SUMUDU TRANSFORM FUNDAMENTAL PROPERTIES INVESTIGATIONS AND APPLICATIONS

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The Sumudu transform, whose fundamental properties are presented in this paper, is still not widely known, nor used. Having scale and unit-preserving properties, the Sumudu transform may be used to solve problems without resorting to a new frequency domain. In 2003, Belgacem et al. have shown it to be the theoretical dual to the Laplace transform, and hence ought to rival it in problem solving. Here, using the Laplace-Sumudu duality (LSD), we avail the reader with a complex formulation for the inverse Sumudu transform. Furthermore, we generalize all existing Sumudu differentiation, integration, and convolution theorems in the existing literature. We also generalize all existing Sumudu shifting theorems, and introduce new results and recurrence results, in this regard. Moreover, we use the Sumudu shift theorems to introduce a paradigm shift into the thinking of transform usage, with respect to solving differential equations, that may be unique to this transform due to its unit-preserving properties. Finally, we provide a large and more comprehensive list of Sumudu transforms of functions than is available in the literature.

1. Introduction

Due to its simple formulation and consequent special and useful properties, the Sumudu transform has already shown much promise. It is revealed herein and elsewhere that it can help to solve intricate problems in engineering mathematics and applied sciences. However, despite the potential presented by this new operator, only few theoretical investigations have appeared in the literature, over a fifteen-year period. Most of the available transform theory books, if not all, do not refer to the Sumudu transform. Even in relatively recent well-known comprehensive handbooks, such as Debnath [6] and Poularikas [8], no mention of the Sumudu transform can be found. Perhaps it is because no transform under this name (as such) was declared until the late 80's and early 90's of the

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previous century (see Watugala [10]). On the other hand, for historical accountability, we must note that a related formulation, called *s*-multiplied Laplace transform, was announced as early as 1948 (see Belgacem et al. [5] and references), if not before.

The Weerakoon [13] paper, showing Sumudu transform applications to partial differential equations, immediately followed Watugala's [10] seminal work. Watugala's [11] work showed that the Sumudu transform can be effectively used to solve ordinary differential equations and engineering control problems. Once more, Watugala's work was followed by Weerakoon [14] introducing a complex inversion formula for the Sumudu transform (see Theorem 3.1 in Section 3). The relatively recent sequence [1–3], indicated how to use the transform to solve integrodifferential equations, with emphasis on dynamic systems. Watugala [12] extended the transform to two variables with emphasis on solutions to partial differential equations.

Belgacem et al. [5] presented applications to convolution type integral equations with focus on production problems. There, a Laplace-Sumudu duality was highlighted, and used to establish or corroborate many fundamental useful properties of this new transform. In particular a two-page table of the transforms of some of the basic functions was provided. At the end of this paper, analogous to the lengthy Laplace transform list found in Spiegel [9], we provide Sumudu transforms for a more comprehensive list of functions. Aside from a paradigm change into the thought process of Sumudu transform usage with respect to applications to differential equations, we introduce more general shift theorems that seem to have combinatorial connections to generalized stirling numbers. Furthermore, we establish more general Sumudu differentiation, integration, and convolution theorems than already established in the literature. Moreover, we use the Laplace-Sumudu duality (LSD) to invoke a complex inverse Sumudu transform, as a Bromwhich contour integral formula.

2. The discrete Sumudu transform

Over the set of functions,

$$A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \, | f(t) | < M e^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \}, \tag{2.1}$$

the Sumudu transform is defined by

$$G(u) = \mathbb{S}[f(t)] = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \tag{2.2}$$

Among others, the Sumudu transform was shown to have units preserving properties, and hence may be used to solve problems without resorting to the frequency domain. As will be seen below, this is one of many strength points for this new transform, especially with respect to applications in problems with physical dimensions. In fact, the Sumudu transform which is itself linear, preserves linear functions, and hence in particular does not change units (see for instance Watugala [11] or Belgacem et al. [5]). Theoretically, this point may perhaps best be illustrated as an implication of this more global result.

THEOREM 2.1. *The Sumudu transform amplifies the coefficients of the power series function,*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \tag{2.3}$$

by sending it to the power series function,

$$G(u) = \sum_{n=0}^{\infty} n! a_n u^n. \tag{2.4}$$

Proof. Let $f(t)$ be in A. If $f(t) = \sum_{n=0}^{\infty} a_n t^n$ in some interval $I \subset \mathbb{R}$, then by Taylor's function expansion theorem,

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n. \tag{2.5}$$

Therefore, by (2.2), and that of the gamma function Γ (see Table 2.1), we have

$$\begin{aligned} \mathbb{S}[f(t)] &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} (ut)^n e^{-t} dt = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n \int_0^{\infty} t^n e^{-t} dt \\ &= \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} u^n \Gamma(n+1) = \sum_{k=0}^{\infty} f^{(n)}(0) u^n. \end{aligned} \tag{2.6}$$

Consequently, it is perhaps worth noting that since

$$\mathbb{S}[(1+t)^m] = \mathbb{S} \sum_{n=0}^{m} C_n^m t^n = \mathbb{S} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} u^n = \sum_{n=0}^{m} \frac{m!}{(m-n)!} u^n = \sum_{n=0}^{m} P_n^m u^n, \tag{2.7}$$

the Sumudu transform sends combinations, C_n^m, into permutations, P_n^m, and hence may seem to incur more order into discrete systems.

Also, a requirement that $\mathbb{S}[f(t)]$ converges, in an interval containing $u = 0$, is provided by the following conditions when satisfied, namely, that

$$\begin{aligned} &\text{(i) } f^{(n)}(0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ &\text{(ii) } \lim_{n\to\infty} \left| \frac{f^{(n+1)}(0)}{f^{(n)}(0)} u \right| < 1. \end{aligned} \tag{2.8}$$

This means that the convergence radius r of $\mathbb{S}[f(t)]$ depends on the sequence $f^{(n)}(0)$, since

$$r = \lim_{n\to\infty} \left| \frac{f^{(n)}(0)}{f^{(n+1)}(0)} \right|. \tag{2.9}$$

Clearly, the Sumudu transform may be used as a signal processing or a detection tool, especially in situations where the original signal has a decreasing power tail. However,

Table 2.1. Special functions.

(1) Gamma function	$\Gamma(n)=\int_0^\infty u^{n-1}e^{-u}du,\ n>0$
(2) Beta function	$B(m,n)=\int_0^1 u^{m-1}(1-u)^{n-1}du=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
(3) Bessel function	$J_n(x)=\dfrac{x^n}{2^n\Gamma(n+1)}\times\left\{1-\dfrac{x^2}{2(2n+2)}+\dfrac{x^4}{2.4(2n+2)(2n+4)}-\cdots\right\}$
(4) Modified Bessel function	$I_n(x)=i^{-n}J_n(ix)=\dfrac{x^n}{2^n\Gamma(n+1)}\times\left\{1+\dfrac{x^2}{2(2n+2)}+\dfrac{x^4}{2.4(2n+2)(2n+4)}-\cdots\right\}$
(5) Error function	$\operatorname{erf}(t)=\dfrac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}du$
(6) Complementary error function	$\operatorname{erf}(t)=1-\operatorname{erf}(t)=\dfrac{2}{\sqrt{\pi}}\int_t^\infty e^{-u^2}du$
(7) Exponential integral	$\operatorname{Ei}(t)=\int_t^\infty \dfrac{e^{-u}}{u}du$
(8) Sine integral	$\operatorname{Si}(t)=\int_0^t \dfrac{\sin u}{u}du$
(9) Cosine integral	$\operatorname{Ci}(t)=\int_0^t \dfrac{\cos u}{u}du$
(10) Fresnel sine integral	$S(t)=\int_0^t \sin u^2du$
(11) Fresnel cosine integral	$C(t)=\int_0^t \cos u^2du$
(12) Laguerre polynomials	$L_n(t)=\dfrac{e^t}{n!}\dfrac{d^n}{dt^n}(t^ne^{-t}),\quad n=0,1,2,\ldots$

care must be taken, especially if the power series is not highly decaying. This next example may instructively illustrate the stated concern. For instance, consider the function

$$f(t)=\begin{cases}\ln(t+1) & \text{if } t\in(-1,1],\\ 0 & \text{otherwise.}\end{cases} \tag{2.10}$$

Since $f(t)=\sum_{n=1}^{\infty}(-1)^{n-1}(t^n/n)$, then except for $u=0$,

$$\mathbb{S}[f(t)]=\sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!u^n \tag{2.11}$$

diverges throughout, because its convergence radius

$$r=\lim_{n\to\infty}\left|\frac{(-1)^{n-1}(n-1)!}{(-1)^n n!}\right|=\lim_{n\to\infty}\frac{1}{n}=0. \tag{2.12}$$

Theorem 2.1 implies a transparent inverse transform in the discrete case, that of getting the original function from its given transform, in the obvious manner. □

Corollary 2.2. *Up to null functions, the inverse discrete Sumudu transform, $f(t)$, of the power series $G(u) = \sum_{n=0}^{\infty} b_n u^n$, is given by*

$$\mathbb{S}^{-1}[G(u)] = f(t) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) b_n t^n. \tag{2.13}$$

In the next section, we provide a general inverse transform formula, albeit in a complex setting.

3. The Laplace-Sumudu duality and the complex Sumudu inversion formula

In Belgacem et al. [5], the Sumudu transform was shown to be the theoretical dual of the Laplace transform. Hence, one should be able to rival it to a great extent in problem solving. Defined for $\mathrm{Re}(s) > 0$, the Laplace transform is given by

$$F(s) = \$(f(t)) = \int_0^{\infty} e^{-st} f(t)dt. \tag{3.1}$$

In consideration of the definition in (2.2), the Sumudu and Laplace transforms exhibit a duality relation expressed as follows:

$$G\left(\frac{1}{s}\right) = sF(s), \qquad F\left(\frac{1}{u}\right) = uG(u). \tag{3.2}$$

Equation (3.2), which from now on shall be referred as the LSD, which is short for the Laplace-Sumudu duality, is illustrated by the fact that the Sumudu and Laplace transforms interchange the images of the Dirac function, $\delta(t)$, and the Heaviside function, $H(t)$, since

$$\mathbb{S}[H(t)] = \$(\delta(t)) = 1, \qquad \mathbb{S}[\delta(t)] = \$(H(t)) = \frac{1}{u}. \tag{3.3}$$

Similarly, this duality is also illustrated by the Sumudu and Laplace transforms interchange of the images of $\sin(t)$ and $\cos(t)$:

$$\mathbb{S}[\cos(t)] = \$(\sin(t)) = \frac{1}{1+u^2}, \qquad \mathbb{S}[\sin(t)] = \$(\cos(t)) = \frac{u}{1+u^2}. \tag{3.4}$$

This is also consistent with the established differentiation and integration formulas:

$$\mathbb{S}[f'(t)] = \frac{\mathbb{S}[f(t)] - f(0)}{u}, \tag{3.5}$$

$$\mathbb{S}\left[\int_0^t f(\tau)d\tau\right] = u\mathbb{S}[f(t)]. \tag{3.6}$$

In terms of applications, the LSD is further demonstrated by the following example. To obtain the solution,

$$x(t) = 1 - e^{-t}, \tag{3.7}$$

for the initial value problem,

$$\frac{dx}{dt} + x = 1, \quad x(0) = 0, \tag{3.8}$$

we traditionally resort to the Laplace transform to form the auxiliary equation, $F(s)(s+1) = 1/s$, from which we get $F(s) = 1/s(1+s) = 1/s - 1/(1+s)$, and then invert.

Alternatively, using the Sumudu transform, we use (3.5) to get the auxiliary equation, $[G(u)/u] + G(u) = 1$, which exhibits the transform $G(u)$:

$$G(u) = \mathbb{S}[x(t)] = \frac{u}{(u+1)} = 1 - \frac{1}{(1+u)}. \tag{3.9}$$

Once again, using entries (1) and (5) in Table 3.1, we get $x(t) = \mathbb{S}^{-1}[G(u)]$, as expected in (3.7).

Nevertheless, while it is well to rely on lists and tables of functions transforms, to find solutions to differential equations such as in the previous example, it is always a far superior position for practicing engineers and applied mathematicians to have available formula for the inverse transform (see Weerakoon [14]). Luckily, the LSD in (3.2) helps us to establish one such useful tool. Indeed, by virtue of the Cauchy theorem, and the residue theorem, the following is a Bromwich contour integration formula for the complex inverse Sumudu transform.

Theorem 3.1. *Let $G(u)$ be the Sumudu transform of $f(t)$ such that*

(i) *$G(1/s)/s$ is a meromorphic function, with singularities having* $\operatorname{Re}(s) < \gamma$, *and*

(ii) *there exists a circular region Γ with radius R and positive constants, M and K, with*

$$\left|\frac{G(1/s)}{s}\right| < MR^{-k}, \tag{3.10}$$

then the function $f(t)$, is given by

$$\mathbb{S}^{-1}[G(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} G\left(\frac{1}{s}\right)\frac{ds}{s} = \sum \textit{residues}\left[e^{st}\frac{G(1/s)}{s}\right]. \tag{3.11}$$

Proof. Let $F(s) = \$(f(t))$, and $G(u) = \mathbb{S}[f(t)]$ be the Laplace and Sumudu transforms of $f(t)$, respectively, then by the complex inversion formula for the Laplace transform (see for instance Spiegel [9, Chapter 7]), for $t > 0$, the function $f(t)$ is given by

$$f(t) = \$^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds. \tag{3.12}$$

Here s is comlex, $s = x + iy$, and the integration is performed in the complex plane along the line $s = \gamma$. The real number γ is chosen, albeit otherwise arbitrarily, to supersede the real part of any and all singularities of $F(s)$ (this is condition (i) in the theorem). This includes branch points, essential singularities, and poles.

In practice, the previous integral is computed in the light of a suitable Bromwich contour. The Bromwhich contour BC is composed of the segment $[A,B] = [\gamma - iT, \gamma + iT]$,

Table 3.1. Special Sumudu transforms.

	$f(t)$	$G(u) = \mathbb{S}(f(t))$
1	1	1
2	t	u
3	$\dfrac{t^{n-1}}{(n-1)!},\ n = 1,2,\ldots$	u^{n-1}
4	$\dfrac{t^{n-1}}{\Gamma(n)},\ n > 0$	u^{n-1}
5	e^{at}	$\dfrac{1}{1-au}$
6	$\dfrac{t^{n-1}e^{at}}{(n-1)!},\ n = 1,2,\ldots$	$\dfrac{u^{n-1}}{(1-au)^n}$
7	$\dfrac{t^{n-1}e^{at}}{\Gamma(n)}$	$\dfrac{u^{n-1}}{(1-au)^n}$
8	$\dfrac{\sin at}{a}$	$\dfrac{u}{1+a^2u^2}$
9	$\cos at$	$\dfrac{1}{1+a^2u^2}$
10	$\dfrac{e^{bt}\sin at}{a}$	$\dfrac{u}{(1-bu)^2+a^2u^2}$
11	$e^{bt}\cos at$	$\dfrac{1-bu}{(1-bu)^2+a^2u^2}$
12	$\dfrac{\sinh at}{a}$	$\dfrac{u}{1-a^2u^2}$
13	$\cosh at$	$\dfrac{1}{1-a^2u^2}$
14	$\dfrac{e^{bt}\sinh at}{a}$	$\dfrac{u}{(1-bu)^2-a^2u^2}$
15	$e^{bt}\cosh at$	$\dfrac{1-bu}{(1-bu)^2-a^2u^2}$
16	$\dfrac{e^{bt}-e^{at}}{b-a},\ a \neq b$	$\dfrac{u}{(1-bu)(1-au)}$
17	$\dfrac{be^{bt}-ae^{at}}{b-a},\ a \neq b$	$\dfrac{1}{(1-bu)(1-au)}$
18	$\dfrac{\sin at - at\cos at}{2a^3}$	$\dfrac{u^3}{(1+a^2u^2)^2}$
19	$\dfrac{t\sin at}{2a}$	$\dfrac{u^2}{(1+a^2u^2)^2}$
20	$\dfrac{\sin at + at\cos at}{2a}$	$\dfrac{u}{(1+a^2u^2)^2}$
21	$\cos at - \dfrac{1}{2}at\sin at$	$\dfrac{1}{(1+a^2u^2)^2}$
22	$t\cos at$	$\dfrac{u(1-a^2u^2)}{(1+a^2u^2)^2}$

Table 3.1. Continued.

	$f(t)$	$G(u)=\mathbb{S}(f(t))$
23	$\dfrac{at\cosh at-\sinh at}{2a^3}$	$\dfrac{u^3}{(1-a^2u^2)^2}$
24	$\dfrac{t\sinh at}{2a}$	$\dfrac{u^2}{(1-a^2u^2)^2}$
25	$\dfrac{\sinh at+at\cosh at}{2a}$	$\dfrac{u}{(1-a^2u^2)^2}$
26	$\cosh at+\dfrac{1}{2}at\sinh at$	$\dfrac{1}{(1-a^2u^2)^2}$
27	$t\cosh at$	$\dfrac{u(1+a^2u^2)}{(1-a^2u^2)^2}$
28	$\dfrac{(3-a^2t^2)\sin at-3at\cos at}{8a^5}$	$\dfrac{u^5}{(1+a^2u^2)^3}$
29	$\dfrac{t\sin at-at^2\cos at}{8a^3}$	$\dfrac{u^4}{(1+a^2u^2)^3}$
30	$\dfrac{(1+a^2t^2)\sin at-at\cos at}{8a^3}$	$\dfrac{u^3}{(1+a^2u^2)^3}$
31	$\dfrac{3t\sin at+at^2\cos at}{8a}$	$\dfrac{u^2}{(1+a^2u^2)^3}$
32	$\dfrac{(3-a^2t^2)\sin at+5at\cos at}{8a}$	$\dfrac{u}{(1+a^2u^2)^3}$
33	$\dfrac{(8-a^2t^2)\cos at-7at\sin at}{8}$	$\dfrac{1}{(1+a^2u^2)^3}$
34	$\dfrac{t^2\sin at}{2a}$	$\dfrac{u^3(3-a^2u^2)}{(1+a^2u^2)^3}$
35	$\dfrac{1}{2}t^2\cos at$	$\dfrac{u^2(1-3a^2u^2)}{(1+a^2u^2)^3}$
36	$\dfrac{1}{6}t^3\cos at$	$\dfrac{u^3(1-6a^2u^2+a^4u^4)}{(1+a^2u^2)^4}$
37	$\dfrac{t^3\sin at}{24a}$	$\dfrac{u^4(1-a^2u^2)}{(1+a^2u^2)^4}$
38	$\dfrac{(3+a^2t^2)\sinh at-3at\cosh at}{8a^5}$	$\dfrac{u^5}{(1-a^2u^2)^3}$
39	$\dfrac{at^2\cosh at-t\sinh at}{8a^3}$	$\dfrac{u^4}{(1-a^2u^2)^3}$

Table 3.1. Continued.

	$f(t)$	$G(u)=\mathbb{S}(f(t))$
40	$\dfrac{at\cosh at+(a^2t^2-1)\sinh at}{8a^3}$	$\dfrac{u^3}{(1-a^2u^2)^3}$
41	$\dfrac{3t\sinh at+at^2\cosh at}{8a}$	$\dfrac{u^2}{(1-a^2u^2)^3}$
42	$\dfrac{(3+a^2t^2)\sinh at+5at\cosh at}{8a}$	$\dfrac{u}{(1-a^2u^2)^3}$
43	$\dfrac{(8+a^2t^2)\cosh at+7at\sinh at}{8}$	$\dfrac{1}{(1-a^2u^2)^3}$
44	$\dfrac{t^2\sinh at}{2a}$	$\dfrac{u^3(3+a^2u^2)}{(1-a^2u^2)^3}$
45	$\dfrac{1}{2}t^2\cosh at$	$\dfrac{u^2(1+3a^2u^2)}{(1-a^2u^2)^3}$
46	$\dfrac{1}{6}t^3\cosh at$	$\dfrac{u^3(1+6a^2u^2+a^4u^4)}{(1-a^2u^2)^4}$
47	$\dfrac{t^3\sinh at}{24a}$	$\dfrac{u^4(1+a^2u^2)}{(1-a^2u^2)^4}$
48	$\dfrac{e^{at/2}}{3a^2}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}at}{2}-\cos\dfrac{\sqrt{3}at}{2}+e^{-3at/2}\right\}$	$\dfrac{u^2}{1+a^3u^3}$
49	$\dfrac{e^{at/2}}{3a}\left\{\cos\dfrac{\sqrt{3}at}{2}+\sqrt{3}\sin\dfrac{\sqrt{3}at}{2}-e^{-3at/2}\right\}$	$\dfrac{u}{1+a^3u^3}$
50	$\dfrac{1}{3}\left(e^{-at}+2e^{at/2}\cos\dfrac{\sqrt{3}at}{2}\right)$	$\dfrac{1}{1+a^3u^3}$
51	$\dfrac{e^{-at/2}}{3a^2}\left\{e^{3at/2}-\cos\dfrac{\sqrt{3}at}{2}-\sqrt{3}\sin\dfrac{\sqrt{3}at}{2}\right\}$	$\dfrac{u^2}{1-a^3u^3}$
52	$\dfrac{e^{-at/2}}{3a}\left\{\sqrt{3}\sin\dfrac{\sqrt{3}at}{2}-\cos\dfrac{\sqrt{3}at}{2}+e^{3at/2}\right\}$	$\dfrac{u}{1-a^3u^3}$
53	$\dfrac{1}{3}\left(e^{at}+2e^{-at/2}\cos\dfrac{\sqrt{3}at}{2}\right)$	$\dfrac{1}{1-a^3u^3}$
54	$\dfrac{1}{4a^3}(\sin at\cosh at-\cos at\sinh at)$	$\dfrac{u^3}{1+4a^4u^4}$
55	$\dfrac{1}{2a^2}\sin at\sinh at$	$\dfrac{u^2}{1+4a^4u^4}$
56	$\dfrac{1}{2a}(\sin at\cosh at+\cos at\sinh at)$	$\dfrac{u}{1+4a^4u^4}$
57	$\cos at\cosh at$	$\dfrac{1}{1+4a^4u^4}$

Table 3.1. Continued.

	$f(t)$	$G(u) = \mathbb{S}(f(t))$
58	$\frac{1}{2a^3}(\sinh at - \sin at)$	$\frac{u^3}{1-a^4u^4}$
59	$\frac{1}{2a^2}(\cosh at - \cos at)$	$\frac{u^2}{1-a^4u^4}$
60	$\frac{1}{2a}(\sinh at + \sin at)$	$\frac{u}{1-a^4u^4}$
61	$\frac{1}{2}(\cosh at + \cos at)$	$\frac{1}{1-a^4u^4}$
62	$\frac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$	$\frac{1}{\sqrt{u}[\sqrt{1+au}+\sqrt{1+bu}]}$
63	$\frac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$	$\frac{\sqrt{u}}{\sqrt{1+au}}$
64	$\frac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$	$\frac{\sqrt{u}}{1-au}$
65	$e^{at}\left\{\frac{1}{\sqrt{\pi t}} - be^{b^2t}\operatorname{erf}c(b\sqrt{t})\right\}$	$\frac{1}{\sqrt{u}(\sqrt{1-au}+b)}$
66	$J_0(at)$	$\frac{1}{\sqrt{1+a^2u^2}}$
67	$I_0(at)$	$\frac{1}{\sqrt{1-au^2}}$
68	$a^nJ_n(at);\ n>-1$	$\frac{(\sqrt{1+a^2u^2}-1)^n}{u^n\sqrt{1+au^2}}$
69	$a^nI_n(at);\ n>-1$	$\frac{(1-\sqrt{1-a^2u^2})^n}{u^n\sqrt{1-a^2u^2}}$
70	$J_0(a\sqrt{t(t+2b)})$	$\frac{e^{(b/u)\left(1-\sqrt{1+a^2u^2}\right)}}{\sqrt{1+a^2u^2}}$
71	$J_0(a\sqrt{t^2-b^2}),\ t>b$ $0,\ t<b$	$\frac{e^{(b/u)\sqrt{1+a^2u^2}}}{\sqrt{1+a^2u^2}}$
86	$\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)$	$1-e^{-a/\sqrt{u}}$
87	$\operatorname{erf}c\left(\frac{a}{2\sqrt{t}}\right)$	$e^{-a/\sqrt{u}}$
88	$e^{b(bt+a)}\operatorname{erf}c\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a/\sqrt{u}}}{1+b\sqrt{u}}$

Table 3.1. Continued.

	$f(t)$	$G(u)=\mathbb{S}(f(t))$
89	$\frac{1}{\sqrt{\pi t}a^{2n+1}}\int_0^\infty u^n e^{-u^2/4a^2t}J_{2n}(2\sqrt{u})du;\ n>-1$	$u^n e^{-a\sqrt{u}}$
90	$\frac{e^{-bt}-e^{-at}}{t}$	$\frac{1}{u}\ln\left(\frac{1+au}{1+bu}\right)$
91	$\mathrm{Ci}(at)$	$\frac{1}{2}\ln\left(\frac{1+a^2u^2}{a^2u^2}\right)$
92	$\mathrm{Ei}(at)$	$\ln\left(\frac{1+au}{au}\right)$
93	$\ln t$	$-\gamma+\ln u$
94	$\frac{2(\cos at-\cos bt)}{t}$	$\frac{1}{u}\ln\left(\frac{1+a^2u^2}{1+b^2u^2}\right)$
95	$\ln^2 t$	$\frac{\pi^2}{6}+(\gamma-\ln u)^2$
96	$\ln t+\gamma;\ \gamma=\text{Euler's constant}=0.5772156\ldots$	$-\ln u$
97	$(\ln t+\gamma)^2-\frac{1}{6}\pi^2;\ \gamma=\text{Euler's constant}=0.5772156\ldots$	$\ln^2 u$
98	$t^n\ln t;\ n>-1$	$u^n[\Gamma'(n+1)+\Gamma(n+1)\ln u]$
99	$\frac{\sin at}{t}$	$\frac{\tan^{-1}au}{u}$
100	$\mathrm{Si}(at)$	$\tan^{-1}au$
101	$\frac{1}{\sqrt{\pi t}}e^{-2\sqrt{at}}$	$\frac{1}{\sqrt{u}}e^{au}\,\mathrm{erf}\,c\sqrt{au}$
102	$\frac{2a}{\sqrt{\pi}}e^{-a^2t^2}$	$\frac{1}{u}e^{1/4a^2u^2}\,\mathrm{erf}\,c\left(\frac{1}{2au}\right)$
103	$\mathrm{erf}(at)$	$e^{1/4a^2u^2}\,\mathrm{erf}\,c\left(\frac{1}{2au}\right)$
104	$\frac{1}{\sqrt{\pi(t+a)}}$	$\frac{1}{\sqrt{u}}e^{a/u}\,\mathrm{erf}\,c\sqrt{\frac{a}{u}}$
105	$\frac{1}{t+a}$	$\frac{1}{u}e^{a/u}\,\mathrm{Ei}\left(\frac{a}{u}\right)$
106	$\frac{1}{t^2+a^2}$	$\frac{\cos(a/u)\{(\pi/2)-\mathrm{Si}(a/u)\}-\sin(a/u)\mathrm{Ci}(a/u)}{au}$
107	$\frac{t}{t^2+a^2}$	$\frac{\sin(a/u)\{(\pi/2)-\mathrm{Si}(a/u)\}+\cos(a/u)\mathrm{Ci}(a/u)}{u}$

Table 3.1. Continued.

	$f(t)$	$G(u)=\mathbb{S}(f(t))$
108	$\tan^{-1}\left(\frac{t}{a}\right)$	$\cos\left(\frac{a}{u}\right)\left\{\frac{\pi}{2}-\mathrm{Si}\left(\frac{a}{u}\right)\right\}-\sin\left(\frac{a}{u}\right)\mathrm{Ci}\left(\frac{a}{u}\right)$
109	$\frac{1}{2}\ln\left(\frac{t^2+a^2}{a^2}\right)$	$\sin\left(\frac{a}{u}\right)\left\{\frac{\pi}{2}-\mathrm{Si}\left(\frac{a}{u}\right)\right\}+\cos\left(\frac{a}{u}\right)\mathrm{Ci}\left(\frac{a}{u}\right)$
110	$\frac{1}{t}\ln\left(\frac{t^2+a^2}{a^2}\right)$	$\frac{[(\pi/2)-\mathrm{Si}(a/u)]^2+\mathrm{Ci}^2(a/u)}{u}$
111	$N(t)$	0
112	$\delta(t)$	$\frac{1}{u}$
113	$\delta(t-a)$	$\frac{e^{-a/u}}{u}$
114	$u(t-a)$	$e^{-a/u}$
115	$\frac{x}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{a}\cos\frac{n\pi t}{a}$	$\frac{\sinh(x/u)}{\sinh(a/u)}$
116	$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\sin\frac{(2n-1)\pi x}{2a}\sin\frac{(2n-1)\pi t}{2a}$	$\frac{\sinh(x/u)}{\cosh(a/u)}$
117	$\frac{t}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos\frac{n\pi x}{a}\sin\frac{n\pi t}{a}$	$\frac{\cosh(x/u)}{\sinh(a/u)}$
118	$1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\frac{(2n-1)\pi x}{2a}\cos\frac{(2n-1)\pi t}{2a}$	$\frac{\cosh(x/u)}{\cosh(a/u)}$
119	$\frac{xt}{a}+\frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\frac{n\pi x}{a}\sin\frac{n\pi t}{a}$	$\frac{u\sinh(x/u)}{\sinh(a/u)}$
120	$x+\frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2a}\cos\frac{(2n-1)\pi t}{2a}$	$\frac{u\sinh(x/u)}{\cosh(a/u)}$
121	$\frac{t^2}{2a}+\frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{a}\left(1-\cos\frac{n\pi t}{a}\right)$	$\frac{u\cosh(x/u)}{\sinh(a/u)}$
122	$t+\frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2a}\sin\frac{(2n-1)\pi t}{2a}$	$\frac{u\cosh(x/u)}{\cosh(a/u)}$
123	$\frac{1}{2}(t^2+x^2-a^2)-\frac{16a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}$ $\times\cos\frac{(2n-1)\pi x}{2a}\cos\frac{(2n-1)\pi t}{2a}$	$\frac{u^2\cosh(x/u)}{\cosh(a/u)}$
124	$\frac{2\pi}{a^2}\sum_{n=1}^{\infty}(-1)^n n e^{-n^2\pi^2 t/a^2}\sin\frac{n\pi x}{a}$	$\frac{\sinh(x/\sqrt{u})}{u\sinh(a/\sqrt{u})}$
125	$\frac{\pi}{a^2}\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)e^{-(2n-1)^2\pi^2 t/4a^2}\cos\frac{(2n-1)\pi x}{2a}$	$\frac{\cosh(x/\sqrt{u})}{u\cosh(a/\sqrt{u})}$

Table 3.1. Continued.

	$f(t)$	$G(u) = \mathbb{S}(f(t))$
126	$\frac{2}{a}\sum_{n=1}^{\infty}(-1)^{n-1}e^{-(2n-1)^2\pi^2 t/4a^2}\sin\frac{(2n-1)\pi x}{2a}$	$\frac{\sinh(x/\sqrt{u})}{\sqrt{u}\cosh(a/\sqrt{u})}$
127	$\frac{1}{a}+\frac{2}{a}\sum_{n=1}^{\infty}(-1)^{n}e^{-n^2\pi^2 t/a^2}\cos\frac{n\pi x}{a}$	$\frac{\cosh(x/\sqrt{u})}{\sqrt{u}\sinh(a/\sqrt{u})}$
128	$\frac{x}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t/a^2}\sin\frac{n\pi x}{a}$	$\frac{\sinh(x/\sqrt{u})}{\sinh(a/\sqrt{u})}$
129	$1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2 t/4a^2}\cos\frac{(2n-1)\pi x}{2a}$	$\frac{\cosh(x/\sqrt{u})}{\cosh(a/\sqrt{u})}$
130	$\frac{xt}{a}+\frac{2a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(1-e^{-n^2\pi^2 t/a^2})\sin\frac{n\pi x}{a}$	$\frac{u\sinh(x/\sqrt{u})}{\sinh(a/\sqrt{u})}$
131	$\frac{1}{2}(x^2-a^2)+t-\frac{16a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}e^{-(2n-1)^2\pi^2 t/4a^2}\cos\frac{(2n-1)\pi x}{2a}$	$\frac{u\cosh(x/\sqrt{u})}{\cosh(a/\sqrt{u})}$
132	$1-2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t/a^2}J_0(\lambda_n x/a)}{\lambda_n J_1(\lambda_n)}$ where $\lambda_1,\lambda_2,\ldots$ are the positive roots of $J_0(\lambda)=0$	$\frac{J_0(ix/\sqrt{u})}{J_0(ia/\sqrt{u})}$
133	$\frac{1}{4}(x^2-a^2)+t+2a^2\sum_{n=1}^{\infty}\frac{e^{-\lambda_n^2 t/a^2}J_0(\lambda_n x/a)}{\lambda_n^3 J_1(\lambda_n)}$ where $\lambda_1,\lambda_2,\ldots$ are the positive roots of $J_0(\lambda)=0$	$\frac{uJ_0(ix/\sqrt{u})}{J_0(ia/\sqrt{u})}$
134	t if $0\le t\le a$; $2a-t$ if $a<t<2a$; $f(t+2a)=f(t)$	$u\tanh\left(\frac{a}{2u}\right)$
135	1 if $0<t<a$; -1 if $a<t<2a$; $f(t+2a)=f(t)$	$\tanh\left(\frac{a}{2u}\right)$
136	$\left\lvert\sin\left(\frac{\pi t}{a}\right)\right\rvert$	$\frac{\pi a u}{a^2+\pi^2u^2}\coth\left(\frac{a}{2u}\right)$
137	$\sin\left(\frac{\pi t}{a}\right)$ if $0\le t\le a$; 0 if $a<t<2a$; $f(t+2a)=f(t)$	$\frac{\pi a u}{(a^2+\pi^2u^2)(1-e^{-a/u})}$
138	$\frac{t}{a}$, $0\le t\le a$, $f(t+a)=f(t)$	$\frac{u}{a}-\frac{e^{-a/u}}{1-e^{-a/u}}$
139	0 if $0\le t<a$; 1 if $t\ge a$; $f(t+2a)=f(t)$	$e^{-a/u}$

Table 3.1. Continued.

	$f(t)$	$G(u) = \mathbb{S}(f(t))$
140	$1; \quad a \le t \le a+\varepsilon$	$e^{-a/u}(1-e^{-\varepsilon/u})$
141	$n+1, \quad na \le t < (n+1)a,\ n = 0,1,2,\dots$	$\dfrac{1}{1-e^{-a/u}}$
142	$n^2, \quad n \le t < n+1,\ n = 0,1,2,\dots$	$\dfrac{e^{-1/u}+e^{-2/u}}{(1-e^{-1/u})^2}$
143	$r^n, \quad n \le t < n+1,\ n = 0,1,2,\dots$	$\dfrac{1-e^{-1/u}}{1-re^{-1/u}}$
144	$\sin\left(\dfrac{\pi t}{a}\right) \quad \text{if } 0 \le t \le a$ $0 \quad \text{if } t > a$	$\dfrac{\pi a u(1+e^{-a/u})}{a^2+\pi^2u^2}$

of the line $s = \gamma$, and the complementing arc Γ of a circle centred at the origin, with radius R, such that $T = \sqrt{R^2 - \gamma^2}$. Simply put, we have

$$\mathrm{BC} = [A,B] \cup \Gamma. \tag{3.13}$$

In all cases, including infinitely many singularities for $F(s)$, a Bromwich contour can be modified to accommodate all situations. Hence and without loss of generality, assuming that the only singularities of $F(s)$ are poles, all of which lie to the left of the real line $s = \gamma$, and that condition (ii) holds, insures that (see Spiegel [9, Theorem 7.1])

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma} e^{st} F(s)\,ds = 0, \tag{3.14}$$

and consequently that

$$f(t) = \$^{-1}[F(s)] = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\,ds = \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{\mathrm{BC}} e^{st} F(s)\,ds. \tag{3.15}$$

Therefore, by virtue of the residue theorem, we have

$$f(t) = \$^{-1}[F(s)] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,ds = \sum \text{residues } [e^{st} F(s)]. \tag{3.16}$$

Finally, by invoking into the previous relation, the LSD between the transforms F and G, namely that

$$F(s) = \frac{G(1/s)}{s}, \tag{3.17}$$

we get the desired conclusion of the theorem (3.10).

Obviously, the previous theorem can be readily applied to the function in (3.9), to get the solution in (3.7) for (3.8). Indeed, one easily recognizes that the residues of $e^{st}/s(s+1)$ do occur at the poles $s=-1$ and $s=0$, with respective values $-e^{-t}$ and 1. □

4. Sumudu theorems for multiple differentiations, integrations, and convolutions

In Belgacem et al. [5], this next theorem was proved by virtue of the LSD between the Sumudu and Laplace transform. While we include it to make this paper autonomous, here we use an induction argument to prove the result.

Theorem 4.1. *Let $f(t)$ be in A, and let $G_n(u)$ denote the Sumudu transform of the nth derivative, $f^{(n)}(t)$ of $f(t)$, then for $n \geqslant 1$,*

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}. \tag{4.1}$$

Proof. For $n=1$, (3.5) shows that (4.1) holds. To proceed to the induction step, we assume that (4.1) holds for n and prove that it carries to $n+1$. Once more by virtue of (3.5), we have

$$\begin{aligned} G_{n+1}(u) = \mathbb{S}\left[\left(f^{(n)}(t)\right)'\right] &= \frac{\mathbb{S}\left[f^{(n)}(t)\right] - f^{(n)}(0)}{u} \\ &= \frac{G_n(u) - f^{(n)}(0)}{u} = \frac{G(u)}{u^{n+1}} - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{u^{n+1-k}}. \end{aligned} \tag{4.2}$$

In particular, this means that the Sumudu transform, $G_2(u)$, of the second derivative of the function, $f(t)$, is given by

$$G_2(u) = \mathbb{S}\left(f''(t)\right) = \frac{G(u) - f(0)}{u^2} - \frac{f'(0)}{u}. \tag{4.3}$$

For instance, the general solution of the second-order equation,

$$\frac{d^2 y(t)}{dt^2} + w^2 y(t) = 0, \tag{4.4}$$

can easily be transformed into its Sumudu equivalent,

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} - w^2 G(u) = 0, \tag{4.5}$$

with general Sumudu solution,

$$G(u) = \frac{y(0) + u y'(0)}{1 + w^2 u^2}, \tag{4.6}$$

and upon inverting, by using Theorem 3.1 (or see Table 3.1), we get the general time solution:

$$y(t) = y(0)\cos(wt) + \frac{y'(0)}{w}\sin(wt). \tag{4.7}$$

Obviously, Theorem 4.1 shows that the Sumudu transform can be used just like the Laplace transform, as in the previous example, to solve both linear differential equations of any order. The next theorem allows us to use the Sumudu transform as efficiently to solve differential equations involving multiple integrals of the dependent variable as well, by rendering them into algebraic ones. □

THEOREM 4.2. *Let $f(t)$ be in A, and let $G^n(u)$ denote the Sumudu transform of the nth antiderivative of $f(t)$, obtained by integrating the function, $f(t)$, n times successively:*

$$W^n(t) = \iint_0^t \cdots \int_0^t f(\tau)(d\tau)^n, \tag{4.8}$$

then for $n \geqslant 1$,

$$G^n(u) = \mathbb{S}(W^n(t)) = u^n G(u). \tag{4.9}$$

Proof. For $n = 1$, (3.6) shows that (4.9) holds. To proceed to the induction step, we assume that (4.9) holds for n, and prove it carries to $n+1$. Once more, by virtue of (3.6), we have

$$G^{n+1}(u) = \mathbb{S}(W^{n+1}(t)) = \mathbb{S}\left[\int_0^t W^n(\tau)d\tau\right] = u\mathbb{S}[W^n(t)] = u[u^n G(u)] = u^{n+1}G(u). \tag{4.10}$$

This theorem generalizes the Sumudu convolution Theorem 4.1 in Belgacem et al. [5], which states that the transform of

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \tag{4.11}$$

is given by

$$\mathbb{S}((f * g)(t)) = uF(u)G(u). \tag{4.12}$$

□

COROLLARY 4.3. *Let $f(t), g(t), h(t), h_1(t), h_2(t), \ldots$, and $h_n(t)$ be functions in A, having Sumudu transforms, $F(u), G(u), H(u), H_1(u), H_2(u), \ldots$, and $H_n(t)$, respectively, then the Sumudu transform of*

$$(f * g)^n(t) = \iint_0^t \cdots \int_0^t f(\tau)g(t-\tau)(d\tau)^n \tag{4.13}$$

is given by

$$\mathbb{S}((f * g)^n(t)) = u^n F(u)G(u). \tag{4.14}$$

Moreover, for any integer $n \geqslant 1$,

$$\mathbb{S}[(h_1 * h_2 * \cdots * h_n)(t)] = u^{n-1}H_1(u)H_2(u)\cdots H_n(u). \tag{4.15}$$

In particular, the Sumudu transform of $(f * g * h)$, with f, g, h in A, is given by

$$\mathbb{S}[(f * g * h)(t)] = u^2 F(u)G(u)H(u). \tag{4.16}$$

Proof. Equation (4.14) is just a direct consequence of Theorem 4.2 due to the property of associativity of the convolution operator, (4.12), that implies (4.15). Finally, (4.16) is just an implication of (4.15) with $n = 3$.

The previous results can be used in a powerful manner to solve integral, differential, and integrodifferential equations. Such applications can be found in the indicated references in the literature review in the paper introduction section. In particular, we remind the reader that Belgacem et al. [5] used such results to solve convolution type equations. □

5. Sumudu multiple shift theorems

The discrete Sumudu transform can be used effectively to discern some rules on how the general transform affects various functional operations. Based on what was already established in Belgacem et al. [5] such as (see Table 5.1), we have

$$\mathbb{S}[tf'(t)] = u\frac{dG(u)}{du}, \tag{5.1}$$

and that

$$\mathbb{S}[t\exp(t)] = \frac{u}{(1-u)^2}, \tag{5.2}$$

one may ask how the Sumudu transform acts on $t^n f(t)$. Clearly, if $f(t) = \sum_{n=0}^{\infty} a_n t^n$, then

$$\mathbb{S}[tf(t)] = \sum_{n=0}^{\infty}(n+1)!a_n u^{n+1} = u\sum_{n=0}^{\infty}(n+1)!a_n u^n = u\frac{d}{du}\sum_{n=0}^{\infty} n!a_n u^{n+1}. \tag{5.3}$$

This result helps us first answer this question when $n = 1$ (also see Belgacem et al. [5]).

Theorem 5.1. *Let $G(u)$ be the Sumudu transform of the function $f(t)$ in A, then the Sumudu transform of the function $tf(t)$ is given by*

$$\mathbb{S}[tf(t)] = u\frac{d(uG(u))}{du}. \tag{5.4}$$

Proof. The function, $tf(t)$, is in A, since f is so; and by integration by parts, (5.1) implies

$$\begin{aligned}\mathbb{S}[tf(t)] &= \int_0^{\infty} utf(ut)e^{-t}dt = u\int_0^{\infty}\frac{d}{dt}[tf(ut)]e^{-t}dt - u[tf(ut)e^{-t}]_0^{\infty}\\ &= u\int_0^{\infty}\frac{d}{dt}[tf(ut)]e^{-t}dt = u\int_0^{\infty}[f(ut)+utf'(ut)]e^{-t}dt\\ &= u[G(u)+u\frac{d}{du}G(u)] = u\frac{d[uG(u)]}{du}.\end{aligned} \tag{5.5}$$

□

Table 5.1. Basic Sumudu transform properties.

Formula	Comment
$G(u)=\mathbb{S}(f(t))=\int_0^\infty f(ut)e^{-t}dt,\ -\tau_1<u<\tau_2$	Definition of Sumudu transform for $f\in A$
$G(u)=\dfrac{F(1/u)}{u},\ F(s)=\dfrac{G(1/s)}{s}$	Duality with Laplace transform
$\mathbb{S}[af(t)+bg(t)]=a\mathbb{S}[f(t)]+b\mathbb{S}[g(t)]$	Linearity property
$G_1(u)=\mathbb{S}[f'(t)]=\dfrac{G(u)-f(0)}{u}=\dfrac{G(u)}{u}-\dfrac{f(0)}{u}$ $G_2(u)=\mathbb{S}[f''(t)]=\dfrac{G(u)}{u^2}-\dfrac{f(0)}{u^2}-\dfrac{f'(0)}{u}$ $G_n(u)=\mathbb{S}[f^{(n)}(t)]=\dfrac{G(u)}{u^n}-\dfrac{f(0)}{u^n}-\cdots-\dfrac{f^{(n-1)}(0)}{u}$	Sumudu transforms of function derivatives
$\mathbb{S}\left[\int_0^t f(\tau)d\tau\right]=uG(u)$	Sumudu transform of an integral of a function
$\mathbb{S}[f(at)]=G(au)$	First scale preserving theorem
$\mathbb{S}\left(t\dfrac{df(t)}{dt}\right)=u\dfrac{dG(u)}{du}$	Second scale preserving theorem
$\mathbb{S}[e^{at}f(t)]=\dfrac{1}{1-au}G\left(\dfrac{u}{1-au}\right)$	First shifting theorem
$\mathbb{S}[f(t-a)H(t-a)]=e^{-a/u}G(u)$	Second shifting theorem
$\mathbb{S}\left[\dfrac{1}{t}\int_0^t f(\tau)d\tau\right]=\dfrac{1}{u}\int_0^t f(v)dv$	Average preserving theorem
$\lim_{u\to 0}G(u)=\lim_{t\to 0}f(t)$	Initial value theorem
$\lim_{u\to\pm\infty}G(u)=\lim_{t\to\pm\infty}f(t)$	Final values theorem
$\mathbb{S}(f(t))=\dfrac{\int_0^{T/u}f(ut)e^{-t}dt}{1-e^{-T/u}}$	Sumudu transform of a T-periodic function
$\mathbb{S}(f*g)=u\mathbb{S}(f(t))\mathbb{S}(g(t))$ $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$	Sumudu convolution theorem

This leads us to an induction argument about the transform of the product of f with a positive integer power of t.

Theorem 5.2. *Let $G(u)$ denote the Sumudu transform of $f(t)$ in A, and let $G_k(u)$ denote the kth derivative of $G(u)$ with respect to u, then the Sumudu transform of the function, $t^n f(t)$, is given by*

$$\mathbb{S}[t^n f(t)]=u^n\sum_{k=0}^{n}a_k^n u^k G_k(u), \tag{5.6}$$

where $a_0^n = n!$, $a_n^n = 1$, $a_1^n = n!n$, $a_{n-1}^n = n^2$, *and for* $k = 2,3,\ldots,n-2$,

$$a_k^n = a_{k-1}^{n-1} + (n+k)a_k^{n-1}. \tag{5.7}$$

Proof. Let $G(u)$ be the Sumudu transform of $f(t)$, then $a_0^0 = 1$. Furthermore, Theorem 5.1 shows that the transform of $tf(t)$ is $uG + u^2G_1$, hence $a_0^1 = 1$ and $a_1^1 = 1$ as expected. To perform the induction step, we assume that (5.7) holds for n, and show it carries to $n+1$. Setting $W(u) = \mathbb{S}[t^n f(t)]$, then W is given by (5.6) with coefficients in (5.7), and by Theorem 5.1, we have

$$\begin{aligned}
\mathbb{S}[t^{n+1}f(t)] &= \mathbb{S}[t[t^n f(t)]] = u\frac{d(uW(u))}{du} = uW(u) + u^2\frac{dW(u)}{du} \\
&= uW(u) + u^2W_1(u) = u^{n+1}\sum_{k=0}^{n} a_k^n u^k G_k(u) + u^2\frac{d}{du}\sum_{k=0}^{n} a_k^n u^{n+k}G_k(u) \\
&= u^{n+1}\sum_{k=0}^{n} a_k^n u^k G_k(u) + u^2\sum_{k=0}^{n} a_k^n\left[(n+k)u^{n+k-1}G_k(u) + u^{n+k}G_{k+1}(u)\right] \\
&= u^{n+1}\sum_{k=0}^{n} a_k^n u^k G_k(u) + u^{n+1}\sum_{k=0}^{n} a_k^n\left[(n+k)u^k G_k(u) + u^{k+1}G_{k+1}(u)\right] \\
&= u^{n+1}\sum_{k=0}^{n}(n+k+1)a_k^n u^k G_k(u) + u^{n+1}\sum_{k=0}^{n} a_k^n u^{k+1}G_k(u).
\end{aligned} \tag{5.8}$$

Noting that for $k < 0$, or $k > n$, $a_k^n = 0$, we can rewrite the previous equation as

$$\begin{aligned}
\mathbb{S}[t^{n+1}f(t)] &= u^{n+1}\sum_{k=0}^{n+1}(n+k+1)a_k^n u^k G_k(u) + u^{n+1}\sum_{k=0}^{n+1} a_{k-1}^n u^k G_k(u) \\
&= u^{n+1}\sum_{k=0}^{n+1}\left[(n+k+1)a_k^n + a_{k-1}^n\right]u^k G_k(u) = u^{n+1}\sum_{k=0}^{n+1} a_k^{n+1}u^k G_k(u).
\end{aligned} \tag{5.9}$$

In particular, from (5.7), we observe that the coefficient a_{n-1}^n, initialized at $a_0^1 = 1 = 1^2$, satisfies

$$a_{n-1}^n = a_{n-2}^{n-1} + (2n-1)a_{n-1}^{n-1} = a_{n-2}^{n-1} + (2n-1), \tag{5.10}$$

which is the same as the relation consecutive squares of integers, namely,

$$n^2 = (n-1)^2 + (2n-1). \tag{5.11}$$

Theorem 5.2 generalizes to an arbitrary positive integer n, Asiru [1, Theorem 2.5] establishing cases $n = 1$ & $n = 2$. Furthermore, Theorem 5.2 establishes a recurrence relation that predicts the coefficients a_k^n for any feasible nonnegative integer pair (n,k). The next table shows all coefficients of those values for $n = 0, 1, 2, 3, 4$, and 5.

$n\backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	2	4	1			
3	6	18	9	1		
4	24	96	72	16	1	
5	120	600	600	200	25	1

(5.12)

For $n = 6$, we can use the table last row and (5.7) to get the coefficients a_k^6:
$a_0^6 = 720$, $a_1^6 = 4320$, $a_2^6 = 5400$, $a_3^6 = 2400$, $a_4^6 = 450$, $a_5^6 = 6^2 = 36$, and $a_6^6 = 1$. □

Theorem 5.3. *Let $G(u)$ denote the Sumudu transform of the function $f(t)$ in A, let $f^{(n)}(t)$ denote the nth derivative of $f(t)$ with respect to t, and let $G_n(u)$ denote the nth derivative of $G(u)$ with respect to u, then the Sumudu transform of the function, $t^n f^{(n)}(t)$ is given by*

$$\mathbb{S}\left[t^n f^{(n)}(t)\right] = u^n G_n(u). \tag{5.13}$$

Proof. Being the Sumudu transform of $f(t)$,

$$G(u) = \int_0^\infty f(ut)e^{-t}dt. \tag{5.14}$$

Therefore, for $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} G_n(u) &= \int_0^\infty \frac{d^n}{du^n} f(ut)e^{-t}dt = \int_0^\infty t^n f^{(n)}(ut)e^{-t}dt \\ &= \frac{1}{u^n}\int_0^\infty (ut)^n f^{(n)}(ut)e^{-t}dt = \frac{1}{u^n}\mathbb{S}\left[t^n f^{(n)}(t)\right]. \end{aligned} \tag{5.15}$$

Upon multiplying both sides of the previous equation by u^n, we obtain (5.13). □

Now, for low values of n, combining Theorems 5.2 and 5.3 yields the following.

Corollary 5.4. *Let $G_n(u)$ denote the nth derivative of $G(u) = \mathbb{S}[(t)]$, then,*

$$\begin{aligned} \mathbb{S}\left[t^2 f'(t)\right] &= u^2\left[2G_1(u) + uG_2(u)\right], \\ \mathbb{S}\left[t^3 f'(t)\right] &= u^3\left[6G_1(u) + 6uG_2(u) + u^2G_3(u)\right], \\ \mathbb{S}\left[t^4 f''(t)\right] &= u^4\left[12G_2(u) + 8uG_3(u) + u^2G_4(u)\right]. \end{aligned} \tag{5.16}$$

The previous results in this section are likely to build up to a global Sumudu shift theorem about $\mathbb{S}[t^n f^{(m)}(t)]$. Furthermore, patterns in Theorem 5.2, and Corollary 5.4, seem to exude beautiful connections with Stirling-like numbers, or some of their generalized versions (see, for instance, Belgacem [4], and Merris [7] and their references).

6. Applications to differential equations

Theorem 5.3 does indicate that the general Euler equation,

$$\sum_{k=0}^{n} c_k u^k y^{(k)}(t) = R(t), \tag{6.1}$$

with second-order prototype,

$$t^2 \frac{d^2 y(t)}{dt^2} + bt \frac{dy(t)}{dt} + ay(t) = R(t), \tag{6.2}$$

is kept unchanged by the Sumudu transform if the equation right hand side, $R(t)$, is a linear function. Hence, there seems to be no direct advantage in using the Sumudu transform in this case, except possibly if there is a discernible benefit in transforming particular forms of $R(t)$. Of course, just like with the Laplace transform, the Sumudu transform can always be indirectly applied to such equations after effecting the change of variable, $t = \exp(x)$, which for instance renders (6.2) into the constant coefficients linear nonhomogeneous equation of second order:

$$\frac{d^2 y(x)}{dx^2} + (b-1) \frac{dy(x)}{dx} + ay(x) = R(x). \tag{6.3}$$

On the other hand, much more mileage may be obtained from the Sumudu transform in such instances if we recall that since the Sumudu transform preserves units, in applications the arguments t and u can be used "somewhat interchangeably." So, the paradigm shift in terms of transform techniques usage here is to possibly look at the given equation as the Sumudu transform of some other equation, and look for the inverse sumudu transform of the equation, rather than the other way around. To illustrate this idea by a simple a priori cooked up example. Consider the second-order Euler differential equation

$$2H(w) + 4wH'(w) + w^2 H''(w) = 2. \tag{6.4}$$

Clearly, one obvious solution of (6.4) is $H(w) = 1$. As stated above, it is futile to try to solve the equation by directly Sumudu transforming as it will remain unchanged. Instead, may be we should find a suitable format for the equation so that it can be readily inverted. This comes down to a sort of finding a suitable (not integrating) but an "inverting factor" for the problem.

Recalling that taking $n = 2$, in Theorem 5.2 (see row $n = 2$ in the a_k^n coefficients table in the previous section), and setting $H(w) = \mathbb{S}[h(s)]$, in (6.4), yields

$$\mathbb{S}[s^2 h(s)] = w^2 [2H(w) + 4wH_1(w) + w^2 H_2(w)] = 2w^2 = \mathbb{S}[s^2] \tag{6.5}$$

Clearly in this case, up to null functions, $H(w) = h(s) = 1$.

At the first glance, the triviality of the example may look like an apparent let-down by this technique of Sumudu transform usage, but this is not really the case as solving the example was not the issue here, as it was mainly set up to convey the following idea. The main point here is that unlike other transforms, the units-preservation property in combination with other properties of the Sumudu transform may allow us, according to the situation at hand, to transform the equation studied from the t domain to the u domain if the obtained equation is believed to be more accessible; or if necessary to consider the given equation as the Sumudu transform of another more readily solvable equation in the t domain, begotten by u inverse Sumudu transforming the equation at hand. So, the Sumudu transform may be used either way, but the successful usage remains dependent on the astuity of the user and familiarity with the properties and technical rules governing the behavior of this transform.

Seeking the discovery and establishment of such governing rules, and finding interesting applications, especially where other techniques (such as using the Laplace transform) may not be adequate or turn out to be more cumbersome, remains our quest. Towards this goal, for which we hope the present effort is another step in the right direction, convergence, albeit still slightly elusive, now seems to be highly feasible. This work, which uncovered many characteristic features of an expanding puzzle, is due to illiciting the spring-up of many more theoretical investigations, various mathematical interconnections, and ramifications.

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