

## Sums of lens spaces bounding rational balls

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We classify connected sums of three-dimensional lens spaces which smoothly bound rational homology balls. We use this result to determine the order of each lens space in the group of rational homology 3–spheres up to rational homology cobordisms, and to determine the concordance order of each 2–bridge knot.

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### 1 Introduction

This paper is a continuation of [7]. Both papers can be viewed as providing partial answers to the question of Andrew Casson [6, Problem 4.5] asking which rational homology 3–spheres bound rationally acyclic 4–manifolds. The main result of [7] characterizes lens spaces which smoothly bound rational homology balls, while here we deal more generally with connected sums of lens spaces. Our proof relies on Donaldson’s theorem about intersection forms of closed, smooth and negative 4–manifolds as well as on some technical results from [7].

Given coprime integers  $p > q > 0$ , we denote by  $L(p, q)$  the oriented 3–manifold obtained by performing  $(-p/q)$ –surgery on the unknot in  $S^3$ . Let  $q'$  be the unique integer satisfying  $p > q' > 0$  and  $qq' \equiv 1 \pmod{p}$ . By [7, Theorem 1.2] the lens space  $L(p, q)$  smoothly bounds a rational homology ball if and only if

- (1)  $p = m^2$  for some  $m \in \mathbb{N}$ , and
- (2)  $q$ ,  $p - q$  or  $q'$  is of one of the following types:
  - (a)  $mk \pm 1$  with  $m > k > 0$  and  $(m, k) = 1$ ;
  - (b)  $d(m \pm 1)$ , where  $d > 1$  divides  $2m \mp 1$ ;
  - (c)  $d(m \pm 1)$ , where  $d > 1$  is odd and divides  $m \pm 1$ .

Let  $\mathcal{R} \subset \mathbb{Q}$  denote the set – described above – of rational numbers  $p/q > 1$  such that the lens space  $L(p, q)$  smoothly bounds a rational homology ball, and define

$$\mathcal{F}_n := \left\{ \frac{m^2 n}{mnk + 1} \mid m > k > 0, (m, k) = 1 \right\} \subset \mathbb{Q}, \quad n \geq 2.$$

**Theorem 1.1** *Let  $Y$  be an oriented 3–manifold homeomorphic to a connected sum of three–dimensional lens spaces. Then,  $Y$  smoothly bounds a rational homology ball if and only if  $Y$  is orientation-preserving homeomorphic to a connected sum in which each summand is (possibly orientation-reversing) homeomorphic to one of the following oriented 3–manifolds:*

- (1)  $L(p, q)$ ,  $p/q \in \mathcal{R}$ ;
- (2)  $L(p, q) \# L(p, p - q)$ ,  $p/q > 1$ ;
- (3)  $L(p_1, q_1) \# L(p_2, q_2)$ ,  $p_i/q_i \in \mathcal{F}_2$ ,  $i = 1, 2$ ;
- (4)  $L(p, q) \# L(n, n - 1)$ ,  $p/q \in \mathcal{F}_n$  for some  $n \geq 2$ ;
- (5)  $L(p_1, q_1) \# L(p_2, p_2 - q_2)$ ,  $p_i/q_i \in \mathcal{F}_n$ ,  $i = 1, 2$ , for some  $n \geq 2$ .

Let  $\Theta_{\mathbb{Q}}^3$  be the group of rational homology 3–spheres up to rational homology cobordism. Using [Theorem 1.1](#) we can determine the order of each lens space viewed as an element of the group  $\Theta_{\mathbb{Q}}^3$  of rational homology 3–spheres up to rational homology cobordism. Define

$$\mathcal{S} := \{p/q \mid p > q > 0, (p, q) = 1 \text{ and } p = q + q'\} \subset \mathbb{Q}.$$

**Corollary 1.2** *The order of the lens space  $L(p, q)$  in  $\Theta_{\mathbb{Q}}^3$  is:*

- 1 if and only if  $p/q \in \mathcal{R}$ ,
- 2 if and only if  $p/q \in (\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{F}_2$ ,
- $\infty$  if and only if  $p/q \notin \mathcal{R} \cup \mathcal{S} \cup \mathcal{F}_2$ .

Since the sets  $\mathcal{R}$ ,  $\mathcal{S}$  and  $\mathcal{F}_2$  have elementary, explicit definitions, [Corollary 1.2](#) reduces the determination of the order of each lens space in  $\Theta_{\mathbb{Q}}^3$  to an elementary calculation.

It is a well-known fact that the lens space  $L(p, q)$  can be viewed as the 2–fold cover of  $S^3$  branched along the 2–bridge link  $K(p, q)$  (see [Section 2](#) for the definition of  $K(p, q)$ ). As a consequence, [Theorem 1.1](#) gives information on the relation of link concordance amongst 2–bridge links  $K(p, q) \subset S^3$ . Here we content ourselves with deriving a corollary of [Theorem 1.1](#) in the knot case (that is, when  $p$  is odd), leaving the case of links to the interested reader. Recall that the *concordance order* of a knot in  $S^3$  is the order of its class in the smooth knot concordance group  $\mathcal{C}_1$ . The survey paper of Livingston [\[8\]](#) describes what was known about the group  $\mathcal{C}_1$  until 2004. More recent papers contain results on knot concordance obtained via Ozsváth–Szabó’s Heegaard Floer homology, Grigsby, Ruberman and Strle [\[4\]](#), Jabuka and Naik [\[5\]](#) and Manolescu and Owens [\[9\]](#). In spite of all the efforts made so far, the structure of the group  $\mathcal{C}_1$  remains quite mysterious. For instance, the basic question asking whether  $\mathcal{C}_1$  contains

nontrivial elements of finite order different from two is still wide open. Restricting to 2–bridge knots, results from [4; 5] determine the corresponding concordance orders in finitely many cases.

**Corollary 1.3** *Let  $p > q > 0$  with  $p$  odd. Then, the 2–bridge knot  $K(p, q)$  has concordance order:*

- 1 if and only if  $p/q \in \mathcal{R}$ ,
- 2 if and only if  $p/q \in \mathcal{S} \setminus \mathcal{R}$ ,
- $\infty$  if and only if  $p/q \notin \mathcal{R} \cup \mathcal{S}$ .

Moreover,  $K(p, q)$  is simultaneously smoothly slice and amphicheiral if and only if

$$(p, q) = \left( (2k^2 + 2k + 1)^2, 2(2k + 1)(k^2 + k + 1) \right)$$

for some  $k \geq 1$ .

Corollary 1.3 corroborates [7, Conjecture 9.4], stating that each knot of order two is concordant to a negative amphicheiral knot. In fact, since  $K(p, q)$  is isotopic to  $K(p, q')$  while  $K(p, p - q)$  is isotopic to the mirror image of  $K(p, q)$ , by Corollary 1.3 each 2–bridge knot of order two is amphicheiral<sup>1</sup>.

The paper is organized as follows. In Section 2 we establish Corollaries 1.2 and 1.3 assuming Theorem 1.1. In Section 3 we prove that the manifolds (1)–(5) of Theorem 1.1 bound rational homology balls. Due to previous results, the only cases which need explicit treatment are (3)–(5). In these cases we show by ribbon moves that each knot bounds a ribbon surface  $\Sigma$  with  $\chi(\Sigma) = 1$ . The rational ball is then the 2–fold cover of  $B^4$  branched along  $\Sigma$  with pushed-in interior. Section 4 is the technical core of the paper. We recall some results from [7] concerning embeddings of certain intersection lattices into the standard diagonal lattice, and we prove a few results with a similar flavor. In Section 5 we use the results of Section 4 and Donaldson’s theorem on the intersection form of smooth 4–manifolds to finish the proof of Theorem 1.1. Apart from technical issues, the strategy of the proof is very simple. If a connected sum of lens spaces  $Y = L_1 \# \cdots \# L_h$  bounds a rational ball  $W$  one can construct a closed, negative definite smooth 4–manifold of the form  $P_Y \cup -W$ , where  $P_Y$  is a certain plumbing canonically associated with  $Y$ . Donaldson’s theorem gives an orthogonal embedding of the intersection lattice of  $P_Y$  into the standard diagonal intersection lattice. A similar conclusion holds for the intersection lattice of  $P_{-Y}$ , where  $-Y$

<sup>1</sup>Since 2–bridge knots are reversible, Siebenmann [13], for such knots the notions of negative and positive amphicheirality coincide.

denotes the 3–manifold  $Y$  with reversed orientation. The rest of the proof consists of applying the results of [Section 4](#) to deduce from the existence of such embeddings that  $Y$  must be of the form stated in [Theorem 1.1](#).

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## 2 Proof of Corollaries 1.2 and 1.3

In this section we prove [Corollaries 1.2](#) and [1.3](#) assuming [Theorem 1.1](#).

**Proof of [Corollary 1.2](#)** By [[7](#), [Theorem 1.2](#)] the lens space  $L(p, q)$  has order 1 in  $\Theta_{\mathbb{Q}}^3$ , that is, it represents the trivial class, if and only if  $p/q \in \mathcal{R}$ .

If  $L(p, q) \# L(p, q)$  bounds a rational ball, a quick inspection of [Theorem 1.1](#) shows that one of the following must hold:

$$(1) \ p/q \in \mathcal{R}, \quad (2) \ p = q + q', \quad \text{or} \quad (3) \ p/q \in \mathcal{F}_2.$$

On the other hand, if  $L(p, q)$  has order 2 in  $\Theta_{\mathbb{Q}}^3$  then (1) does not hold. Hence, by the definition of  $\mathcal{S}$  we have  $p/q \in (\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{F}_2$ .

Conversely, suppose  $p/q \in (\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{F}_2$ . If  $p/q \in \mathcal{S}$  then  $q' = p - q$ , therefore

$$L(p, q) \# L(p, q) = L(p, q) \# L(p, q') = L(p, q) \# L(p, p - q) = L(p, q) \# -L(p, q)$$

bounds a rational ball, while  $L(p, q)$  does not bound such a ball because  $p/q \notin \mathcal{R}$ . Therefore  $L(p, q)$  has order 2 in  $\Theta_{\mathbb{Q}}^3$ . If  $p/q \in \mathcal{F}_2$ , by [Theorem 1.1\(3\)](#) the connected sum  $L(p, q) \# L(p, q)$  bounds a rational ball.

To finish the proof it suffices to show that if  $p/q \notin \mathcal{R} \cup \mathcal{S} \cup \mathcal{F}_2$  then  $L(p, q)$  has infinite order in  $\Theta_{\mathbb{Q}}^3$ . But [Theorem 1.1](#) immediately implies that if  $L(p, q)$  has finite order then it has either order 1 or 2, and therefore by what we have already proved  $p/q \in \mathcal{R} \cup \mathcal{S} \cup \mathcal{F}_2$ .  $\square$

Before proving [Corollary 1.3](#) we need to recall the definitions of the links  $K(p, q)$  and the set  $\mathcal{R}$ . A link in  $S^3$  is called *2–bridge* if it can be isotoped so it has exactly two local maxima with respect to a standard height function. [Figure 1](#) represents the 2–bridge link  $\mathcal{L}(c_1, \dots, c_n)$ , where  $c_i \in \mathbb{Z}$ ,  $i = 1, \dots, n$ . Given coprime integers

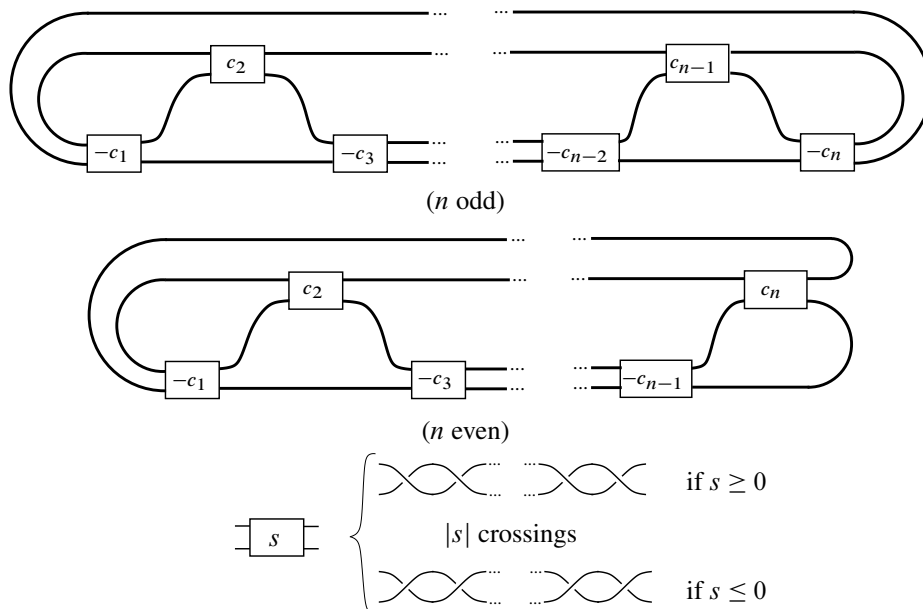


Figure 1: The 2-bridge link  $\mathcal{L}(c_1, \dots, c_n)$

$p > q > 0$ , we can always write

$$p/q = [c_1, \dots, c_n]^+ := c_1 + \frac{1}{c_2 + \frac{1}{\dots + \frac{1}{c_n}}}, \quad \text{with } c_i > 0, i = 1, \dots, n.$$

The 2-bridge link  $K(p, q)$  is, by definition,  $\mathcal{L}(c_1, \dots, c_n)$ . When  $p$  is even,  $K(p, q)$  is a 2-component link, when  $p$  is odd  $K(p, q)$  is a knot. It is well-known (see Burde and Zieschang [1, Chapter 12]) that  $K(p, q)$  and  $K(\bar{p}, \bar{q})$  are isotopic if and only if  $p = \bar{p}$  and either  $q = \bar{q}$  or  $q\bar{q} \equiv 1 \pmod{p}$ , and that every 2-bridge link is isotopic to some  $K(p, q)$ . Moreover,  $K(p, p - q)$  is isotopic to the mirror image of  $K(p, q)$ .

**Proof of Corollary 1.3** If  $K(p, q)$  has finite order in  $\mathcal{C}_1$  then  $L(p, q)$  has finite order in  $\Theta_{\mathbb{Q}}^3$ . Therefore by Corollary 1.2 we have  $p/q \in \mathcal{R} \cup \mathcal{S} \cup \mathcal{F}_2$ . If  $p/q \in \mathcal{F}_2$  then  $p$  would be even, while in our case  $p$  is odd because we are assuming that  $K(p, q)$  is a knot. Hence,  $p/q \in \mathcal{R} \cup \mathcal{S}$ . Moreover,  $K(p, q)$  is smoothly slice if and only if  $p/q \in \mathcal{R}$  by [7, Corollary 1.3], while if  $p/q \in \mathcal{S}$  then  $q' = p - q$  and therefore

$K(p, q) = K(p, q') = K(p, p - q)$ , so

$$K(p, q) \# K(p, q) = K(p, q \# K(p, p - q)) = K(p, q) \# -K(p, q)$$

is smoothly slice. This shows that  $K(p, q)$  has finite order in  $\mathcal{C}_1$  if and only if either (i)  $p/q \in \mathcal{R}$  and  $K(p, q)$  has order 1 or (ii)  $p/q \in \mathcal{S} \setminus \mathcal{R}$  and  $K(p, q)$  has order 2. This concludes the proof of the part of the statement about concordance orders.

For the last statement, recall that  $K(p, q)$  is amphicheiral if and only if  $q + q' = p$ . If

$$(p, q) = ((2k^2 + 2k + 1)^2, 2(2k + 1)(k^2 + k + 1))$$

then, setting  $m = 2k^2 + 2k + 1$  and  $d = 2k + 1$  we have  $(p, q) = (m^2, d(m + 1))$  and  $d$  divides  $2m - 1 = d^2$ . Therefore  $p/q \in \mathcal{R}$ . Moreover,

$$p(p - q) \equiv -q^2 = -d^2(m + 1)^2 \equiv (1 - 2m)(1 + 2m) \equiv 1 \pmod{m^2},$$

hence  $q' = p - q$ . This shows that if  $(p, q)$  is of the stated form then  $K(p, q)$  is smoothly slice and amphicheiral.

Conversely, suppose that  $K(p, q)$  is smoothly slice and amphicheiral, that is  $p/q \in \mathcal{R}$  and  $q' = p - q$ . Since  $K(p, p - q) = K(p, q)$ , we may assume without loss that  $q$  is of the form (a), (b) or (c) from the definition of  $\mathcal{R}$ . Case (a) is impossible because it is easy to check that  $q' = m(m - k) \pm 1$ , hence  $q + q' = m^2 \pm 2 \neq p$ . In Case (b), if  $q = d(m - 1)$  then  $q' = h(m - 1)$ , where  $h$  is defined by  $dh = 2m + 1$ . Then,  $q + q' = (d + h)(m - 1) = m^2 = p$  is impossible. If  $q = d(m + 1)$  we have  $q' = m^2 - h(m + 1)$ , with  $dh = 2m - 1$ . Thus  $q + q' = p$  if and only if  $d = h$ , which implies  $m = (d^2 + 1)/2$ , so  $(p, q)$  is of the form  $((d^2 + 1)^2/4, d(d^2 + 3)/2)$ . Observing that  $d > 1$  is odd and setting  $d = 2k + 1$  one obtains the statement of the lemma. In Case (c), if  $q = d(m - 1)$  then  $q' = h(2m + 1)$ , with  $dh = m + 1$ , and the equality  $q + q' = p$  gives a contradiction as in Case (b). If  $q = d(m + 1)$  then  $q' = m^2 - h(2m - 1)$ , and the condition  $q + q' = p$  implies  $m = (d^2 + 1)/2$  as before, to the effect that  $(p, q)$  must be of the stated form.  $\square$

### 3 Proof of Theorem 1.1: first half

The purpose of this section is to prove the first half of Theorem 1.1, that is, the following result.

**Proposition 3.1** *Each of the manifolds (1)–(5) listed in Theorem 1.1 bounds a rational homology ball.*

**Remark 3.2** According to Siebenmann [13], Casson, Gordon and Conway showed that every knot of the form  $K(p, q)$  with  $p/q \in \mathcal{R}$  is ribbon. This implies Case (1) of Proposition 3.1 by a standard argument (see Lemma 3.6 or [7] for complete details). Also, in Case (a) of  $\mathcal{R}$  Casson and Harer [2] showed that  $L(p, q)$  bounds a contractible 4-manifold of Mazur-type.

We start with some preparation. For any integer  $t \geq 0$  we will use the notation

$$(\dots, 2^{[t]}, \dots) := (\dots, \overbrace{2, \dots, 2}^t, \dots).$$

For example,

$$(3, 2^{[3]}, 4) = (3, 2, 2, 2, 4).$$

Recall (see eg Orlik and Wagreich [10, Appendix]) that if  $p/q > 1$  then

$$\frac{p}{q} = [a_1, a_2, \dots, a_{h-1}, a_h]^- := a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ddots - \frac{1}{a_h}}}}$$

for some  $a_i \geq 2$ ,  $i = 0, \dots, h$ , and

$$(3-1) \quad \frac{p}{q'} = [a_h, a_{h-1}, \dots, a_2, a_1]^-,$$

where  $p > q' > 0$  and  $qq' \equiv 1 \pmod{p}$ . Let  $a_1, \dots, a_{2d}$  be positive integers and  $d \geq 2$ . Then, the following identity holds (Popescu-Pampu [11, Proposition 2.3]):

$$(3-2) \quad [a_1, \dots, a_{2d}]^+ = [a_1 + 1, 2^{[a_2-1]}, a_3 + 2, 2^{[a_4-1]}, \dots, a_{2d-1} + 2, 2^{[a_{2d}-1]}]^-$$

We define the *reverse* of a string  $(a_1, \dots, a_k)$  to be  $(a_k, \dots, a_1)$ , and the *negative* (respectively *positive*) *fraction* of  $(a_1, \dots, a_k)$  to be the rational number  $[a_1, \dots, a_k]^-$  (respectively  $[a_1, \dots, a_k]^+$ ).

**Lemma 3.3** Suppose that a string  $\mathcal{S}$  is obtained from  $(2, n+1, 2)$ ,  $n \geq 2$ , by a finite sequence of operations each of which is of one of the following types:

- (1)  $(n_1, n_2, \dots, n_{b-1}, n_b) \mapsto (2, n_1, n_2, \dots, n_{b-1}, n_b + 1)$ ,
- (2)  $(n_1, n_2, \dots, n_{b-1}, n_b) \mapsto (n_1 + 1, n_2, \dots, n_{b-1}, n_b, 2)$ .

Then, the negative fraction of  $\mathcal{S}$  is of the form

$$\frac{m^2 n}{mnk + 1}, \quad m > k > 0, \quad (m, k) = 1.$$

Moreover, the negative fraction of either  $S$  or the reverse of  $S$  is equal to

$$(3-3) \quad [c_s, c_{s-1}, \dots, c_2, c_1, 1, n-1, c_1+1, c_2, \dots, c_s]^+$$

for some integers  $c_1, \dots, c_s \geq 1$ , where for  $s = 1$  the above formula is to be interpreted as  $[c_1, 1, n-1, c_1+1]^+$ .

**Proof** The negative fraction of  $(2, n+1, 2)$  is  $4n/(2n+1)$ , which is of the form  $m^2n/(mnk+1)$  for  $m=2$  and  $k=1$ . Therefore it suffices to show that if the negative fraction of a string  $S$  is of the form  $m^2n/(mnk+1)$ , then the negative fractions of the strings obtained from  $S$  by applying the operations (1) and (2) are of the same form. Let

$$(3-4) \quad p := m^2n, \quad q := mnk+1, \quad q' := mn(m-k)+1,$$

$$(3-5) \quad t := nk(m-k)+1, \quad q'' := 2q - nk^2 = q+t.$$

Then, we have  $0 < q' < p$ ,  $(p, q') = 1$ ,

$$qq' = 1 + tp \quad \text{and} \quad q''q' = 1 + t(p+q').$$

Therefore if  $p/q = [a_1, \dots, a_h]^-$ , by Equation (3-1)

$$[a_h+1, a_{h-1}, \dots, a_2, a_1]^- = \frac{p+q'}{q'}.$$

Since  $0 < q'' = q+t < q+q' < p+q'$  and  $(q'', p+q') = 1$ , we have

$$[a_1, a_2, \dots, a_{h-1}, a_h+1]^- = \frac{p+q'}{q''}$$

and therefore

$$(3-6) \quad [2, a_1, a_2, \dots, a_{h-1}, a_h+1]^- = 2 - \frac{q''}{p+q'} = \frac{(2m-k)^2n}{(2m-k)nm+1},$$

with  $2m-k > m$  and  $(2m-k, m) = 1$ . Applying Equation (3-6) to

$$[a_h, \dots, a_1]^- = \frac{m^2n}{mn(m-k)+1}$$

gives

$$[2, a_h, \dots, a_1+1]^- = \frac{(m+k)^2n}{(m+k)nm+1}.$$

By Equation (3-1) and the formula for  $q'$  in Equation (3-4) we have

$$(3-7) \quad [a_1+1, a_2, \dots, a_{h-1}, a_h, 2]^- = \frac{(m+k)^2n}{(m+k)nk+1}.$$



This proves the first part of the lemma. To prove the second part, observe that a straightforward induction using the definition shows that either  $\mathcal{S}$  or its reverse is of one of the forms:

$$(c_s + 1, 2^{[c_s-1]}, c_{s-2} + 2, \dots, c_3 + 2, 2^{[c_2-1]}, c_1 + 2, n + 1, 2^{[c_1]}, c_2 + 2, \dots, c_{s-1} + 2, 2^{[c_s-1]})$$

or

$$(c_1 + 1, n + 1, 2^{[c_1]}).$$

for some positive integers  $c_1, \dots, c_s$ . Applying Formula (3-2) we obtain Expansion (3-3).  $\square$

**Lemma 3.4** *Let  $m > k > 0$  and  $n \geq 1$  be integers, with  $(m, k) = 1$ . Then,*

$$\frac{m^2 n}{mnk + 1} = [a_1, \dots, a_h]^-,$$

where the string  $(a_1, \dots, a_h)$  is obtained from  $(2, n + 1, 2)$  by a sequence of operations of types (1) or (2) as in Lemma 3.3.

**Proof** We argue by induction on  $m \geq 2$ . For  $m = 2$  we have  $k = 1$  and

$$\frac{4n}{2n + 1} = [2, n + 1, 2]^-.$$

Now assume  $m > 2$  and the conclusion of the lemma for any fraction

$$\frac{p^2 n}{pnl + 1}, \quad p > l > 0, \quad (p, l) = 1,$$

with  $p < m$ . If  $m = 2k$  then  $m = 2$ , therefore we have either  $m > 2k$  or  $m < 2k$ . Since

$$(mnk + 1)(mn(m - k) + 1) \equiv 1 \pmod{m^2 n},$$

by Equation (3-1)

$$\frac{m^2 n}{mnk + 1} = [a_1, \dots, a_h]^- \iff \frac{m^2 n}{mn(m - k) + 1} = [a_h, \dots, a_1]^-.$$

Since string reversal turns the operations (1) and (2) of Lemma 3.3 into each other, up to replacing  $k$  with  $m - k$  we may assume  $m > 2k$  without losing generality. Setting  $p := m - k$  and  $l := k$  we have  $m > p$ ,  $p > l > 0$  and  $(p, l) = 1$ . By the induction hypothesis

$$\frac{p^2 n}{pnl + 1} = [b_1, \dots, b_q]^-,$$

where  $(b_1, \dots, b_q)$  is obtained from  $(2, n+1, 2)$  by a sequence of operations of types (1) or (2) as in [Lemma 3.3](#). Then, by Equation (3–7) we have

$$(a_1, \dots, a_h) = (b_1 + 1, \dots, b_q, 2).$$

This concludes the proof.  $\square$

**Lemma 3.5** *Let  $p/q \in \mathcal{F}_n$ ,  $n \geq 2$ . Then, there exists a ribbon move which turns the link  $K(p, q)$  into a split link given by the union of  $K(n, 1)$  and an unknot.*

**Proof** By [Lemmas 3.3](#) and [3.4](#), up to isotopy we have the equality

$$K(m^2n, mnk + 1) = \mathcal{L}(c_s, \dots, c_1, 1, n-1, c_1 + 1, c_2, \dots, c_s)$$

for some integers  $c_1, \dots, c_s \geq 1$ . The rest of the proof consists of [Figure 2](#).  $\square$

The following lemma can be extracted from [[7](#), Proof of Theorem 1.2]. We include the proof for the reader's convenience.

**Lemma 3.6** *Let  $Y$  be an oriented rational homology 3–sphere given as the 2–fold cover of  $S^3$  branched along a link  $L$  which bounds a ribbon surface  $\Sigma \looparrowright S^3$  with  $\chi(\Sigma) = 1$ . Then,  $Y$  smoothly bounds a rational homology 4–ball.*

**Proof** The 4–manifold  $W$  given as the 2–fold cover of the 4–ball  $B^4$  branched along a pushed-in copy of  $\Sigma$  smoothly bounds  $Y$ . Moreover,  $W$  has a handle decomposition with only 0–, 1– and 2–handles (see eg [Casson and Harer \[2, lemma at pages 30–31\]](#)). Therefore, from

$$b_0(W) - b_1(W) + b_2(W) = \chi(W) = 2\chi(B^4) - \chi(\Sigma) = 1$$

we deduce  $b_1(W) = b_2(W)$ . On the other hand, since  $b_1(Y) = b_1(\partial W) = 0$  and  $H_1(W, \partial W; \mathbb{Q}) \cong H^3(W; \mathbb{Q}) = 0$ , the homology exact sequence of the pair  $(W, \partial W)$  gives  $b_1(W) = 0$ , so it follows that  $H_*(W; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q})$ .  $\square$

**Proof of Proposition 3.1** As explained in [Remark 3.2](#), Case (1) of the proposition follows from the results of [[13](#)]. In Case (2) the result is well-known. To treat Cases (3)–(5) we apply [Lemma 3.6](#). First observe that, as shown in [Figure 3](#), there is a connected sum  $K(n, 1)\#K(n, n-1)$  which can be turned into an unlink of 2 components by a ribbon move. On the other hand, applying [Lemma 3.5](#) we see that if  $p_i/q_i \in \mathcal{F}_2$  then there is a connected sum  $K(p_1, q_1)\#K(p_2, q_2)$  which can be turned by 2 ribbon moves into the connected sum  $K(2, 1)\#K(2, 1)$  of [Figure 3](#) (for  $n = 2$ ) plus a 2–component unlink. Therefore, a connected sum  $K(p_1, q_1)\#K(p_2, q_2)$  with  $p_i/q_i \in \mathcal{F}_2$  can be

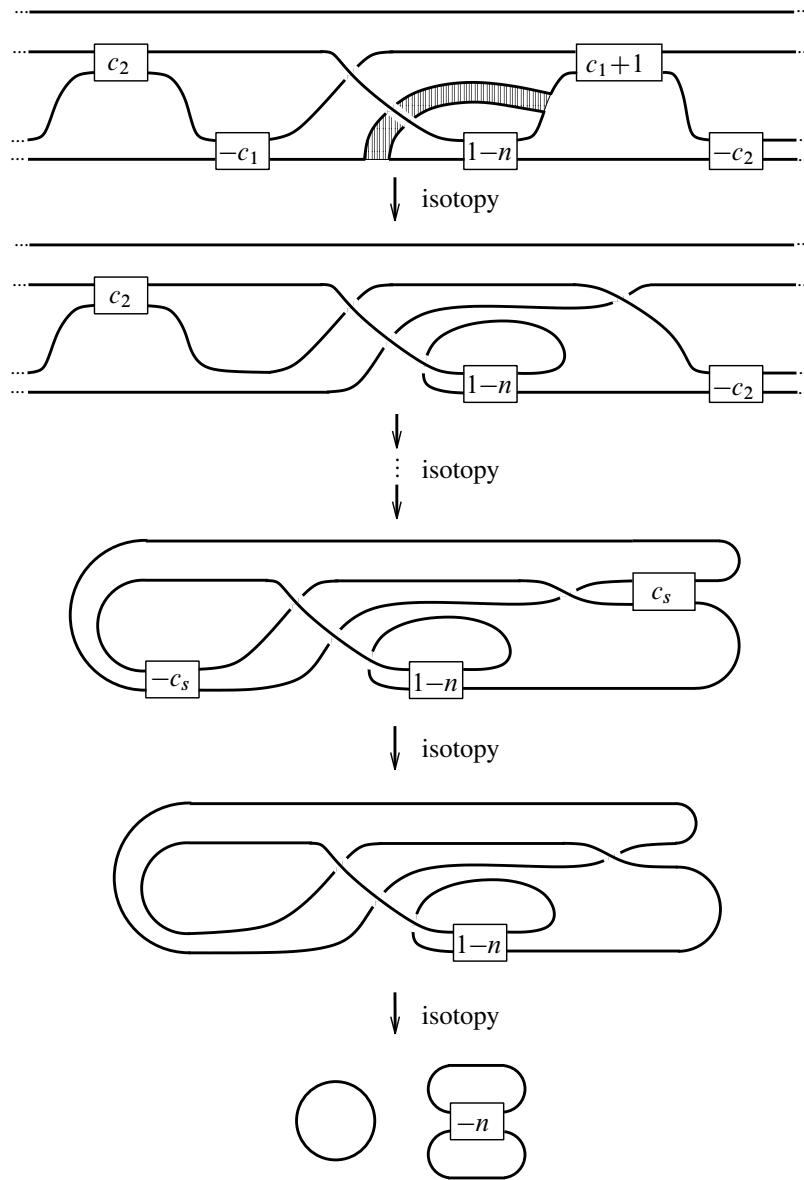
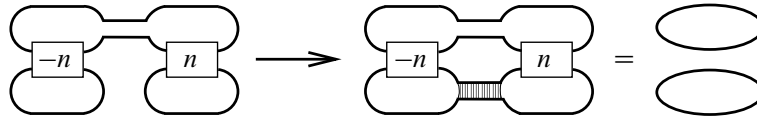


Figure 2: Ribbon move on  $K(m^2n, mnk + 1)$  and isotopies

turned into a 4-component unlink by 3 ribbon moves. So we conclude that such a link bounds a ribbon surface  $\Sigma$  with  $\chi(\Sigma) = 4 - 3 = 1$ . The same argument applies to a connected sum  $K(p_1, q_1) \# K(p_2, p_2 - q_2)$  with  $p_i/q_i \in \mathcal{F}_n$ ,  $n \geq 2$ . Since

Figure 3: Ribbon move on  $K(n, 1)\#K(n, n-1)$ 

the 2-fold cover of  $S^3$  branched along any connected sum  $K(p_1, q_1)\#K(p_2, q_2)$  is diffeomorphic to  $L(p_1, q_1)\#L(p_2, q_2)$ , by [Lemma 3.6](#) this suffices to prove the proposition in Cases (3) and (5). A similar construction shows that if  $p/q \in \mathcal{F}_n$  a connected sum  $K(p, q)\#K(n, n-1)$  can be reduced to a 3-component unlink by 2 ribbon moves. Therefore the same connected sum bounds a ribbon surface  $\Sigma$  with  $\chi(\Sigma) = 3 - 2 = 1$ , and applying [Lemma 3.6](#) as before proves the proposition in Case (4).  $\square$

## 4 Algebraic interlude

In this section we recall some definitions and results from [\[7\]](#) and we establish some new results having a similar flavor. This material will be used in [Section 5](#).

### Recollection of previous results

Let  $\mathbf{D}$  denote the intersection lattice  $(\mathbb{Z}, (-1))$ , and  $\mathbf{D}^n$  the orthogonal direct sum of  $n$  copies of  $\mathbf{D}$ . Elements of  $\mathbf{D}^n$  will also be called *vectors*. Fix generators  $e_1, \dots, e_n \in \mathbf{D}^n$  such that

$$e_i \cdot e_j = -\delta_{ij}, \quad i, j = 1, \dots, n.$$

Given a subset  $S = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$ , define

$$E_i^S := \{j \in \{1, \dots, n\} \mid v_j \cdot e_i \neq 0\}, \quad i = 1, \dots, n,$$

$$V_i := \{j \in \{1, \dots, n\} \mid e_j \cdot v_i \neq 0\}, \quad i = 1, \dots, n,$$

and

$$p_i(S) := |\{j \in \{1, \dots, n\} \mid |E_j^S| = i\}|, \quad i = 1, \dots, n.$$

Let  $v_1, \dots, v_n \in \mathbf{D}^n$  be elements such that, for  $i, j \in \{1, \dots, n\}$ ,

$$(4-1) \quad v_i \cdot v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j, \\ 0 \text{ or } 1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1, \end{cases}$$

for some integers  $a_i$ ,  $i = 1, \dots, n$ .

Let  $S = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$  be a subset which satisfies (4–1). Define the *intersection graph* of  $S$  as the graph having as vertices  $v_1, \dots, v_n$ , and an edge between  $v_i$  and  $v_j$  if and only if  $v_i \cdot v_j = 1$  for  $i, j = 1, \dots, n$ . The number of connected components of the intersection graph of  $S$  will be denoted by  $c(S)$ . An element  $v_j \in S$  is *isolated*, *final* or *internal* if the quantity

$$\sum_{\substack{i=1 \\ i \neq j}}^n (v_i \cdot v_j)$$

is equal to, respectively, 0, 1 or 2. In other words,  $v_j$  is isolated or final if it is, respectively, an isolated vertex or a leaf of the intersection graph, and it is internal otherwise. Two elements  $v, w \in \mathbf{D}^n$  are *linked* if there exists  $e \in \mathbf{D}^n$  with  $e \cdot v = -1$  such that

$$v \cdot e \neq 0, \quad \text{and} \quad w \cdot e \neq 0.$$

A set  $S \subseteq \mathbf{D}^n$  is *irreducible* if, given two elements  $v, w \in S$ , there exists a finite sequence of elements of  $S$

$$v_0 = v, v_1, \dots, v_k = w,$$

such that  $v_i$  and  $v_{i+1}$  are linked for  $i = 0, \dots, k-1$ . A set which is not irreducible is *reducible*.

A subset  $S = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$  is *good* if it is irreducible and its elements satisfy (4–1). If moreover  $v_i \cdot v_j = 1$  whenever  $|i - j| = 1$ , we say that  $S$  is *standard*.

Given a subset  $S = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$ , define

$$I(S) := \sum_{i=1}^n (-v_i \cdot v_i - 3) \in \mathbb{Z}.$$

Let  $\text{Aut}(\mathbf{D}^n)$  be the group of automorphisms of  $\mathbf{D}^n$  as an intersection lattice.

**Lemma 4.1** [7, Lemma 2.4] *Let  $S = \{v_1, v_2, v_3\} \subseteq \mathbf{D}^3 = \langle e_1, e_2, e_3 \rangle$  be a good subset with  $I(S) < 0$ . Then, up to applying to  $S$  an element of  $\text{Aut}(\mathbf{D}^3)$  and possibly replacing  $(v_1, v_2, v_3)$  with  $(v_3, v_2, v_1)$ , one of the following holds:*

- (1)  $(v_1, v_2, v_3) = (e_1 - e_2, e_2 - e_3, -e_2 - e_1)$ ,
- (2)  $(v_1, v_2, v_3) = (e_1 - e_2, e_2 - e_3, e_1 + e_2 + e_3)$ ,
- (3)  $(v_1, v_2, v_3) = (e_1 + e_2 + e_3, -e_1 - e_2 + e_3, e_1 - e_2)$ .

Moreover,

$$(p_1(S), p_2(S), c(S), I(S)) = \begin{cases} (1, 1, 1, -3) & \text{in case (1),} \\ (0, 2, 2, -2) & \text{in case (2),} \\ (0, 1, 2, -1) & \text{in case (3).} \end{cases}$$

In particular,  $(a_1, a_2, a_3) \in \{(2, 2, 2), (2, 2, 3), (3, 3, 2)\}$ .  $\square$

**Lemma 4.2** [7, Lemma 2.6] *Let  $p > q \geq 1$  be coprime integers, and suppose that*

$$\frac{p}{q} = [a_1, \dots, a_n]^-, \quad \frac{p}{p-q} = [b_1, \dots, b_m]^-,$$

with  $a_1, \dots, a_n \geq 2$  and  $b_1, \dots, b_m \geq 2$ . Then,

$$\sum_{i=1}^n (a_i - 3) + \sum_{j=1}^m (b_j - 3) = -2. \quad \square$$

Given elements  $e, v \in \mathbf{D}^n$  with  $e \cdot e = -1$ , we denote by  $\pi_e(v)$  the projection of  $v$  in the direction orthogonal to  $e$ :

$$\pi_e(v) := v + (v \cdot e)e \in \mathbf{D}^n.$$

**Definition 4.3** Let  $S = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$  be a subset satisfying (4–1) and such that  $|v_i \cdot e_j| \leq 1$  for every  $i, j = 1, \dots, n$ . Suppose that there exist  $1 \leq h, s, t \leq n$  such that

$$E_h^S = \{s, t\} \quad \text{and} \quad a_t > 2.$$

Then, we say that the subset  $S' \subseteq \langle e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_n \rangle \cong \mathbf{D}^{n-1}$  defined by

$$S' := S \setminus \{v_s, v_t\} \cup \{\pi_{e_h}(v_t)\}$$

is obtained from  $S$  by a *contraction*, and we write  $S \searrow S'$ . We also say that  $S$  is obtained from  $S'$  by an *expansion*, and we write  $S' \nearrow S$ .

**Definition 4.4** Let  $S' = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$ ,  $n \geq 3$ , be a good subset, and suppose there exists  $1 < s < n$  such that  $C' = \{v_{s-1}, v_s, v_{s+1}\} \subseteq S'$  is a connected component of the intersection graph of  $S'$ , with  $v_{s-1} \cdot v_{s-1} = v_{s+1} \cdot v_{s+1} = -2$ ,  $v_s \cdot v_s < -2$  and  $E_j^{S'} = \{s-1, s, s+1\}$  for some  $j$ . Let  $S \subseteq \mathbf{D}^m$  be a subset of cardinality  $m \geq n$  obtained from  $S'$  by a sequence of expansions by final  $(-2)$ -vectors attached to  $C'$ , so that  $c(S) = c(S')$  and there is a natural one-to-one correspondence between the sets of connected components of the intersection graphs of  $S$  and  $S'$ . Then, the connected component  $C \subseteq S$  corresponding to  $C' \subseteq S'$  is a *bad component* of  $S$ . The number of bad components of  $S$  will be denoted by  $b(S)$ .

In the arguments of the next two sections we shall need the following [Proposition 4.5](#) and [Lemmas 4.6](#) and [4.7](#).

**Proposition 4.5** [[7](#), Corollary 5.4] Suppose  $n \geq 3$ , and let  $S_n = \{v_1, \dots, v_n\} \subseteq \mathbf{D}^n$  be a good subset with no bad components and such that  $I(S_n) < 0$ . Then  $I(S_n) \in \{-1, -2, -3\}$ , there exists a sequence of contractions

$$(4-2) \quad S_n \searrow S_{n-1} \searrow \cdots \searrow S_3$$

such that, for each  $k = 3, \dots, n-1$  the set  $S_k$  is good, has no bad components and we have either

$$(4-3) \quad (I(S_k), c(S_k)) = (I(S_{k+1}), c(S_{k+1}))$$

or

$$(4-4) \quad I(S_k) \leq I(S_{k+1}) - 1 \quad \text{and} \quad c(S_k) \leq c(S_{k+1}) + 1.$$

Moreover:

- (1) If  $p_1(S_n) > 0$  then  $I(S_n) = -3$ ,  $S_n$  is standard and one can choose the above sequence so that  $I(S_k) = -3$  and  $S_k$  is standard for every  $k = 3, \dots, n-1$ .
- (2) If  $I(S_n) + c(S_n) \leq 0$  then  $S_3$  is given, up to applying an automorphism of  $\mathbf{D}^3$ , by either (1) or (2) in [Lemma 4.1](#); if  $I(S_n) + c(S_n) < 0$  then the former case occurs.

**Lemma 4.6** [[7](#), Lemma 6.3] Let  $S_3 \subset \mathbf{D}^3$  be a good subset with  $I(S_3) = -3$  and  $c(S_3) = 1$ . Suppose that  $S_3 \nearrow \cdots \nearrow S_k$  is a sequence of expansions such that, for each  $h = 3, \dots, k$ ,  $S_h$  is good, has no bad component and  $(I(S_h), c(S_h)) = (-3, 1)$ . Then, it is not possible to expand  $S_k$  by an isolated  $(-3)$ -vector.  $\square$

**Lemma 4.7** [[7](#), Lemma 6.2] Let  $S_3 \nearrow \cdots \nearrow S_n$  be a sequence of expansions such that, for each  $k = 3, \dots, n$ ,  $S_k$  is good, has no bad component and  $(I(S_k), c(S_k)) = (-2, 2)$ . Then,

- (1)  $p_1(S_n) = 0$ ,  $p_2(S_n) = 2$  and  $p_3(S_n) = n - 2$ .
- (2) If  $E_i^{S_n} = \{t, t'\}$  then  $v_t$  and  $v_{t'}$  are not internal and exactly one of them has square  $-2$ .
- (3) If  $v_t \in S_n$  is not internal then there exists  $i \in V_t$  such that  $|E_i^{S_n}| = 2$ .  $\square$

## New results

**Lemma 4.8** *Let  $S_3 \nearrow \cdots \nearrow S_n$  be a sequence of expansions of good subsets without bad components such that  $(I(S_k), c(S_k)) = (-2, 2)$  for every  $k = 3, \dots, n$ . Then, it is not possible to expand  $S_n$  by an isolated  $(-3)$ -vector.*

**Proof** Let  $S_n = \{\bar{v}_1, \dots, \bar{v}_n\} \subset \langle e_1, \dots, e_n \rangle \cong \mathbf{D}^n$ , and suppose that

$$\mathbf{D}^{n+1} \cong \langle e_1, \dots, e_{n+1} \rangle \supset S_{n+1} = \{v_1, \dots, v_{n+1}\} \searrow S_n$$

is a contraction obtained by eliminating the isolated vector  $v_{n+1} \in S_{n+1}$  of square  $-3$ . Since  $S_n$  is good, has no bad components and  $I(S_n) < 0$ , by [7, Proposition 5.2] we have  $|\bar{v}_i \cdot e_j| \leq 1$  for every  $i, j = 1, \dots, n$ . Moreover, we may assume without loss  $v_{n+1} = e_1 + e_2 + e_{n+1}$ , with  $|E_{n+1}^{S_{n+1}}| = 2$ . Then, since

$$\begin{aligned} |E_1^{S_n}| + |E_2^{S_n}| + 1 &= |E_1^{S_{n+1} \setminus \{v_{n+1}\}}| + |E_2^{S_{n+1} \setminus \{v_{n+1}\}}| + |E_{n+1}^{S_{n+1} \setminus \{v_{n+1}\}}| \\ &\equiv \sum_{i \neq n+1} v_i \cdot v_{n+1} \pmod{2} = 0 \pmod{2}, \end{aligned}$$

we see that  $|E_1^{S_n}| + |E_2^{S_n}|$  is odd. Therefore, in view of Lemma 4.7(1) we may assume  $|E_1^{S_n}| = 2$  and  $|E_2^{S_n}| = 3$ . We claim that this is impossible. In fact, observe that by Lemma 4.7(2) we have  $E_1^{S_n} = \{t, t'\}$ , with  $\bar{v}_t$  and  $\bar{v}_{t'}$  not internal and  $\bar{v}_t \cdot \bar{v}_{t'} = -2$ , say. Moreover, it is easy to check that  $\bar{v}_t$  is final and  $v_t \cdot e_{n+1} = 0$ . Therefore, since  $v_{n+1} \cdot v_t = 0$  we have  $v_t = \pm(e_1 - e_2)$ . Since  $E_2^{S_n} = \{s, r\}$  for some  $s, r$  and  $\bar{v}_t$  is not internal, we can assume  $\bar{v}_t \cdot \bar{v}_s = 0$ . This implies  $e_1 \cdot \bar{v}_s \neq 0$ , therefore  $\bar{v}_s = \bar{v}_{t'}$ . It follows that  $e_1 \cdot \bar{v}_r = 0$ , hence  $\bar{v}_r \cdot \bar{v}_t = 1$ . We conclude  $\bar{v}_t \cdot \bar{v}_{t'} = 0$ , so  $v_{t'} = \pm(e_1 + e_2) + \cdots$ , which is incompatible with the fact that  $v_{n+1} = e_1 + e_2 + e_{n+1}$  is isolated in  $S_{n+1}$ .  $\square$

**Lemma 4.9** *Let  $S \subseteq \mathbf{D}^n$  be a good subset with no bad components and  $I(S) < 0$ . Then,  $c(S) \leq 2$ .*

**Proof** If  $n \leq 2$  the conclusion is obvious, so we assume  $n \geq 3$ . By contradiction, suppose  $c(S) \geq 3$ . In view of Proposition 4.5 we have  $I(S) \in \{-1, -2, -3\}$  and there is a sequence of contractions

$$(4-5) \quad S =: S_n \searrow S_{n-1} \searrow \cdots \searrow S_3 \subseteq \mathbf{D}^3,$$

where for each  $k = 3, \dots, n$  the subset  $S_k$  is good, has no bad components and either (4-3) or (4-4) holds. In particular,  $I(S_3) \leq I(S_4) \leq \cdots \leq I(S_n)$ .



**First case:**  $I(S_n) = -3$  Since by Lemma 4.1 we have  $-3 \leq I(S_3)$ , in this case  $I(S_k) = -3$  for every  $k$ . Hence (4-3) holds for every  $k$ , implying  $c(S_3) = c(S_n) \geq 3$ , and contradicting Lemma 4.1.

**Second case:**  $I(S_n) = -2$  If  $I(S_3) = -2$  then arguing as in the previous paragraph one gets a contradiction, so we may assume  $I(S_3) = -3$ . In this case (4-3) must hold for all but at most one index  $k$ . This clearly implies  $c(S_3) \geq c(S_n) - 1 \geq 2$ , which contradicts Lemma 4.1.

**Third case:**  $I(S_n) = -1$  If  $I(S_3) = -1$  one gets a contradiction as before. If  $I(S_3) = -2$  then in Sequence (4-5) there would be an expansion by an isolated  $(-3)$ -vector, going against Lemma 4.8. If  $I(S_3) = -3$  then, since  $c(S_n) \geq 3$  and by Lemma 4.1  $c(S_3) = 1$ , in Sequence (4-5) there would be two expansions by isolated  $(-3)$ -vectors, one of which would contradict Lemma 4.6.  $\square$

**Proposition 4.10** *Let  $S \subseteq \mathbf{D}^n$  be a good subset such that  $I(S) + b(S) < 0$ . Then,  $c(S) \leq 2$ .*

**Proof** By definition of bad component, there is a “minimal” good subset  $S'$  associated to  $S$  with  $I(S') = I(S)$ ,  $b(S') = b(S)$  and  $c(S') = c(S)$ . The set  $S'$  is obtained from  $S$  via contractions by final  $(-2)$ -vectors, and each bad component of  $S'$  is of the form  $\{v_{s-1}, v_s, v_{s+1}\}$ , with  $v_{s-1} \cdot v_{s-1} = v_{s+1} \cdot v_{s+1} = -2$ ,  $v_s \cdot v_s < -2$  and  $E_j^{S'} = \{s-1, s, s+1\}$  for some  $j$  and  $s$ . Then, it is easy to check that if for each bad component of  $S'$  we apply the transformation

$$S' \mapsto S' \setminus \{v_s, v_{s+1}\} \cup \{\pi_{e_j}(v_s)\},$$

the resulting subset  $S''$  is good, has no bad components and satisfies  $c(S'') = c(S)$  and  $I(S'') = I(S') + b(S') = I(S) + b(S) < 0$ . Then, by Lemma 4.9 we have  $c(S'') = c(S) \leq 2$ .  $\square$

## 5 Proof of Theorem 1.1: second half

In this section we use the results of Section 4 to show that if an oriented 3-manifold  $Y$  homeomorphic to a connected sum of lens spaces  $L_1 \# \cdots \# L_h$  smoothly bounds a rational homology 4-ball, then  $Y$  is homeomorphic to a connected sum where each summand is (up to orientation) homeomorphic to one of the manifolds listed in Theorem 1.1(1)–(5). The reader is referred to Section 4 for the algebraic terminology.

To start with, we observe that if  $h = 1$  then it follows from [7, Theorem 1.2] that  $Y = L(p, q)$  with  $p/q \in \mathcal{R}$ . Therefore in this case the statement of Theorem 1.1 is established, and from now on we assume  $h > 1$ .

Recall that the lens space  $L(p, q)$  is orientation-preserving diffeomorphic to the oriented boundary of the 4-dimensional plumbing  $P(p, q)$  given by the weighted graph of Figure 4, where  $p/q = [a_1, \dots, a_n]^-$ . It is easy to check that the intersection form



Figure 4: The weighted graph prescribing the plumbing  $P(p, q)$

of the 4-dimensional plumbing  $P(p, q)$  is negative definite. To simplify the notation, we shall also denote by  $P_L$  the plumbing associated as above with a lens space  $L$ . Likewise, if  $Y$  is a connected sum of lens spaces  $L_1 \# \dots \# L_h$ , we define  $P_Y$  as the boundary connected sum

$$P_{L_1} \natural \dots \natural P_{L_h}.$$

Then,  $-Y = (-L_1) \# \dots \# (-L_h)$ . Suppose that  $Y = L_1 \# \dots \# L_h$  smoothly bounds a rational homology 4-ball  $W_Y$ . Consider the smooth, closed, negative 4-manifolds

$$X_Y := P_Y \cup_{\partial} (-W_Y), \quad X'_Y := P_{-Y} \cup_{\partial} W_Y.$$

By Donaldson’s theorem on the intersection form of definite 4-manifolds [3], the intersection forms of  $X_Y$  and  $X'_Y$  are both standard diagonal.

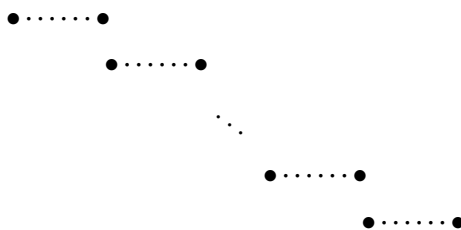
Suppose that the intersection lattice of  $X_Y$  is isomorphic to  $\mathbf{D}^n$  (as defined in Section 4) and the intersection lattice of  $X'_Y$  is isomorphic to  $\mathbf{D}^{n'}$ . Clearly, the groups  $H_2(P_Y; \mathbb{Z}) \cong \mathbb{Z}^n$  and  $H_2(P_{-Y}; \mathbb{Z}) \cong \mathbb{Z}^{n'}$  have integral bases which satisfy Equations (4-1). Therefore, via the embeddings  $P_Y \subset X_Y$  and  $P_{-Y} \subset X'_Y$  we can view such bases as subsets  $S = S_1 \cup \dots \cup S_h \subset \mathbf{D}^n$  and  $S' = S'_1 \cup \dots \cup S'_h \subset \mathbf{D}^{n'}$ , where  $S_i$ , respectively  $S'_i$ , originates from a basis of the corresponding summand  $P_{L_i}$  of  $P_Y$ , respectively  $P_{-L_i}$  of  $P_{-Y}$ . Observe that  $c(S) = c(S') = h$  and  $c(S_i) = c(S'_i) = 1$  for  $i = 1, \dots, h$ .

**Definition 5.1** Let  $S = \{v_1, \dots, v_k\} \subset \mathbf{D}^n$ . The *string* of  $S$  is  $(a_1, \dots, a_k)$ , where  $a_i = -v_i \cdot v_i$  for  $i = 1, \dots, k$ .

We now briefly recall Riemenschneider’s point rule [12]. Let  $p > q > 0$  be coprime integers, and suppose

$$\frac{p}{q} = [a_1, \dots, a_l]^-, \quad a_i \geq 2, \quad \frac{p}{p-q} = [b_1, \dots, b_m]^-, \quad b_j \geq 2.$$

Then, the coefficients  $a_1, \dots, a_l$  and  $b_1, \dots, b_m$  are related by a diagram of the form



where the  $i$ -th row contains  $a_i - 1$  “points” for  $i = 1, \dots, l$ , and the first point of each row is vertically aligned with the last point of the previous row. The point rule says that there are  $m$  columns, and the  $j$ -th column contains  $b_j - 1$  points for every  $j = 1, \dots, m$ . For example if  $7/5 = [2, 2, 3]^-$  and  $7/2 = [4, 2]^-$  the corresponding diagram is given by



**Lemma 5.2** We have  $b(S) + b(S') \leq c(S)$ .

**Proof** Suppose that  $S_i$  is a bad component of  $S$ . Then, by Definition 4.4 the string of  $S_i$  is obtained from the string  $(2, a, 2)$ ,  $a \geq 3$ , via a finite sequence of operations of type Lemma 3.3(1) or Lemma 3.3(2). Applying Riemenschneider’s point rule (cf [7, Proof of Lemma 2.6]) one can easily check that, similarly, the string of  $S'_i$  is obtained from the string

$$(3, \overbrace{2, \dots, 2}^{a-3}, 3)$$

via a sequence of operations of type Lemma 3.3(1) or Lemma 3.3(2). Clearly, such a string is not the string of a bad component. This shows that, for each  $i = 1, \dots, h$ , at most one set among  $S_i$  and  $S'_i$  can be a bad component, which implies the statement of the lemma.  $\square$

**Lemma 5.3** Up to replacing  $S$  with  $S'$  we have  $I(S) + c(S) \leq 0$  and  $I(S) + b(S) < 0$ .

**Proof** We have  $c(S) = c(S')$  and by Lemma 4.2 it follows that  $I(S) + I(S') = -2c(S)$ . Therefore, up to replacing  $S$  with  $S'$  we may assume  $I(S) \leq -c(S)$ . If  $I(S) < -c(S)$  then, since  $b(S) \leq c(S)$ , we have

$$(5-1) \quad I(S) + b(S) \leq I(S) + c(S) < 0$$

and the lemma holds. Now assume  $I(S) = -c(S) = I(S')$ . By [Lemma 5.2](#)  $b(S) + b(S') \leq c(S)$ . Therefore, up to replacing  $S$  with  $S'$  we may assume  $I(S) = -c(S)$  and  $b(S) < c(S)$ , so it follows that

$$(5-2) \quad I(S) + b(S) < I(S) + c(S) = 0.$$

This concludes the proof.  $\square$

In view of [Lemma 5.3](#), from now on we will assume:

$$I(S) + c(S) \leq 0 \quad \text{and} \quad I(S) + b(S) < 0.$$

Now there are two possibilities. Either  $S$  is irreducible or it is reducible. We consider the two cases separately.

### First case: $S$ irreducible

In this case  $S$  is a good subset by definition. Since  $I(S) + b(S) < 0$ , by [Proposition 4.10](#), we have  $h = c(S) \leq 2$ . But we are assuming  $h > 1$ , so we conclude  $h = c(S) = 2$ . Since  $b(S) \leq c(S)$ , we must analyze the three subcases  $b(S) = 0$ ,  $b(S) = 1$  and  $b(S) = 2$ .

**First subcase:**  $b(S) = 0$  By [Proposition 4.5](#) there is a sequence of contractions of good subsets without bad components as in (4-2). Since  $I(S_3) \leq I(S) \leq -c(S) = -2$ , applying (4-3), (4-4) and [Lemma 4.1](#) it is easy to check that  $(I(S), c(S)) = (-2, 2)$  and either  $(I(S_3), c(S_3)) = (-2, 2)$  or  $(I(S_3), c(S_3)) = (-3, 1)$ . The latter case is excluded by (4-3), (4-4) and [Lemma 4.6](#). So we are left with the case  $(I(S_3), c(S_3)) = (-2, 2)$ . Since  $I(S) = I(S_3) = -2$ , by (4-3) and (4-4) Sequence (4-2) satisfies the assumptions of [Lemma 4.7](#). Applying the lemma to  $S_k$  for each  $k = 3, \dots, n$  it is easy to check that the string associated with  $S$  is a union of two strings  $s_1 \cup s_2$  related to each other by Riemenschneider's point rule. We conclude that, up to orientation,  $Y = L_1 \# L_2$  with  $L_1 = L(p, q)$  and  $L_2 = L(p, p - q)$  for some  $p > q > 0$ , so Case (2) of [Theorem 1.1](#) holds.

**Second subcase:**  $b(S) = 1$  By definition of bad component there is a sequence of contractions of good subsets

$$S \searrow \dots \searrow T$$

obtained by erasing final  $(-2)$ -vectors, such that  $c(T) = c(S) = 2$ ,  $b(T) = b(S) = 1$ ,  $I(T) = I(S) \leq -c(S) = -2$  and such that the bad component of  $T$  is of the form  $\{v_{s-1}, v_s, v_{s+1}\}$  with  $v_{s-1} \cdot v_{s-1} = v_{s+1} \cdot v_{s+1} = -2$  and  $v_s \cdot v_s < -2$ . We can further reduce  $T$  to a good subset without bad components  $U$  as in the proof of [Proposition 4.10](#), so that  $I(U) = I(T) + 1 \leq -1$ ,  $c(U) = c(T) = 2$  and one connected component

of the intersection graph of  $U$  consists of a single element. Moreover, by [Proposition 4.5](#) there is a sequence of contractions

$$U \searrow \cdots \searrow S_3 \subset \mathbf{D}^3$$

satisfying [\(4-3\)](#) and [\(4-4\)](#). In particular,  $I(U) \geq I(S_3) \geq -3$ . The case  $I(U) = I(S_3) = -3$  is impossible by [\(4-3\)](#) and [\(4-4\)](#) because  $c(U) = 2$  while by [Lemma 4.1](#) the equality  $I(S_3) = -3$  implies  $c(S_3) = 1$ .

If  $I(U) = -2$ , the same analysis made in the first subcase above implies  $(I(S_3), c(S_3)) = (-2, 2)$  and the string of  $U$  must be of the form

$$(n) \cup \overbrace{(2, \dots, 2)}^{n-1}, \quad n \geq 2,$$

therefore the string of  $T$  is of the form

$$(2, n+1, 2) \cup \overbrace{(2, \dots, 2)}^{n-1}, \quad n \geq 2.$$

We conclude that the string of  $S$  is of the form

$$s_n \cup \overbrace{(2, \dots, 2)}^{n-1}, \quad n \geq 2,$$

where the string  $s_n$  is obtained from  $(2, n+1, 2)$  by a finite sequence of operations as in [Lemma 3.3](#). Applying [Lemma 3.3](#) we conclude that, up to orientation,  $Y = L_1 \# L_2$  with  $L_1 = L(m^2n, mnk+1)$ ,  $L_2 = L(n, n-1)$  for some  $n \geq 2$ ,  $m > k > 0$  and  $(m, k) = 1$ . In other words, [Case \(4\)](#) of [Theorem 1.1](#) holds.

If  $I(U) = -1$  then  $I(S) = I(T) = I(U) - 1 = -2$ . By [Lemma 4.2](#) the subset  $S' \subset \mathbf{D}^{n'}$  originating from the integral basis of  $H_2(P_{-Y}; \mathbb{Z})$  satisfies  $(I(S'), c(S')) = (-2, 2)$  and by [Lemma 5.2](#) we have  $b(S') \leq 1$ . If  $b(S') = 0$ , the argument given in the first subcase shows that  $Y = L(p, q) \# L(p, p-q)$ , so [Case \(2\)](#) of [Theorem 1.1](#) holds. If  $b(S') = 1$ , the same argument just used for  $S$  shows that there is a sequence of contractions of good subsets

$$S' \searrow \cdots \searrow T'$$

obtained by erasing final  $(-2)$ -vectors, such that  $c(T') = c(S') = 2$ ,  $b(T') = b(S') = 1$ ,  $I(T') = I(S') = -2$ . By Riemenschneider's point rule, the string of the bad component of  $T'$  is dual to the string of the non-bad component of  $T$ , and vice-versa. This immediately implies that the intersection graph of  $S$  is a union of connected components  $S_1 \cup S_2$ , where (i) the string of  $S_1$  is obtained from  $(2, a, 2)$ , for some  $a \geq 3$ , by a finite sequence of operations as in [Lemma 3.3](#) and (ii) the string of  $S_2$  is the Riemenschneider dual of a string obtained in a similar way from  $(2, b, 2)$ , for

some  $b \geq 3$ . Clearly  $I(S_1) = a - 5$  and by [Lemma 4.2](#) we have  $I(S_2) = -b + 3$ . But  $I(S) = I(S_1) + I(S_2) = -2$  implies  $a = b$ , therefore applying [Lemma 3.3](#) we conclude that, up to orientation,  $Y = L_1 \# L_2$  with  $L_1, -L_2 \in \mathcal{F}_{a-1}$ . Hence, Case (5) of [Theorem 1.1](#) holds.

**Third subcase:**  $b(S) = 2$  As in the previous subcase, there is a sequence of contractions of good subsets by final  $(-2)$ -vectors

$$S \searrow \cdots \searrow T$$

such that  $c(T) = c(S) = 2$ ,  $b(T) = 2$  and  $I(T) = I(S)$ . Erasing final  $(-2)$ -vectors as before we get a good subset  $U \subseteq \mathbf{D}^2$ , and it is easy to check that the string of  $U$  must be equal to  $(2) \cup (2)$ . Therefore the string of  $T$  is

$$(2, 3, 2) \cup (2, 3, 2)$$

and the string of  $S$  is of the form  $s_n \cup t_n$ , where each one of  $s_n$  and  $t_n$  is obtained from  $(2, 3, 2)$  by a finite sequence of operations as in [Lemma 3.3](#). Applying [Lemma 3.3](#) as in the previous subcase we see that, up to orientation,  $Y = L_1 \# L_2$  with  $L_1 = L(2m^2, 2mk + 1)$  and  $L_2 = L(2p^2, 2ph + 1)$ , for some  $m > k > 0$ ,  $(m, k) = 1$  and  $p > h > 0$ ,  $(p, h) = 1$ . Therefore Case (3) of [Theorem 1.1](#) holds. Summarizing, so far we have proved:

**Lemma 5.4** *If  $h \geq 2$  and the subset  $S$  is irreducible, then  $h = 2$  and  $Y$  is (possibly after reversing its orientation) homeomorphic to one of the manifolds given in [Theorem 1.1\(2\)–\(5\)](#).  $\square$*

### Second case: $S$ reducible

In this case the set  $S$  can be written as a disjoint union  $S = \cup_j T_j$  of maximal irreducible subsets  $T_j \subset S$ . As observed in [[7](#), Remark 2.1], the elements of  $S$  are linearly independent over  $\mathbb{Z}$  because they satisfy Equation ([4-1](#)). We claim that, for each index  $j$ ,

$$(5-3) \quad |T_j| = |\cup_{v_i \in T_j} V_i|.$$

In fact, by the linear independence we have

$$|\cup_{v_i \in T_j} V_i| \geq |T_j|,$$

while by the maximality of each  $T_j$  the union  $\cup_j (\cup_{v_i \in T_j} V_i)$  is disjoint. Hence

$$|\cup_j (\cup_{v_i \in T_j} V_i)| = \sum_j |\cup_{v_i \in T_j} V_i|.$$

So we have

$$\sum_j \left| \bigcup_{v_i \in T_j} V_i \right| \geq \sum_j |T_j| = |S| \geq \left| \bigcup_j (\cup_{v_i \in T_j} V_i) \right| = \sum_j \left| \bigcup_{v_i \in T_j} V_i \right|,$$

and the claim follows. Next, we claim that there exists at least one index  $i$  such that  $I(T_i) + c(T_i) \leq 0$  and  $I(T_i) + b(T_i) < 0$ . Since  $b(T_j) \leq c(T_j)$  for every  $j$ , it suffices to show that either (i) there is an index  $i$  such that  $I(T_i) + c(T_i) < 0$  or (ii) there is an index  $i$  such that  $I(T_i) + c(T_i) = 0$  and  $I(T_i) + b(T_i) < 0$ . Since

$$\sum_j (I(T_j) + c(T_j)) = I(S) + c(S) \leq 0,$$

if (i) does not hold then  $I(T_j) + c(T_j) = 0$  for every  $j$ . In this case, since we also have

$$\sum_j (I(T_j) + b(T_j)) = I(S) + b(S) < 0,$$

we conclude that  $I(T_i) + b(T_i) < 0$  for some  $i$ , that is, that (ii) holds. By (5-3) and the last claim, we may apply Lemma 5.4 to  $T_i$ , viewed as a subset of the span of the  $e_k$ 's hit by the vectors of  $T_i$ . Therefore, after possibly renaming the  $L_i$ 's we conclude that either  $L_h$  or  $L_{h-1} \# L_h$  smoothly bounds a rational homology ball  $Z$ . Let  $p \in \partial Z$ , let  $B \subseteq \partial Z$  a regular neighborhood of  $p$ , and let  $W$  be a smooth rational homology ball with boundary  $L_1 \# \cdots \# L_h$ . Clearly, the 3-manifold with boundary  $\partial Z \setminus B$  can be viewed as a subset of  $\partial W$  as well as  $\partial Z$ . Then, the space

$$W \cup_{(\partial Z) \setminus B} (-Z)$$

obtained by gluing  $W$  and  $-Z$  together along  $\partial Z \setminus B$ , is (after smoothing corners) a smooth rational homology ball with boundary  $L_1 \# \cdots \# L_{h'}$ , where  $h'$  is equal to  $h - 1$  or  $h - 2$ , respectively. Thus, we have proved:

**Lemma 5.5** *If  $h \geq 2$  and  $S$  is reducible, after possibly renaming the  $L_i$ 's one of the following holds:*

- $L_1 \# \cdots \# L_{h-1}$  and  $L_h$  smoothly bound a rational homology ball.
- $L_1 \# \cdots \# L_{h-2}$  and  $L_{h-1} \# L_h$  smoothly bound a rational homology ball. □

At the beginning of this section we observed that when  $h = 1$  the statement of Theorem 1.1 was proved in [7]. Therefore, the second half of Theorem 1.1 follows combining Lemmas 5.4 and 5.5.

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