Summing up Open String Instantons and N=1 String Amplitudes

P. Mayr

CERN Theory Division
CH-1211 Geneva 23
Switzerland

Abstract

We compute the instanton expansions of the holomorphic couplings in the effective action of certain $\mathcal{N} = 1$ supersymmetric four-dimensional open string vacua. These include the superpotential $W(\phi)$, the gauge kinetic function $f(\phi)$ and a series of other holomorphic couplings which are known to be related to amplitudes of topological open strings at higher world-sheet topologies. The results are in full agreement with the interpretation of the holomorphic couplings as counting functions of BPS domain walls. Similar techniques are used to compute genus one partition function for the closed topological string on Calabi–Yau 4-fold which gives rise to a theory with the same number of supercharges in two dimensions.

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1. Introduction

The low energy effective action of many $\mathcal{N} = 1$ supersymmetric string vacua is described by a standard four-dimensional supergravity action defined by the two functions

$$f(\phi), \qquad G = K(\phi, \bar{\phi}) + \ln W(\phi) + \ln \bar{W}(\bar{\phi}),$$

where ϕ denotes a chiral $\mathcal{N} = 1$ multiplet. The Kähler potential K is a real function, whereas the superpotential $W(\phi)$ and the field dependent gauge coupling $f(\phi)$ are holomorphic in the chiral multiplets. The computation and interpretation of instanton corrections to the physical amplitudes in these phenomenologically relevant string vacua has experienced a remarkable development over the last few years. The non-perturbative $\mathcal{N} = 1$ superpotential from genus zero world-sheet instantons can be determined by now for quite a few classes of $\mathcal{N} = 1$ supersymmetric backgrounds by generalizations of the idea of mirror symmetry. These include type II closed string backgrounds with fluxes [1][2][3] and type II backgrounds with open string sectors from background D-branes.

In this paper we study the open string case with an focus on the other holomorphic coupling, the gauge kinetic function $f(\phi)$, as well as some natural generalizations. The open string vacua that we consider are type II compactifications on Calabi–Yau manifolds with additional background D-branes. In this case, a powerful framework for couplings other then the superpotential exists. In fact it has been known for quite some time [4][5][6][3] that an infinite number of physical $\mathcal{N} = 1$ amplitudes in these vacua are computed by the topological version [7] of open type II strings. These are the holomorphic F-terms:

$$\int d^2\theta h\, N\, \mathcal{F}_{g,h}(\phi)\, \mathcal{W}^{2g} S^{h-1}. \tag{1.1}$$

Here g and h denote the genus and number of boundaries of the string world-sheet, respectively, $\mathcal{W}$ is the superfield for the graviphoton field strength and $S = tr W_\alpha W_\alpha$ is the chiral superfield for the gauge field on N coinciding D-branes. In particular the superpotential W and the f-function are related to the genus zero partition functions $\mathcal{F}_{g=0,h=1}$ and $\mathcal{F}_{g=0,h=2}$ with one and two boundaries, respectively. Other world-sheet topologies may contribute to W and f in the presence of vev's for the graviphoton field strength and the gaugino bilinear.

The all genus partition function is related in a beautiful way to a 1-loop Schwinger integral of a dual M-theory compactification [8][6]. In this context the partition function is the weighted counting function of M-theory BPS states. As the number of BPS states is clearly integer, eq.(1.1) leads to the intriguing prediction that the coefficients

1

of the instanton expansion of the holomorphic terms in the $\mathcal{N} = 1$ four-dimensional string effective action are essentially integral. The unraveling of this structure involves a careful treatment of multiple wrappings and their bound states, studied in [6][9][10].

Despite the remarkable progress in open string mirror symmetry, there are still many open problems. The computation of the superpotential $\mathcal{F}_{0,1}$ involves a generalization of mirror symmetry to open strings [11][12][6], which has been a subject of intense studies over the last two years [13] -[25]. For other topologies, an explicit open string recursion relation in g and h for the partition functions $\mathcal{F}_{g,h}$ along the lines of [4] would be desirable[1].

There is another way to compute the partition functions $\mathcal{F}_{g,h}$, proposed originally by Kontsevich [26] and generalized recently to world-sheets with boundaries in [15][16][20]. The purpose of this note is to explore further the structure of the holomorphic $\mathcal{N} = 1$ string amplitudes by an explicit computation of topological partition functions $\mathcal{F}_{g,h}$ in this framework. One of the basic hopes is that the knowledge of the instanton expansions, apart from being an important physical quantity, will serve as a starting point to develop more general principles that provide a closed form of the amplitudes and apply more globally in the space of perturbative moduli.

The organization of this note is as follows. In sect. 2 we review the relation of the physical couplings in the string effective action to the partition functions of the topologial string and recall the interpretation of the latter as weighted counting functions of BPS states in type IIA/M-theory. In sect. 3 we describe the D-brane geometry in terms of a linear sigma model (LSM). We discuss an ambiguity in the definition of the non-perturbative D-brane geometry which amounts to a choice of a $U(1)$ direction in the $U(1)^2$ global symmetry of the LSM. An geometric interpretation is given that is directly related to the framing in Chern-Simons theory and predicts a specific dependence of the A-model computation on the integral parameter ν that labels the $U(1)$ direction. This is the A-model version of the ambiguity discovered in the mirror B-model in [14] and studied in the context of framings in [10]. In sect. 4 we describe the localization computation in the A-model for the two basic one-moduli cases and compute some partition functions $\mathcal{F}_{g,h}$ for various phases of D-branes. In particular we compute the superpotential $W(\phi)$ and the gauge kinetic functions $f(\phi)$ of the string effective $\mathcal{N} = 1$ action, as well as some higher genus generalizations thereof. Re-summing the fractional coefficients of the instanton expansions $\mathcal{F}_{g,h}$ as predicted by M-theory, leads to an integral expansion in an extremely non-trivial way, giving a further verification on the ideas of [6], as well as the techniques proposed

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[1] An explicit expression for the holomorphic anomaly is known for $g = 0, h = 2$ [4].

in [15][16][20]. In sect. 5 we discuss closed string vacua with the same amount of supersymmetry, obtained from a Calabi–Yau 4-fold compactification. We describe the localization computation for the 4-fold, which is slightly different due to the different ghost number of the vacuum, and compute the genus zero and genus one partition functions for the basic one-modulus case. The genus zero result is in agreement with a computation by local mirror symmetry for the 4-fold. Some results on the open string amplitudes are collected in the appendix, an expanded version of which can be found at [27].

2. Open topological strings and counting of BPS domain walls

In this section we review a few facts and describe the setup used in the following sections. We will mainly consider four-dimensional string vacua with $\mathcal{N} = 1$ supersymmetry obtained from type IIA compactification on Calabi–Yau 3-folds X with background D-branes wrapping a 3-cycle L in X and filling space-time. If L is a special Lagrangian (sL) 3-cycle, the effective four-dimensional theory on the brane preserves perturbatively $\mathcal{N} = 1$ supersymmetry. Instanton effects may generate a non-zero superpotential [7][4][28][6][13].

To avoid questions of global flux conservation, we consider non-compact models which describe the local neighborhood of a D-brane in a compact manifold X. The non-compact manifolds will be defined by a linear sigma model [29], which allows also for a simple description of a class of sL 3-cycles found in [12].

The type II D-brane configuration alluded to above is compatible with a topological twist of the world-sheet theory of the type IIA string, the so-called A-model [30]. The correlation functions of the topological A-model on X receive contributions from holomorphic maps from the world-sheet Σ to X, with the boundary $\partial\Sigma$ mapped to L [7]. The topological partition functions $\mathcal{F}_{g,h}$ for world-sheets with genus g and h boundaries, are related to physical amplitudes in the four-dimensional type IIA theory by (1.1)[4][5][3][6].

The partition functions $\mathcal{F}_{g,h}$ of the A-model may be computed by "counting" the holomorphic maps from Σ to X [30][7]. A single world-sheet instanton that maps Σ to a holomorphic curve C in X contributes a term $\exp(-Vol(C))$ times a phase factor. More precisely, there are often families of maps and the "number" of maps is replaced by the virtual Euler number of the appropriately compactified moduli space. This has been made precise for world-sheets without boundaries in [26][31]:

$$\mathcal{F}_{g,0} = \sum_{g,\beta} F_{g,\beta}\, q(\beta), \qquad F_{g,\beta} = \int_{\overline{\mathcal{M}}_{g,0}^{vir}(\beta,X)} c_{top}(U_\beta).$$

(2.1)

3

Here $\overline{\mathcal{M}}_{g,k}^{vir}(\beta, X)$ is the virtual moduli cycle for stable maps from genus g curves with k marked points into X, $\beta \in H_2(X, \mathbf{Z})$ the class of the image and $q(\beta)$ its complexified Kähler volume. Moreover $c_{top}(U_\beta)$ is the Euler class of the relevant obstruction bundle U_β. The fractional coefficients $F_{g,\beta}$ are the Gromov–Witten (GW) invariants. A, somewhat preliminary, generalization of these definitions to world-sheets with boundaries has been developed recently in the papers [15][16][20]. It will be used in the following to determine the partition functions $\mathcal{F}_{g,h}$. The highly non-trivial consistency of the results may serve as a further verification of these generalizations.

A strong consistency condition arises from the interpretation of $\mathcal{F}_{g,h}$ as a type IIA/M-theory 1-loop amplitude that computes the effective couplings (1.1) in a constant self-dual graviphoton background in two dimensions [8][6]. The amplitude receives contributions only from short BPS multiplets and predicts a re-summation of $\mathcal{F}_{g,h}$ into a weighted counting function of integral degeneracies of BPS super-multiplets in a given representation[2]. For a single stack of N D-branes on a sL 3-cycle L with $h_1(L) = 1$, the instanton expansion of the partition function (for a fixed number of boundaries) is of the form

$$\mathcal{F}_h(t_i, r', V) = \sum_{g;w_\alpha} g_s^{2g-2+h} \frac{1}{h!} F_{g;w_\alpha}(q_i) e^{-2\pi r' w} \prod_{\alpha=1}^{h} \mathrm{Tr}\, V^{w_\alpha}, \qquad (2.2)$$

where g_s is the string coupling constant. Moreover the $q_i = exp(2\pi i t_i)$ are the exponentials of the closed string moduli, r' is the Kähler volume of the primitive disc with boundary on L and V is a diagonal $N \times N$ matrix with entries $V_{aa} = e^{i\phi_a}$, where ϕ_a is the holonomy in the a-th $U(1)$ factor along the non-trivial cycle in L. Moreover the integer w_α is the winding number of the α-th boundary along the $S^1 \subset L$ and $w = \sum_\alpha w_\alpha$.

The re-summation of the partition function $\mathcal{F}_{g,h}$ in terms of multiplicities of BPS states has the form [9][10]:

$$\sum_{g=0}^{\infty} g_s^{2g-2+h} F_{g,\beta} = \frac{(-1)^h}{\prod_\alpha w_\alpha} \sum_{\beta'=\beta/d} n_{g,\beta'} \, d^{h-1} \left(2 \sin \frac{dg_s}{2}\right)^{2g-2} \prod_\alpha \left(2 \sin \frac{w_\alpha g_s}{2}\right) q(\beta), \quad (2.3)$$

where $q(\beta)$ is again the exponentiated complex Kähler volume of the class $\beta \in H_2(X, L)$. The class β is specified by the winding numbers w_α of the boundaries of disc components and $h^{1,1}(X)$ degrees n_i for a basis of holomorphic 2-cycles C_i for $H_2(X, \mathbf{Z})$. The coefficients $n_{g,\beta}$ are integral linear combinations of the number of

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[2] For earlier work along these lines, see [32].

4

$\mathcal{N} = 1$ supersymmetric BPS multiplets of a given bulk charge determined by the class β. The terms with $d > 1$ in the above formula are the contributions from multiply wrapped branes, which have an interpretation as the momentum states along the extra circle in the M-theory compactification [32][8][3].

The partition function $\mathcal{F}_h$ as written in (2.2) is the specialization $\operatorname{Im} t_a' = r' \, \forall \, a$ of a holomorphic section $\mathcal{F}_h(t_i', t_a')$, where t_i and t_a' are the bottom components of the moduli superfields from the closed and open string sector, respectively. It comprises amplitudes from various types of world-sheets that involve $N' \leq N$ branes. To extract these different sectors and to make the relation to the results obtained from the following A-model computations explicit[4], it is useful to restore the dependence on the complex open string moduli t_a'. The moduli t_i and t_a' are defined classically by the action of the primitive world-sheet instantons [30][11]:

$$t_i = \int_{C_i} b + iJ, \, i = 1, ..., h^{1,1}(X), \qquad t_a' = \int_{\gamma_a} A + \int_{D_a} iJ, \, a = 1, ..., N. \qquad (2.4)$$

Here J is the Kählerform, b the anti-symmetric 2-form and A the gauge field on the D-brane. Moreover the D_a are N discs ending at the N D-branes on the 1-cycles γ_a. The imaginary parts $\operatorname{Im} t_a'$ parametrize the N independent positions of the D-branes that determine also the size of the discs D_a. The dependence of the sections $\mathcal{F}_{g,h}$ on the complex moduli is determined by holomorphicity:

$$\mathcal{F}_{g,h}(t_i, t_a') = \sum_{g; w_\alpha} g_s^{2g-2+h} \frac{1}{h!} F_{g; w_\alpha}(q_i) \sum_{\vec{a}} \prod_{\alpha=1}^{h} (v_{a_\alpha})^{w_\alpha}. \qquad (2.5)$$

In the above $v_a = exp(2\pi i t_a')$ and the last sum is over $N \cdot h$ choices $a_\alpha \in \{1, ..., N\}$. The partition functions $\mathcal{F}_{g,h}$ for $N' \leq h$ branes can be obtained by moving $N - N'$ of the branes to infinity. This is the limit $v_a = 0$, $a = N'+1, ..., N$. From the above it is clear that the independent GW invariants are related to world-sheets with h distinguished boundaries landing on $N = h$ different branes. This is the siutation which we will consider in the A-model computation. The multi-cover structure will be extracted from the specialized form (2.3) for a stack of $h = N$ D-branes. Partition functions $\mathcal{F}_{g,h}$ and their invariants for less then h branes are obtained from those for $N = h$ branes by the appropriate identifications amongst the open string moduli.

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[3] The class $\beta' = \beta/d \in H_2(X, L; \mathbf{Z})$ exists, if all the degrees n_i and windings w_α specifying the class β are are divisible by d.

[4] This has been also studied by M. Mariño and E. Zaslow [33].

3. A-type D-branes and torus actions

In the following we discuss the definition of the A-model D-brane geometry and its moduli space in terms of a gauged LSM. An important detail is the existence of an integral parameter ν that enters the non-perturbative definition of the string vacuum. This is the A-model version of the framing ambiguity in the Chern-Simons theory [10], and the dependence of the B-model on a certain boundary condition [14]. Here we propose a simple geometric interpretation of the ambiguity in the A-model which implies the specific dependence of the partition function on this parameter.

The classical D-brane geometry for the A-model will be defined as follows. We consider Calabi–Yau 3-folds X defined as a direct sum of concave line bundles[5] over $\mathbf{P}^n$. Much of the following discussion holds generally for D-branes of the same topology. In concrete we will study the two one-moduli cases $\mathcal{K}_{\mathbf{P}^2}$, the canonical bundle of $\mathbf{P}^2$ and the bundle $\mathcal{O}(-1)^{\oplus 2}_{\mathbf{P}^1}$ that describes the local geometry of the blow up of the conifold. The manifold X may be defined as a 2d linear sigma model [29], a (2,2) supersymmetric $U(1)$ theory with four matter fields Z_i of charges $(1, 1, 1, -3)$ for $\mathcal{K}_{\mathbf{P}^2}$ and $(1, 1, -1, -1)$ for $\mathcal{O}(-1)^{\oplus 2}_{\mathbf{P}^1}$, respectively. The solution to the D-term vacuum equations in the scalar components z_i

$$\mathcal{K}_{\mathbf{P}^2} : |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 - r = 0,$$
$$\mathcal{O}(-1)^{\oplus 2}_{\mathbf{P}^1} : |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 - r = 0.$$

describe a smooth Calabi–Yau 3-fold X for large positive values of the real FI parameter r. The single complex Kähler modulus of X is $t = b + ir$ with b the value of the B-field on the fundamental sphere in X.

In addition the type II string vacuum includes D-branes on sL 3-cycles L in X. As in [12] we consider 3-cycles of topology $S^1 \times \mathbf{C} \simeq S^1 \times \tilde{S}^1 \times \mathbf{R}_{\geq 0}$, defined by the equations

$$|z_3|^2 - |z_2|^2 = c^1, \quad |z_3|^2 - |z_4|^2 = c^2, \quad Arg(z_1 z_2 z_3 z_4) = 0. \tag{3.1}$$

Here c^a are two complex constants that are chosen such that one of the S^1 factors (representing $\tilde{S}^1 \subset \mathbf{C}$) shrinks over some edge $z_i = z_j = 0$. In the relevant patch of X, let z_a, z_b denote the two gauge invariant coordinates that vanish at the origin of $\mathbf{C} \subset L$ and z_D the third gauge invariant coordinate on the 3-fold X. The primitive disc in this geometry, that enters the classical definition of the open string modulus (2.4) is defined as

$$D : \quad z_a = z_b = 0, \ |z_D|^2 \leq c. \tag{3.2}$$

with c some complex constant.

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[5] For a discussion and classification of more general bundles, see [34].

The sL cycle L is a cone over T^2, where the 1-cycle $\tilde{S}^1 \in \mathbf{C}$ shrinks at the tip of the cone. The complex torus base can be identified with an $U(1)^2$ subgroup of the global symmetry group $U(1)^3$ of the gauged LSM. The latter acts by the rotations $(z_a, z_b, z_D) \to (z_a\, e^{i\lambda_a \varphi}, z_b\, e^{i\lambda_b \varphi}, z_D\, e^{i\lambda_D \varphi})$ on the inhomogeneous coordinates. The sL condition (3.1) selects the $U(1)^2$ subgroup,

$$\lambda_a + \lambda_b + \lambda_D = 0, \tag{3.3}$$

which is the same as the condition of anomaly freedom for a gauged $U(1)$ symmetry.

There is an inherent ambiguity in the definition of the non-perturbative D-brane geometry which amounts to the choice of a $U(1)$ subgroup of the global symmetry group. We choose coordinates z_a and $\arg(z_D)$ on $L \subset X$ and denote by (γ_0, γ_1) a basis of $H_1(T^2)$ parametrized by the phases (θ_a, θ_D). Although γ_0 is homologically trivial, it will become relevant when defining a pair of non-trivial loops in L. E.g. the geometry of a single world-sheet instanton ending on the D-brane wrapped on L is defined by two maps $f : (\Sigma, \partial\Sigma) \to (\Sigma, L)$ and $g : W \to L$, where Σ is the world-sheet and W the part of the D-brane world-volume mapping into X. In the definition of the pair of maps f and g one has to specify how often the image of a fixed generator in $H_1(W)$, defined by the non-trivial Wilson line, wraps around the image of $\partial\Sigma$. The image of the boundary $\partial\Sigma$ of a minimal volume world-sheet is in the class $[\gamma_1]$ and the image of $H_1(W)$ is in the class

$$[\gamma(\nu)] = 1 \cdot [\gamma_1] + \nu \cdot [\gamma_0], \qquad \nu \in \mathbf{Z}, \tag{3.4}$$

where the coefficient ν of the trivial cycle γ_0 is well-defined in the presence of the world-sheet instanton. The shifts of the origin of the 1-cycle $\gamma(\nu) \simeq S^1$ define a $U(1)$ subgroup of the global $U(1)^2$ symmetry with weights

$$\lambda_a = \nu\lambda_D, \qquad \lambda_b = -(\nu+1)\lambda_D. \tag{3.5}$$

The above geometric picture is closely related to the definition of a framing in the context of Chern-Simons theory. The latter is a choice of UV regularization for the product of Wilson line operators on the same knot k which preserves general covariance. It is specified by a choice of a normal vector to k and the inequivalent choices are labeled by an integer ν that counts how often the first knot wraps the second one in transverse space. In the above, the transverse space (within W) to the non-trivial homology cycle γ_1 is $\mathbf{C}$ and thus the integer ν indeed defines a framing for a Wilson loop on it.

The following A-model computations of the partition functions $\mathcal{F}_{g,h}$ are based on counting the "number" of discs with boundary on L. More precisely this number is

defined as an Euler number of the appropriately defined moduli space of discs. The phases of the open string moduli (2.4) are the Wilson lines along $H_1(W)$ and the definition of the instanton corrected moduli space depends on the invariant ν for the maps f and g described above[6]. On the other hand the computation is technically performed by localizing to the fixed points of the torus action on the moduli space induced by an $U(1) \subset U(1)^2$ action on X. The deformations for a given choice of non-perturbative D-brane geometry are consistent with the specific torus action (3.5). Thus the A-model computation with torus action (3.5) computes the partition function of the D-brane geometry defined by ν, which in turn is directly identified with a framing in CS theory. The existence of an integral ambiguity in the A-model computation had been already verified in [16], guided by the B-model result of [14].

The previous geometric interpretation of the parameter ν in the A-model is similar to the one discussed in [14] for a dual type IIB compactification on a web of (p, q) 5-branes. It is likely that the geometry of the A-model can be used to derive also the functional form of the ν dependence of the special coordinate t', obtained in [14] in the mirror B-model.

4. Localization in the open string A-model and graph sums

The method for the computation of $\mathcal{F}_g$ proposed in [26] uses a group $\mathbf{T}$ of torus actions on the target space X to localize the integrals (2.1) to the fixed points of $\mathbf{T}$. The toric manifolds arising from the linear sigma model have, by definition, a sufficient number of torus actions. Specifically, the torus $\mathbf{T} \simeq (\mathbf{C}^*)^4$ acts on the bundle $X \to \mathbf{P}^n$ by phase rotations of the homogeneous coordinates z_i.

Let M denote some moduli space of stable maps involved in the integral (2.1), ϕ the relevant top form on it and $M^{\mathbf{T}}$ the fixed locus of the induced action of $\mathbf{T}$ on M. An application of the Atiyah-Bott fixed point theorem yields the formula (derived for $g > 0$ and $h = 0$ in [31])

$$K = \int_M \phi = \sum_{\mathcal{M}^\Gamma} \int_{\mathcal{M}^\Gamma} \frac{i^* \phi}{e(N^{vir})}. \tag{4.1}$$

It localizes the integral (2.1) to the components $\mathcal{M}^\Gamma$ of the fixed set $M^{\mathbf{T}}$ of the torus action. In the above, $i : \mathcal{M}^\Gamma \hookrightarrow M$ is the embedding map and $e(N^{vir})$ the Euler class of the virtual normal bundle $N_{\mathcal{M}^\Gamma / M}$.

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[6] A similar comment applies to several loops arising from world-sheet instantons with more boundaries. One may think of the topological instantons as the BPS limit of non-minimal world-sheet wrappings with boundary in the class γ.

The space of maps fixed under the torus action has the following structure [26][16][20]. Let p_i denote the fixed points $z_k = 0$, $k \neq i$ in the base $\mathbf{P}^n$ of the bundle X. Consider a domain curve Σ which is a union of irreducible components $(\cup_\alpha C_\alpha) \cup (\cup_\alpha D_\alpha)$, where C_α is a genus g_α Riemann surface with n_α marked points and D_α are disc components with one marked point in the interior. The irreducible components attach to each other at marked points to form the nodal domain curve Σ. An invariant map contracts all components of Σ to the fixed points p_i with two exceptions: i) a genus zero component C_α with two marked points maps to the line l_{ij} : $z_k = 0$, $k \neq i, j$ connecting the fixed points p_i and p_j with the marked points mapped to the fixed points; ii) at generic moduli of X, the map f restricted to a disc component is non-constant and has as its image the disc $\tilde{l}_{ia}$: $z_k = 0$, $k \neq i, a$; $|z_a|^2 \leq 1$. Here z_a may be either a coordinate z_i on the base $\mathbf{P}^n$, so $\tilde{l}_{ij}$ lies on the line l_{ij} for some j, or a coordinate on the fiber of the bundle X over $\mathbf{P}^n$. The circle $\gamma_{a,I}$ at $|z_a|^2 = 1$ parametrized by the phase of z_a carries a label I that specifies the I-th D-brane on which the boundary of the disc lands. Note that the D_α are the only components of Σ which may map outside the compact base of the bundle X. The above discussion slightly generalizes the set of fixed loci with respect to [16][20].

The irreducible components $\mathcal{M}^\Gamma$ of the fixed locus $\mathcal{M}^\mathbf{T}$ may be characterized by decorated graphs Γ for world-sheets without boundaries [26] and a generalization that includes disc components has been proposed in [20]. A graph Γ will be defined by three sets, the vertices V, connected by edges $e \in E$ and in addition a set of legs $l \in L$ originating at the vertices. A vertex $v \in V$ represents a contracted component or a pole of a $\mathbf{P}^1$ component and carries two labels $i(v)$ and $g(v)$ that specify its image in X, namely the fixed point p_i, and the genus of a contracted component. An edge e represents a non-constant mapping from a genus zero component C_α and carries the labels $i(e), j(e)$ and $\vec{n}$ that specify the image of the south and north poles in X and the class β_e in $H_2(X, \mathbf{Z})$ of the image. The legs represent the disc components and carry labels $i(l)$ and $\vec{w}$ specifying the image of the disc center in X and the class of the image as an element of the relative homology $H_2(X, L)$. In particular $\vec{w}$ specifies the D-brane label I and the circle $\gamma_{a,I}$.

According to the above discussion, we consider the following graphs for the computation of the Gromov–Witten invariant $F_{g,\beta}$. A graph Γ represents a component of the moduli space of maps from world-sheets of genus g, with k marked points and h disc components D_α with image in the class β if:

$$
\begin{aligned}
&1)\ 1 + |E| + \sum_V (g(v) - 1) = g; \qquad &&4)\ \sum_{e \in E} \beta_e + \sum_{l \in L} \beta_L = \beta; \\
&2)\ e \in E \ \Rightarrow\ i(e) \neq j(e); \qquad &&5)\ \cup_v S(v) = \{1, ..., k\}. \\
&3)\ l \in L \ \Rightarrow\ \exists \tilde{l}_{i(l)a};
\end{aligned}
\qquad (4.2)
$$

Here $S(v)$ is the set of marked points on the component C_v and β_e and β_l are the classes in $H_2(X, L)$ of the images of the components C_e and C_l, respectively.

The moduli space $\mathcal{M}^\Gamma$ is the quotient $\prod_{v \in V} \overline{\mathcal{M}}_{g(v), val(v)}/Aut(\Gamma)$ of the product of moduli spaces of Riemann surfaces of genus g and with $val(v)$ marked points. Here $val(v)$ is the number of marked points of the component C_v associated to the vertex v, including the points of intersection with the edge and leg components. As described in the following section, the integrand in (4.1) is a formal sum in the classes c_i and ψ_i in $H^*(\overline{\mathcal{M}}_{g(v), val(v)}, \mathbf{Q})$. Here c_i are the Chern classes of the dual of the Hodge bundle and the ψ_i the first Chern classes of the line bundles associated to the marked points x_i, with fiber the cotangent space at x_i. The integrals over this sum are computable by Faber's algorithm [35]. The result has to be divided by a symmetry factor A_Γ that takes into account the quotient structure of $\mathcal{M}^\Gamma$. It is $A_\Gamma = \prod_{l \in L} w_l^{-1} \times \prod_{e \in E} n_e^{-1} \times a_\Gamma$ where a_Γ is the order of the automorphism group of Γ as a dressed graph with distinguished legs[7] and $n_e = gcd(\vec{n}(e))$.

4.1. The integrand on $\mathcal{M}^\Gamma$

It remains to determine the $\mathbf{T}$ equivariant class of the integrand in the integrals of the graph sum (4.1). The obstruction sequence is

$$0 \to Ext^0(\Omega_\Sigma(E), \mathcal{O}_\Sigma) \to H^0(\Sigma, \partial\Sigma; f^*TX, f^*TL) \to \mathcal{T}^1 \to$$
$$Ext^1(\Omega_\Sigma(E), \mathcal{O}_\Sigma) \to H^1(\Sigma, \partial\Sigma; f^*TX, f^*TL) \to \mathcal{T}^2 \to 0,$$

where E is the divisor of marked points on Σ. The equivariant class of the integrand $i^*\phi/e(N_{\mathcal{M}^\Gamma/M})$ is equal to the class $\mathcal{T}^2/\mathcal{T}^1$ [31][15][16][20]. For an explicit computation one may use the normalization of Σ in terms of its irreducible components

$$0 \to \mathcal{O}_\Sigma \to \oplus_\alpha \mathcal{O}_{C_\alpha} \oplus_\alpha \mathcal{O}_{D_\alpha} \to \oplus_i TX|_{f(x_i)} \to 0,$$

where x_i are the nodes on Σ. The case without boundaries has been considered in [26][31], and the contribution from the disc components in [15][16][20]. In particular the leg contribution to the integrand of a specific sL 3-cycle L in the conifold [6] has been derived in [15][16], and for a phase of a D-brane on $\mathcal{K}_{\mathbf{P}^2}$ in [20]. The integrands needed in the following computations are obtained by a straightforward though somewhat lengthy variation of the arguments in [31][15][16][20] and we refer to these references for more details on the computation. Here we limit ourselves to point out the universal

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[7] Note that a_Γ is not equal to the symmetry factor for the graph Γ with the distinguished legs deleted.

form of the disc contributions which leads to a quick way to determine the open string integrand for any D-brane phase in any toric Calabi–Yau 3-fold.

Indeed it is straightforward to see that the addition of disc components to Σ leads to an essentially universal modification of the closed string computation. Adding discs to Σ is very similar to adding marked points (as discs can not concatenate), up to some extra contributions from the characteristic classes related to the disc components[8]. However the latter modifications are essentially fixed by the degree zero result $\vec{n}(e) = 0 \,\forall\, e \in E$, which is universal. This is because in the limit of large Kähler moduli of X, any D-brane configuration reduces to that in $\mathbf{C^3}$ and this is what is described by the partition function of X at degree zero.

The contribution to the closed string integrand from the normal bundle is [31][9]

$$
\frac{1}{e(N^{vir})} = \prod_{e \in E} \left(\frac{(-)^d}{d!^2} \left(\frac{d}{\lambda_i - \lambda_j} \right)^{2d} \prod_{\substack{k \neq i,j \\ a=0}}^{a=d} \left(\frac{a}{d}\lambda_i + \frac{d-a}{d}\lambda_j - \lambda_k \right)^{-1} \right) \times
$$
$$
\prod_{v \in V} \left(\prod_{k \neq i} \lambda_i - \lambda_k \right)^{val-1} \times
\begin{cases}
\left(\sum_{F \ni v} w_F^{-1} \right)^{val-3} \prod_{F \ni v} w_F^{-1} & g = 0 \\
\frac{\prod_{k \neq i} P_g(\lambda_i - \lambda_k)}{\prod_{F \ni v} w_F - \psi_F} & g > 0
\end{cases}
\tag{4.3}
$$

The weights λ_i specify the torus action $z_i \to e^{i\lambda_i \alpha} z_i$ on the homogeneous coordinates of the $\mathbf{P}^n$ base. The polynomial $P_g(\lambda) = \sum_{k=0}^{g} \lambda^k c_{g-k}$ is the equivariant top Chern class of the dual of the Hodge bundle, twisted by $U(1)$ with weight λ. Moreover a flag $F : (v, e)$ is defined as an oriented edge e with origin $i(e) = i(v)$ and its weight under the torus action is defined as $w_F = (\lambda_{i(e)} - \lambda_{j(e)})/n_e$. A similar expression describes the pull-back $i^*\phi$ of the Euler class of the obstruction bundle, which depends however on X and will thus be stated later.

Adding disc components to the world-sheet amounts to replacing a closed string graph Γ by a related graph Γ' with some legs added. The modified integrand that includes the contributions from the disc components can be conveniently written in two factors. The first one is identical to $i^*\phi \cdot (4.3)$, with $val(v)$ counting the marked points, edges *and* legs at the vertex $v \in V$ of the graph Γ' and similarly the set $\{F \ni v\}$ of flags runs over flags associated with both, edges and legs, attached to v.[10]

---

[8] This was used in [20] to derive a mirror identity for $g = 0$, $h = 1$.

[9] Below it is understood that a subscript i refers to an edge label $i(e)$ under the first product and a vertex label $i(v)$ under the second product. A similar convention will be used in the following formulae.

[10] The weight of a disc flag $F : (v, l)$ is defined as $w_F = -\lambda_D/w(l)$, where λ_D is the weight of the gauge invariant coordinate z_D on the disc in X.

This is the result of [20] for $\mathcal{K}_{\mathbf{P}^2}$, which holds generally for toric 3-folds by the previous universality argument. The second factor of the open string integrand is the following product of universal disc contributions

$$\prod_L \left(\frac{-1}{w!}\left(\frac{w}{\lambda_D}\right)^w\right) \times \left(\prod_{k=1}^{w-1} \lambda_\perp + \frac{k}{w}\lambda_D\right)$$
$$= (-1)^h \prod_L \frac{1}{\lambda_D} \prod_{k=1}^{w-1} 1 + \frac{w}{k}\frac{\lambda_\perp}{\lambda_D},$$

(4.4)

and it describes the contribution from $H^k(D_{\alpha(l)}, f^*TX)$, $k = 0, 1$. The above expression summarizes the results of [15][16][20] in a universal form, the important point being that the contribution from the cohomology groups that contribute to (4.4) depends only on the local disc geometry and is therefore universal, that is independent of the bundle V.

The weight λ_D in (4.4) is the weight of the coordinate z_D on the primitive disc in X defined as in (3.2). It remains to specify the weight $\lambda_\perp$. First note that on dimensional grounds, the integral in (4.1) is of total degree zero in the torus weights λ_i. The partition function $\mathcal{F}_{g,h}$ depends therefore only on ratios of the λ_i. The definition of the sL cycle L requires the sum of the weights to be zero and reduces the free parameter to a single ratio of weights.[11]

As discussed in sect. 3 , the relevant ratio originates in the definition of the geometry of the world-sheet instantons, where we have to specify how often the image of $H_1(W)$ wraps around the origin of $\mathbf{C} \subset L$ when it wraps once around the non-trivial $S^1 \subset L$. With the coordinates z_a, z_b defined as in (3.2), the geometry with fixed winding ν leads to two possible choices for the $U(1)$ action, depending on the orientation of the D-brane:

$$\lambda_\perp = \lambda_{z_a} = \nu\lambda_D, \qquad \lambda_\perp = \lambda_{z_b} = \nu'\lambda_D, \qquad \nu \in \mathbf{Z}.$$

The two choices are related by the invariance condition as in (3.5). There will be a symmetry of the invariants $n_{g,\beta}(\nu)$ for the moduli spaces (not the partition functions) under the exchange $\nu \to -(\nu + 1)$, if the exchange of coordinates $z_a \leftrightarrow z_b$ induces also a symmetry transformation of the closed string background.

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[11] The closed string partition functions $\mathcal{F}_{g,h=0}$ are in fact completely independent of a choice of torus action, essentially since the closed string observables are related to the ordinary cohomology of X.

4.2. Case I: The blow up of the conifold $\mathcal{O}(-1)^{\oplus 2}_{\mathbf{P}^1}$

The first one modulus case we consider is the blow up of the conifold. The degree zero contribution to $\mathcal{F}_{g,h}$ has been studied in [16][15]; the same result had been previously obtained in [6] by a Chern-Simons computation for the unknot. Substantial generalizations of the CS computation to other world-sheet topologies and other knots and links have appeared in [36][9][10][37]. The partition function $\mathcal{F}_{0,1}$ has also been obtained in [12] by mirror symmetry.

In the patch $z_1 \neq 0$, the gauge invariant coordinates are $z = z_2/z_1$ for the $\mathbf{P}^1$ and two coordinates $a = z_3 z_1$ and $b = z_4 z_1$ on the $\mathcal{O}(-1)$ fibers. With the above conventions, the weight of z is $\lambda_z = \lambda_2 - \lambda_1$. The weights $\lambda_{a,i}, \lambda_{b,i}$ of the torus action on the fiber can be chosen arbitrary at the fixed point p_1 and are related by the projective action on $\mathbf{P}^1$ to the weights at the second fixed point. Specifically $\lambda_{a,2} = \lambda_{a,1} + \lambda_z$, and similarly for b.

The class of the pull back $i^*\phi$ of the Euler class of the obstruction bundle computes to

$$
i^*\phi = \prod_{v \in V} P_g(-\lambda_{a,i}) P_g(-\lambda_{b,i}) (\lambda_{a,i} \cdot \lambda_{b,i})^{val(v)-1}
$$
$$
\times \prod_{e \in E} \prod_{m=1}^{d-1} \left(\lambda_{a,i} - \frac{m}{d}(\lambda_i - \lambda_j) \right) \left(\lambda_{b,i} - \frac{m}{d}(\lambda_i - \lambda_j) \right). \tag{4.5}
$$

The open string integrand is the product $(4.3)\cdot (4.4)\cdot (4.5)$. The invariance condition for the sL cycle (3.1) reads $\lambda_{a,1} + \lambda_{b,1} + \lambda_z = 0$. There are two different phases of D-branes, depending on whether the center of $\mathbf{C} \subset L$ maps to a point on the compact $\mathbf{P}^1$ or not. In the first case, the primitive disc is $D : a = b = 0, |z|^2 \leq c$ and thus $z_D = z, z_\perp = a, \nu = \frac{\lambda_{a,1}}{\lambda_z}$. In the second case, $D : b = z = 0, |a|^2 \leq c$ and $z_D = a, z_\perp = z, \nu = \frac{\lambda_z}{\lambda_{a,1}}$. Although the invariants are significantly different in the two phases[12], the the general structure is similar and the following discussion will be limited to the second phase. Results for the first phase can be found in the appendix.

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[12] The two phases are related to the two sides of a flop transition in a Calabi–Yau 4-fold by an open/closed string duality [18].

4.2.1. The superpotential and higher genus generalizations

We consider first the superpotential $W = \mathcal{F}_{0,1}$ on X and its higher genus generalizations described in (1.1). The topological partition functions $\mathcal{F}_{g,h=1}$ are obtained by evaluating the graph sum (4.1) for graphs with one leg. Let $q = exp(2\pi i t)$ denote the exponential of the complexified Kähler volume of the primitive sphere in X and $v_1 = exp(2\pi i t')$ the exponential of the single open string modulus. The first terms of the instanton expansion of the partition functions $\mathcal{F}_{g,1}$ for $g < 5$ are:

$$
\begin{aligned}
\mathcal{F}_{0,1} =& \left(v_1 + \tfrac{1}{4}(1-2\nu)v_1^2 + \tfrac{1}{18}((-1+3\nu)(-2+3\nu))v_1^3 + \tfrac{1}{48}(-(-1+2\nu)(-1+4\nu)(-3+4\nu))v_1^4\right)+ \\
& q_1\left(-v_1 + \nu v_1^2 + \tfrac{1}{2}(-(-1+3\nu)\nu)v_1^3 + \tfrac{1}{3}((-1+2\nu)(-1+4\nu)\nu)v_1^4\right)+ \\
& q_1^2\left(\tfrac{1}{4}(-2\nu-1)v_1^2 + \tfrac{1}{2}(\nu(3\nu+1))v_1^3 + \tfrac{1}{4}(-(-1+4\nu)\nu(4\nu+1))v_1^4\right) + ... \\
\mathcal{F}_{1,1} =& \left(\tfrac{1}{24}v_1 + \tfrac{1}{24}((\nu^2-\nu-1)(-1+2\nu))v_1^2 + \tfrac{1}{48}(-(2\nu^2-2\nu-1)(-1+3\nu)(-2+3\nu))v_1^3\right)+ \\
& q_1\left(-\tfrac{1}{24}v_1 + \tfrac{1}{12}(-(-1+2\nu^2)\nu)v_1^2 + \tfrac{1}{48}((-5-6\nu+18\nu^2)(-1+3\nu)\nu)v_1^3\right)+ \\
& q_1^2\left(\tfrac{1}{24}((2\nu+1)(\nu^2+\nu-1))v_1^2 + \tfrac{1}{48}(-\nu(3\nu+1)(18\nu^2+6\nu-5))v_1^3\right) \\
\mathcal{F}_{2,1} =& \left(\tfrac{7}{5760}v_1 + \tfrac{1}{1440}(-(3\nu^4+11\nu-6\nu^3-8\nu^2+7)(-1+2\nu))v_1^2\right)+ \\
& q_1\left(-\tfrac{7}{5760}v_1 + \tfrac{1}{720}((-10\nu^2+3+6\nu^4)\nu)v_1^2 + \tfrac{1}{3840}(-(108\nu-276\nu^2+53-288\nu^3+432\nu^4)(-1+3\nu)\nu)v_1^3\right)+ \\
& q_1^2\left(\tfrac{1}{1440}(-(2\nu+1)(3\nu^4+6\nu^3-8\nu^2-11\nu+7))v_1^2 + \tfrac{1}{3840}(\nu(3\nu+1)(432\nu^4+288\nu^3-276\nu^2-108\nu+53))v_1^3\right) + ... \\
\mathcal{F}_{3,1} =& \left(\tfrac{31}{967680}v_1 + \tfrac{1}{60480}((-57\nu-9\nu^5-31+45\nu^3+3\nu^6-15\nu^4+33\nu^2)(-1+2\nu))v_1^2\right)+ \\
& q_1\left(-\tfrac{31}{967680}v_1 + \tfrac{1}{30240}(-(21\nu^2-5-21\nu^4+6\nu^6)\nu)v_1^2\right)+ \\
& q_1^2\left(\tfrac{1}{60480}((2\nu+1)(3\nu^6+9\nu^5-15\nu^4-45\nu^3+33\nu^2+57\nu-31))v_1^2\right) + ... \\
\mathcal{F}_{4,1} =& \left(\tfrac{127}{154828800}v_1 + \tfrac{1}{7257600}(-(741\nu+190\nu^5+381-656\nu^3-20\nu^7-40\nu^6+158\nu^4+5\nu^8-378\nu^2)(-1+2\nu))v_1^2\right)+ \\
& q_1\left(-\tfrac{127}{154828800}v_1 + \tfrac{1}{3628800}((126\nu^4-100\nu^2+21-60\nu^6+10\nu^8)\nu)v_1^2\right)+ \\
& q_1^2\left(0v_1 + \tfrac{1}{7257600}(-(2\nu+1)(5\nu^8+20\nu^7-40\nu^6-190\nu^5+158\nu^4+656\nu^3-378\nu^2-741\nu+381))v_1^2\right) + ...,
\end{aligned}
$$

$$\tag{4.6}$$

The prediction of [6] is that the above expansions in q_1 and v_1 are integral when rewritten in the form (2.3) for any choice of an integer $\nu \in \mathbf{Z}$. The invariants $n_{g,n,w}^{h=1}$ for general framing ν are collected in the appendix. As expected on general grounds, these invariants are independent of the framing ν for $w = 1$, except for a phase factor $\epsilon = (-1)^\nu$. For $w > 1$ the $n_{g,n,w}$ are polynomials $p_\delta(\nu, \epsilon)$ of degree $\delta = 2g + w - 1$ in ν. A closed proof of integrality of all the polynomials $p_\delta(\nu, \epsilon)$ would be formidable. We contented ourselves to a verification of the integrality of the $p_\delta^{h=1}(\nu, \epsilon)$ for a large number of values of ν. For framing $\nu = 0$, the only nonzero invariants are $n_{0,0,1} = -1$

14

and $n_{0,1,1} = 1$[13]. The result for framing $\nu = \pm 1$ is:

Framing $\nu = 1$ (left):

							$g=0$
1	−1	1	−2	5	−13	35	−100
−1	1	−2	5	−14	42	−132	429
0	0	1	−4	14	−52	198	−752
0	0	0	1	−6	31	−150	693
							$g=1$
0	0	−1	6	−32	156	−718	3220
0	0	1	−10	70	−420	2310	−12012
0	0	0	4	−49	406	−2838	17840
0	0	0	0	11	−166	1650	−13398
							$g=2$
0	0	0	−5	76	−772	6356	−45990
0	0	0	6	−133	1743	−17556	150150
0	0	0	−1	63	−1300	17655	−189260
0	0	0	0	−6	351	−7785	115269
							$g=3$
0	0	0	1	−85	2059	−32037	386484
0	0	0	−1	121	−3926	76571	−1111682
0	0	0	0	−37	2241	−63063	1191808
0	0	0	0	1	−382	20825	−586146
							$g=4$
0	0	0	0	45	−3225	102243	−2138540
0	0	0	0	−55	5291	−213785	5460026
0	0	0	0	10	−2297	144430	−4992704
0	0	0	0	0	232	−35221	1974995

Framing $\nu = -1$ (right):

							$g=0$
1	0	0	0	0	0	0	0
−1	−1	−1	−1	−1	−1	−1	−1
0	1	2	4	6	9	12	16
0	0	−1	−5	−14	−31	−60	−105
							$g=1$
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	−1	−4	−11	−24	−46	−80
0	0	1	10	49	166	450	1050
							$g=2$
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	6	22	62	148
0	0	0	−6	−63	−351	−1392	−4431
							$g=3$
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	−1	−8	−37	−128
0	0	0	1	37	382	2333	10424
							$g=4$
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	1	10	56
0	0	0	0	−10	−232	−2343	−15233

Table 1 : Low degree Ooguri–Vafa (OV) invariants $n_{g,n,w}^{h=1}$ for framing $\nu = 1$ (left) and $\nu = -1$ (right). The degree $n \geq 0$ (winding $w \geq 1$) corresponds to the vertical (horizontal) direction.

The integrality of the above invariants provides a highly non-trivial verification on the M-theory predictions of [6][9], and the localization methods proposed in refs.[15][16][20].

4.2.2. The gauge kinetic function and higher genus generalizations

The other holomorphic coupling in the standard $\mathcal{N} = 1$ supergravity is the gauge kinetic f-function related to the partition function $\mathcal{F}_{0,2}$. Higher genus partition functions contribute to f with a coefficient proportional to the $2g$-th power of the vev of the graviphoton field strength.

To compute $\mathcal{F}_{g,2}$ we consider the sum over graphs with two distinguished legs which computes the Gromov-Witten invariants $F_{g,n;w_1,w_2}$, associated to world-sheets with two boundaries landing on two distinguished parallel branes. As discussed in

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[13] Here and in the following tables there is a factor $(-1)^h$ in our conventions relative to [12][20].

sect. 2 these GW invariants are the coefficients of the monomial $v_1^{w_1} v_2^{w_2}$. Restricting to the terms with $w_1 \leq w_2$, the first terms in the instanton expansion of the partition function $\mathcal{F}_{0,2}$ are

$$
\begin{aligned}
\mathcal{F}_{0,2}^* =& \tfrac{1}{2}(-n(-1+n))v_1 v_2 + \tfrac{1}{3}(n(-1+n)(-1+2n))v_1 v_2^2 + \tfrac{1}{4}(-n(-1+n)(-1+2n)^2)v_1^2 v_2^2+ \\
& q_1\big(n^2 v_1 v_2 + -n^2(-1+2n)v_1 v_2^2 + n^2(-1+2n)^2 v_1^2 v_2^2\big)+ \\
& q_1^2\big(\tfrac{1}{2}(-(1+n)n)v_1 v_2 + n^2(2n+1)v_1 v_2^2 + \tfrac{1}{2}(-n^2(12n^2-1))v_1^2 v_2^2\big)+ \\
& q_1^3\big(\tfrac{1}{3}(-n(1+n)(2n+1))v_1 v_2^2 + n^2(2n+1)^2 v_1^2 v_2^2\big)+ \\
& q_1^4\big(\tfrac{1}{4}(-n(1+n)(2n+1)^2)v_1^2 v_2^2\big) + \dots ,
\end{aligned}
\tag{4.7}
$$

where the star is to remind that there are other terms in $\mathcal{F}_{0,2}$ following from the general form (2.2). The partition function $\mathcal{F}_{0,2}$ for a single brane is obtained by setting $v_1 = v_2$ in the above. Similar statements apply to the partition functions $\mathcal{F}_{g,2}$ for $g > 0$, which can be found in the appendix.

Taking into account the multi-coverings leads to the integral invariants $n_{g,\beta}$. We restrict to quote the low degree invariants for two particular choices of framings and refer to the appendix for more detailed results. The partition functions $\mathcal{F}_{g,2}$ are identically zero for zero framing

$$
\nu = 0: \qquad n_{g,n,w_1,w_2}^{h=2} = 0, \qquad \forall\, g, n, w_1, w_2.
$$

This extends the findings of [15][16] for degree zero, $n = 0$ to all other degrees. For framing $\nu = -1$ we find

						g=0	
0 0 0 0	1 1 1 1	-1 -3 -6 -10		0	2	10	30
0 0 0 0	1 1 1 1	-3 -6 -9 -14		2	9	26	60
0 0 0 0	1 1 1 1	-6 -9 -13 -18		10	26	57	112
0 0 0 0	1 1 1 1	-10 -14 -18 -24		30	60	112	195
						g=1	
0 0 0 0	0 0 0 0	0 1 5 15		0	-1	-15	-85
0 0 0 0	0 0 0 0	1 4 12 28		-1	-12	-65	-240
0 0 0 0	0 0 0 0	5 12 26 51		-15	-65	-220	-616
0 0 0 0	0 0 0 0	15 28 51 88		$-85 -$	240	-616	-1430
						g=2	
0 0 0 0	0 0 0 0	0 0 -1 -7		0	0	7	91
0 0 0 0	0 0 0 0	0 -1 -6 -23		0	6	70	427
0 0 0 0	0 0 0 0	-1 -6 -22 -63		7	70	395	1622
0 0 0 0	0 0 0 0	-7 -23 -63 -150		91	427	1622	5159
						g=3	
0 0 0 0	0 0 0 0	0 0 0 1		0	0	-1	-46
0 0 0 0	0 0 0 0	0 0 1 8		0	-1	-38	-421
0 0 0 0	0 0 0 0	0 1 8 37		-1	-38	-398	-2501
0 0 0 0	0 0 0 0	1 8 37 128		-46	-421	-2501	-11192
						g=4	
0 0 0 0	0 0 0 0	0 0 0 0			$w_1 \rightarrow$		
0 0 0 0	0 0 0 0	0 0 0 -1			w_2		
0 0 0 0	0 0 0 0	0 0 -1 -10			$\downarrow$		
0 0 0 0	0 0 0 0	0 -1 -10 -56					

Table 2 : Low degree OV invariants $n_{g,n,w_1,w_2}^{h=2}$ for degree $n \geq 0$ (increasing to the right) and framing $\nu = -1$.

Again these and the other results reported in the appendix are in impressive agreement with the integrality prediction.

4.2.3. More boundaries and general structures

It is interesting to study also the general structure of the partition functions with more boundaries, which describe higher dimension operators in the D-brane gauge theory and couplings involving powers of the graviphoton superfield[14]. The computation of the graph sums with h distinguished legs results in the following expression for the

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[14] See [3] for an interpretation of these couplings in the D-brane gauge theory.

leading terms in the expansion of the genus zero partition functions $\mathcal{F}_{0,h}$:

$$\mathcal{F}_{0,3}^* = -\nu^3(-2+3\nu)q_1v_1v_2v_3 + \nu^3(-1+2\nu)(-3+4\nu)q_1v_1v_2v_3^2 + -\nu^3(-4+5\nu)(-1+2\nu)^2 q_1v_1v_2^2v_3^2 +$$

$$\nu^3(-5+6\nu)(-1+2\nu)^3 q_1v_1^2v_2^2v_3^2 + (2+3\nu)\nu^3 q_1^2v_1v_2v_3 + -2\nu^3(6\nu^2-1)q_1^2v_1v_2v_3^2 +$$

$$2\nu^3(-1+2\nu)(10\nu^2-\nu-1)q_1^2v_1v_2^2v_3^2 + -(-1+2\nu)^2\nu^3(30\nu^2-5\nu-2)q_1^2v_1^2v_2^2v_3^2 + \ldots$$

$$\mathcal{F}_{1,3}^* = \tfrac{1}{24}((10-9\nu-24\nu^2+24\nu^3)\nu^3)q_1v_1v_2v_3 + \tfrac{1}{12}(-(15-65\nu^2+52\nu^3)\nu^3(-1+2\nu))q_1v_1v_2v_3^2 +$$

$$\tfrac{1}{24}((68+51\nu-392\nu^2+280\nu^3)\nu^3(-1+2\nu)^2)q_1v_1v_2^2v_3^2 +$$

$$\tfrac{1}{12}(-5(13+18\nu-90\nu^2+60\nu^3)\nu^3(-1+2\nu)^3)q_1v_1^2v_2^2v_3^2 + \ldots$$

$$\mathcal{F}_{0,4}^* = \nu^4(-3+4\nu)^2 q_1v_1v_2v_3v_4 + -\nu^4(-4+5\nu)^2(-1+2\nu)q_1v_1v_2v_3v_4^2 +$$

$$\nu^4(-5+6\nu)^2(-1+2\nu)^2 q_1v_1v_2v_3^2v_4^2 + -\nu^4(-6+7\nu)^2(-1+2\nu)^3 q_1v_1v_2^2v_3^2v_4^2 +$$

$$\nu^4(-7+8\nu)^2(-1+2\nu)^4 q_1v_1^2v_2^2v_3^2v_4^2 + \ldots$$

The polynomial dependence of the GW invariants on the framing ν is described by the simple relation[15]

$$n_{g,n,\vec{w}} = p(\nu,\epsilon), \qquad \deg_\nu(p) = 2g - 2 + h + w, \tag{4.8}$$

with $w = \sum_\alpha w_\alpha$. The above expansions vanish again at $\nu = 0$, as do all partition functions $\mathcal{F}_{g,h}$ considered so far, except for the two non-zero instantons contributing to $\mathcal{F}_{0,1}$. As for non-zero framing, subtracting the multi-cover contributions one obtains the invariants defined in [6]; for all classes and framings we have considered they are integral. Some low degree invariants for framing $\nu = -1$ are:

								$h=3$	$g=0$
0	0	0	0	0	0	0	0	0	0
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
5	10	17	16	24	33	24	32	42	53
-4	-21	-66	-60	-141	-276	-130	-258	-453	-729
0	12	90	81	328	964	292	881	2136	4502
								$h=3$	$g=1$
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
-1	-6	-20	-17	-41	-80	-36	-73	-128	-205
1	22	145	119	462	1302	394	1148	2718	5598
0	-16	-275	-222	-1536	-6602	-1282	-5692	-18792	-50951

Table 3 : Low degree OV invariants $n_{g,n,\vec{w}}^{h=3}$ for framing $\nu = -1$. The degree $n \geq 0$ corresponds to the vertical direction and the windings $w_a \leq 3$ to the horizontal direction with 3-tuples sorted in increasing numerical order.

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[15]   The degree in ν coincides with the dimension of the moduli space $\mathcal{M}_g^{rel}(\mathbf{P}^1, \beta)$ defined in [15].

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
−18	−30	−46	−44	−62	−82	−60	−80	−102	−126	−80	−100	−124	−150	−178
49	152	375	346	708	1269	662	1200	1980	3051	1135	1890	2934	4320	6102
−32	−243	−1060	−960	−2993	−7512	−2736	−6936	−15131	−29580	−6368	−14104	−27808	−50409	−85584

Table 4 : Low degree OV invariants $n_{g,n,\vec{w}}^{h=4}$ for framing $\nu = -1$. The degree $n \geq$ 0 corresponds to the vertical direction and the windings $w_a \leq 3$ to the horizontal direction with 4-tuples sorted in increasing numerical order.

4.3. Case II: The canonical bundle $K_{\mathbf{P}^2}$

A similar computation leads to the open string partition functions $\mathcal{F}_{g,h}$ for the two phases of D-branes on the second one moduli case, the canonical bundle on $\mathbf{P}^2$. One phase was already considered in the paper of Graber and Zaslow [20] and the one boundary partition functions $F_{g,1}$ contributing to the superpotential were discussed in detail. The extension of these computation to other world-sheet topologies and D-brane phases is a straightforward elaboration on their work. Moreover the arguments and computations are similar to those in the discussion of $\mathcal{O}(-1)_{\mathbf{P}^1}^{\oplus 2}$ and we can thus be brief in the following.

The equivariant class of the pull-back $i^*\phi$ computes to [38]

$$\prod_{e \in E} \prod_{a=1}^{3d-1} (\Lambda_i + \frac{a}{d}(\lambda_i - \lambda_j)) \times \prod_{v \in V} \left(\Lambda_i^{val-1} P_g(\Lambda_i) \right) \tag{4.9}$$

where $\Lambda_i = \sum_{a=1}^3 \lambda_a - 3\lambda_i$ serves as one possible linearization. There are essentially again two distinguished phases for the D-branes, depending on whether the center of **C** lands on the compact divisor $\mathbf{P}^2$ (I) or not (II). The primitive disc $D : z_a = z_b = 0, |z_D|^2 \leq c$ and the choice of a direction in the charge lattice of $U(1)^2$ for these two phases are:

$$I: \quad z_D = z_1/z_3 \;, z_a = z_2/z_3 \;, z_b = z_4 z_3^3 \qquad \nu = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3},$$

$$II: \quad z_D = z_4 z_3^3 \quad , z_a = z_1/z_3 \;, z_b = z_2/z_3 \qquad \nu = \frac{\lambda_1 - \lambda_3}{-\Lambda_3}, \tag{4.10}$$

The open string integrand for the integrals in (4.1) is given by the product (4.3)·(4.4)·(4.9). Results for the partition functions and OV invariants are collected in the appendix.

5. The A-model and local mirror symmetry on a Calabi–Yau 4-fold

It is for several reasons interesting to study the topological closed string on Calabi–Yau 4-folds Z. The type II string compactified on Z has the same number of supercharges as the type II theory on a Calabi–Yau 3-fold with supersymmetric branes. A much stronger connection exists for the class of open string theories considered above: there is a proposal for an open/closed string duality [18] which associates to a D-brane geometry on a chosen Calabi–Yau 3-fold X a closed string theory on a specific Calabi–Yau 4-fold Z such that the superpotential of the 2d closed string is identical to that of the four-dimensional $\mathcal{N} = 1$ supersymmetric open string theory. The superpotential of the two-dimensional closed string is proportional to the genus zero partition function and the equality of the superpotentials amounts to the relation $\mathcal{F}_{0,0}(Z) = \mathcal{F}_{0,1}(X)$, which has been shown on the level of the full world-sheet instanton expansion. This is a very strong indication for a deeper relation between the two theories and possibly a true string duality, perhaps involving F-theory compactified on Z. It would be interesting to study this relation further on the level of other world-sheet genera.

Whereas the genus zero partition function can be computed by mirror symmetry for Calabi–Yau 4-folds [39][40], and can be interpreted in terms of "numbers" of holomorphic curves in Z, the higher genus case has not been considered so far. In the following we outline the use of localization techniques for the computation of the genus zero and one partition functions for the basic non-compact Calabi–Yau 4-fold $\mathcal{O}(-4)_{\mathbf{P}^2}$ with $h^{1,1} = 1$. It would be interesting to extend these computations to the moderately more complicated Calabi–Yau 4-folds relevant for the open/closed string duality of [18].

Genus zero computation and comparison with local mirror symmetry

The basic genus zero GW invariants on a Calabi–Yau d-fold Z are defined as the coefficients in the expansion of the topological 3-point amplitude

$$\langle \mathcal{O}^{(1)}\mathcal{O}^{(1)}\mathcal{O}^{(d-2)}\rangle, \tag{5.1}$$

where the operators $\mathcal{O}^{(k)}$ are associated to elements $\gamma^{(d-k)}$ in $H_*(Z)$ of complex codimension k [30]. The new aspect of the 4-fold is that the last operator in (5.1) has codimension two in Z and thus the generic holomorphic curve does not intersect the class $\gamma^{(d-k)}$. The genus zero partition function for the basic GW invariants has the formal instanton expansion [41]

$$\mathcal{F}_0 = \sum_{\beta \neq 0} \sum_a \tilde{\gamma}_a^{(2)} F_{0,a,\beta}\, q(\beta)$$

where β denotes again a class in $H_2(Z, \mathbf{Z})$, $q(\beta)$ its complexified Kähler volume and moreover the last sum runs over a basis of formal coordinates $\tilde{\gamma}_a^{(2)}$ dual to the homology classes of codimension two. The genus zero GW invariants $F_{0,a,\beta}$ can be interpreted as the "number" of holomorphic curves that intersect the 4-cycle $\gamma_a^{(2)}$. The multi-cover structure that exposes the integral invariants $n_{0,a,\beta}$ associated to the virtual Euler number of moduli spaces of such curves is [40][39]

$$F_{0,a,\beta} = \sum_{\beta' \cdot d = \beta} d^{-2} \, n_{0,a,\beta'}. \tag{5.2}$$

Note that the multi-cover structure of the closed string theory compactification to two dimensions is identical to that of the open string theory (2.3), at $(g, h) = (0, 1)$.[16] For the relation between the open/closed string duals considered in [18] this amounts to the statement that the integral open string disc invariants $n_{0,\beta}^{h=1}$ map 1-1 to the closed string integral invariants $n_{0,a,\hat{\beta}}$, after an appropriate identification of the classes in $\beta \in H_2^{rel}(X, L; \mathbf{Z})$ with classes in $\hat{\beta} \in H(Z, \mathbf{Z})$.

What makes it necessary to impose an condition on the location of the curves in the A-model computation is the dimension of the moduli space of curves in general position, which is too large to find an appropriate form on it. To obtain a moduli space of the right dimension with a top form that can be integrated over it, one may add one marked point x on the domain curve Σ and requires the image $f(x)$ in Z to lie within $\gamma_a^{(2)}$. The computation of the weights of the Euler class of the obstruction bundle and the virtual normal bundle leads to the result

$$\prod_{e \in E} \left[\frac{(-)^d}{d!^2} \left(\frac{d}{\lambda_i - \lambda_j} \right)^{2d} \prod_{\substack{k \neq i,j \\ a=0}}^{a=d} \left(\frac{a}{d} \lambda_i + \frac{d-a}{d} \lambda_j - \lambda_k \right)^{-1} \prod_{a=1}^{4d-1} \left(\Lambda_i + \frac{a}{d}(\lambda_i - \lambda_j) \right) \right] \times$$
$$\prod_{v \in V} \left[\left(\Lambda_i \prod_{k \neq i} (\lambda_i - \lambda_k) \right)^{val-1} (\lambda_i)^{2p} \times \left\{ \left(\sum_{F \ni v} w_F^{-1} \right)^{val+p-3} \prod_{F \ni v} w_F^{-1} \right\} \right], \tag{5.3}$$

where $p(v)$ is the number of marked points at a vertex p and the indices of the weights run over the four fixed points p_i on $\mathbf{P}^3$. Moreover, in a particular linearization, $\Lambda_i = \sum_{k=1}^{4} \lambda_k - \lambda_i$. The graph sum (2.1) leads to the GW invariants collected in the genus zero partition function

$$\mathcal{F}_0 = -20q_1 - 825q_1^2 - \frac{612560}{9} q_1^3 - \frac{29946585}{4} q_1^4 - \frac{4825194504}{5} q_1^5 - \frac{412709577260}{3} q_1^6 + \dots \tag{5.4}$$

---

[16] In fact also the open string result is derived in [6] using a two-dimensional variant of the open string background.

21

Here q_1 is the exponentiated Kähler volume of the single primitive class $\beta \in H_2(Z, \mathbf{Z})$ and we have omitted the label for the single class $\gamma^{(2)}$ in $H_4(Z)$. From the genus zero multi-cover formula (5.2) for the 4-fold one obtains the invariants $n_{0,n\cdot\beta}$

$$-20, \ -820, \ -68060, \ -7486440, \ -965038900, \ -137569841980, \ ...$$

which are indeed all integers.

The above result agrees with the one obtained from mirror symmetry for Calabi–Yau 4-folds. The GW invariants associated to the compact divisor $\mathbf{P}^3$ in $\mathcal{O}(-4)_{\mathbf{P}^2}$ can be computed from an application of local mirror symmetry as in [42]. The non-compact toric 4-fold Z may be defined as the gauged $U(1)$ linear sigma model with five matter fields of charges $l = (-4, 1, 1, 1, 1)$. The periods of the local mirror manifold are the solutions to the (GKZ) system of differential equations associated to the vector l:

$$\mathcal{L} = \theta^4 - 8z\theta(4\theta + 1)(2\theta + 1)(4\theta + 3),$$

where z is the single complex structure modulus of the mirror and $\theta = z\frac{d}{dz}$. Apart from the constant solution $\omega_0 =$const., the solutions ω_k of the differential operator $\mathcal{L}$ are of the form $\omega_k = ln(z)^k + S_k(z)$, $k = 1, 2, 3$, where $S_k(z)$ are power series in z. The single logarithmic solution describes the mirror map

$$2\pi it = \omega_1(z) = \ln(z) + 24z + 1260z^2 + 123200z^3 + 15765750z^4 + ...$$

relating the Kähler modulus t of $\mathcal{O}(-4)_{\mathbf{P}^2}$ and the complex structure modulus z of its mirror. The partition function $\mathcal{F}_0$ is given by the double logarithmic solution $\omega_2(z)$ [40]. Inverting the mirror map and inserting the result $z(t)$ into ω_2 leads to a power series in $q_1 = exp(2\pi it)$ that agrees with the result from the localization computation (5.4).

Genus one

At genus one, the moduli space of generic curves is of the right dimension for any dimension d of the Calabi–Yau target space and the computation involves maps without extra marked points. The genus one integrand is obtained from that at genus zero by replacing the expression $\{(\sum_{F \ni v} w_F^{-1})^{val+p-3} \prod_{F \ni v} w_F^{-1}\}$ in the last line of (5.3) by the class

$$\{\frac{P_1(\Lambda_i) \cdot \prod_{i \neq k} P_1(\lambda_i - \lambda_k)}{\prod_{F \ni v} w_F - \psi_F}\} \tag{5.5}$$

The graph sum (2.1) leads to the following expansion of the genus one partition function:

$$\mathcal{F}_1 = -\frac{25}{3}q_1 - \frac{2425}{6}q_1^2 - \frac{204700}{9}q_1^3 + \frac{688375}{12}q_1^4 + 492685322q_1^5 + \frac{1433052348850}{9}q_1^6 + ... \tag{5.6}$$

It would be interesting to relate this partition function to the number of elliptic curves in Z.

6. Discussion

The improved techniques for $\mathcal{N} = 1$ open strings developed over the last years have put the study of non-perturbative properties of $\mathcal{N} = 1$ string vacua on a new level. Important non-perturbative aspects, such as supersymmetry breaking and a lift of the vacuum degeneracy, have been discussed in a plethora of papers in the past, mainly based on qualitative arguments and some reasonable working hypotheses. It will be interesting to check some of these ideas now and to redo the analysis with the new quantitative techniques that emerged from the D-brane techniques.

The computations in the previous sections provide the instanton expansions of the basic holomorphic couplings $W(\phi)$ and $f(\phi)$ in the low energy string effective action and are thus a modest first step into this direction. One of the primary motivations of the computation of the explicit instanton expansions in this paper has been the wish to use them as a starting point for the development of methods that govern also the global structure on the $\mathcal{N} = 1$ "moduli space". This is the parameter space of scalar vev's which are flat directions in the perturbative sense. The instanton expansions do not lead immediately to information about the global behavior of the couplings. One promising approach to uncover the principles that govern this global structure would be a generalization of mirror symmetry arguments along the lines of [4][11][12]. Another possible strategy would be to derive a system of differential equations satisfied by the instanton expansions for $\mathcal{F}_{g,h}$, such as (4.6) and (4.7) for the conifold. For the superpotential $W = \mathcal{F}_{0,1}$, such equations are known [18][22].

Appendix A. Integral invariants for general framings

In the following we restrict to describe the low degree OV invariants for general framing in the phase of D-branes on $\mathcal{O}(-1)_{\mathbf{P}^1}^{\oplus 2}$ studied in sect. 4 . An extended appendix including results for other phases and D-branes on the Calabi–Yau 3-fold $\mathcal{K}_{\mathbf{P}^2}$ can be found at [27].

The results will be presented in terms of generating functions $\mathcal{A}_{g,h}$ similar to the partition functions $\mathcal{F}_{g,h}$, but with the coefficients being the integral OV invariants

$n_{g,h,\beta}$ instead of the fractional GW invariants $F_{g,h,\beta}$. The first terms in the generating functions for world-sheets with $h=1$ boundary are:

$$\mathcal{A}_{0,1} = -\epsilon v_1 + \tfrac{1}{4}(-2\nu-1+\epsilon)v_1^2 + \tfrac{1}{2}(-\epsilon\nu(1+\nu))v_1^3 + \tfrac{1}{3}(-\nu(1+2\nu)(1+\nu))v_1^4 +$$
$$q_1\big(\epsilon v_1 + \nu v_1^2 + \tfrac{1}{2}(\epsilon\nu(1+3\nu))v_1^3 + \tfrac{1}{3}(\nu(1+2\nu)(1+4\nu))v_1^4\big) +$$
$$q_1^2\big(\tfrac{1}{4}(-2\nu+1-\epsilon)v_1^2 + \tfrac{1}{2}(-\epsilon\nu(-1+3\nu))v_1^3 + -4\nu^3 v_1^4\big) +$$
$$q_1^3\big(\tfrac{1}{2}(\epsilon\nu(-1+\nu))v_1^3 + \tfrac{1}{3}((-1+4\nu)(-1+2\nu)\nu)v_1^4\big) + \ldots$$

$$\mathcal{A}_{1,1} =$$
$$\tfrac{1}{48}(-4\nu-3+6\nu^2+4\nu^3+3\epsilon)v_1^2 + \tfrac{1}{8}(\epsilon\nu(1+\nu)(3\nu^2+3\nu-2))v_1^3 + \tfrac{1}{3}(\nu(1+2\nu)(1+\nu)(2\nu^2+2\nu-1))v_1^4 +$$
$$q_1\big(\tfrac{1}{6}(-\nu(-1+\nu)(1+\nu))v_1^2 + \tfrac{1}{8}(-\epsilon\nu(1+3\nu)(-2+3\nu)(1+\nu))v_1^3 + \tfrac{1}{3}(-\nu(1+4\nu)(1+2\nu)(-1+2\nu)(1+\nu))v_1^4\big) +$$
$$q_1^2\big(\tfrac{1}{48}(-4\nu+3-6\nu^2+4\nu^3-3\epsilon)v_1^2 + \tfrac{1}{8}(\epsilon\nu(-1+\nu)(3\nu+2)(-1+3\nu))v_1^3 + 4\nu^3(-1+2\nu^2)v_1^4\big) +$$
$$q_1^3\big(\tfrac{1}{8}(-\epsilon\nu(-1+\nu)(3\nu^2-3\nu-2))v_1^3 + \tfrac{1}{3}(-\nu(-1+\nu)(1+2\nu)(-1+2\nu)(-1+4\nu))v_1^4\big) + \ldots$$

$$\mathcal{A}_{2,1} =$$
$$\tfrac{1}{960}(-16\nu-15+40\nu^2+20\nu^3-10\nu^4-4\nu^5+15\epsilon)v_1^2 + \tfrac{1}{80}(-\epsilon\nu(-1+\nu)(-1+3\nu)(3\nu+4)(2+\nu)(1+\nu))v_1^3 +$$
$$q_1\big(\tfrac{1}{120}(\nu(-1+\nu)(\nu-2)(2+\nu)(1+\nu))v_1^2 + \tfrac{1}{80}(\epsilon\nu(-1+\nu)(3\nu+4)(1+3\nu)(-2+3\nu)(1+\nu))v_1^3\big) +$$
$$q_1^2\big(\tfrac{1}{960}(-16\nu+15-40\nu^2+20\nu^3+10\nu^4-4\nu^5-15\epsilon)v_1^2 + \tfrac{1}{80}(-\epsilon\nu(-1+\nu)(3\nu+2)(-1+3\nu)(3\nu-4)(1+\nu))v_1^3\big) +$$
$$q_1^3\big(\tfrac{1}{80}(\epsilon\nu(-1+\nu)(\nu-2)(3\nu-4)(1+3\nu)(1+\nu))v_1^3\big) + \ldots$$

$$\mathcal{A}_{3,1} =$$
$$\tfrac{1}{80640}(28\nu^6+8\nu^7-288\nu-315+952\nu^2+392\nu^3-350\nu^4-112\nu^5+315\epsilon)v_1^2 +$$
$$q_1\big(\tfrac{1}{5040}(-\nu(-1+\nu)(\nu-2)(-3+\nu)(3+\nu)(2+\nu)(1+\nu))v_1^2\big) +$$
$$q_1^2\big(\tfrac{1}{80640}(-28\nu^6+8\nu^7-288\nu+315-952\nu^2+392\nu^3+350\nu^4-112\nu^5-315\epsilon)v_1^2\big) + \ldots$$

$$\mathcal{A}_{4,1} =$$
$$\tfrac{1}{2903040}(504\nu^6+120\nu^7-2304\nu-2835-18\nu^8-4\nu^9+9216\nu^2+3280\nu^3-4032\nu^4-1092\nu^5+2835\epsilon)v_1^2 +$$
$$q_1\big(\tfrac{1}{362880}(\nu(-1+\nu)(\nu-2)(-3+\nu)(-4+\nu)(4+\nu)(3+\nu)(2+\nu)(1+\nu))v_1^2\big) +$$
$$q_1^2\big(\tfrac{1}{2903040}(-504\nu^6+120\nu^7-2304\nu+2835+18\nu^8-4\nu^9-9216\nu^2+3280\nu^3+4032\nu^4-1092\nu^5-2835\epsilon)v_1^2\big) + \ldots$$

The predicted integrality of all coefficients in the above expansions for any choice of $\nu \in \mathbf{Z}$ is quite impressive given the complicated denominators. The expansions of the generating functions for $h=2$ are:

$$\mathcal{A}_{0,2} = \tfrac{1}{2}(-\nu(\nu+1))v_1 v_2 + \tfrac{1}{3}(-\epsilon\nu(\nu+1)(2\nu+1))v_1 v_2^2 - \nu^2(\nu+1)^2 v_1^2 v_2^2 +$$
$$q_1\big(\nu^2 v_1 v_2 + \epsilon\nu^2(2\nu+1)v_1 v_2^2 + \nu^2(2\nu+1)^2 v_1^2 v_2^2\big) +$$
$$q_1^2\big(\tfrac{1}{2}(-\nu(-1+\nu))v_1 v_2 - \epsilon\nu^2(-1+2\nu)v_1 v_2^2 - 6\nu^4 v_1^2 v_2^2\big) +$$
$$q_1^3\big(0 v_1 v_2 + \tfrac{1}{3}(\epsilon\nu(-1+2\nu)(-1+\nu))v_1 v_2^2 + \nu^2(-1+2\nu)^2 v_1^2 v_2^2\big) + \ldots$$

$$\mathcal{A}_{1,2} =$$

$\frac{1}{24}(\nu(-1+\nu)(\nu+2)(\nu+1))v_1v_2 + \frac{1}{6}(\epsilon\nu(\nu+1)(2\nu+1)(\nu^2+\nu-1))v_1v_2^2 + \frac{1}{3}(2\nu^2(2\nu^2+2\nu-1)(\nu+1)^2)v_1^2v_2^2 +$

$q_1\big(\frac{1}{12}(-\nu^2(-1+\nu)(\nu+1))v_1v_2 + \frac{1}{6}(-\epsilon\nu^2(2\nu+1)(3\nu-2)(\nu+1))v_1v_2^2 + \frac{1}{3}(-2\nu^2(\nu+1)(-1+2\nu)(2\nu+1)^2)v_1^2v_2^2\big) +$

$q_1^2\big(\frac{1}{24}(\nu(-1+\nu)(\nu-2)(\nu+1))v_1v_2 + \frac{1}{6}(\epsilon\nu^2(-1+\nu)(3\nu+2)(-1+2\nu))v_1v_2^2 + 4\nu^4(-1+2\nu^2)v_1^2v_2^2\big) +$

$q_1^3\big(\frac{1}{6}(-\epsilon\nu(-1+2\nu)(-1+\nu)(\nu^2-\nu-1))v_1v_2^2 + \frac{1}{3}(-2\nu^2(2\nu+1)(-1+\nu)(-1+2\nu)^2)v_1^2v_2^2\big) + \dots$

$$\mathcal{A}_{2,2} =$$

$\frac{1}{720}(-\nu(-1+\nu)(\nu-2)(\nu+3)(\nu+2)(\nu+1))v_1v_2 + \frac{1}{360}(-\epsilon\nu(-1+\nu)(2\nu+1)(\nu+2)(\nu+1)(13\nu^2+13\nu-12))v_1v_2^2 +$

$q_1\big(\frac{1}{360}(\nu^2(-1+\nu)(\nu-2)(\nu+2)(\nu+1))v_1v_2 + \frac{1}{360}(\epsilon\nu^2(-1+\nu)(2\nu+1)(\nu+1)(39\nu^2+26\nu-34))v_1v_2^2\big) +$

$q_1^2\big(\frac{1}{720}(-\nu(-1+\nu)(\nu-2)(\nu-3)(\nu+2)(\nu+1))v_1v_2 + \frac{1}{360}(-\epsilon\nu^2(-1+\nu)(-1+2\nu)(\nu+1)(39\nu^2-26\nu-34))v_1v_2^2\big) +$

$q_1^3\big(\frac{1}{360}(\epsilon\nu(-1+\nu)(\nu-2)(-1+2\nu)(\nu+1)(13\nu^2-13\nu-12))v_1v_2^2\big) + \dots$

$$\mathcal{A}_{3,2} =$$

$\frac{1}{40320}(\nu(-1+\nu)(\nu-2)(\nu-3)(\nu+4)(\nu+3)(\nu+2)(\nu+1))v_1v_2 +$

$q_1\big(\frac{1}{20160}(-\nu^2(-1+\nu)(\nu-2)(\nu-3)(\nu+3)(\nu+2)(\nu+1))v_1v_2\big) +$

$q_1^2\big(\frac{1}{40320}(\nu(-1+\nu)(\nu-2)(\nu-3)(\nu-4)(\nu+3)(\nu+2)(\nu+1))v_1v_2\big) +$

$q_1^3\big(\frac{1}{45360}(-\epsilon\nu(-1+\nu)(\nu-2)(-1+2\nu)(\nu+1)(205\nu^4-410\nu^3-614\nu^2+819\nu+558))v_1v_2^2\big) + \dots$

$$\mathcal{A}_{4,2} =$$

$\frac{1}{3628800}(-\nu(-1+\nu)(\nu-2)(\nu-3)(\nu-4)(\nu+5)(\nu+4)(\nu+3)(\nu+2)(\nu+1))v_1v_2 +$

$q_1\big(\frac{1}{1814400}(\nu^2(-1+\nu)(\nu-2)(\nu-3)(\nu-4)(\nu+4)(\nu+3)(\nu+2)(\nu+1))v_1v_2\big) +$

$q_1^2\big(\frac{1}{3628800}(-\nu(-1+\nu)(\nu-2)(\nu-3)(\nu-4)(\nu-5)(\nu+4)(\nu+3)(\nu+2)(\nu+1))v_1v_2\big) + \dots$

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