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RESEARCH ARTICLE

Sums of Σ-strictly diagonally dominant matrices

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In this paper a generalization of a known result about the subdirect sum of two S-SDD matrices is obtained for Σ-SDD matrices. The class of Σ-SDD matrices is a generalization of S-SDD matrices, and it is also a subclass of H-matrices. More precisely, the question of when the subdirect sum, and consequently, the usual sum, of two Σ-SDD matrices is an Σ-SDD matrix is studied.

Keywords: Subdirect sum, H-matrices, overlapping blocks

1. Introduction

As in [7], if A and B are two square matrices of order n_1 and n_2, respectively, and A_{22} and B_{11} are square matrices of order k, $1 \le k \le \min(n_1, n_2)$, then the k-subdirect sum of A and B, denoted by $C = A \oplus_k B$ is defined to be

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{1}$$

Let $n = n_1 + n_2 - k$ and let us define the following set of indices

$$S_1 = \{1, \dots, n_1 - k\},\ S_2 = \{n_1 - k + 1, \dots, n_1\},\ S_3 = \{n_1 + 1, \dots, n\}. \tag{2}$$

Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, we define the following deleted row sums:

$$r_i(A) = \sum_{j \ne i,\, j=1}^{n} |a_{ij}|, \ \ r_i^S(A) = \sum_{j \ne i,\, j \in S} |a_{ij}|, \ \ i \in N,$$

where $N = \{1, 2, ..., n\}$ is the set of indices and $S \subseteq N$. If S is the empty set, then $r_i^S(A)$ is considered to be zero. Finally, $\overline{S} := N \backslash S$.

Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \ge 2$, and a nonempty proper subset S of N, we say that A is an S-strictly diagonally dominant (S-SDD) matrix if

- $|a_{ii}| > r_i^S(A)$, for all $i \in S$ and
- $|a_{jj}| > r_j^{\overline{S}}(A)$, for all $j \in \overline{S}$ and

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- $\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) > r_i^{\overline{S}}(A) r_j^S(A)$, for all $i \in S$, $j \in \overline{S}$.

By convention, if S is either the empty set or the whole set of indices, then we identify the classes $\emptyset$-SDD matrices and N-SDD matrices with the class of SDD matrices.

The class of S-SDD can be characterized in the following way. For an arbitrary nonempty proper set of indices S, let us define the interval

$$J_A(S) := \Big(\max_{i \in S} \frac{r_i^{\overline{S}}(A)}{|a_{ii}| - r_i^S(A)}, \min_{j \in \overline{S}, r_j^S(A) \neq 0} \frac{|a_{jj}| - r_j^{\overline{S}}(A)}{r_j^S(A)} \Big),$$

where the last fraction is defined to be $+\infty$ if $r_j^S(A) = 0$ for all $j \in \overline{S}$. For $S = \emptyset$ or $S = N$, we define $J_A(S) = (0, +\infty)$. With the usual notation $A[S]$ for the principal submatrix of A with indices from the set S, it is easy to show that for a given $S \subseteq N$, the matrix A is S-SDD matrix if and only if $A[S]$ and $A[\overline{S}]$ are strictly diagonally dominant matrices and the interval $J_A(S)$ is nonempty; the proof easily follows from the results obtained in [6] and [3].

The following characterization of S-SDD matrices is also known (see, for example, a related statement in the eigenvalue localization field given in [5]): A matrix $A \in \mathbb{C}^{n,n}$ is an S-SDD matrix if and only if there exists a diagonal matrix

$$X_n(S, x) = diag(x_1, \ldots, x_n), \text{ where } x_i = x > 0 \text{ for } i \in S \text{ and } x_i = 1 \text{ otherwise},$$

such that $AX_n(S, x)$ is an SDD matrix. Moreover, $x \in J_A(S)$.

As in [3], a matrix A belongs to the class of Σ-SDD matrices if there is a subset S of N, such that A is an S-SDD matrix.

2. Subdirect sum of Σ-SDD matrices

The above characterization of an S-SDD matrix in terms of a scaling matrix allows us to simplify the proof of an existing result. Indeed, this characterization provides a more general result for the subdirect sum of Σ-SDD matrices, which is the main goal of this paper.

We recall now the following result of [2].

Theorem 2.1 : *Let A and B be matrices of order n_1 and n_2, respectively. Let $n_1 \geq 2$, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$, which defines the sets S_1, S_2, S_3 as in* (2)*. Let A and B be partitioned as in* (1)*. Let S be a set of indices of the form $S = \{1, 2, \ldots\}$. Let A be S-strictly diagonally dominant, with $card(S) \leq card(S_1)$, and let B be strictly diagonally dominant. If the diagonal entries of A_{22} and B_{11} are all positive (or all negative), then the k-subdirect sum $C = A \oplus_k B$ is S-strictly diagonally dominant, and therefore, nonsingular.*

We give a straightforward proof of this theorem in the following way. Since A is an S-SDD matrix and $S \subseteq S_1 := \{1, 2, \ldots, n_1 - k\}$ we know that there exists a scaling matrix $X_{n_1}(S, x)$, such that $AX_{n_1}(S, x)$ is an SDD matrix. Now we construct the matrix $X_n(S, x)$ with $n = n_1 + n_2 - k$. From (1) we have that $CX_n(S, x) = AX_{n_1}(S, x) \oplus_k B$. Since $AX_{n_1}(S, x)$ and B are SDD matrices with the same sign pattern of diagonal entries of the overlapping blocks A_{22} and B_{11}, it is easy to show that their subdirect sum $CX_n(S, x)$ is also an SDD matrix, which means that C is also an S-SDD matrix.

Using the same technique, we obtain a more general result.

Theorem 2.2: *Let $A \in \mathbb{C}^{n_1,n_1}$, $B \in \mathbb{C}^{n_2,n_2}$, $n_1 \geq 2$, $1 \leq k \leq \min(n_1, n_2)$, and let the sets of indices S_1, S_2, S_3 be defined as in* (2)*. For A and B partitioned as in* (1)*, let the corresponding diagonal entries of A_{22} and B_{11} have the same sign pattern. For an arbitrary set of indices $S \subseteq \{1, 2, \ldots, n\}$, where $n = n_1 + n_2 - k$, let us define $S_A := S \cap (S_1 \cup S_2)$ and $S_B := \{i - t : i \in S \cap (S_2 \cup S_3)\}$. If*

- *A is an S_A-SDD matrix,*
- *B is S_B-SDD matrix, and*
- *$J_A(S_A) \cap J_B(S_B) \neq \emptyset$,*

then the k-subdirect sum $C = A \oplus_k B$ is an S-SDD matrix.

Proof: Let $x \in J_A(S_A) \cap J_B(S_B)$ (which is nonempty). We construct the following scaling matrices: $X_{n_1}(S_A, x)$ and $X_{n_2}(S_B, x)$. Since A is an S_A-SDD matrix and B is an S_B-SDD matrix, it follows that $AX_{n_1}(S_A, x)$ and $BX_{n_2}(S_B, x)$ are SDD matrices. Now, building the matrix $X_n(S, x)$, it is easy to see that $CX_n(S, x) = AX_{n_1}(S_A, x) \oplus_k BX_{n_2}(S_B, x)$. Since the k-subdirect sum of SDD matrices with the same sign pattern of diagonal entries of the overlapped blocks is again an SDD matrix, we conclude that $CX_n(S, x)$ is an SDD matrix, which means that C is an S-SDD matrix. $\square$

Example 2.3 Let

$$A_1 = B_1 = \begin{bmatrix} 1.0 & 0.3 & 0.4 & 0.5 \\ 0.9 & 1.6 & 0.4 & 0.7 \\ 0.1 & 0.4 & 1.3 & 0.4 \\ 0.1 & 0.9 & 0.1 & 2.0 \end{bmatrix}, \quad C_1 = A_1 \oplus_2 B_1 = \begin{bmatrix} 1.0 & 0.3 & 0.4 & 0.5 & 0 & 0 \\ 0.9 & 1.6 & 0.4 & 0.7 & 0 & 0 \\ 0.1 & 0.4 & 2.3 & 0.7 & 0.4 & 0.5 \\ 0.1 & 0.9 & 1.0 & 3.6 & 0.4 & 0.7 \\ 0 & 0 & 0.1 & 0.4 & 1.3 & 0.4 \\ 0 & 0 & 0.1 & 0.9 & 0.1 & 2.0 \end{bmatrix}.$$

A_1 and B_1 are both $\{1, 2\}$-SDD matrices and $\{3, 4\}$-SDD matrices. But C_1 is not an S-SDD matrix for any $S \subseteq \{1, 2, \ldots, 6\}$. Thus, the answer to the question of whether the subdirect sum of two Σ-SDD matrices is or is not an Σ-SDD matrix is, in general, negative. Observing the conditions of Theorem 2.2 and taking $S = \{3, 4\}$, we have that $S_{A_1} = \{3, 4\}$ and $S_{B_1} = \{1, 2\}$; thus, the intervals $J_{A_1}(S_{A_1})$ and $J_{B_1}(S_{B_1})$ are well defined. But, obviously, $J_{A_1}(S_{A_1}) \cap J_{B_1}(S_{B_1}) = (0.56, 0.64) \cap (1.57, 1.80) = \emptyset$.

The sufficient condition of Theorem 2.2 is not necessary as the following example shows.

Example 2.4 Let

$$A_2 = B_2 = \begin{bmatrix} 2.0 & 0.9 & 0.3 & 0.1 \\ 0.8 & 2.9 & 0.2 & 0.5 \\ 0.5 & 0.1 & 1.4 & 0.9 \\ 0.6 & 0.8 & 0.8 & 2.3 \end{bmatrix}, \quad C_2 = A_2 \oplus_2 B_2 = \begin{bmatrix} 2.0 & 0.9 & 0.3 & 0.1 & 0 & 0 \\ 0.8 & 2.9 & 0.2 & 0.5 & 0 & 0 \\ 0.5 & 0.1 & 3.4 & 1.8 & 0.3 & 0.1 \\ 0.6 & 0.8 & 1.6 & 5.2 & 0.2 & 0.5 \\ 0 & 0 & 0.5 & 0.1 & 1.4 & 0.9 \\ 0 & 0 & 0.6 & 0.8 & 0.8 & 2.3 \end{bmatrix}.$$

For $S = \{3, 4\}$, we have $S_{A_2} = \{3, 4\}$ and $S_{B_2} = \{1, 2\}$. Computing the corresponding intervals $J_{A_2}(S_{A_2}) = (1.20, 2.75)$ and $J_{B_2}(S_{B_2}) = (0.36, 0.83)$, we find that $J_{A_2}(S_{A_2}) \cap J_{B_2}(S_{B_2}) = \emptyset$, but C is still an S-SDD matrix, since $J_{C_2}(S) = (0.63, 0.83)$ is nonempty and $C_2[S]$ and $C_2[\overline{S}]$ are both SDD matrices.

Note that the usual sum of two matrices A and B of the same order, $n = n_1 = n_2$, is, in fact, a k-subdirect sum with $k = n$. In this case, the sets of indices given by (2) reduce to $S_2 = \{1, 2, \ldots, n\}$. These remarks lead to the following corollary of Theorem 2.2.

Corollary 2.5 Let $A, B \in \mathbb{C}^{n,n}$ have all diagonal entries with the same sign pattern and let $S \subseteq \{1, 2, \ldots, n\}$. If A and B are S-SDD matrices and $J_A(S) \cap J_B(S) \neq \emptyset$, then the sum $A + B$ is an S-SDD matrix.

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