_X \tag{A-20}$$

Y-axis force coefficients and derivatives.-

$$C'_Y = C_Y \tag{A-21}$$

$$C'_{Y_\beta} = C_{Y_\beta} \tag{A-22}$$

$$C'_{Y_{\dot{\beta}}} = C_{Y_{\dot{\beta}}} \tag{A-23}$$

$$C'_{Y_p} = C_{Y_p} + \frac{2\bar{z}}{\ell} C_{Y_\beta} \tag{A-24}$$

$$C'_{Y_r} = C_{Y_r} - \frac{2\bar{x}}{\ell} C_{Y_\beta} + \frac{4\bar{y}}{\ell} C_Y \tag{A-25}$$

Z-axis force coefficients and derivatives.-

$$C'_Z = C_Z \tag{A-26}$$

$$C'_{Z_\alpha} = C_{Z_\alpha} \tag{A-27}$$

$$C'_{Z_{\dot{\alpha}}} = C_{Z_{\dot{\alpha}}} \tag{A-28}$$

$$C'_{Z_q} = C_{Z_q} + \frac{2\bar{x}}{\ell} C_{Z_\alpha} - \frac{4\bar{z}}{\ell} C_Z \tag{A-29}$$

X-axis moment (roll) coefficients and derivatives.-

$$C'_{l} = C_{l} + \frac{\bar{z}}{\ell} C_{Y} - \frac{\bar{y}}{\ell} C_{Z} \tag{A-30}$$

$$C'_{l_\beta} = C_{l_\beta} + \frac{\bar{z}}{\ell} C_{Y_\beta} \tag{A-31}$$

$$C'_{l_{\dot{\beta}}} = C_{l_{\dot{\beta}}} + \frac{\bar{z}}{\ell} C_{Y_{\dot{\beta}}} \tag{A-32}$$

$$C'_{l_p} = C_{l_p} + \frac{\bar{z}}{\ell} C_{Y_p} + \frac{2\bar{z}}{\ell} C_{l_\beta} + \frac{2\bar{y}^2}{\ell^2} C_{Z_\alpha} + \frac{2\bar{z}^2}{\ell^2} C_{Y_\beta} \tag{A-33}$$

$$C'_{l_r} = C_{l_r} + \frac{\bar{z}}{\ell} C_{Y_r} - \frac{2\bar{x}}{\ell} C_{l_\beta} - \frac{2\bar{x}\bar{z}}{\ell^2} C_{Y_\beta} + \frac{4\bar{y}}{\ell} C_{l} + \frac{4\bar{y}\bar{z}}{\ell^2} C_{Y} \tag{A-34}$$

Y-axis moment (pitch) coefficients and derivatives.-

$$C'_{m} = C_{m} + \frac{\bar{x}}{\ell} C_{Z} - \frac{\bar{z}}{\ell} C_{X} \tag{A-35}$$

$$C'_{m_\alpha} = C_{m_\alpha} + \frac{\bar{x}}{\ell} C_{Z_\alpha} - \frac{\bar{z}}{\ell} C_{X_\alpha} \tag{A-36}$$

$$C'_{m_{\dot{\alpha}}} = C_{m_{\dot{\alpha}}} + \frac{\bar{x}}{\ell} C_{Z_{\dot{\alpha}}} - \frac{\bar{z}}{\ell} C_{X_{\dot{\alpha}}} \tag{A-37}$$

$$\begin{aligned} C'_{m_q} = C_{m_q} &+ \frac{\bar{x}}{\ell} C_{Z_q} - \frac{\bar{z}}{\ell} C_{X_q} + \frac{2\bar{x}}{\ell} C_{m_\alpha} \frac{2\bar{x}^2}{\ell^2} C_{Z_\alpha} \\ &- \frac{2\bar{x}\bar{z}}{\ell^2} C_{X_\alpha} - \frac{4\bar{z}}{\ell} C_{m} - \frac{4\bar{x}\bar{z}}{\ell^2} C_{Z} + \frac{4\bar{z}^2}{\ell^2} C_{X} \end{aligned} \tag{A-38}$$

Z-axis moment (yaw) coefficients and derivatives.-

$$C'_{n} = C_{n} + \frac{\bar{y}}{\ell} C_{Z} - \frac{\bar{x}}{\ell} C_{Y} \tag{A-39}$$

$$C'_{n_\beta} = C_{n_\beta} - \frac{\bar{x}}{\ell} C_{Y_\beta} \tag{A-40}$$

$$C'_{n_{\dot{\beta}}} = C_{n_{\dot{\beta}}} - \frac{\bar{x}}{\ell} C_{Y_{\dot{\beta}}} \tag{A-41}$$

$$C'_{n_p} = C_{n_p} - \frac{\bar{x}}{\ell} C_{Y_p} - \frac{2\bar{y}^2}{\ell^2} C_{X_\alpha} + \frac{2\bar{z}}{\ell} C_{n_\beta} - \frac{2\bar{x}\bar{z}}{\ell^2} C_{Y_\beta} \tag{A-42}$$

$$C'_{n_r} = C_{n_r} - \frac{\bar{x}}{\ell} C_{Y_r} - \frac{2\bar{x}}{\ell} C_{n_\beta} + \frac{2\bar{x}^2}{\ell^2} C_{Y_\beta} + \frac{4\bar{y}}{\ell} C_{n} - \frac{4\bar{x}\bar{y}}{\ell^2} C_{Y} \tag{A-43}$$

Transfer of Coefficients and Derivatives From Body to Wind-Tunnel Stability Axes

These equations are the most frequently used axes transformations. They convert the coefficients C_X, C_Y, C_Z, etc., measured about body axes in the wind tunnel, into the coefficients about wind-tunnel stability axes C_L, C_Y, C_D', etc. They can be obtained from the general inverse equations (III-55) to (III-107) of section III by replacing the angle A by α and letting the angle B equal zero.

$$C_D' = -C_X \cos\alpha - C_Z \sin\alpha \tag{A-44}$$

$$C_Y = C_Y \tag{A-45}$$

$$C_L = C_X \sin\alpha - C_Z \cos\alpha \tag{A-46}$$

$$C_{l,wt} = C_l \cos\alpha + C_n \sin\alpha \tag{A-47}$$

$$C_m = C_m \tag{A-48}$$

$$C_{n,wt} = -C_l \sin\alpha + C_n \cos\alpha \tag{A-49}$$

$$C_{D_\alpha}' = -C_{X_\alpha} \cos\alpha - C_{Z_\alpha} \sin\alpha + C_L \tag{A-50}$$

$$C_{D_{\dot\alpha}}' = -C_{X_{\dot\alpha}} \cos\alpha - C_{Z_{\dot\alpha}} \sin\alpha \tag{A-51}$$

$$C_{Y_\beta} = C_{Y_\beta} \tag{A-52}$$

$$C_{Y_{\dot\beta}} = C_{Y_{\dot\beta}} \tag{A-53}$$

$$C_{L_\alpha} = C_{X_\alpha} \sin\alpha - C_{Z_\alpha} \cos\alpha - C_D' \tag{A-54}$$

$$C_{L_{\dot\alpha}} = C_{X_{\dot\alpha}} \sin\alpha - C_{Z_{\dot\alpha}} \cos\alpha \tag{A-55}$$

$$C_{l_{\beta,wt}} = C_{l_\beta} \cos\alpha + C_{n_\beta} \sin\alpha \tag{A-56}$$

$$C_{l_{\dot\beta,wt}} = C_{l_{\dot\beta}} \cos\alpha + C_{n_{\dot\beta}} \sin\alpha \tag{A-57}$$

$$C_{l_{p,wt}} = C_{l_p} \cos^2\alpha + C_{n_r} \sin^2\alpha + \left(C_{l_r} + C_{n_p}\right) \sin\alpha \cos\alpha \tag{A-58}$$

$$C_{l_{r,wt}} = C_{l_r} \cos^2\alpha - C_{n_p} \sin^2\alpha + \left(C_{n_r} - C_{l_p}\right) \sin\alpha \cos\alpha \tag{A-59}$$

$$C_{l_{\delta,wt}} = C_{l_\delta} \cos\alpha + C_{n_\delta} \sin\alpha \tag{A-60}$$

$$C_{m_\alpha} = C_{m_\alpha} \quad \text{(A-61)}$$

$$C_{m_{\dot{\alpha}}} = C_{m_{\dot{\alpha}}} \quad \text{(A-62)}$$

$$C_{m_q} = C_{m_q} \quad \text{(A-63)}$$

$$C_{m_\delta} = C_{m_\delta} \quad \text{(A-64)}$$

$$C_{n_{\beta,wt}} = -C_{l_\beta} \sin\alpha + C_{n_\beta} \cos\alpha \quad \text{(A-65)}$$

$$C_{n_{\dot{\beta},wt}} = -C_{n_{\dot{\beta}}} \sin\alpha + C_{n_{\dot{\beta}}} \cos\alpha \quad \text{(A-66)}$$

$$C_{n_{p,wt}} = C_{n_p} \cos^2\alpha - C_{l_r} \sin^2\alpha + \left(C_{n_r} - C_{l_p}\right) \sin\alpha \cos\alpha \quad \text{(A-67)}$$

$$C_{n_{r,wt}} = C_{n_r} \cos^2\alpha + C_{l_p} \sin^2\alpha - \left(C_{l_r} + C_{n_p}\right) \sin\alpha \cos\alpha \quad \text{(A-68)}$$

$$C_{n_{\delta,wt}} = -C_{l_\delta} \sin\alpha + C_{n_\delta} \cos\alpha \quad \text{(A-69)}$$

Transfer of Coefficients and Derivatives From Wind-Tunnel Stability to Body Axes

These transformations convert coefficients about wind-tunnel stability axes C_L, C_Y, C'_D, etc., into coefficients about body axes C_X, C_Y, C_Z, etc. They can be obtained from the general direct transformations in equations (III-1) to (III-36) of section III by replacing the angle A by α and letting the angle B equal zero.

$$C_X = -C'_D \cos\alpha + C_L \sin\alpha = -C_A \quad \text{(A-70)}$$

$$C_Y = C_Y \quad \text{(A-71)}$$

$$C_Z = -C'_D \sin\alpha - C_L \cos\alpha = -C_N \quad \text{(A-72)}$$

$$C_l = C_{l,wt} \cos\alpha - C_{n,wt} \sin\alpha \quad \text{(A-73)}$$

$$C_m = C_m \quad \text{(A-74)}$$

$$C_n = C_{l,wt} \sin\alpha + C_{n,wt} \cos\alpha \quad \text{(A-75)}$$

$$C_{X_\alpha} = -C'_{D_\alpha} \cos\alpha + C_{L_\alpha} \sin\alpha + C_N = -C_{A_\alpha} \quad \text{(A-76)}$$

$$C_{X_{\dot{\alpha}}} = -C'_{D_{\dot{\alpha}}} \cos\alpha + C_{L_{\dot{\alpha}}} \sin\alpha = -C_{A_{\dot{\alpha}}} \quad \text{(A-77)}$$

$$C_{Y_\beta} = C_{Y_\beta} \tag{A-78}$$

$$C_{Y_{\dot\beta}} = C_{Y_{\dot\beta}} \tag{A-79}$$

$$C_{Z_\alpha} = -C'_{D_\alpha} \sin\alpha - C_{L_\alpha} \cos\alpha - C_A = -C_{N_\alpha} \tag{A-80}$$

$$C_{Z_{\dot\alpha}} = -C'_{D_{\dot\alpha}} \sin\alpha - C_{L_{\dot\alpha}} \cos\alpha = -C_{N_{\dot\alpha}} \tag{A-81}$$

$$C_{l_\beta} = C_{l_{\beta,wt}} \cos\alpha - C_{n_{\beta,wt}} \sin\alpha \tag{A-82}$$

$$C_{l_{\dot\beta}} = C_{l_{\dot\beta,wt}} \cos\alpha - C_{n_{\dot\beta,wt}} \sin\alpha \tag{A-83}$$

$$C_{l_p} = C_{l_{p,wt}} \cos^2\alpha + C_{n_{r,wt}} \sin^2\alpha - \left(C_{n_{p,wt}} + C_{l_{r,wt}}\right)\sin\alpha\cos\alpha \tag{A-84}$$

$$C_{l_r} = C_{l_{r,wt}} \cos^2\alpha - C_{n_{p,wt}} \sin^2\alpha + \left(C_{l_{p,wt}} - C_{n_{r,wt}}\right)\sin\alpha\cos\alpha \tag{A-85}$$

$$C_{l_\delta} = C_{l_{\delta,wt}} \cos\alpha - C_{n_{\delta,wt}} \sin\alpha \tag{A-86}$$

$$C_{m_\alpha} = C_{m_\alpha} \tag{A-87}$$

$$C_{m_{\dot\alpha}} = C_{m_{\dot\alpha}} \tag{A-88}$$

$$C_{m_q} = C_{m_q} \tag{A-89}$$

$$C_{m_\delta} = C_{m_\delta} \tag{A-90}$$

$$C_{n_\beta} = C_{l_{\beta,wt}} \sin\alpha + C_{n_{\beta,wt}} \cos\alpha \tag{A-91}$$

$$C_{n_{\dot\beta}} = C_{l_{\dot\beta,wt}} \sin\alpha + C_{n_{\dot\beta,wt}} \cos\alpha \tag{A-92}$$

$$C_{n_p} = C_{n_{p,wt}} \cos^2\alpha - C_{l_{r,wt}} \sin^2\alpha + \left(C_{l_{p,wt}} - C_{n_{r,wt}}\right)\sin\alpha\cos\alpha \tag{A-93}$$

$$C_{n_r} = C_{n_{r,wt}} \cos^2\alpha + C_{l_{p,wt}} \sin^2\alpha + \left(C_{l_{r,wt}} + C_{n_{p,wt}}\right)\sin\alpha\cos\alpha \tag{A-94}$$

$$C_{n_\delta} = C_{l_{\delta,wt}} \sin\alpha + C_{n_{\delta,wt}} \cos\alpha \tag{A-95}$$

TRANSFORMATION EQUATIONS FOR MOMENTS AND PRODUCTS OF INERTIA

Body to Flight Stability Axes

$$I_{X,s} = I_X \cos^2 \alpha_0 + I_Z \sin^2 \alpha_0 - 2I_{XZ} \sin \alpha_0 \cos \alpha_0 \tag{A-96}$$

$$I_{Y,s} = I_Y \tag{A-97}$$

$$I_{Z,s} = I_X \sin^2 \alpha_0 + I_Z \cos^2 \alpha_0 + 2I_{XZ} \sin \alpha_0 \cos \alpha_0 \tag{A-98}$$

$$I_{XZ,s} = \left(I_X - I_Z\right) \sin \alpha_0 \cos \alpha_0 + I_{XZ}\left(\cos^2 \alpha_0 - \sin^2 \alpha_0\right) \tag{A-99}$$

$$I_{XY,s} = I_{XY} \cos \alpha_0 + I_{YZ} \sin \alpha_0 \tag{A-100}$$

$$I_{YZ,s} = I_{YZ} \cos \alpha_0 - I_{XY} \sin \alpha_0 \tag{A-101}$$

Body to Principal Axes

$$I_{X,P} = I_X \cos^2 \epsilon + I_Z \sin^2 \epsilon - 2I_{XZ} \sin \epsilon \cos \epsilon \tag{A-102}$$

$$I_{Y,P} = I_Y \tag{A-103}$$

$$I_{Z,P} = I_Z \cos^2 \epsilon + I_X \sin^2 \epsilon + 2I_{XZ} \sin \epsilon \cos \epsilon \tag{A-104}$$

Flight Stability to Body Axes

$$I_X = I_{X,s} \cos^2 \alpha_0 + I_{Z,s} \sin^2 \alpha_0 + 2I_{XZ,s} \sin \alpha_0 \cos \alpha_0 \tag{A-105}$$

$$I_Y = I_{Y,s} \tag{A-106}$$

$$I_Z = I_{X,s} \sin^2 \alpha_0 + I_{Z,s} \cos^2 \alpha_0 - 2I_{XZ,s} \sin \alpha_0 \cos \alpha_0 \tag{A-107}$$

$$I_{XZ} = \left(I_{Z,s} - I_{X,s}\right) \sin \alpha_0 \cos \alpha_0 + I_{XZ,s}\left(\cos^2 \alpha_0 - \sin^2 \alpha_0\right) \tag{A-108}$$

$$I_{YZ} = I_{YZ,s} \cos \alpha_0 + I_{XY,s} \sin \alpha_0 \tag{A-109}$$

$$I_{XY} = I_{XY,s} \cos \alpha_0 - I_{YZ,s} \sin \alpha_0 \tag{A-110}$$

Flight Stability to Principal Axes

$$I_{X,P} = I_{X,s} \cos^2 \eta + I_{Z,s} \sin^2 \eta + 2I_{XZ,s} \sin \eta \cos \eta \tag{A-111}$$

$$I_{Y,P} = I_{Y,s} \tag{A-112}$$

$$I_{Z,P} = I_{X,s} \sin^2 \eta + I_{Z,s} \cos^2 \eta - 2I_{XZ,s} \sin \eta \cos \eta \tag{A-113}$$

Principal to Body Axes

$$I_X = I_{X,P} \cos^2 \epsilon + I_{Z,P} \sin^2 \epsilon \tag{A-114}$$

$$I_Y = I_{Y,P} \tag{A-115}$$

$$I_Z = I_{X,P} \sin^2 \epsilon + I_{Z,P} \cos^2 \epsilon \tag{A-116}$$

$$I_{XZ} = \left(I_{Z,P} - I_{X,P}\right) \sin \epsilon \cos \epsilon \tag{A-117}$$

Principal to Flight Stability Axes

$$I_{X,s} = I_{X,P} \cos^2 \eta + I_{Z,P} \sin^2 \eta \tag{A-118}$$

$$I_{Y,s} = I_{Y,P} \tag{A-119}$$

$$I_{Z,s} = I_{X,P} \sin^2 \eta + I_{Z,P} \cos^2 \eta \tag{A-120}$$

$$I_{XZ,s} = \left(I_{X,P} - I_{Z,P}\right) \sin \eta \cos \eta \tag{A-121}$$

SPECIAL FORMS OF EQUATIONS OF MOTION

Coupled Linear Equations of Motion

These are the linearized equations that describe the small perturbation motions of a vehicle about a steady-state flight condition. In steady-state flight, the components of thrust, aerodynamic force, and other forces in a given direction are balanced by the component of vehicle weight and, hence, the initial components do not appear in the equations. Other specific assumptions made in deriving these equations are

(1) Total-velocity components, angular rates, and angles are equal to a steady-state value plus a small perturbation value; for example, the velocity component along the X-axis is $u = U_o + \Delta u$, where u is the total-velocity component, U_o is the initial steady-state velocity, and Δu is the perturbation velocity

(2) Products and squares of perturbations can be neglected

(3) The vehicle has a plane of symmetry $\left(I_{XY} = I_{YZ} = 0\right)$

(4) Pitch and roll perturbation rates are given in terms of the initial pitch and roll angles θ_o and ϕ_o and not in terms of the total pitch and roll angles $\theta_o + \Delta\theta$ and $\phi_o + \Delta\phi$; that is,

$$\dot{\theta} = \Delta q \cos\phi_O - \Delta r \sin\phi_O$$

and

$$\dot{\phi} = \Delta p + \Delta q \sin\phi_O \frac{\sin\theta_O}{\cos\theta_O} + \Delta r \cos\phi_O \frac{\sin\theta_O}{\cos\theta_O}$$

(5) Aerodynamic cross-coupling coefficients are zero

(6) Mass-damping and jet-reaction forces and moments can be neglected

(7) The vehicle is flying at constant thrust $\left(\frac{\partial T}{\partial u} = \frac{\partial T}{\partial \delta_{RPM}} = 0\right)$

These equations are written about the body axes but can be converted to the flight stability axes by letting $W_O = 0$ and $\theta_O = \gamma_O$. Also, to reduce the number of variables to six, the incremental velocity components Δu, Δv, and Δw can be expressed in terms of resultant velocity and incremental angle of attack by using the following equations (the alternative would be to expand the aerodynamic forces on the right-hand side in terms of u,v,w derivatives):

For body axis,

$$\Delta u = \Delta V \cos\alpha_0 - V_\infty \Delta\alpha \sin\alpha_0$$

$$\Delta v = V_\infty \Delta\alpha$$

$$\Delta w = \Delta V \sin\alpha_0 + V_\infty \Delta\alpha \cos\alpha_0$$

For flight-stability axis,

$$\Delta u = \Delta V$$

$$\Delta v = V_\infty \Delta\alpha$$

$$\Delta w = V_\infty \Delta\alpha$$

The six equations are

$$\Delta\dot{u} + Q_O \Delta w + W_O \Delta q - R_O \Delta v - V_O \Delta r + g\left(\int \Delta q\, dt \cos\theta_O \cos\phi_O - \int \Delta r\, dt \cos\theta_O \sin\phi_O\right)$$
$$= \frac{\rho V_\infty^2 S}{2m}\left(C_{X_\alpha} \Delta\alpha + C_{X_{\dot{\alpha}}} \frac{\Delta\dot{\alpha}\,\ell}{2V_\infty} + C_{X_V} \frac{\Delta V}{V_\infty} + C_{X_q} \frac{\Delta q\,\ell}{2V_\infty} + \sum C_{X_\delta} \Delta\delta + \ldots\right) \qquad \text{(A-122)}$$

$$\Delta\dot{v} + R_O \Delta u + U_O \Delta r - P_O \Delta w - W_O \Delta p - g\left(\int \Delta p\, dt \cos\theta_O \cos\phi_O + \int \Delta r\, dt \sin\theta_O\right)$$
$$= \frac{\rho V_\infty^2 S}{2m}\left(C_{Y_\beta} \Delta\beta + C_{Y_{\dot{\beta}}} \frac{\Delta\dot{\beta}\,\ell}{2V_\infty} + C_{Y_p} \frac{\Delta p\,\ell}{2V_\infty} + C_{Y_r} \frac{\Delta r\,\ell}{2V_\infty}\right.$$
$$\left. + \sum C_{Y_\delta} \Delta\delta + \ldots\right) \qquad \text{(A-123)}$$

$$\Delta\dot{w} + P_o\,\Delta v + V_o\,\Delta p - Q_o\,\Delta u - U_o\,\Delta q + g\left(\int \Delta p\,dt \cos\theta_o \sin\phi_o + \int \Delta q\,dt \sin\theta_o\right)$$

$$= \frac{\rho V_\infty^2 S}{2m}\left(C_{Z_\alpha}\,\Delta\alpha + C_{Z_{\dot{\alpha}}}\frac{\Delta\dot{\alpha}\,\ell}{2V_\infty} + C_{Z_V}\frac{\Delta V}{V_\infty} + C_{Z_q}\frac{\Delta q\,\ell}{2V_\infty} + \sum C_{Z_\delta}\,\Delta\delta + \ldots\right) \tag{A-124}$$

$$\Delta\dot{p}\,I_X + \left(Q_o\,\Delta r + R_o\,\Delta q\right)\left(I_Z - I_Y\right) - \left(P_o\,\Delta q + Q_o\,\Delta p + \Delta\dot{r}\right)I_{XZ}$$

$$= \frac{1}{2}\rho V_\infty^2 S\ell\left(C_{l_\beta}\,\Delta\beta + C_{l_{\dot{\beta}}}\frac{\Delta\dot{\beta}\,\ell}{2V_\infty} + C_{l_p}\frac{\Delta p\,\ell}{2V_\infty} + C_{l_r}\frac{\Delta r\,\ell}{2V_\infty} + \sum C_{l_\delta}\,\Delta\delta + \ldots\right) \tag{A-125}$$

$$\Delta\dot{q}\,I_Y + \left(P_o\,\Delta r + R_o\,\Delta p\right)\left(I_X - I_Z\right) + 2\left(P_o\,\Delta p - R_o\,\Delta r\right)I_{XZ}$$

$$= \frac{1}{2}\rho V_\infty^2 S\ell\left(C_{m_\alpha}\,\Delta\alpha + C_{m_{\dot{\alpha}}}\frac{\Delta\dot{\alpha}\,\ell}{2V_\infty} + C_{m_V}\frac{\Delta V}{V_\infty} + C_{m_q}\frac{\Delta q\,\ell}{2V_\infty} + \sum C_{m_\delta}\,\Delta\delta + \ldots\right) \tag{A-126}$$

$$\Delta\dot{r}\,I_Z + \left(P_o\,\Delta q + Q_o\,\Delta p\right)\left(I_Y - I_X\right) + \left(Q_o\,\Delta r + R_o\,\Delta q - \Delta\dot{p}\right)I_{XZ}$$

$$= \frac{1}{2}\rho V_\infty^2 S\ell\left(C_{n_\beta}\,\Delta\beta + C_{n_{\dot{\beta}}}\frac{\Delta\dot{\beta}\,\ell}{2V_\infty} + C_{n_p}\frac{\Delta p\,\ell}{2V_\infty} + C_{n_r}\frac{\Delta r\,\ell}{2V_\infty} + \sum C_{n_\delta}\,\Delta\delta + \ldots\right) \tag{A-127}$$

Uncoupled Equations of Motion

The linearized equations of motion can be uncoupled (lateral motions made independent of longitudinal motions) by assuming that the vehicle is in straight and level flight and that there are no components of initial velocity except U_o, W_o, and Q_o in the initial steady-state condition $\left(\text{i.e., } V_o = P_o = R_o = \psi_o = \phi_o = 0\right)$. Under these assumptions, longitudinal equations contain only the variables Δu, $\Delta\theta$, and Δw $(q = \dot{\theta})$; and lateral equations contain only the variables Δv, Δr, and Δp.

Longitudinal equations.-

$$\Delta\dot{u} + Q_o\,\Delta w + W_o\,\Delta q + g\theta_g \cos\theta_{g,o} = \frac{\rho V_\infty^2 S}{2m}\left(C_{X_\alpha}\,\Delta\alpha + C_{X_{\dot{\alpha}}}\frac{\Delta\dot{\alpha}\,\ell}{2V_\infty} + C_{X_V}\frac{\Delta V}{V_\infty}\right.$$

$$\left. + C_{X_q}\frac{\Delta q\,\ell}{2V_\infty} + \sum C_{X_\delta}\,\Delta\delta\right) \tag{A-128}$$

$$\Delta\dot{w} - Q_o\,\Delta u - U_o\,\Delta q + g\theta_g \sin\theta_{g,o} = \frac{\rho V_\infty^2 S}{2m}\left(C_{Z_\alpha}\,\Delta\alpha + C_{Z_{\dot{\alpha}}}\frac{\Delta\dot{\alpha}\,\ell}{2V_\infty} + C_{Z_V}\frac{\Delta V}{V_\infty}\right.$$

$$\left. + C_{Z_q}\frac{\Delta q\,\ell}{2V_\infty} + \sum C_{Z_\delta}\,\Delta\delta\right) \qquad \text{(A-129)}$$

$$\Delta\dot{q}\,I_Y = \frac{\rho V_\infty^2 S \ell}{2}\left(C_{m_\alpha}\,\Delta\alpha + C_{m_{\dot{\alpha}}}\frac{\Delta\dot{\alpha}\,\ell}{2V_\infty} + C_{m_V}\frac{\Delta V}{V_\infty} + C_{m_q}\frac{\Delta q\,\ell}{2V_\infty} + \sum C_{m_\delta}\,\Delta\delta\right) \qquad \text{(A-130)}$$

Lateral equations.-

$$\Delta\dot{v} + U_o\,\Delta r - W_o\,\Delta p - g\left(\phi_g \cos\theta_{g,o} + \psi_g \sin\theta_{g,o}\right)$$

$$= \frac{\rho V_\infty^2 S}{2m}\left(C_{Y_\beta}\,\Delta\beta + C_{Y_{\dot{\beta}}}\frac{\Delta\dot{\beta}\,\ell}{2V_\infty} + C_{Y_p}\frac{\Delta p\,\ell}{2V_\infty} + C_{Y_r}\frac{\Delta r\,\ell}{2V_\infty} + \sum C_{Y_\delta}\,\Delta\delta\right) \qquad \text{(A-131)}$$

$$\Delta\dot{p}\,I_X + Q_o\,\Delta r\left(I_Z - I_Y\right) - \left(Q_o\,\Delta p + \Delta\dot{r}\right)I_{XZ}$$

$$= \frac{\rho V_\infty^2 S \ell}{2}\left(C_{l_\beta}\,\Delta\beta + C_{l_{\dot{\beta}}}\frac{\Delta\dot{\beta}\,\ell}{2V_\infty} + C_{l_p}\frac{\Delta p\,\ell}{2V_\infty} + C_{l_r}\frac{\Delta r\,\ell}{2V_\infty} + \sum C_{l_\delta}\,\Delta\delta\right) \qquad \text{(A-132)}$$

$$\dot{r}\,I_Z + Q_o\,\Delta p\left(I_Y - I_X\right) + \left(Q_o\,\Delta r - \Delta\dot{p}\right)I_{XZ}$$

$$= \frac{\rho V_\infty^2 S \ell}{2}\left(C_{n_\beta}\,\Delta\beta + C_{n_{\dot{\beta}}}\frac{\Delta\dot{\beta}\,\ell}{2V_\infty} + C_{n_p}\frac{\Delta p\,\ell}{2V_\infty} + C_{n_r}\frac{\Delta r\,\ell}{2V_\infty} + \sum C_{n_\delta}\,\Delta\delta\right) \qquad \text{(A-133)}$$

Wind-Axes Equations for a Point Mass

The wind-axes equations used in ballistic trajectory studies in which the vehicle is considered to be a point mass are, along the flight path,

$$m\frac{dV}{dt} = -\frac{\rho V^2 S}{2} C_D - W \sin\gamma + T \qquad \text{(A-134)}$$

and, normal to the flight path,

$$mV\frac{d\gamma}{dt} = \frac{\rho V^2 S}{2} C_L - W \cos\gamma \qquad \text{(A-135)}$$

These equations assume a constant thrust acting parallel to the flight path and neglect jet damping and reaction control forces.

APPENDIX B

DERIVATION OF EQUATIONS PRESENTED IN SECTION II FOR TRANSFER OF COEFFICIENTS TO NEW REFERENCE CENTER

The aerodynamic forces and moments acting on a flight vehicle can be considered to be functions of six independent variables: α,β,V,p,q,r in one system, or u,v,w,p,q,r in the alternate system. To evaluate force and moment derivatives at a new reference center, such as at a new c.g., the derivatives at the new center, which represent changes in forces or moments with changes in the independent variables, both evaluated at the new center, are written, for example,

$$\left.\begin{aligned}
\frac{\partial \overline{X}'}{\partial \alpha'} &= \frac{\partial \overline{X}'}{\partial \alpha}\frac{\partial \alpha}{\partial \alpha'} + \frac{\partial \overline{X}'}{\partial \beta}\frac{\partial \beta}{\partial \alpha'} + \frac{\partial \overline{X}'}{\partial V}\frac{\partial V}{\partial \alpha'} + \frac{\partial \overline{X}'}{\partial p}\frac{\partial p}{\partial \alpha'} + \frac{\partial \overline{X}'}{\partial q}\frac{\partial q}{\partial \alpha'} + \frac{\partial \overline{X}'}{\partial r}\frac{\partial r}{\partial \alpha'} \\
\frac{\partial \overline{X}'}{\partial \beta'} &= \frac{\partial \overline{X}'}{\partial \alpha}\frac{\partial \alpha}{\partial \beta'} + \frac{\partial \overline{X}'}{\partial \beta}\frac{\partial \beta}{\partial \beta'} + \frac{\partial \overline{X}'}{\partial V}\frac{\partial V}{\partial \beta'} + \frac{\partial \overline{X}'}{\partial p}\frac{\partial p}{\partial \beta'} + \frac{\partial \overline{X}'}{\partial q}\frac{\partial q}{\partial \beta'} + \frac{\partial \overline{X}'}{\partial r}\frac{\partial r}{\partial \beta'} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial \overline{X}'}{\partial r'} &= \frac{\partial \overline{X}'}{\partial \alpha}\frac{\partial \alpha}{\partial r'} + \frac{\partial \overline{X}'}{\partial \beta}\frac{\partial \beta}{\partial r'} + \frac{\partial \overline{X}'}{\partial V}\frac{\partial V}{\partial r'} + \frac{\partial \overline{X}'}{\partial p}\frac{\partial p}{\partial r'} + \frac{\partial \overline{X}'}{\partial q}\frac{\partial q}{\partial r'} + \frac{\partial \overline{X}'}{\partial r}\frac{\partial r}{\partial r'}
\end{aligned}\right\} \qquad \text{(B-1)}$$

where the forces $\overline{X}',\overline{Y}',\overline{Z}'$ and moments $\overline{L}',\overline{M}',\overline{N}'$ at the new reference center are related to those at the original reference center by

$$\left.\begin{aligned}
\overline{X}' &= \overline{X} & \qquad \overline{L}' &= \overline{L} + \bar{z}\overline{Y} - \bar{y}\overline{Z} \\
\overline{Y}' &= \overline{Y} & \qquad \overline{M}' &= \overline{M} + \bar{x}\overline{Z} - \bar{z}\overline{X} \\
\overline{Z}' &= \overline{Z} & \qquad \overline{N}' &= \overline{N} + \bar{y}\overline{X} - \bar{x}\overline{Y}
\end{aligned}\right\} \qquad \text{(B-2)}$$

The variables α,β,V,p,q,r evaluated at the original reference center can now be expressed as dependent functions of the independent variables $\alpha',\beta',V',p',q',r'$ of the new system through the equations

$$\left.\begin{aligned}
\alpha &= \tan^{-1}\frac{w}{u} = \tan^{-1}\frac{V'\sin\alpha'\cos\beta' - p\bar{y} + q\bar{x}}{V'\cos\alpha'\cos\beta' - q\bar{z} + r\bar{y}} \\
\beta &= \sin^{-1}\frac{v}{V} = \sin^{-1}\frac{V'\sin\beta' - r\bar{x} + p\bar{z}}{V} \\
V' &= \sqrt{u^2 + v^2 + w^2} \\
&= \sqrt{\left(V'\cos\alpha'\cos\beta' - q\bar{z} + r\bar{y}\right)^2 + \left(V'\sin\beta' - r\bar{x} + p\bar{y}\right)^2 + \left(V'\sin\alpha'\cos\beta' - p\bar{y} + q\bar{x}\right)^2}
\end{aligned}\right\} \qquad \text{(B-3)}$$

$$\left.\begin{aligned}
p &= p' \\
q &= q' \\
r &= r'
\end{aligned}\right\} \qquad \text{(B-4)}$$

Since from equations (B-4) $\frac{\partial}{\partial p'} = \frac{\partial}{\partial p}$, $\frac{\partial}{\partial q'} = \frac{\partial}{\partial q}$, $\frac{\partial}{\partial r'} = \frac{\partial}{\partial r}$, and derivatives such as $\frac{\partial p}{\partial q'} = \frac{\partial p'}{\partial q'} = 0$ (they are derivatives of one independent variable with respect to another), the derivatives in equations (B-1) can be evaluated from equations (B-3) and (B-4) as

$$\left.\begin{aligned}
&\frac{\partial\alpha}{\partial\alpha'} = 1 && \frac{\partial\beta}{\partial\alpha'} = \frac{\partial V}{\partial\alpha'} = \frac{\partial p}{\partial\alpha'} = \frac{\partial q}{\partial\alpha'} = \frac{\partial r}{\partial\alpha'} = 0 && \\
&\frac{\partial\beta}{\partial\beta'} = 1 && \frac{\partial\alpha}{\partial\beta'} = \frac{\partial V}{\partial\beta'} = \frac{\partial p}{\partial\beta'} = \frac{\partial q}{\partial\beta'} = \frac{\partial r}{\partial\beta'} = 0 && \\
&\frac{\partial V}{\partial V'} = 1 && \frac{\partial\alpha}{\partial V'} = \frac{\partial\beta}{\partial V'} = \frac{\partial p}{\partial V'} = \frac{\partial q}{\partial V'} = \frac{\partial r}{\partial V'} = 0 && \\
&\frac{\partial\alpha}{\partial p'} = \frac{-\bar{y}\cos\alpha}{V\cos\beta} && \frac{\partial\beta}{\partial p'} = \frac{\bar{z}\cos\beta + \bar{y}\sin\alpha\sin\beta}{V} && \frac{\partial V}{\partial p'} = \bar{z}\sin\beta - \bar{y}\sin\alpha\cos\beta \\
&\frac{\partial p}{\partial p'} = 1 && \frac{\partial q}{\partial p'} = \frac{\partial r}{\partial p'} = 0 && \\
&\frac{\partial\alpha}{\partial q'} = \frac{\bar{x}\cos\alpha + \bar{z}\sin\alpha}{V\cos\beta} && \frac{\partial\beta}{\partial q'} = \frac{(\bar{z}\cos\alpha - \bar{x}\sin\alpha)\sin\beta}{V} && \frac{\partial V}{\partial q'} = (\bar{x}\sin\alpha - \bar{z}\cos\alpha)\cos\beta \\
&\frac{\partial q}{\partial q'} = 1 && \frac{\partial p}{\partial q'} = \frac{\partial r}{\partial q'} = 0 && \\
&\frac{\partial\alpha}{\partial r'} = \frac{-\bar{y}\sin\alpha}{V\cos\beta} && \frac{\partial\beta}{\partial r'} = \frac{-\bar{x}\cos\beta - \bar{y}\cos\alpha\sin\beta}{V} && \frac{\partial V}{\partial r'} = \bar{y}\cos\alpha\cos\beta - \bar{x}\sin\beta \\
&\frac{\partial r}{\partial r'} = 1 && \frac{\partial p}{\partial r'} = \frac{\partial q}{\partial r'} = 0 &&
\end{aligned}\right\} \qquad \text{(B-5)}$$

Substituting equations (B-5) and (B-2) into equations (B-1) and nondimensionalizing yields the transformation equations given in section II.

The transformations for the u,v,w derivatives (eqs. (II-41) to (II-77) of Sec. II) are derived in the same way except that in this system the derivatives are of the form

$$\left.\begin{aligned}
\frac{\partial \bar{X}'}{\partial u'} &= \frac{\partial \bar{X}'}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \bar{X}'}{\partial v}\frac{\partial v}{\partial u'} + \frac{\partial \bar{X}'}{\partial w}\frac{\partial w}{\partial u'} + \frac{\partial \bar{X}'}{\partial p}\frac{\partial p}{\partial u'} + \frac{\partial \bar{X}'}{\partial q}\frac{\partial q}{\partial u'} + \frac{\partial \bar{X}'}{\partial r}\frac{\partial r}{\partial u'} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial \bar{X}'}{\partial r'} &= \frac{\partial \bar{X}'}{\partial u}\frac{\partial u}{\partial r'} + \frac{\partial \bar{X}'}{\partial v}\frac{\partial v}{\partial r'} + \frac{\partial \bar{X}'}{\partial w}\frac{\partial w}{\partial r'} + \frac{\partial \bar{X}'}{\partial p}\frac{\partial p}{\partial r'} + \frac{\partial \bar{X}'}{\partial q}\frac{\partial q}{\partial r'} + \frac{\partial \bar{X}'}{\partial r}\frac{\partial r}{\partial r'}
\end{aligned}\right\} \tag{B-6}$$

The derivatives in equations (B-6) are evaluated from equations expressing velocity components at the original reference center in terms of those at the new reference center as

$$\left.\begin{aligned}
u &= u' - q\bar{z} + r\bar{y} \\
v &= v' - r\bar{x} + p\bar{z} \\
w &= w' - p\bar{y} + q\bar{x}
\end{aligned}\right\} \tag{B-7}$$

and are given as

$$\left.\begin{aligned}
&\frac{\partial u}{\partial u'} = \frac{\partial v}{\partial v'} = \frac{\partial w}{\partial w'} = \frac{\partial p}{\partial p'} = \frac{\partial q}{\partial q'} = \frac{\partial r}{\partial r'} = 1 \\
&\frac{\partial u}{\partial p'} = 0 \qquad \frac{\partial v}{\partial p'} = \bar{z} \qquad \frac{\partial w}{\partial p'} = -\bar{y} \\
&\frac{\partial u}{\partial q'} = -\bar{z} \qquad \frac{\partial v}{\partial q'} = 0 \qquad \frac{\partial w}{\partial q'} = \bar{x} \\
&\frac{\partial u}{\partial r'} = \bar{y} \qquad \frac{\partial v}{\partial r'} = -\bar{x} \qquad \frac{\partial w}{\partial r'} = 0
\end{aligned}\right\} \tag{B-8}$$

All other derivatives are zero.

Substituting these derivatives from equations (B-8) along with equations (B-2) into equations of the form (B-6) and nondimensionalizing then yield the transformations for u,v,w derivatives given in section II.

In nondimensionalizing, forces are divided by $q_\infty S$; moments, by $q_\infty S\ell$; p, q, r, $\dot{\alpha}$, and $\dot{\beta}$ are multiplied by $1/2V$ and u,v,w by $1/V$. The velocity derivatives, such as C_{X_V}, C_{Y_V}, and C_{m_V}, are written in terms of static forces and moments by making use of the following relationship:

$$\bar{X} = \frac{1}{2}\rho V^2 S C_X$$

therefore,

$$\frac{\partial \bar{X}}{\partial V} = \rho V S C_X + \frac{1}{2}\rho V^2 S \frac{\partial C_X}{\partial V} \tag{B-9}$$

The derivative $\partial C_X/\partial V$ represents the change in C_X brought about by changes in velocity V by itself as an independent variable and not by changes in velocity due to angular-velocity components p,q,r for the rotating body. They are the result of such things as aeroelastic and Mach number effects and are usually neglected, so that equation (B-9) becomes

$$\frac{\partial \bar{X}}{\partial V} = \rho V S C_X \tag{B-10}$$

The relationship in equation (B-10) is also used in nondimensionalizing the derivatives.

The transformations given in section II for the static force and moment coefficients and the static-stability derivatives (derivatives with respect to α or β) are simplified forms that apply only if there is no vehicle rotation and p, q, and r are zero. Although it probably is not practical to use such forms, more general forms can be derived that account for significant rotation; for example, it can be shown that the complete form of $\partial\alpha/\partial\alpha'$ from equations (B-3) is

$$\frac{\partial \alpha}{\partial \alpha'} = 1 + (q\bar{z} - r\bar{y}) \cos \alpha + (p\bar{y} - q\bar{x}) \sin \alpha$$

and, by similarly evaluating $\partial\beta/\partial\alpha'$ and $\partial V/\partial\alpha'$, it can be shown that the more complete transformation for C_{X_α} is

$$\begin{aligned} C'_{X_\alpha} = {} & C_{X_\alpha}\left[1 + \frac{(q\bar{z} - r\bar{y}) \cos \alpha + (p\bar{y} - q\bar{x}) \sin \alpha}{V \cos \beta}\right] \\ & - C_{X_\beta}\left[\frac{(q\bar{z} - r\bar{y}) \sin \alpha - (p\bar{y} - q\bar{x}) \cos \alpha}{V}\right] \sin \beta \\ & + 2C_X\left[\frac{(q\bar{z} - r\bar{y}) \sin \alpha - (p\bar{y} - q\bar{x}) \cos \alpha}{V}\right] \cos \beta \end{aligned} \tag{B-11}$$

which reduces to $C'_{X_\alpha} = C_{X_\alpha}$ (the form given in Sec. II) if $p = q = r = 0$.

APPENDIX C

METHODS OF MEASURING CENTER-OF-GRAVITY LOCATIONS AND MOMENTS OF INERTIA OF MODELS AND FLIGHT VEHICLES

The methods presented here are a summary of methods used at the Langley Research Center to measure moments of inertia of rocket-propelled models and the methods presented in reference 3 for determining c.g. locations and moments of inertia of full-scale airplanes. In these methods all oscillations are assumed to be small. Some other limitations and precautions that must be taken in using these methods, and which apply to all methods, are discussed in reference 3.

CENTER-OF-GRAVITY LOCATION

Longitudinal c.g. Location

The longitudinal c.g. location can be determined by mounting the aircraft or rocket on weighing scales. For a typical aircraft installation, two scales measuring the forces R_1 and R_2 are located at wing jack points and a third scale measuring the force R_3 is located at some distance $\bar{l}$ forward or aft of the jack points. The longitudinal distance from the jack points to the c.g. is then given by

$$\bar{x} = \frac{R_3\bar{l}}{R_1 + R_2 + R_3} \tag{C-1}$$

Vertical c.g. Location

In order to determine the vertical c.g. location the vehicle is mounted on a knife edge as shown in figure 10. The weighing cradle has a weight W_c and a centroid $\bar{z}_c$, and the vehicle is supported in various roll and pitch attitudes by a vertical reaction force R_1 acting at a distance y_1. The distance from the knife edge to the airplane c.g. is obtained from measurements of R_1, y_1, and ϕ by using the equation

$$\bar{z} = \frac{R_1 y_1 - W_c\bar{z}_c \sin\phi}{W \sin\phi} \tag{C-2}$$

MOMENTS OF INERTIA

Compound-Pendulum Method (Fig. 11)

This method is used mainly to determine pitch and roll moments of inertia on small models that can easily be mounted with a single attachment point. The moment of inertia of the model about an axis through the model c.g. is given by

$$I = \frac{W_{M+S}\bar{z}_{M+S}}{4\pi^2} P_{M+S}^2 - \frac{W_S\bar{z}_S}{4\pi^2} P_S^{\;2} - \frac{W_M\bar{z}_M^{\;2}}{g} \tag{C-3}$$

The subscript M+S refers to the model plus the supporting hardware, S to the supporting hardware alone, and M to the model alone. The distance $\bar{z}_{M+S}$, from the knife edge to the c.g. of the model plus supporting hardware, is obtained from

$$\bar{z}_{M+S} = \frac{W_S\bar{z}_S + W_M\bar{z}_M}{W_{M+S}} \tag{C-4}$$

Spring Method

This method is usually used to measure pitch and roll moments of inertia when the model is so large that more than one support is needed. The model is mounted on a knife edge and springs are attached at equal distances on both sides of the knife edge, as shown in figure 12. The moment of inertia can be determined from the equation

$$I = \frac{K\bar{\ell}^2}{4\pi^2} P^2 + \frac{mg\bar{z}}{4\pi^2} P^2 - m\bar{z}^2 \tag{C-5}$$

Spring Method for Full-Scale Vehicles

This method, which is described in detail in reference 3, uses two sets of springs with each set having a different spring constant, so that both the moments of inertia and the vertical c.g. location can be determined. The springs are arranged as shown in figure 13. The moment of inertia about the point axis through the knife edge is given by

$$I = \frac{\left(K_{t,1} - K_{t,2}\right)P_1^2}{4\pi^2\left[1 - \left(\frac{P_1}{P_2}\right)^2\right]} - m\bar{z}^2 - I_c - m_c\bar{z}_c^{\;2} \tag{C-6}$$

The vertical distance from the knife edge to the c.g. of the model is then determined from

$$\bar{z} = \frac{K_{t,2} - K_{t,1}\left(\frac{P_1}{P_2}\right)^2}{W\left[1 - \left(\frac{P_1}{P_2}\right)^2\right]} - \frac{W_c\bar{z}_c}{W} \tag{C-7}$$

where W_c and $\bar{z}_c$ are the weight and c.g. vertical displacement (measured from the knife edge) of the weighing cradle that is used. The constants $K_{t,1}$ and $K_{t,2}$ are determined from the constants for the springs shown in figure 13 by the equations

$$K_{t,1} = 2K_{s,1}a_s^2$$

$$K_{t,2} = 2K_{s,2}a_s^2$$

Torsion-Pendulum Method

Used for measuring yaw moments of inertia, this method involves mounting the model on a torsion shaft as shown in figure 14(a). The moment of inertia about the longitudinal axis of the shaft is given by

$$I = \frac{kP_{M+S}^2}{4\pi^2} - \frac{kP_S^2}{4\pi^2} \tag{C-8}$$

Multifilar-Pendulum Methods

Essentially the same as the torsion spring method except that the vertical wires or rods provide the restraining spring moment, an example of a bifilar pendulum is shown in figure 14(b). The moments of inertia are determined from

$$I = \frac{W_{M+S}}{16\pi^2}\frac{P_{M+S}^2}{\ell'}\bar{a}^2 - \frac{W_S}{16\pi^2}\frac{P_S^2}{\ell'}\bar{a}^2 \tag{C-9}$$

where $\bar{a}$ is the diameter of the circle around which the wires or rods are attached and ℓ' is the length of one of the wires or rods. A system with two wires is called a bifilar pendulum; a system with three is called a trifilar pendulum. In using these methods, the wires or rods must be centered about the system mass center.

APPENDIX D

DETERMINATION OF LONGITUDINAL AND LATERAL STABILITY DERIVATIVES BY USING SIMPLIFIED LINEAR ANALYSIS

Perhaps the simplest and most direct method of analyzing transient flight data is to determine static- and dynamic-stability derivatives from the frequency and the time to damp to half-amplitude of transient motions, such as the damped angle-of-attack oscillation shown in figure 15. This method has been used effectively to determine the control response characteristics of rocket-propelled models in reference 9. Even when a more sophisticated analysis is planned, this method can provide, before the detailed analysis is begun, a fairly accurate quick-look assessment of the vehicle characteristics from the basic flight records.

The oscillation in figure 15 represents the transient response to a step-control input. The method assumes that

(1) The forward velocity and the Mach number of the vehicle are constant

(2) The longitudinal aerodynamic forces vary linearly with angle of attack α, pitch angle θ, elevator deflection δ_e, $d\alpha/dt$, and $d\theta/dt$; lateral aerodynamic forces are assumed to vary linearly with sideslip angle β, yaw angle ψ, rudder deflection δ_r, $d\beta/dt$, and $d\psi/dt$

(3) The vehicle is in level flight before the control deflection is applied

(4) Longitudinal and lateral motions are independent of each other $\left(\text{all aerodynamic cross derivatives such as } C_{l_q} \text{ and } C_{m_p} \text{ are neglected}\right)$

Under these assumptions, the longitudinal equations of motion can be simplified to

$$\frac{mV_\infty}{q_\infty S}\left(\frac{d\theta}{dt} - \frac{d\alpha}{dt}\right) = C_{L,o} + C_{L_\alpha}\alpha + C_{L_{\delta_e}}\delta_e \tag{D-1}$$

$$\frac{I_Y}{q_\infty S\bar{c}}\frac{d^2\theta}{dt^2} = C_{m,o} + C_{m_\alpha}\alpha + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}\bar{c}}{2V_\infty} + C_{m_q}\frac{q\bar{c}}{2V_\infty} + C_{m_{\delta_e}}\delta_e \tag{D-2}$$

where $C_{L,o}$ and $C_{m,o}$ are the values of lift and pitching-moment coefficient in trimmed level flight. The solution to equations (D-1) and (D-2) is of the form

$$\alpha = Ce^{a't}\cos(\omega t - \xi) + \alpha_t \tag{D-3}$$

In equation (D-3), C is a constant that is determined from initial conditions, α_t is the trim angle of attack,

$$a' = -\frac{1}{2}\left[\frac{C_{L_\alpha}}{m'} - \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\frac{\bar{c}}{2V_\infty I'}\right] \tag{D-4}$$

$$\omega = \sqrt{-\frac{C_{m_\alpha}}{I'} - \frac{\bar{c}}{2V_\infty}\frac{C_{m_q}C_{L_\alpha}}{I'm'} - (a')^2} \tag{D-5}$$

and

$$\left.\begin{aligned} m' &= \frac{mV_\infty}{q_\infty S} \\ I' &= \frac{I_Y}{q_\infty S\bar{c}} \end{aligned}\right\} \tag{D-6}$$

The damping constant a' and the frequency ω are determined from measured quantities; a' can be calculated from the rate of decay of the oscillation (see fig. 15) as

$$a' = \frac{\log_e \frac{\Delta\alpha_2}{\Delta\alpha_1}}{t_2 - t_1} = \frac{\log_e \frac{1}{2}}{t_{1/2}} = \frac{-0.693}{t_{1/2}} \tag{D-7}$$

where $t_{1/2}$ is the time for the oscillation to damp to one-half its initial amplitude. The frequency ω can be calculated from the period of the oscillation P as

$$\omega = \frac{2\pi}{P} \tag{D-8}$$

Equation (D-5) can be rewritten as

$$C_{m_\alpha} = -I'\left[(a')^2 + \omega^2\right] - \frac{\bar{c}}{2V_\infty}\frac{C_{m_q}C_{L_\alpha}}{m'} \tag{D-9}$$

and equation (D-4), as

$$C_{m_q} + C_{m_{\dot{\alpha}}} = \frac{4I'V_\infty}{\bar{c}}\left(a' + \frac{C_{L_\alpha}}{2m'}\right) \tag{D-10}$$

The last term of equation (D-9) is usually very small compared to the first term (usually less than 1 percent) and may be omitted in most cases. (See ref. 9.) Omitting this term and substituting equations (D-7) and (D-8) give the equations for the longitudinal static- and dynamic-stability derivatives in terms of the period and the time to damp to half-amplitude of the oscillation as

$$C_{m_\alpha} = \frac{-4\pi^2 I_Y}{q_\infty S\bar{c}}\left[\frac{1}{P^2} + \frac{1}{4\pi^2}\left(\frac{0.693}{t_{1/2}}\right)^2\right] \tag{D-11}$$

$$\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right) = \frac{-2I_Y}{\bar{c}^2}\left(\frac{1.386V_\infty}{q_\infty S t_{1/2}} - \frac{g}{W} C_{L_\alpha}\right) \tag{D-12}$$

Similar equations for the lateral derivatives can be developed as

$$C_{n_\beta} = \frac{4\pi^2 I_Z}{q_\infty Sb}\left[\frac{1}{P^2} + \frac{1}{4\pi^2}\left(\frac{0.693}{t_{1/2}}\right)^2\right] \tag{D-13}$$

$$\left(C_{n_r} - C_{n_{\dot{\beta}}}\right) = \frac{-2I_Z}{b^2}\left(\frac{1.386V_\infty}{q_\infty S t_{1/2}} + \frac{g}{W} C_{Y_\beta}\right) \tag{D-14}$$

For the lateral-derivative equations a small angle of attack is assumed. The values of C_{n_β} and $\left(C_{n_r} - C_{n_{\dot{\beta}}}\right)$ given by equations (D-13) and (D-14) are those about the stability axes.

It should be noted that the mean values of angle of attack or sideslip do not necessarily correspond to the trim value, so that the mean value that is used to determine $t_{1/2}$ (eq. (D-7)) must be obtained by selecting positive and negative amplitudes from the angle-of-attack or sideslip envelopes and determining the mean of these positive and negative amplitudes.

APPENDIX E

USE OF DIRECTION COSINES AND QUATERNIONS IN MOTION CALCULATIONS

In the equations of motion and throughout this report, vehicle orientation with respect to given axes is specified in terms of the Euler angles ψ, θ, ϕ. Two other methods can be used, however, to specify alinement. One is to give the alinement in terms of direction cosines; the other is to specify alinement in terms of "Euler parameters," which are components of a four-parameter quantity called a quaternion. Whereas there is no particular advantage in one of these methods over the other, both have certain advantages over the use of the Euler angles. Both eliminate the singularities that occur when vehicle attitudes approach $\pm 90^{\circ}$ (a condition known as gimbal lock) and, whereas the equations relating Euler angles to angular rates are nonlinear (eqs. (V-7) to (V-9) of Sec. V), those relating direction cosines and Euler parameters to angular rates are linear, so that the computational procedure is simplified, particularly in analog computations. Either method is preferred to the Euler angle method, therefore, in certain applications. The basic equations needed in applying these two methods are given here.

DIRECTION-COSINE METHOD

In the direction-cosine method, the axes transformation for any vector has the form

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = D \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{E-1}$$

where D is the matrix made up of direction cosines and is defined as

$$D = \begin{bmatrix} \ell_{XX} & \ell_{XY} & \ell_{XZ} \\ \ell_{YX} & \ell_{YY} & \ell_{YZ} \\ \ell_{ZX} & \ell_{ZY} & \ell_{ZZ} \end{bmatrix} \tag{E-2}$$

where the elements of D are given in terms of Euler angles as

$$\left.\begin{aligned}
\ell_{XX} &= \cos\psi\cos\theta \\
\ell_{XY} &= \sin\psi\cos\theta \\
\ell_{XZ} &= -\sin\theta \\
\ell_{YX} &= -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi \\
\ell_{YY} &= \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi \\
\ell_{YZ} &= \cos\theta\sin\phi \\
\ell_{ZX} &= \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\
\ell_{ZY} &= -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\
\ell_{ZZ} &= \cos\theta\cos\phi
\end{aligned}\right\} \tag{E-3}$$

It can be shown that D can be determined from the angular velocities by using equations (E-4) and (E-5) as follows:

$$\dot{D} = \begin{bmatrix} 0 & \omega_{Z'} & -\omega_{Y'} \\ -\omega_{Z'} & 0 & \omega_{X'} \\ \omega_{Y'} & -\omega_{X'} & 0 \end{bmatrix} D \tag{E-4}$$

which is a linear (matrix) differential equation. The angular velocities $\omega_{X'},\omega_{Y'},\omega_{Z'}$ are those of the primed axes system (see eq. (E-1)) with respect to the unprimed axes system. In motion calculations where the primed axes are considered as the vehicle body or other reference axes and the unprimed axes are taken as the gravity-axis system (see fig. 5), $\omega_{X'},\omega_{Y'},\omega_{Z'}$ can be determined from the body-axes angular velocities p,q,r by

$$\begin{bmatrix} \omega_{X'} \\ \omega_{Y'} \\ \omega_{Z'} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} - D \begin{bmatrix} -\dot{L} \\ -\left(\Omega_e + \dot{\lambda}\right)\cos L \\ -\left(\Omega_e + \dot{\lambda}\right)\sin L \end{bmatrix} \tag{E-5}$$

or, for a flat nonrotating earth,

$$\left.\begin{aligned}
\omega_{X'} &= p \\
\omega_{Y'} &= q \\
\omega_{Z'} &= r
\end{aligned}\right\} \tag{E-6}$$

QUATERNION METHOD

Basic Quaternion Relationships

In the quaternion method an axes transformation has the form (ref. 10)

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = G \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{E-7}$$

where G is a matrix composed of the Euler parameters e_0, e_1, e_2, e_3 and is defined as

$$G = \begin{bmatrix} \left(e_0^2 + e_1^2 - e_2^2 - e_3^2\right) & 2\left(e_1e_2 + e_0e_3\right) & 2\left(e_1e_3 - e_0e_2\right) \\ 2\left(e_1e_2 - e_0e_3\right) & \left(e_0^2 - e_1^2 + e_2^2 - e_3^2\right) & 2\left(e_0e_1 + e_2e_3\right) \\ 2\left(e_1e_3 + e_0e_2\right) & 2\left(e_2e_3 - e_0e_1\right) & \left(e_0^2 - e_1^2 - e_2^2 + e_3^2\right) \end{bmatrix} \tag{E-8}$$

The Euler parameters are elements of a four-parameter quantity called a quaternion, defined as

$$q = e_0 + e_1 i + e_2 j + e_3 k \tag{E-9}$$

where e_0, e_1, e_2, e_3 are real numbers and the vectors i,j,k satisfy the following conditions:

$$\left.\begin{aligned} i^2 = j^2 = k^2 = -1 \qquad & jk = -kj = 1 \\ ij = -ji = k \qquad & ki = -ik = j \end{aligned}\right\} \tag{E-10}$$

The quantity e_0 is the real or scalar part of the quaternion; the terms $e_1 i + e_2 j + e_3 k$ make up the imaginary part. The length or norm of the quaternion is defined as

$$|q| = \sqrt{qq^*} = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2} \tag{E-11}$$

The quantity q^* is the conjugate of the quaternion and is defined as

$$q^* = e_0 - e_1 i - e_2 j - e_3 k \tag{E-12}$$

The transformation in equation (E-7) can also be written in terms of the quaternion and its conjugate as

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = q^* \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} q \tag{E-13}$$

In flight-motion calculations, in which the Euler parameters define the alinement of the vehicle body or other reference axes with respect to the gravity axes (see figs. 4 and 5), the differential equation from which the Euler parameters can be determined is

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix}\left\{\begin{bmatrix} p \\ q \\ r \end{bmatrix} - G\begin{bmatrix} -\dot{L} \\ -\left(\Omega_e + \dot{\lambda}\right)\cos L \\ -\left(\Omega_e + \dot{\lambda}\right)\sin L \end{bmatrix}\right\} \tag{E-14}$$

where G is defined by equation (E-8).

Relationship Between Euler Parameters and Euler Angles

Initial values of e_0, e_1, e_2, e_3 for use in performing the calculation indicated by equation (E-14) can be determined from the initial values of the Euler angles. The Euler parameters are related to Euler angles by the following equations:

$$\left.\begin{aligned} e_0 &= \cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} \\ e_1 &= \cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} \\ e_2 &= \cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} \\ e_3 &= \sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} \end{aligned}\right\} \tag{E-15}$$

$$\left.\begin{aligned} \sin\theta &= -2\left(e_1 e_3 - e_0 e_2\right) \\ \tan\phi &= \frac{2\left(e_0 e_1 + e_2 e_3\right)}{e_0^2 + e_1^2 - e_2^2 - e_3^2} \\ \tan\psi &= \frac{2\left(e_1 e_2 + e_0 e_3\right)}{e_0^2 - e_1^2 - e_2^2 + e_3^2} \end{aligned}\right\} \tag{E-16}$$

APPENDIX F

SCALING PARAMETERS

This appendix summarizes parameters for the three types of scaling used in aerodynamic testing: dynamic, aerodynamic, and aeroelastic. These parameters establish the requirements for a given type of scaling in that each significant parameter must have the same value for both model and prototype if the simulation is to be valid. In most cases, it is impossible to satisfy all the scaling requirements at the same time and compromise techniques have to be worked out. A knowledge of the scaling parameters is essential to understanding and applying these techniques.

DYNAMIC-SCALING PARAMETERS

Dynamic-scaling parameters established the conditions under which the motions (accelerations, velocities, and angles) and the forces for the model simulate those for the prototype. These parameters are obtained by requiring that, in the equation describing the motion of the flight vehicle, the ratio of any one term to another has the same value for model and prototype. The equation of motion has the general form

$$\text{Inertial force} = \text{Weight} + \text{Aerodynamic force} \qquad \text{(F-1)}$$

Two scaling parameters determined from equation (F-1) are

$$\text{Froude No.} = \frac{\text{Vehicle inertial force}}{\text{Vehicle weight}} = \frac{mV_\infty^2/\ell}{W} = \frac{V_\infty^2}{g\ell} \qquad \text{(F-2)}$$

$$\text{Mass ratio} = \frac{\text{Vehicle inertial force}}{\text{Aerodynamic force}} = \frac{mV_\infty^2/\ell}{\rho V_\infty^2 \ell^2} = \frac{W}{\rho g \ell^3} \qquad \text{(F-3)}$$

In addition, the scaling assumes geometric similarity between model and prototype as well as similarity in mass distribution. The scale factors given in table VI were obtained by satisfying all these requirements.

AERODYNAMIC-SCALING PARAMETERS

Aerodynamic-scaling parameters must have the same value for model and prototype if the flow field around the model and, hence, the aerodynamic force and moment coefficients, is to be the same as around the prototype. Aerodynamic-scaling parameters determined from the equation of motion of the fluid, in which viscous, pressure, and gravity forces are considered, are

$$\text{Reynolds No.} = \frac{\text{Fluid inertial force}}{\text{Fluid viscous force}} = \frac{\rho V_\infty^2 \ell^2}{\mu V_\infty \ell} = \frac{\rho V_\infty \ell}{\mu} \tag{F-4}$$

$$\text{Mach No.} = \frac{\text{Fluid inertial force}}{\text{Fluid pressure force}} = \left(\frac{\rho V_\infty^2 \ell^2}{\rho a^2 \ell^2}\right)^{1/2} = \frac{V_\infty}{a} \tag{F-5}$$

$$\text{Froude No.} = \frac{\text{Fluid inertial force}}{\text{Fluid gravity force}} = \frac{\rho V_\infty^2 \ell^2}{\rho g \ell^3} = \frac{V_\infty^2}{g\ell} \tag{F-6}$$

In addition, if surface-tension forces are important, a fourth parameter to be considered is

$$\text{Weber No.} = \frac{\text{Fluid surface-tension force}}{\text{Fluid inertial force}} = \frac{\sigma \ell}{\rho V_\infty^2 \ell^2} = \frac{\sigma}{\rho V_\infty^2 \ell} \tag{F-7}$$

where σ is the surface tension per unit length.

The pressure force considered in Mach number (eq. (F-5)) is that due to the pressure differential across a shock wave in compressible flow. The pressure-force parameter considered in incompressible flow is

$$\text{Euler No.} = \frac{\text{Fluid pressure force}}{\text{Fluid inertial force}} = \frac{\bar{p}}{\rho V_\infty^2 \ell^2} \tag{F-8}$$

The Euler number is usually not important because usually the resultant body forces are measured; however, it becomes important when body forces are determined from measurements of pressure distribution.

Two scaling parameters determined from the laws of thermodynamics are

$$\text{Prandtl No.} = \frac{1}{\text{Reynolds No.}} \times \frac{\text{Heat added by convection}}{\text{Heat added by conduction}}$$

$$= \frac{\nu}{V_\infty \ell} \, \frac{c_p \rho V_\infty T/\ell}{kT/\ell^2} = \frac{c_p \mu}{k} \tag{F-9}$$

where $\nu = \mu/\rho$, k is the coefficient of heat conduction (thermal conductivity) of the fluid, and

$$\text{Grashof No.} = \frac{3g\,\Delta T}{V_\infty^2 T_o} \tag{F-10}$$

where ΔT is the temperature difference between two representative points in the fluid and T_o is a representative temperature.

Aeroelastic-Scaling Parameters

The basic aeroelastic-scaling parameters to be satisfied in simulating deformations caused by aerodynamic loads are

For structural elongation:

$$\frac{EA}{\rho V_{\infty}^{2} \ell^{2}} \tag{F-11}$$

For structural bending:

$$\frac{EI}{\rho V_{\infty}^{2} \ell^{2}} \tag{F-12}$$

in which it is assumed that the strains in the model structure are the same as for the prototype; that is,

$$\frac{\epsilon_m}{\epsilon_p} = 1.0 \tag{F-13}$$

It is also assumed that neither model nor prototype materials are stressed beyond their elastic limits and that the ratios of shear-to-tensile stress and of shear-to-elastic modulus are the same for model and prototype. A more thorough discussion of elastic scaling is given in reference 11.

COMMENTS ON SCALING PROCEDURES

Dynamic Scaling

In general, tests of dynamically scaled models are limited to subsonic speeds (below $M = 0.6$) because of the requirement that the force coefficients must be the same on model and prototype. For vehicles designed to fly at transonic or supersonic speeds, the model is usually tested at the correct full-scale Mach number. The motions, loads, and accelerations of the test models are then reduced to nondimensional coefficients by utilizing the known mass and inertial characteristics of the models, and these coefficients are in turn used to calculate the motions, loads, and accelerations of the prototype. In any case, care must be taken to insure that Reynolds number effects do not introduce an unacceptable distortion. Reynolds number effects can be neglected in most tests if the model Reynolds number is above 10^6; however, in general, the Reynolds number effects on skin-friction drag must be accounted for.

Aerodynamic Scaling

In wind-tunnel and free-flight model testing, it is impossible to satisfy all the aerodynamic-scaling parameters at the same time; however, the problem usually reduces to one in which only two of the parameters, Reynolds number and Mach number, are important. In most problems, the effect of the fluid gravity on the motion is negligible, and so the Froude number and the Grashof number of the fluid can be neglected. It can be shown that the Prandtl number is the same for model and prototype if the ratio of specific heats c_p/c_v is the same for both, and this condition is automatically satisfied if air is used as the fluid in the model tests. A still further simplification is possible because Reynolds number effects are limited to the narrow boundary-layer region at the body surface and the flow outside the boundary layer is practically without the influence of viscosity. Thus, Reynolds number and Mach number effects can be considered separately. Most wind-tunnel tests at transonic and supersonic speeds are made at the full-scale Mach number. The Reynolds number effects, called scale effects, are accounted for by making corrections to skin-friction drag coefficient (the coefficient primarily affected by Reynolds number) or by using boundary-layer trips, wires or a rough grit, to cause separation at the proper chordwise position and therefore to simulate the full-scale flow pattern. These techniques are discussed in textbooks on wind-tunnel testing (for example, ref. 12) and reports dealing with their application, such as reference 13.

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TABLE I.- CONVERSION OF U.S. CUSTOMARY UNITS TO SI UNITS

Physical quantity	Multiply U.S. Customary Unit	by conversion factor (a)	to obtain SI Unit (b)
Angle	deg	0.01745329	rad
Length	ft	0.3048	m
Velocity	ft/sec	0.3048	m/sec
	knot (international)	0.5144444	m/sec
	knot (U.S. statute)	0.44704	m/sec
	mph	0.44704	m/sec
Acceleration	ft/sec^2	0.3048	m/sec^2
Pressure	lbf/ft^2	47.88026	N/m^2
	atmosphere	1.01325×10^5	N/m^2
	ft of H_2O (39.2° F)	2.98898×10^3	N/m^2
	in. of Hg (60° F)	3.37685×10^3	N/m^2
Moment of inertia	slug-ft^2	1.355818	kg-m^2
Mass	slugs	14.59390	kg
Force	lbf	4.448222	N
Moment	ft-lbf	1.355818	N-m
Area	ft^2	0.09290304	m^2
Mass flow	slugs/sec	14.59390	kg/sec
Density	slugs/ft^3	515.3788	kg/m^3
Spring constant	lbf/in.	175.1268	N/m
	lbf/ft	14.59390	N/m
Gravitational constant	ft^3/sec^2	0.02831685	m^3/sec^2
Specific heat	$\frac{\text{Btu(thermochemical)}}{\text{lbm-°F}}$	4.184×10^3	joule/kg-°C
Kinematic viscosity	ft^2/sec	0.09290304	m^2/sec
Coefficient of heat conduction	$\frac{\text{Btu(thermochemical)-in.}}{\text{ft}^2\text{-sec-°F}}$	518.87315	J/m-sec-K
Temperature	°F	(c)	K
	°C	(c)	K
	°R	(c)	K
Coefficient of viscosity	slugs/ft-sec	47.880258	N-sec/m^2

[a] Based on values in ref. 14.

[b] Prefixes to indicate multiples of SI Units are as follows:

Prefix (symbol)	Multiple
kilo (k)	10^3
hecto (h)	10^2
deka (da)	10
deci (d)	10^{-1}
centi (c)	10^{-2}
milli (m)	10^{-3}
micro (μ)	10^{-6}

[c] Temperatures related by following formulas:

$$K = \frac{5}{9}(°F + 459.67)$$

$$K = °C + 273.15$$

$$K = \frac{5}{9}\,°R$$

TABLE II.- RELATIONSHIPS BETWEEN α,β,V AND u,v,w DERIVATIVES

$$C_{X_\alpha} = -C_{X_u} \sin\alpha \cos\beta + C_{X_w} \cos\alpha \cos\beta$$

$$C_{X_\beta} = -C_{X_u} \cos\alpha \sin\beta + C_{X_v} \cos\beta - C_{X_w} \sin\alpha \sin\beta$$

$$C_{X_V} = C_{X_u} \cos\alpha \cos\beta + C_{X_v} \sin\beta + C_{X_w} \sin\alpha \cos\beta$$

$$C_{X_u} = -C_{X_\alpha} \frac{\sin\alpha}{\cos\beta} - C_{X_\beta} \cos\alpha \sin\beta + C_{X_V} \cos\alpha \cos\beta$$

$$C_{X_v} = C_{X_\beta} \cos\beta + C_{X_V} \sin\beta$$

$$C_{X_w} = C_{X_V} \frac{\cos\alpha}{\cos\beta} - C_{X_\beta} \sin\alpha \sin\beta + C_{X_V} \sin\alpha \cos\beta$$

TABLE III.- ANGLE DESIGNATIONS FOR DIRECT [a] TRANSFORMATIONS

Axes conversion		Angle designation	
From (Unprimed coefficients)	To (Primed coefficients)	A	B
Principal	Body	ϵ [b]	0
Flight stability	Body	α_0	0
Flight stability	Principal	η [b]	0
Wind-tunnel stability	Body	α	0
Wind-tunnel stability	Principal	$\alpha - \epsilon$	0
Wind-tunnel stability	Flight stability	$\alpha - \alpha_0$	0
Wind	Body	α	β
Wind	Principal	$\alpha - \epsilon$	β
Wind	Flight stability	$\alpha - \alpha_0$	β
Wind	Wind-tunnel stability	0	β

[a] Direct transformations (eqs. (III-I) to (III-54)) represent rotation from reference (unprimed) axes system through angle -B about Z-axis and then through angle +A about Y-axis to new (primed) axes system. Inverse transformations (table IV and eqs. (III-55) to (III-107)) represent reverse of direct.

[b] Angles ϵ and η between the principal axes and the body and flight stability axes, respectively, can be determined from moments of inertia by

$$\tan 2\epsilon = \frac{2I_{XZ}}{I_Z - I_X}$$

$$\tan 2\eta = \frac{2I_{XZ,s}}{I_{X,s} - I_{Z,s}}$$

TABLE IV.- ANGLE DESIGNATIONS FOR INVERSE [a] TRANSFORMATIONS

Axes conversion		Angle designation	
From (Primed coefficients)	To (Unprimed coefficients)	A	B
Body	Principal	ϵ	0
Body	Flight stability	α_0	0
Body	Wind-tunnel stability	α	0
Body	Wind	α	β
Principal	Flight stability	η [b]	0
Principal	Wind-tunnel stability	$\alpha - \epsilon$ [b]	0
Principal	Wind	$\alpha - \epsilon$	β
Flight stability	Wind-tunnel stability	$\alpha - \alpha_0$	0
Flight stability	Wind	$\alpha - \alpha_0$	β
Wind-tunnel stability	Wind	0	β

[a] Inverse transformations (eqs. (III-55) to (III-107)) are reverse of direct (table III and eqs. (III-1) to (III-54)) and represent rotation of primed axes system, through angle $-A$ about Y-axis and then through angle $+B$ about Z-axis, until axes coincide with original unprimed system.

[b] Equations for determining angles ϵ and η given in footnote in table III.

TABLE V.- DESIGNATIONS OF FORCE AND MOMENT COEFFICIENTS FOR DIFFERENT AXES SYSTEMS

Component	Coefficients for axes system –			
	Body or principal (a)	Flight stability	Wind-tunnel stability	Wind
X-axis force	C_X or $-C_A$	$C_{X,s}$	$-C'_D$	$-C_D$
Y-axis force	C_Y	$C_{Y,s}$	C_Y	C_C
Z-axis force	C_Z or $-C_N$	$C_{Z,s}$	$-C_L$	$-C_L$
X-axis moment (roll)	C_l	$C_{l,s}$	$C_{l,wt}$	$C_{l,w}$
Y-axis moment (pitch)	C_m	$C_{m,s}$	$C_{m,wt}$	$C_{m,w}$
Z-axis moment (yaw)	C_n	$C_{n,s}$	$C_{n,wt}$	$C_{n,w}$

[a] Subscript P sometimes used to denote principal axes.

TABLE VI.- SCALE FACTORS[a] FOR DYNAMIC SCALING

Area[b]	R_l^2
Volume	R_l^3
Velocity	$R_g^{1/2} R_l^{1/2}$
Acceleration	R_g
Mass	$R_\rho R_l^3$
Mass-flow	$R_\rho R_g^{1/2} R_l^{5/2}$
Weight	$R_\rho R_g R_l^3$
Force coefficient	1.0
Moment coefficient	1.0
Wing loading	$R_\rho R_g R_l$
Time	$R_l^{1/2} R_g^{-1/2}$
Dynamic pressure	$R_\rho R_g R_l$
Force	$R_\rho R_g R_l^3$
Angular velocity	$R_l^{-1/2} R_g^{1/2}$
Angular acceleration	$R_l^{-1} R_g$
Moment	$R_\rho R_g R_l^4$
Moment of inertia	$R_\rho R_l^5$

[a] Scale factor is ratio of model quantity to prototype quantity; for example,

$$\text{Model area} = R_l^2 \times \text{Prototype area}$$

[b] Definitions of symbols:

$$R_l = \frac{\text{Model length}}{\text{Prototype length}}$$

$$R_g = \frac{\text{Acceleration of gravity at model altitude}}{\text{Acceleration of gravity at prototype altitude}}$$

$$R_\rho = \frac{\text{Air density at model altitude}}{\text{Air density at prototype altitude}}$$

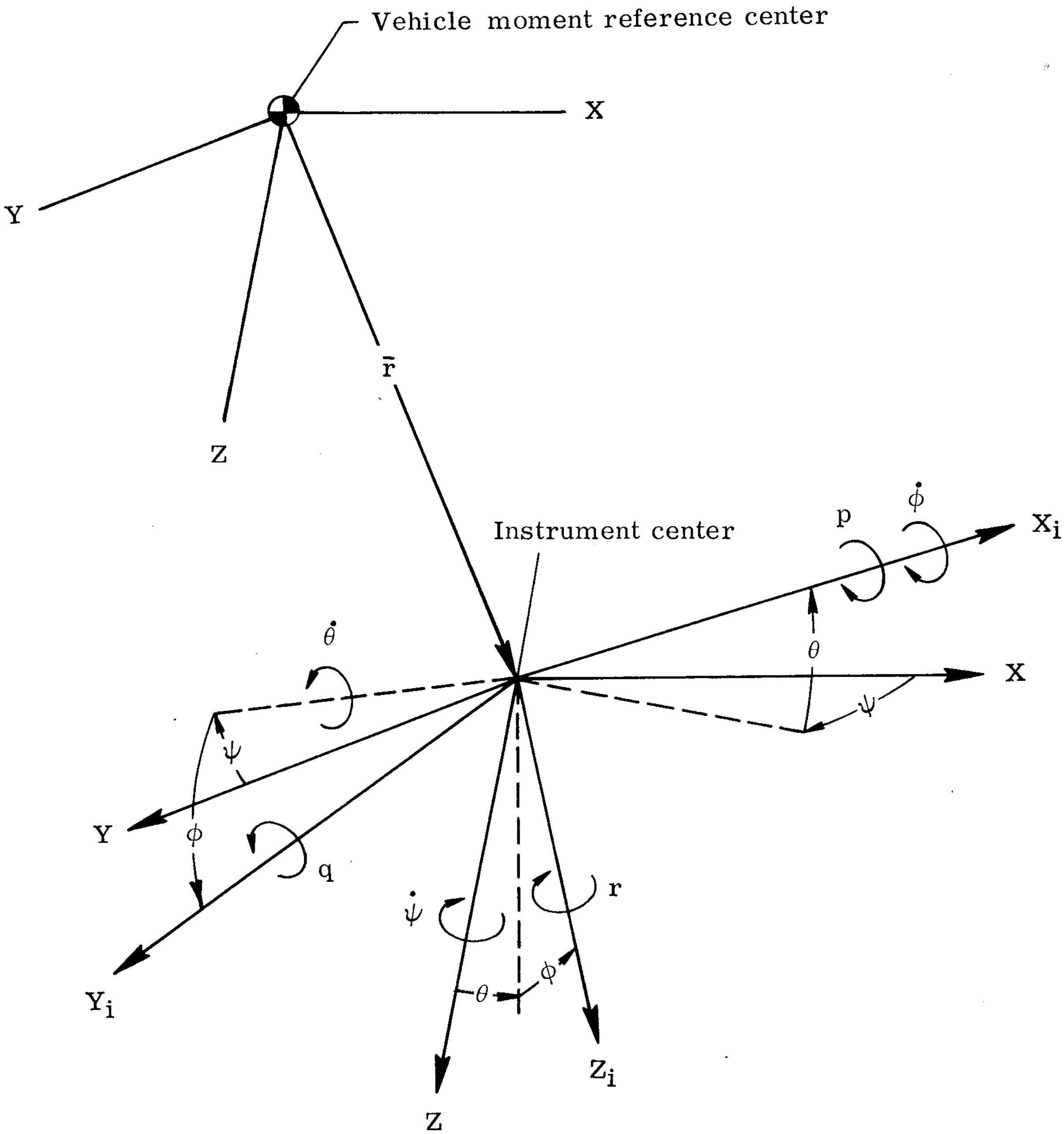


Figure 1.- Systems of vehicle reference axes and instrument axes.

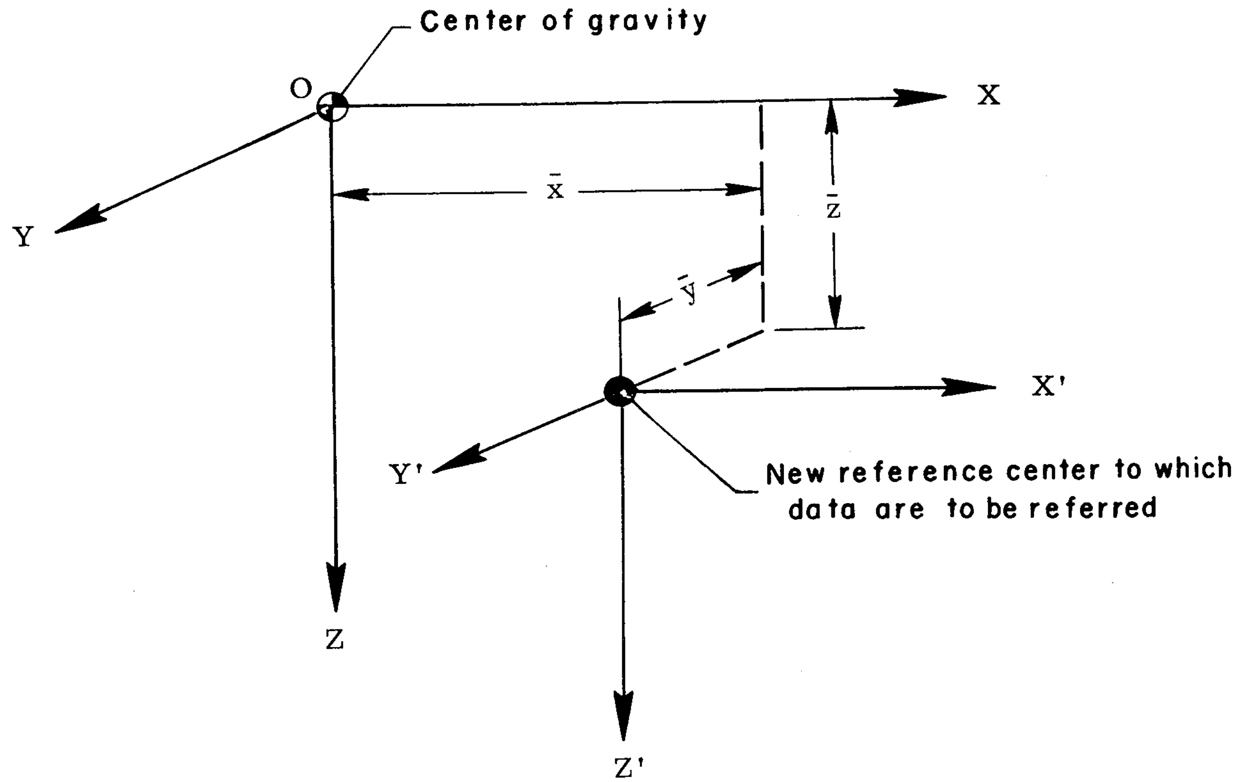


Figure 2.- Axes systems for transfer from vehicle c.g. to new reference center by equations of section II.

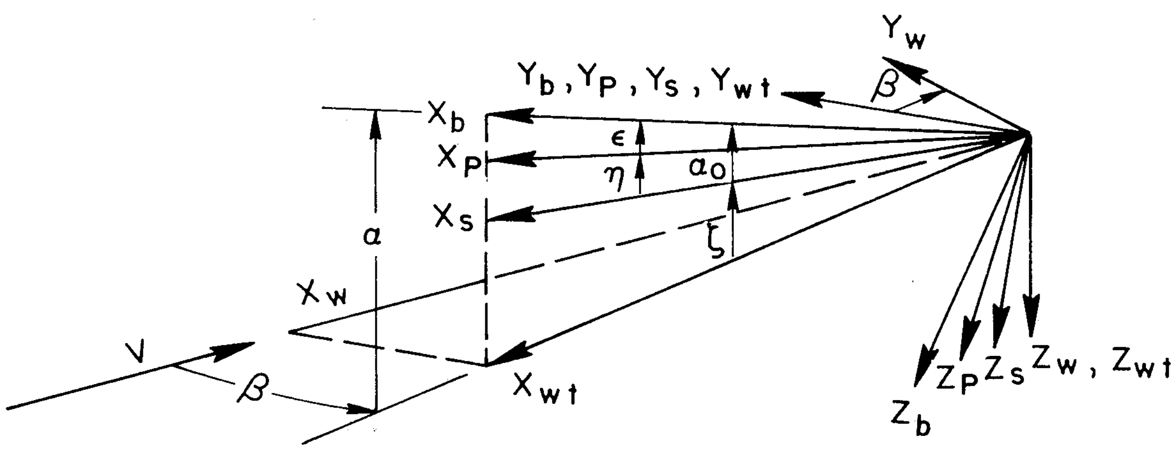


Figure 3.- Systems of vehicle reference axes, including body, principal, wind, flight stability, and wind-tunnel stability.

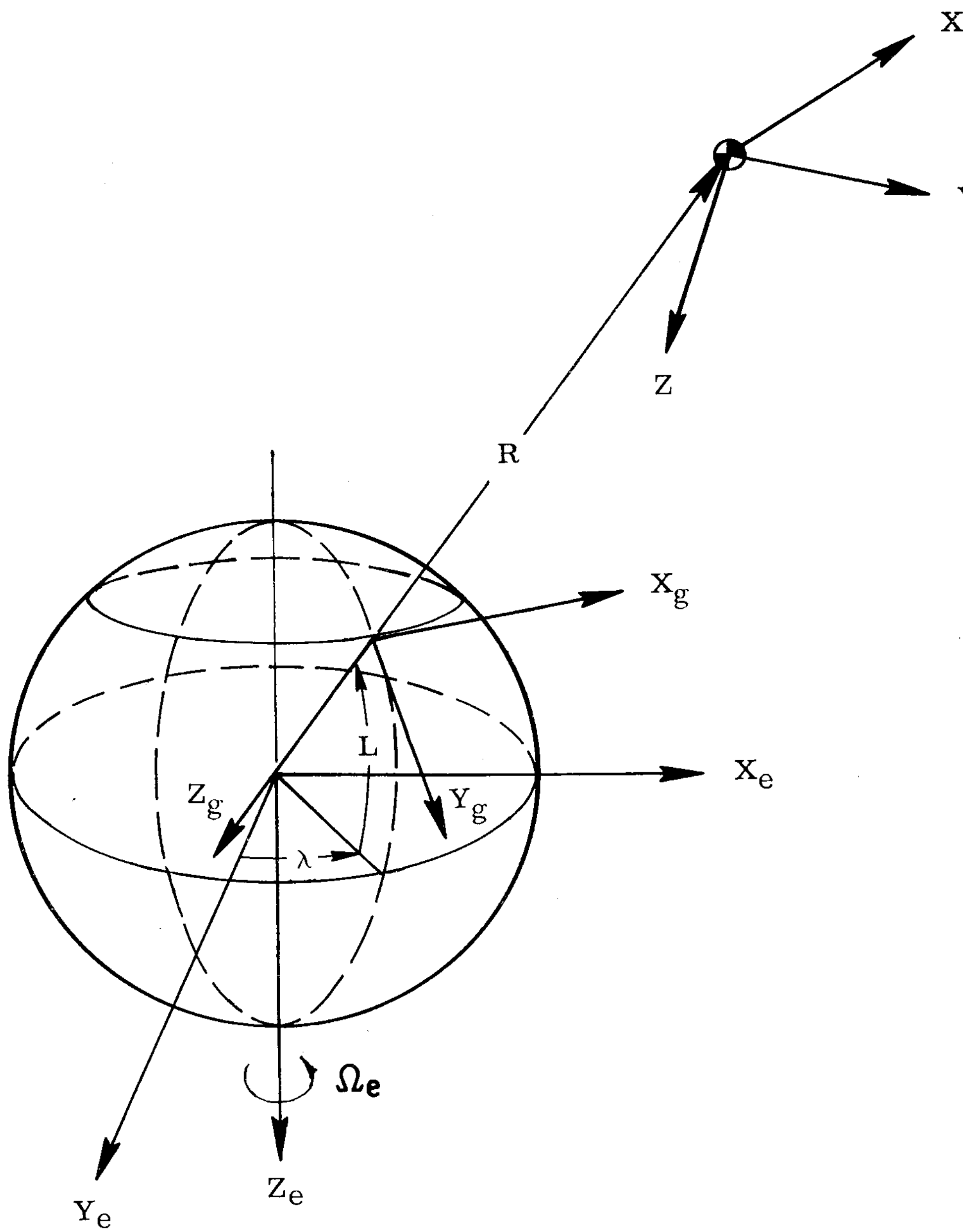


Figure 4.- Relationship between earth-centered inertial axes, gravity axes, and vehicle reference axes.

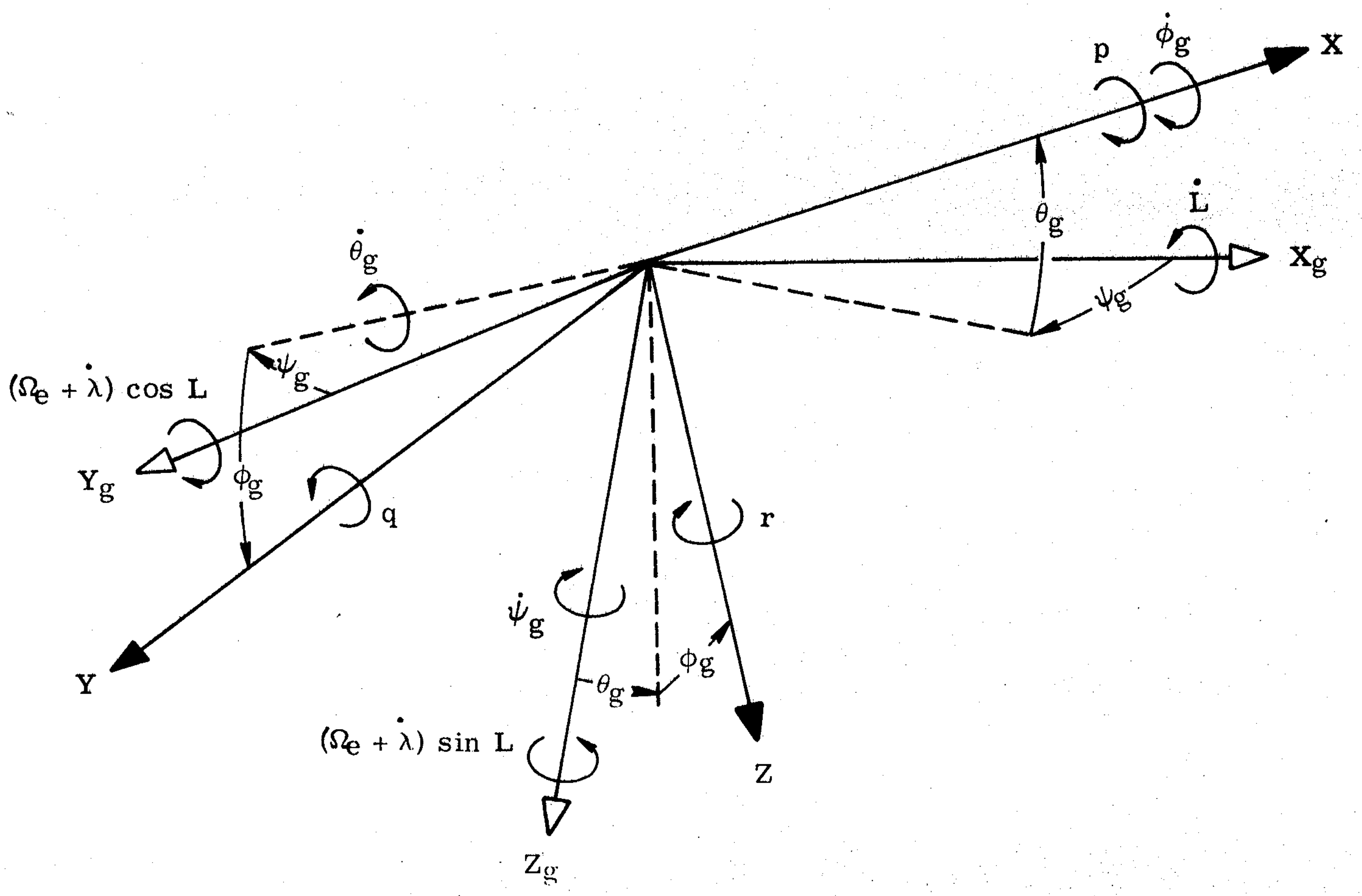


Figure 5.- Systems of gravity axes and vehicle reference axes.

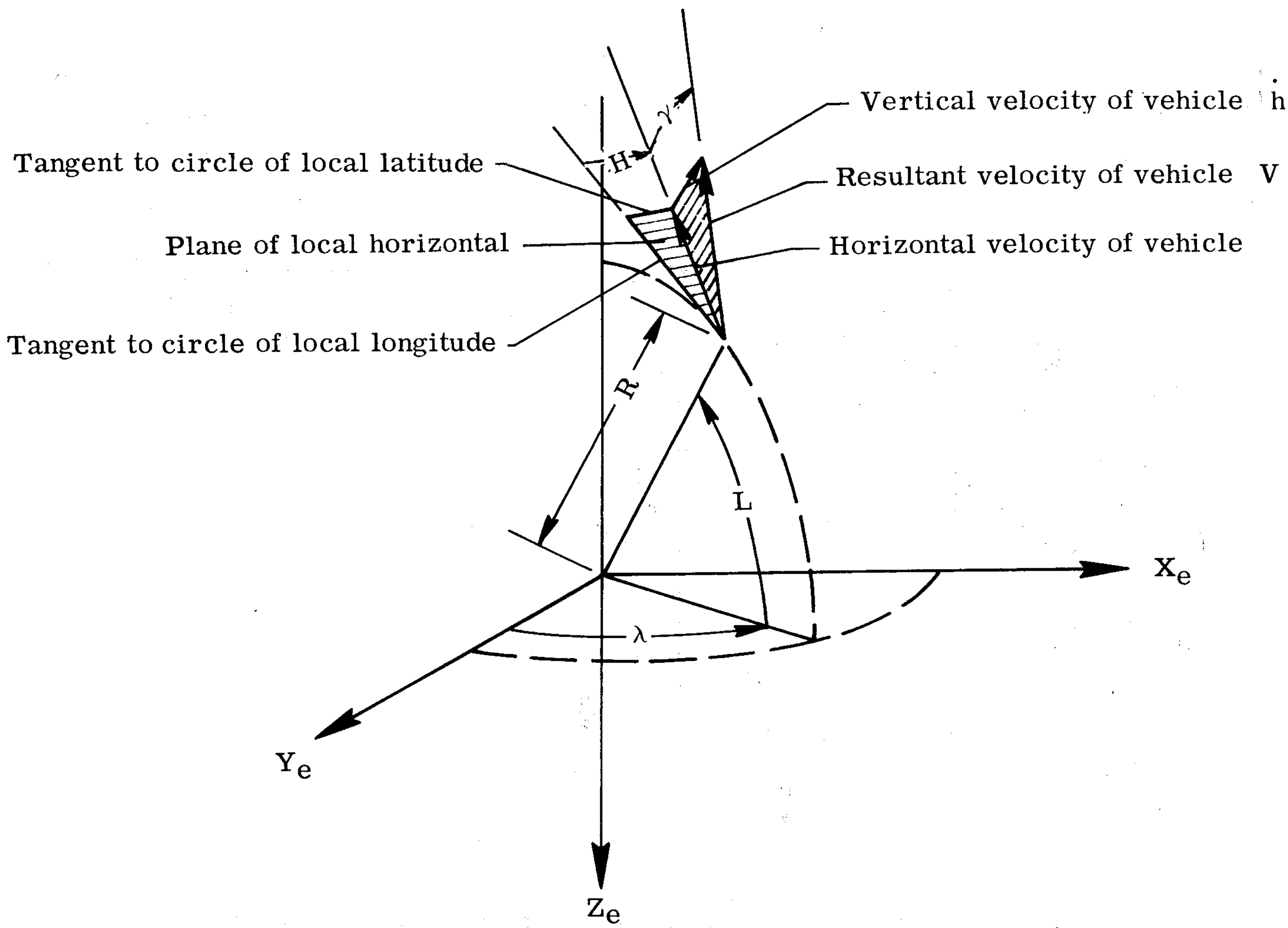


Figure 6.- Relation of heading angle H and flight-path angle γ to earth-centered inertial axes.

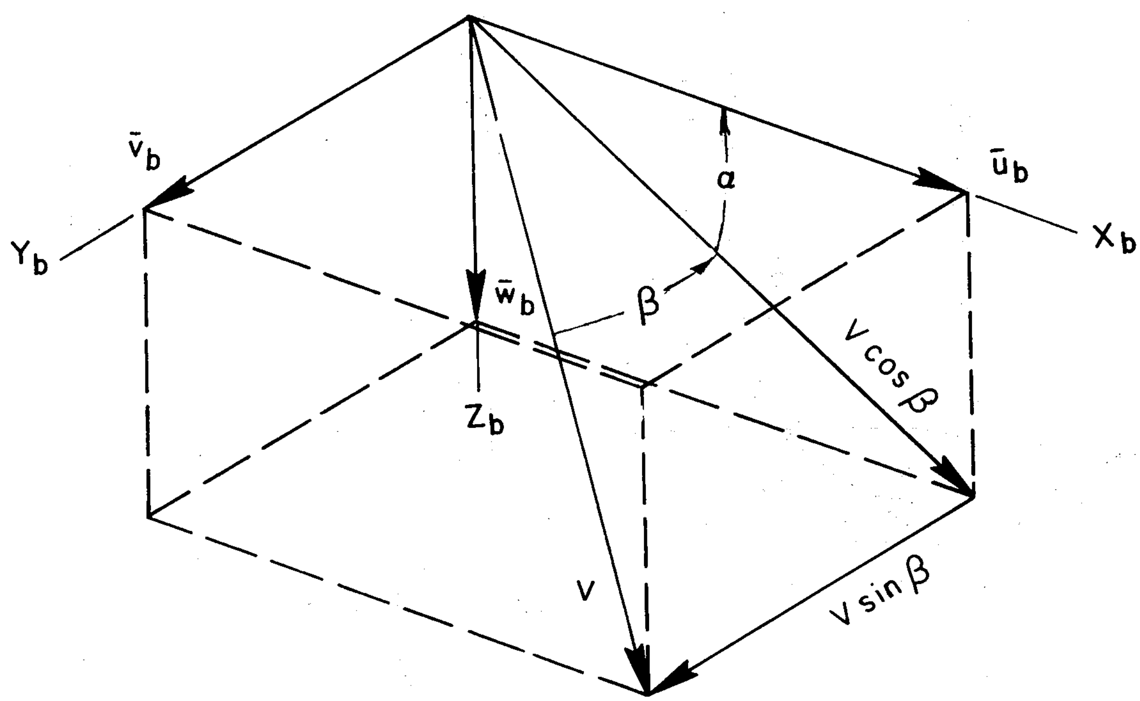


Figure 7.- Resolution of relative velocity into components along vehicle body axes.

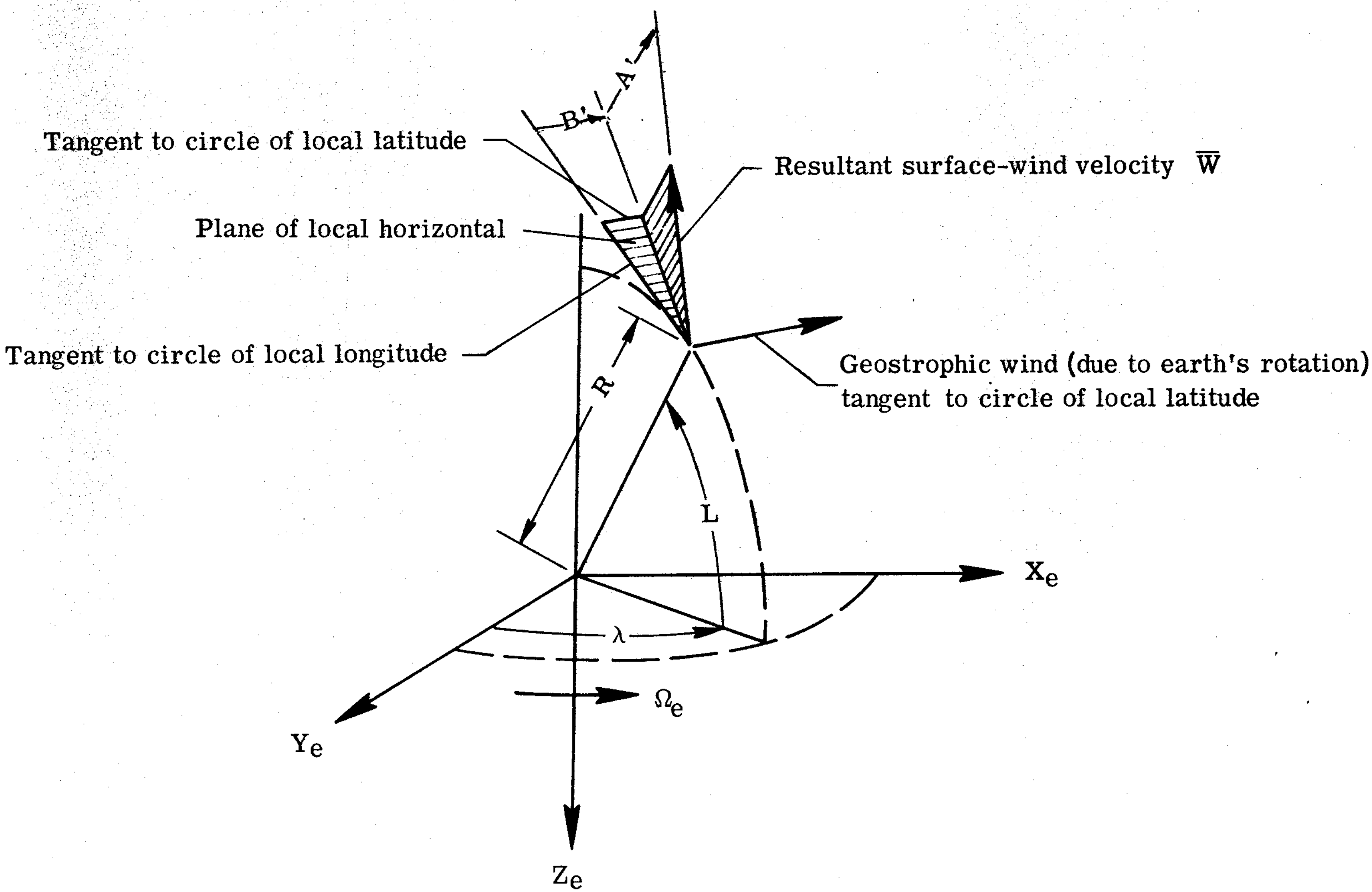


Figure 8.- Directions of surface and geostrophic winds.

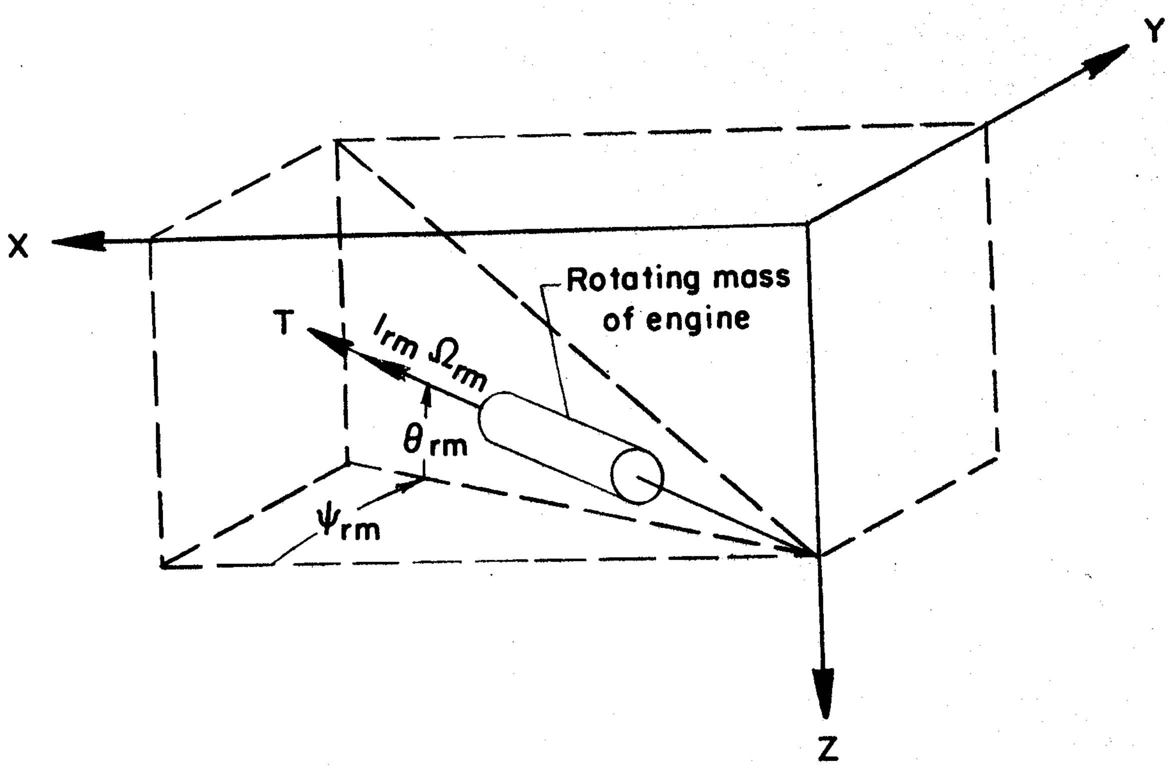


Figure 9.- Alinement with respect to vehicle reference axes of thrust and torque due to rotating mass of engine.

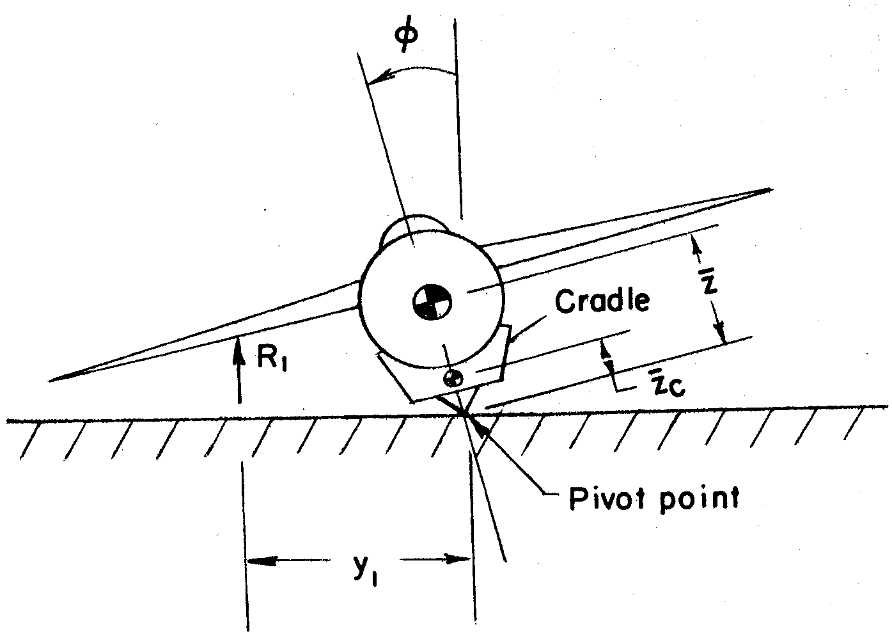


Figure 10.- Determination of vertical location of center of gravity.

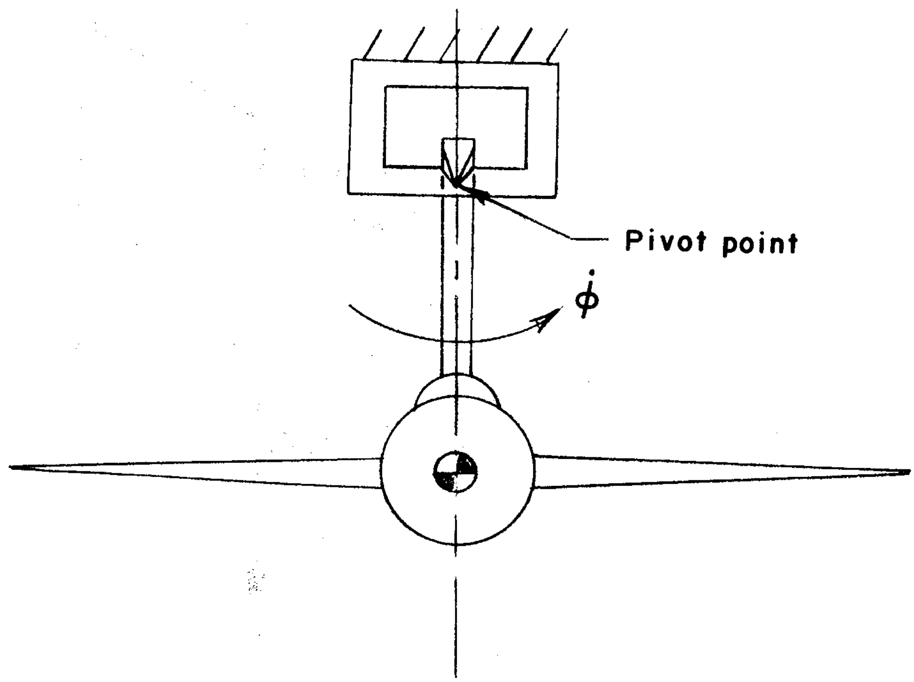


Figure 11.- Measurement of moment of inertia by compound-pendulum method.

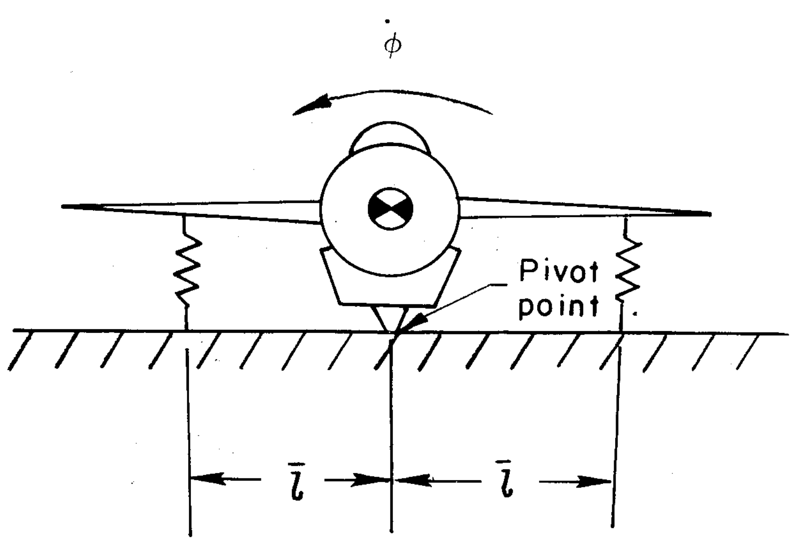


Figure 12.- Measurement of moment of inertia by spring method.

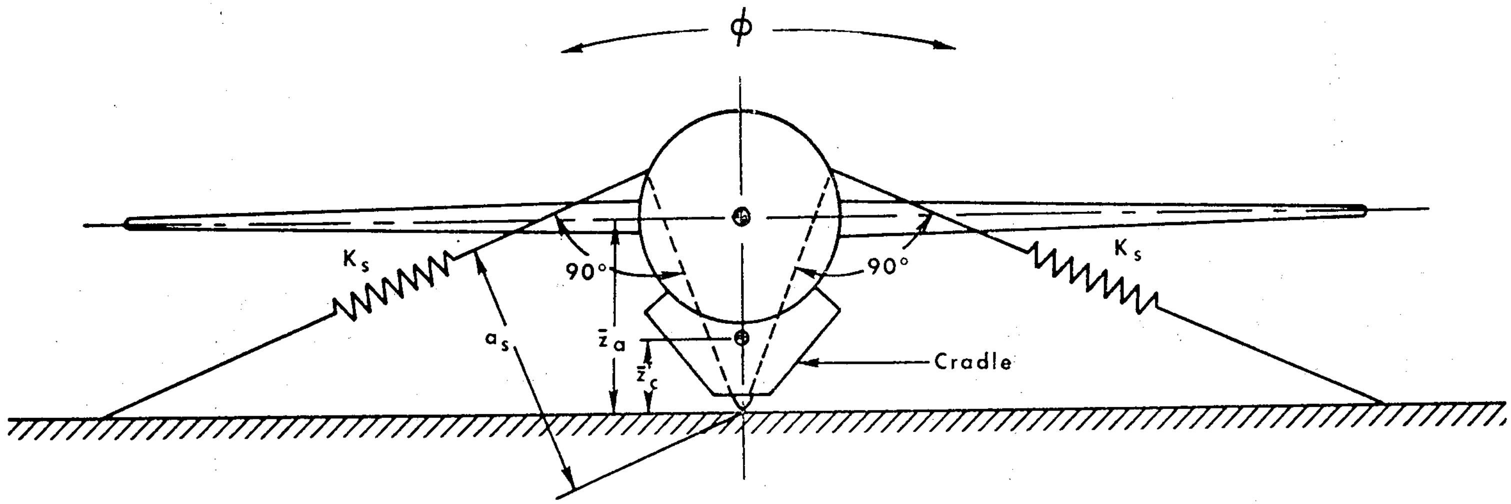
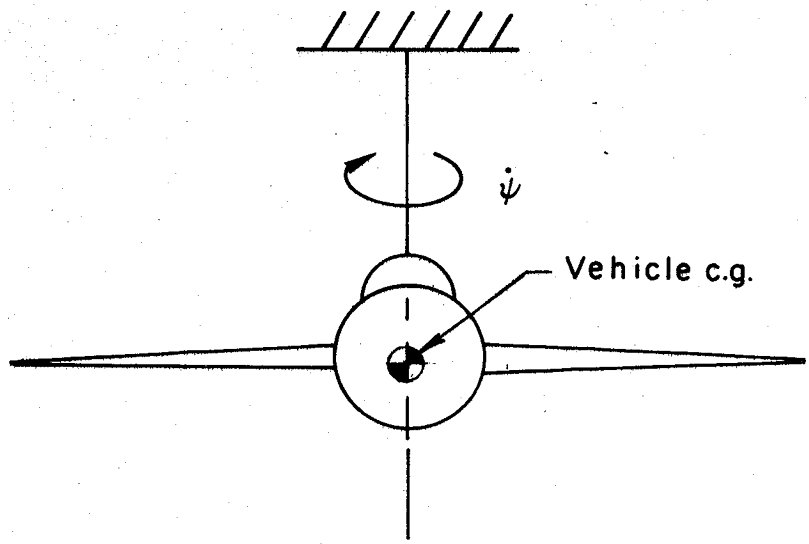
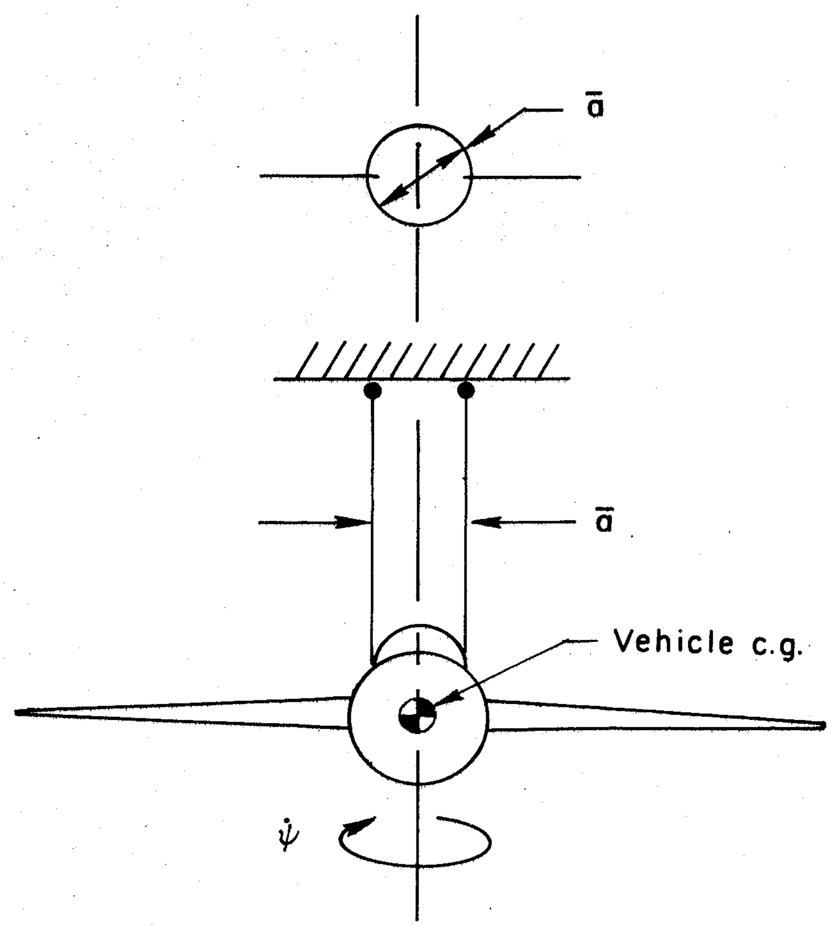


Figure 13.- Determination of vertical location of center of gravity and rolling moments of inertia for full-scale airplanes. (Reproduced from ref. 3.)



(a) Torsion pendulum.



(b) Bifilar pendulum.

Figure 14.- Methods of measuring yawing moments of inertia.

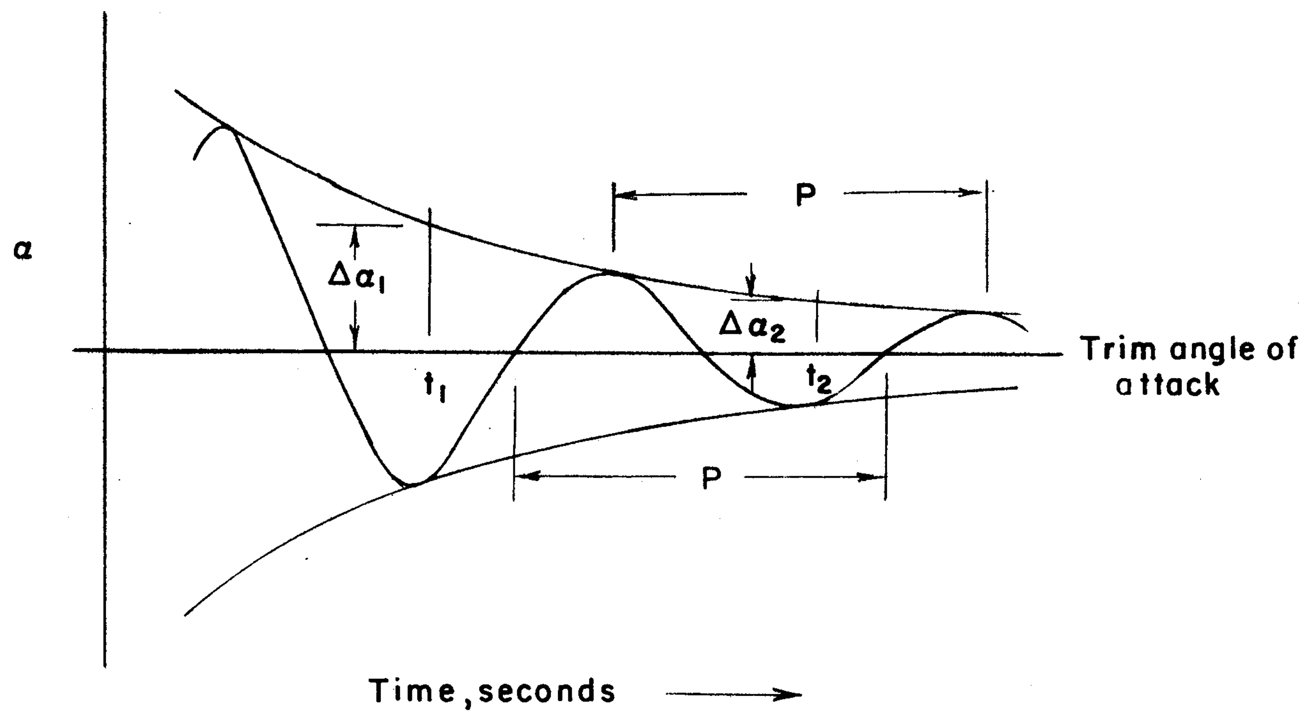


Figure 15.- Damped angle-of-attack oscillation assumed in analysis of appendix D.

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