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Super-Chern–Simons Theory as Superstring Theory

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Superstrings and topological strings with supermanifolds as target space play a central role in the recent developments in string theory. Nevertheless the rules for higher-genus computations are still unclear or guessed in analogy with bosonic and fermionic strings. Here we present a common geometrical setting to develop systematically the prescription for amplitude computations. The geometrical origin of these difficulties is the theory of integration of superforms. We provide a translation between the theory of supermanifolds and topological strings with supertarget space. We show how in this formulation one can naturally construct picture changing operators to be inserted in the correlation functions to soak up the zero modes of commuting ghost and we derive the amplitude prescriptions from the coupling with an extended topological gravity on the worldsheet. As an application we consider a simple model on $\mathbf{R}^{(3|2)}$ leading to super-Chern-Simons theory.

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1. Introduction

The interest on superstrings and topological strings with supermanifold as target space has increased during the last months. This was mainly due to the progresses in formulating the superstrings in 10 dimensions with manifest super-Poincaré invariance [1,2,3] and in formulating 4d $\mathcal{N} = 4$ SYM in the supertwistor space [4,5,6]. The pure spinor appoach allows the consistent quantization of superstrings for any generic background, for example with Ramond-Ramond fields such as the celebrated $AdS_5 \times S^5$ (see for example [7] for a worldsheet formulation), while the twistor formulation allows a direct comparison between Feynman diagram computations in quantum field theory and the correlation functions of a topological B model. The common ground for these models is the super-target space and the problems of constructing the amplitudes have the same geometrical origin: the theory of integration of superforms on supermanifolds.

The study of supermanifolds as string theory target space has to be traced back to the original papers by M. Green and J. Schwarz [8] where a sigma model for 10d superstring with manifest supersymmetry was formulated. Later, A. Schwarz *et al.* started the analyis of topological sigma models with target space as supermanifolds [9]. More recently, mirror symmetry has given a new acceleration to the analysis of super target spaces [6,10,11,12]. Indeed, the formulation of mirror symmetry [13,14] based on Landau-Ginzburg models directly leads to supermanifolds. This is due to the fact that the superpotential of non-linear Landau-Ginzburg theory must be modified by introducing some ghost superfields. These are twisted fermion fields which, together with the bosonic superfields, parametrize a supermanifold and the problem of integration on such manifolds emerges again.

For a bosonic manifold $\mathcal{M}$, the theory of integration is related to the theory of anti-symmetric tensors in the cotangent bundle $T^*\mathcal{M}$. Indeed, the measure for a given simplex is obtained by constructing the top form of the submanifold. However, this cannot be naively generalized to superspaces $\hat{\mathcal{M}}$. The naive space of superforms $\Omega^*\hat{\mathcal{M}}$ can be locally factorized (if the supermanifold is split, *i.e.* is obtained by the total space of a vector bundle by changing the parity of the fibres) into $\Omega^*\mathcal{M} \oplus C^\infty(\Omega^*\mathcal{N})$ where $\mathcal{M}$ is the bosonic body of the supermanifold $\hat{\mathcal{M}}$ and $\mathcal{N}$ is the fermionic extension. Although these superforms are polynomial in the anticommuting coordinated dx^m, they are functions of the commuting superform $d\theta^\alpha$ (we choose the set of coordinates (x^m, θ^α) on $\hat{\mathcal{M}}$ which is locally $\mathcal{M} \times \mathcal{N}$). In the purely bosonic case the form degree can only be equal or less than the dimension of the manifold and the top form transforms as a measure under smooth, orientation preserving coordinates transformations. This allows one to integrate the top forms over the oriented manifold. On the other hand, superforms may have any form degree and none of them transforms as a Berezinian measure. Some generalizations have been proposed by Bernstein and Leites [15] and they introduced the concept of pseudoforms as distributions on $\Omega^*\mathcal{M} \oplus C^\infty(\Omega^*\mathcal{N})$ (in the literature they are also called integral forms, see for example book [16]). But how is this related to superstrings and topological strings on supermanifolds?

In the pioneering paper [17] Friedan, Martinec and Shenker in the case of fermionic strings (superstrings with worldsheet fermions and 2d local supersymmetry) pointed out that one has to insert into correlation functions some suitable BRST-closed operators to soak up the zero modes of bosonic ghosts – the superghosts of 2d supergravity. They indeed conjectured that if one considers the superspace version of the worldsheet (adding some fermionic coordinates) the superghosts can be viewed as the differentials of the fermionic coordinates and therefore the insertions, known also as Picture Changing Operators (PCO), were needed in order to make a sensible integration theory for superforms (the vertex operators) on the super-worldsheet. Later, it was further recognized [18,19,20] that the gauge fixing of the gravitino of the 2d superconformal gravity was directly related to the choice of those PCO insertions. The need of a gauge choice for the gravitino is particularly important in the computations at higher genus [21,22].

Fermionic strings is not the only σ-model yielding a supersymmetric string theory. The Green-Schwarz formalism indeed provides an example of a sigma model with supersymmetry in the target space and this can be quantized using pure spinor formalism (see [23] for a pedagogical account). In that framework, the construction of higher genus Riemann surface amplitudes requires PCO to reabsorb the zero modes of the commuting ghosts [23]. Recently in [24,25], it has been showed that rules similar to those used in [23] can be employed to provide a loop expansion in the point-particle limit of string theory and for a particle description of supergravity. In this case, the PCO insertions are not derived from a gauge fixing of some worldsheet gauge fields, and they are motivated by symmetries (BRST symmetry) and ghost anomaly cancellation. However, the bosonic ghosts λ^α of the pure spinor formalism are BRST partners of the fermionic coordinates θ^α of the target space and this suggests that λ^α are the differential of θ^α where the role of the de Rham differential is played by the BRST charge. The path integral over the zero modes of θ^α and λ^α can then be viewed as an integral of superforms. Following the same logic, the ghosts of the fermionic coordinates in supertwistor space $\mathbf{CP}^{3|4}$ must require corresponding PCO in the amplitudes. Indeed in this context the analogy is rather close as pointed out in footnote 12 of [4].

The connection between PCO and supergeometry is studied in papers [26,27,28]. There the author provides a bridge between the formalism for integration on supermanifolds [9,29,30,31] and the PCO. However, the application of his formalism to target space supermanifold in the context of superparticle and topological strings is lacking and it is discussed in the present work. We should also mention the work by Sethi [10] where some brief comments on integration on supermanifolds were made.

In order to clarify these issues, we consider a simplified model: we take into account a topological worldline model – Sec. 2.1 – (see [32] for a pedagogical account) with the supermanifold $\mathbf{R}^{3|2}$ as target space (this is the easiest supersymmetric generalization of the model taken into consideration by Witten in [33]; it has N=1 supersymmetry in the target space parametrized by x^m with $m = 0, 1, 2$ and θ^α with $\alpha = 1, 2$. The latter is a Majorana spinor in 3d). This model can be identified with the open sector of a topological A model on the worldsheet; in this context, $\mathbf{R}^{3|2}$ is a Lagrangian submanifold which defines the boundary conditions for open strings – see Sec. 2.2. The particle model only deals with the zero modes of the worldsheet theory. The choice of a superparticle model instead of a superstring is due to our interest in the zero modes of the theory and we therefore neglect normal ordering problems and other worldsheet details. A similar analysis is performed in [34]. The target space theory is supersymmetric Chern-Simons theory which is essentially given by the usual Chern-Simons theory of the a 3-manifold plus a mass term for non-dynamical fermions [35,36,37]. It is a supersymmetric model and the equations of motion are given by null curvatures plus vanishing fermions. The quantization is performed along the lines of [3,38] extending the work of [39] by relaxing the pure spinor constraints. In Sec. 3, we discuss the cohomology of the model at any ghost number at zero and at non-zero momentum. The zero momentum cohomology is then used to characterize the tree level measure and to derive the target space action directly from Witten's string-field theory as in [33].

The relation between the cohomology at different ghost numbers and the Batalin-Vilkovisky formalism for the off-shell theory was discovered in [40,41] and applied to RNS formalism in [42]. In those papers, the zero momentum cohomology at the highest ghost number provides a meaningful integration measure for tree level correlation functions. In the present work (see Sec. 3.3), we compute the highest (in the present case is 3) ghost number cohomology group and we found out that there is a unique element which is a polynomial in the ghost fields and in the fermion coordinates. In terms of this polynomial we derive the tree-level measure. The resulting measure is BRST invariant and it contains enough Dirac delta functions to reabsorb the zero modes as expected. As discussed above, those delta functions should appear as insertions by means of PCO in order to guarantee the gauge invariance and the supersymmetry of the model. For that purpose in Sec. 4, we derive the PCO from a complete geometrical method based on supergeometry and we show that in the space of pseudoforms the top form coincides with the measure computed with the zero momentum cohomology. This points out the dependence of the PCO on the gauge choice, and it illustrates some aspects of the formula for higher genus supestring correlators given in [43]. Indeed, we clarify the relation between the zero momentum cohomology and the path integral measure by identifying the correct top form to be integrated, and we develop the Cartan calculus for the supergeometry. We show that a gauge fixed bosonization formula for the commuting ghosts leads to the construction of PCO and the picture number operator (as will be defined later in Sec. 4.6). We show that the anomaly of the ghost number and of the picture number is saturated by the measure constructed in Sec. 5. There, we finally give the prescription for multiloop computations with any number of external states. In Sec. 5.1, we derive the insertions computed in Sec. 5 by coupling the model to an extended topological gravity on the worldline. There are essentailly two symmetries that are important: the Virasoro constraint $P^2 \sim 0$ and the κ-symmetry constraints $\not{P}d \sim 0$. We derive both the insertions for composite antighost field B and for the picture changing operators. In app. A, we collected very few basic ingredients of Batalin-Vilkovisky formalism used in the text.

2. Actions and BRST Symmetry

2.1. Worldline action and BRST symmetry

We consider the Liouville action (the action of the general form $\int Pdq$) on the worldline to which gauge fields are added[3]

$$S = \int d\tau \Big[P_m \Big(\dot{x}^m - l^m(\tau) \Big) - p_\alpha \Big(\dot{\theta}^\alpha - \Lambda^\alpha(\tau) \Big) \Big] \,, \tag{2.1}$$

[3] The action in (2.1) is closely related to the action provided in [44,45] with the main difference that, here, it is supersymmetrized in the target space and the conjugate p_α is added to the theory. The latter is an independent degree of freedom. One can also understand the action in (2.1) as a superparticle moving on a supermanifolds as will be discussed at length in sec. (2.2).

Here x^m are the bosonic coordinates (we consider D-dimensional target space manifolds where $D = (d, 1)$ and $d = 2, 3, 5, 9$. In those particular cases we have the Fierz identities $\eta_{mn}\gamma^n_{(\alpha\beta}\gamma^m_{\gamma\delta)} = 0$ for the Dirac matrices of the corresponding spaces). The coordinates of the corresponding superspaces are (x^m, θ^α) where the index α runs over $\alpha = 1, 2$ for $d = 2$ (Majorana-Weyl spinors); $\alpha = 1, \ldots, 4$ for $d = 3$ (Majorana spinors); $\alpha = 1, \ldots, 4$ for $d = 5$ (symplectic-Majorana spinors [46]) and finally $\alpha = 1, \ldots, 16$ for $d = 9$ (Majorana-Weyl spinors).

The action in (2.1) is invariant under the following gauge transformations with local parameters η^α and ζ^m

$$
\begin{aligned}
&\delta x^m = \zeta^m + \frac{i}{2}(\eta\gamma^m\theta)\,, \quad \delta\theta^\alpha = \eta^\alpha\,, \quad \delta l^m = \dot{\zeta}^m + \frac{i}{2}\,(\eta\gamma^m\Lambda) - \frac{i}{2}(\theta\gamma\dot{\eta})\,, \\
&\delta\Lambda^\alpha = \dot{\eta}^\alpha\,, \quad \delta P_m = 0\,, \quad \delta p_\alpha = \frac{i}{2}P_m(\gamma^m\eta)_\alpha\,,
\end{aligned}
\tag{2.2}
$$

and under rigid super-Poincaré transformations

$$
\begin{aligned}
&\delta_\epsilon x^m = a^m - \frac{i\alpha}{4}(\epsilon\gamma^m\theta)\,, \quad \delta_\epsilon\theta^\alpha = \epsilon^\alpha\,, \quad \delta_\epsilon l^m = +\frac{i\alpha}{4}(\epsilon\gamma^m\dot{\theta}) - \frac{i\beta}{2}(\epsilon\gamma^m\Lambda)\,, \\
&\delta\Lambda^\alpha = 0\,, \quad \delta_\epsilon P_m = 0\,, \quad \delta_\epsilon p_\alpha = +\frac{i\beta}{2}P_m(\gamma^m\epsilon)_\alpha\,,
\end{aligned}
\tag{2.3}
$$

with constant parameters a^m and ϵ^α and $\alpha + \beta = 1$. By redefining $l^m \to l^m + a\dot{x}^m$, we can set $\alpha = 0$. The gauge transformations remove any propagating degrees of freedom and, therefore, this worldline model should describe a topological model in target space whose physical sector is restricted to the zero modes [33].[4] Notice that the non-linear terms in the gauge transformations (2.2) such as $\frac{i}{2}(\eta\gamma^m\theta)$ in δl^m are needed in order that the gauge symmetry commutes with the super-Poincaré transformations (2.3) by assuming that the gauge parameters ζ^m and η^α are supersymmetric invariant.

The action should also be diffeomorphism invariant; the corresponding transformation rules are $\delta x^m = \xi P^m$, $\delta P^m = 0$ and $\delta l^m = \frac{d}{dt}(\xi P^m)$, but comparison with (2.2) shows that these transformation rules are just a special case of the gauge transformations, with $\zeta^m = P^m\xi$. Thus we do not treat diffeomorphisms separately. In addition, the action is also invariant under a fermionic symmetry (Siegel κ symmetry) with $\delta_\kappa\theta^\alpha = \gamma^{\alpha\beta}_m P^m k_\beta$ and $\delta_\kappa x^m = \delta_\kappa\theta^\alpha\gamma^m_{\alpha\beta}\dot{\theta}^\beta$ (the conjugate momenta p_α and P^m transform accordingly). Again,

[4] The main difference with the conventional superparticle action $S = \int d\tau \frac{1}{2e}(\dot{x}^m - \frac{i}{2}\theta\gamma^m\dot{\theta})^2$ of ref. [47] is that for that model the gauge symmetries are the reparametrizations on the worldline and the κ-symmetry.

this symmetry is part of the original symmetry when the gauge parameters are $\eta^\alpha = \gamma_m^{\alpha\beta} P^m k_\beta$. These transformations are easily compensated by changing the gauge fields Λ^α and l^m. It is interesting to note that the reparametrizations and the κ symmetry are a subalgebra of the gauge transformations and they will play a role in the definition of amplitudes.

To render (2.1) explicitly supersymmetric, we introduce the composite fields $\Pi^m = \dot{x}^m + \frac{i}{2}(\theta\gamma^m\dot{\theta})$, $d_\alpha = p_\alpha + \frac{i}{2}P_m(\gamma^m\theta)_\alpha$ and $L^m = l^m + \frac{i}{2}(\theta\gamma^m\Lambda)$. All of them are supersymmetric expressions and in terms of them the action reads

$$S = \int d\tau \Big[P_m \Big(\Pi^m - L^m \Big) - d_\alpha \Big(\dot{\theta}^\alpha - \Lambda^\alpha \Big) \Big] \,. \tag{2.4}$$

By replacing the gauge parameters ξ^m and η^α by ghost fields c^m and λ^α (which are anticommuting and commuting, respectively), we obtain a nilpotent BRST symmetry

$$s\,x^m = c^m + \frac{i}{2}(\lambda\gamma^m\theta)\,, \quad s\theta^\alpha = \lambda^\alpha\,, \quad s\,L^m = \dot{c}^m + i\,(\lambda\gamma^m\Lambda)\,, \quad s\Lambda^\alpha = \dot{\lambda}^\alpha\,, \tag{2.5}$$

$$sP_m = 0\,, \quad s\,d_\alpha = iP_m(\gamma^m\lambda)_\alpha\,, \quad s\,c^m = -\frac{i}{2}(\lambda\gamma^m\lambda)\,, \quad s\lambda^\alpha = 0\,.$$

We notice that the BRST transformations map the fields x^m and θ^α into fields and ghosts, the gauge fields L^m and Λ^α into gauge fields and ghosts and, finally, the composite operators P_m and d_α are mapped into themselves and ghosts.

To gauge-fix the action, we introduce the anti-ghost fields b_m and w_α and the BRST-auxiliary fields ρ_m and ρ_α, which transform under BRST symmetry as follows

$$s\,b_m = -\rho_m\,, \quad s\,\rho_m = 0\,, \quad sw_\alpha = -\rho_\alpha + i\,(\lambda\gamma^m b_m)\,, \quad s\,\rho_\alpha = i\,\rho_m(\gamma^m\lambda)_\alpha\,. \tag{2.6}$$

Notice that $s\,w_\alpha$ contains a nonlinear term with $i(\lambda\gamma^m b_m)$. We could have written the simpler transformation laws $s\hat{w}_\alpha = \hat{\rho}_\alpha$ and $s\hat{\rho}_\alpha = 0$, but in order to have manifest supersymmetry we have shifted w_α as $\hat{w}_\alpha = w_\alpha + ib_m(\gamma^m\theta)_\alpha$, and this leads also to the nonlinear term $i\rho_m(\gamma^m\lambda)_\alpha$ in $s\,\rho_\alpha$.

Next, we add a gauge fermion

$$\begin{aligned} S' =& S + s\int d\tau \Big[b_m \Big(L^m - \frac{1}{2}P^m \Big) + w_\alpha\Lambda^\alpha \Big] = \\ =& S + \int d\tau \left[\rho_m \left(L^m - \frac{1}{2}P^m \right) - b_m\dot{c}^m + \rho_\alpha\Lambda^\alpha + w_\alpha\dot{\lambda}^\alpha \right] \,. \end{aligned} \tag{2.7}$$

Eliminating the auxiliary fields ρ_m and ρ_α and the gauge fields L^m, Λ^α by their algebraic equations of motion, we arrive at the final action and the final BRST charge

$$S' = \int d\tau \Big[P_m \Pi^m - \frac{1}{2} P^2 - d_\alpha \dot{\theta}^\alpha - b_m \dot{c}^m + w_\alpha \dot{\lambda}^\alpha \Big] \,, \tag{2.8}$$

$$Q = \lambda^\alpha d_\alpha - c^m P_m + \frac{i}{2} b_m (\lambda \gamma^m \lambda) \,.$$

Fields transform as $s\,\Phi = i[Q, \Phi\}$, and this reproduces (2.5) and (2.7).

By using the commutators $[P^m, x^n] = -i\,\eta^{mn}$, $\{p_\alpha, \theta^\beta\} = -i\,\delta_\alpha^{\ \beta}$, $\{b_m, c^n\} = -i\,\delta_m^{\ n}$ and $[w_\alpha, \lambda^\beta] = -i\,\delta_\alpha^{\ \beta}$, it is easy to show that the fields d_α satisfy the algebra $\{d_\alpha, d_\beta\} = -P_m \gamma^m_{\alpha\beta}$ and the BRST charge Q is nilpotent. We take $P_m, \lambda^\alpha, \theta^\alpha$ and b_m hermitian, and d_α and c^m antihermitian; then Q is antihermitian. Both the action and the BRST charge are invariant under supersymmetry.[5] The action resembles very closely the superparticle quantized with pure spinors [23](see also [38]), apart from the presence of the anticommuting ghosts and the fact that the spinors λ^α are unconstrained.[6] The main difference between the present model and the $D = (9, 1)$ superstring is that in the latter case the constraints $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ can be solved, thereby reducing the number of independent ghost fields. In lower dimensions , such as in 3,4, and 6 dimensions, there is no solution besides the trivial one (for Majorana spinors). However, in the present context, the fact that there is no solution to the constraint means that there are no physical degrees of freedom. Thus, we obtain a purely topological model.

As in topological strings, the Virasoro constraints $P^2 = 0$ are not present in the BRST charge and correspondingly also the ghost and the antighosts of the reparametrization invariance are absent. Thus it is difficult to establish the correct measure for different worldlines. But in the present context the relation

$$P^2 = \{Q, b_m P^m\} \tag{2.9}$$

[5] Note that the bosonic sector of the action (2.8), which contains x^m and the corresponding ghosts c^m and b^m, resembles the conventional quantized bosonic point-particle whose free action is given $S' = \int d\tau \Big[P_m \dot{x}^m - \frac{1}{2} P^2 - b\dot{c} \Big]$. The main difference is that the ghost b and c are scalar fields with respect to target-space Lorentz transformations.

[6] In [3], the introduction of new ghosts classified by different grading numbers has led to a similar BRST charge and, in order to single out the physical states, one has to consider the restricted functional space with non-negative grading.

is valid. We therefore introduce the composite field $B = b_m P^m$, which plays the role of the antighost in the usual reparametrization invariant theory for the particle. Moreover, we can also use the reparametrizations generated by $P^2 \sim 0$ to set two coordinates of x^m to zero by choosing a light-cone gauge. However, this choice would break the manifest supersymmetry. To avoid this we must invoke another symmetry to remove consistently two fermionic dof's. This can be done by another gauge symmetry generated by $\not{P} d \sim 0$ which forms a subalgebra together with P^2. Its BRST transformation $\{Q, \not{P} d\} = \lambda^\alpha P^2$ does not vanish[7] and therefore $\not{P} d \sim 0$ cannot be put on the same ground as P^2. Nevertheless, we can modify $\not{P} d \sim 0$ by adding $\lambda^\alpha b_m P^m$ to render it BRST close and, in addition, BRST exact. We postpone the discussion on the implications at the end of Sec. 5.

Although it is very convenient to work with supersymmetric quantities, we have to point out that there is a unitary transformation which suitably simplifies the BRST charge

$$Q' = e^{-\frac{1}{2}\lambda^\alpha \gamma^m_{\alpha\beta}\theta^\beta b_m} Q e^{\frac{1}{2}\lambda^\alpha \gamma^m_{\alpha\beta}\theta^\beta b_m} = \lambda^\alpha p_\alpha - c^m P_m \,. \tag{2.10}$$

The new charge is still nilpotent, but it is not supersymmetric. The form of the BRST symmetry in (2.10) may be convenient for other purposes, but we use Q given in (2.8).[8]

2.2. Worldsheet actions and BRST symmetry

In the previous section, we analysed a simple model of a superparticle. In the present section we point out that this model can indeed be interpreted as the reduction of a topological string on a supermanifold. Here, we simply sketch the construction of topological strings on supermanifold. We do not pursue this analysis in the present paper and it will be interesting to see all these ideas been completely developed for A/B models, such the construction of A_∞ algebra [48], open/closed interactions [49,50,51,52,53,54] and the analysis of boundary conditions. We postpone these issues to a future publication [55].

The model consists of maps $\Phi : \Sigma \to X^{m|n}$, from a two-dimensional surface Σ to a Riemannian supermanifold $X^{m|n}$ of bosonic dimension m and fermionic dimension n,

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[7] In pure spinor string theory, there is no negative ghost number operator; due to the gauge invariance generated by the pure spinor constraints, w_α appear only in gauge invariant combinations $w\gamma^{mn}\lambda$ and $w_\alpha\lambda^\alpha$, which have ghost number zero. In that case there is an operator b^α such that $\{Q, b^\alpha\} = P^2\lambda^\alpha$, which has been used in [43] to define the amplitudes on genus g Riemann surfaces.

[8] We acknowledge W. Siegel for a discussion on the relation between the present formalism and the "Big Picture" formalism [42].

equipped with a supermetric g. If we pick local coordinates $z, \bar{z}$ on Σ and x^I, θ^α (where $I = 1, \ldots, m$ and $\alpha = 1, \ldots, n$) on $X^{m|n}$, then Φ can be described locally via functions $(x^I(z, \bar{z}), \theta^\alpha(z, \bar{z}))$. Let us introduce the fields (ψ^I_+, q^α_+), a section of $K^{1/2} \otimes \Phi^*(TX^{m|n})$ (where $TX^{m|n}$ is the complexified tangent bundle of $X^{m|n}$ and $K^{1/2}$ is the spin bundle over Σ). In the same way, we introduce (ψ^I_-, q^α_-), a section of $K^{1/2} \otimes \Phi^*(TX^{m|n})$. The action is

$$
\begin{aligned}
S = 2t \int d^2z \Big(& \frac{1}{2} g_{IJ} \partial_z x^I \partial_{\bar{z}} x^J + g_{I\alpha} \partial_z x^I \partial_{\bar{z}} \theta^\alpha + \frac{1}{2} g_{\alpha\beta} \partial_z \theta^\alpha \partial_{\bar{z}} \theta^\beta + \\
& + \frac{i}{2} g_{IJ} \psi^I_- D_z \psi^J_- + \frac{i}{2} g_{I\alpha} \psi^I_- D_z q^\alpha_- + \frac{i}{2} g_{\alpha\beta} q^\alpha_- D_z q^\beta_- \\
& + \frac{i}{2} g_{IJ} \psi^I_+ D_{\bar{z}} \psi^J_+ + \frac{i}{2} g_{I\alpha} \psi^I_+ D_{\bar{z}} q^\alpha_+ + \frac{i}{2} g_{\alpha\beta} q^\alpha_+ D_{\bar{z}} q^\beta_+ \Big) \\
& + \textit{curvature terms} \,.
\end{aligned} \tag{2.11}
$$

The supermetric $g = (g_{(IJ)}, g_{(I\alpha)}, g_{[\alpha\beta]})$ has been decomposed along the bosonic and fermionic components. The super-Riemann tensor R is also decomposed into several components. The structure of the first line recalls the structure of superstrings in Ramond-Ramond background [7,56,57,58,59,60] where the components of the metric g along the fermionic components $g_{I\alpha}$ are proportional to the gravitinos and $g_{\alpha\beta}$ are proportional the the inverse of the RR field strenghts. To see this, it convenient to decompose the coordinates into $(x^i, \theta^a, x^{\bar{i}}, \theta^{\bar{a}})$ and introduce new fields p_a and $p_{\bar{b}}$. Then, the first line of (2.11) can be rewritten as follows

$$
\hat{g}_{i\bar{j}} \partial_z x^i \partial_{\bar{z}} x^{\bar{j}} + p_a \partial_{\bar{z}} \theta^a + p_{\bar{b}} \partial_z \theta^{\bar{b}} + (p_a - g_{a\bar{j}} \partial_{\bar{z}} x^{\bar{j}}) g^{a\bar{b}} (p_{\bar{b}} - g_{i\bar{b}} \partial x^i) \,. \tag{2.12}
$$

The last term has the same structure of pure spinor string theory on $AdS_5 \times S^5$ constructed in [7]. Notice that there are additional terms coming from the curved manifold that contributes to the curvature terms in (2.11). This analogy with AdS space might be useful to gain some new results in amplitude computations on a curved manifolds for superstrings. As in the bosonic case, we expect the model (2.12) to be conformally invariant if the target space is a super-Calabi-Yau manifold. This condition is apparently weaker than super-Ricci flatness, as has been recently remarked in [61].

We specialize now the previous formulae to the case of a complex super-3 fold $\mathbf{C}^{3|4}$. The fermionic directions are parametrized by a symplectic Majorana spinor. By a suitable choice of the background (choice of D-branes and fluxes) it should be possible to reduce further the number of target space fermions to a single Majorana spinor of 3d and the superspace becomes $\mathbf{R}^{3|2}$. Most likely the mechanism for the reduction would involve choosing a Lagrangian submanifold of X. Then the action for the zero modes should reduce to (2.8). We do not derive this fact here but will work with this assumption in mind. Since we can give consistent rules for the amplitudes (sec. 5), we can *a posteriori* justify this assumption, but clearly a deeper investigation is needed [55].

As is well-known, when the target space is a Kähler manifold the action (2.1) has $N=2$ worldsheet supersymmetry. The fermions can be twisted using the fermion number current $U(1)$, and for the topological A model, one ends up with the sections $\chi^i, \chi^{\bar{i}}, \lambda^a$ and $\lambda^{\bar{a}}$ and with the 1-forms $\psi^i_{\bar{z}}$, $\psi^{\bar{i}}_z$, $w^{\bar{a}}_z$ and $w^a_{\bar{z}}$. As part of the twisted N=2 supersymmetry we can derive the BRST transformations

$$\delta x^i = i(\chi^i + \lambda^a \gamma^i_{ab}\theta^b)\,, \qquad \delta\chi^i = -\lambda^a\gamma^i_{ab}\lambda^b\,, \tag{2.13}$$

$$\delta\theta^a = i\lambda^a\,, \qquad \delta\lambda^a = 0\,,$$

$$\delta\psi^{\bar{i}}_z = -\partial_z x^{\bar{i}} + \lambda^{\bar{a}}\gamma^{\bar{i}}_{\bar{a}\bar{b}} w^{\bar{b}}_z - i\Big(\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_z + \lambda^{\bar{a}}\Gamma^{\bar{i}}_{\bar{a}\bar{m}}\psi^{\bar{m}}_z + \chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{b}} w^{\bar{b}}_z + \lambda^{\bar{a}}\Gamma^{\bar{i}}_{\bar{a}\bar{b}} w^{\bar{b}}_z\Big)\,,$$

$$\delta w^{\bar{a}}_z = -\partial_z\theta^{\bar{a}} - i\Big(\chi^{\bar{j}}\Gamma^{\bar{a}}_{\bar{j}\bar{m}}\psi^{\bar{m}}_z + \lambda^{\bar{a}}\Gamma^{\bar{a}}_{\bar{a}\bar{m}}\psi^{\bar{m}}_z + \chi^{\bar{j}}\Gamma^{\bar{a}}_{\bar{j}\bar{b}} w^{\bar{b}}_z + \lambda^{\bar{a}}\Gamma^{\bar{a}}_{\bar{a}\bar{b}} w^{\bar{b}}_z\Big)\,,$$

They are nilpotent only on the equations of motion of the ghost fields. Using these BRST transformations, one can rewrite the action in the following form

$$S = 2t\int_\Sigma d^2z\{Q, V\} + \int_\Sigma x^*(K) \tag{2.14}$$

and the last term is the pull-back of the Kähler 2-form

$$K = g_{i\bar{j}}dx^i dx^{\bar{j}} + g_{a\bar{b}}d\theta^a d\theta^{\bar{b}} \tag{2.15}$$

and the off-diagonal terms are removed by a θ-dependent diffeomorphism on the manifold.

Even though we are mostly concerned in this paper with the point-particle limit, it is still useful to have the complete theory at hand. Indeed, as we will see in section 5, the rules for calculating the amplitudes cannot be completely justified without appealing to the conformal field theory formalism.

Recently, the relation between WZW model based on a supergroup and twistor spaces have been explored [62]. It is worth pointing out that the pure spinor formulation can be seen to emerge from a WZW model based on supergroup [63,64] and the computation rules can be established on the basis of [65].

3. The $D = (2,1), N = 1$ Chern-Simons model

Now we go back to the worldline model. In this section we compute the vertex operators, both integrated and unintegrated, and the zero momentum cohomology needed to construct the tree level measure.

3.1. Vertex operators

To show that the formalism constructed in the previous section describes a super-Chern-Simons theory, we compute the BRST cohomology: $\{Q, U^{(1)}\} = 0$ with $\delta U^{(1)} = [Q, \Omega]$. The fields of the field theory, namely the gauge field a_m and the gaugino ψ^α, are identified with the BRST cohomology classes at ghost number 1.

Applying the BRST charge to the most general superfield with ghost number one expressed in terms of a maximal set of commuting coordinates

$$U^{(1)} = \lambda^\alpha A_\alpha(x,\theta) - c^m A_m(x,\theta)\,, \tag{3.1}$$

we have the following field equations

$$\{Q, U^{(1)}\} = -\frac{i}{2}(\lambda\gamma^m\lambda)A_m - c^m c^n \partial_n A_m + c^m \lambda^\beta D_\beta A_m - c^m \lambda^\beta \partial_m A_\beta + \lambda^\alpha \lambda^\beta D_\alpha A_\beta = 0\,, \tag{3.2}$$

where $D_\alpha = \partial_\alpha + \frac{i}{2}(\gamma^m\theta)_\alpha \partial_m$, which imply

$$F_{\alpha\beta} \equiv D_{(\alpha}A_{\beta)} - \frac{i}{2}\gamma^m_{\alpha\beta}A_m = 0\,, \qquad F_{\alpha m} \equiv D_\alpha A_m - \partial_m A_\alpha = 0\,, \qquad F_{mn} = 0\,, \tag{3.3}$$

where $F_{[mn]} = \frac{1}{2}(\partial_m A_n - \partial_n A_m)$. The last equation is a consequence of the first two equations by exploiting the Bianchi identities $[\nabla_m, \{\nabla_\alpha, \nabla_\beta\}] + \{\nabla_\alpha, [\nabla_\beta, \nabla_m]\} - \{\nabla_\beta, [\nabla_m, \nabla_\alpha]\} = 0$ where $\nabla_\alpha = D_\alpha + A_\alpha$ and $\nabla_m = \partial_m + A_m$. Equations (3.3) are invariant under $\delta U^{(1)} = [Q, \Omega^{(0)}]$, or, in terms of the components A_α, A_m of the connections, $\delta A_\alpha = D_\alpha \Omega^{(0)}$ and $\delta A_m = \partial_m \Omega^{(0)}$.

From now on we consider the case of 3-dimensional target space: $x = x^m$ with $m = 0,1,2$. Then, $\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}(\theta^\alpha\theta_\alpha)$ where $\theta^\alpha = \theta^\beta\epsilon_{\beta\alpha}$. Decomposing the superfields A_α and A_m in the following way,

$$A_m = a_m + \theta^\beta \hat{\xi}_{m\beta} + (\theta^\alpha\theta_\alpha)\xi_m\,, \qquad A_\alpha = \chi_\alpha + \theta^\beta \hat{\chi}_{\alpha\beta} + \theta^\beta\theta_\beta \psi_\alpha\,. \tag{3.4}$$

and eliminating the auxiliary fields $\hat{\xi}_{m\beta}, \xi_m, \chi_\alpha, \hat{\chi}_{\alpha\beta}$ by using the field equations and the algebraic gauge transformations, one finds the equations of motion $\partial^m a_n - \partial_n a_m = 0$ and

$\psi^\alpha = 0$. These coincide with the equation of motion for super-Chern-Simons theory. After removing the auxiliary fields, the supersymmetry is realized by the usual transformation laws $\delta_\epsilon a_m = (\epsilon\gamma^m\psi)$ and $\delta_\epsilon\psi^\alpha = \frac{1}{2}F_{mn}(\gamma^{mn}\epsilon)^\alpha$.

Another important element is the integrated vertex operator with integrand V. This satisfies the equation $\{Q, V^{(0)}\} = \dot{U}^{(1)}$, and it is defined up to the gauge transformations $\delta V^{(0)} = \dot{\Omega}^{(0)}$, where $\Omega^{(0)}$ is the superfield of the gauge transformations. For on-shell fields $\dot{\theta}^\alpha = 0, \dot{P}_m = 0, \dots$ the vertex V has the generic form

$$V^{(0)} = \dot{x}^m A_m + d_\alpha W^\alpha + w_\alpha \lambda^\beta F^\alpha_{\ \beta} + b_m c^n F^m_{\ n} + w_\alpha c^m F^\alpha_{\ m} + b_m \lambda^\alpha F^m_{\ \alpha} \,. \tag{3.5}$$

However, imposing $\{Q, V^{(0)}\} = \dot{U}^{(1)}$ and using the equations of motion (3.3)(which implies that all the curvatures vanish), the vertex reduces to

$$V = \dot{x}^m A_m \,. \tag{3.6}$$

This coincides exactly with the usual vertex operator in the case of bosonic Chern-Simons in D=(2,1). One can finally define the Wilson loop by setting

$$W(\gamma) = \mathrm{tr}\Big(P e^{\int_\gamma d\tau \dot{x}^m A_m}\Big) \,, \tag{3.7}$$

where γ is a curve in the target space. P denotes the path ordering and the trace is then needed for non-abelian gauge-invariance. Notice that dealing with non-abelian gauge group, the superderivatives in (3.2) and (3.3) should be replaced by covariant superderivatives. In the following section the complete non-abelian action will be discussed. Furthermore, $\{Q, W(\gamma)\} = 0$, and it cannot be written as a BRST exact quantities. As is well-known no local gauge invariant observable can be constructed for Chern-Simons theory. Because A_m is a superfield, $W(\gamma)$ has an expansion in terms of zero modes θ^α.

In order to study the target space field theory, it is useful to compute also the cohomologies at ghost number $0, 2, 3$ and higher. It is easy to see that the equation $\{Q, U^{(0)}\} = 0$ for a ghost number zero superfield $U^{(0)}$ implies that it is a constant. Usually the cohomology at ghost number 2 doubles the cohomology at ghost number 1 if the momentum does not vanish (see for example [42]). The most general superfield at ghost number 2 is given by $U^{(2)} = \lambda^\alpha\lambda^\beta A^*_{\alpha\beta} + \lambda^\alpha c^m A^*_{\alpha m} + c^m c^n A^*_{mn}$, and clearly the number of antifields exceeds the number of corresponding superfields A_α and A_m. Therefore, we expect that the equations of motion will show that some of the antifields of $A^*_{\alpha\beta}$, $A^*_{\alpha m}$ and A^*_{mn} are redundant and can be expressed in terms of the others.

From BRST invariance $\{Q, U^{(2)}\} = 0$, we obtain the following equations

$$
\begin{aligned}
&D_{(\alpha} A^*_{\beta\gamma)} + \frac{i}{2}\gamma^m_{(\alpha\beta} A^*_{\gamma)m} = 0\,, \qquad D_{(\alpha} A^*_{\beta)m} - \frac{i}{2}\gamma^n_{\alpha\beta} A^*_{[mn]} + \partial_m A^*_{(\alpha\beta)} = 0\,, \\
&D_\alpha A^*_{[mn]} - \partial_{[m} A^*_{|\alpha|n])} = 0\,, \qquad \partial_{[m} A^*_{nr]} = 0\,.
\end{aligned}
\tag{3.8}
$$

We can decompose the superfields $A^*_{(\alpha\beta)}, A^*_{\alpha m}$ and A^*_{mn} into irreducible representation of the super-Poincaré group: $A^*_{\alpha\beta} = \gamma^m_{\alpha\beta} B^*_m$, $A^*_{\alpha m} = \gamma_{m\alpha\beta} B^{*\beta} + B^*_{\alpha m}$, where $B^*_{\alpha m}$ is γ-traceless and $A^*_{mn} = \epsilon_{mnr} B^{*r}$. From the equations of motion, one obtains that $B^*_{\alpha m}$ and B^{*r} are algebraically related to one vector B^*_m and one spinor superfield $B^{*\beta}$. Those two superfields are the antifields for A_m and A_α. The gauge transformations which leave the equations of motion of the antifields A^* invariant should be the equations of motion of the superfield A. This is indeed the case since $\delta U^{(2)} = Q\Omega^{(1)}$ with $\Omega^{(1)} = \lambda^\alpha C_\alpha - c^m C_m$ yields

$$
\begin{aligned}
&\delta A^*_{(\alpha\beta)} = D_{(\alpha} C_{\beta)} - \frac{i}{2}\gamma^m_{\alpha\beta} C_m\,, \qquad \delta A^*_{\alpha m} = D_\alpha C_m - \partial_m C_\alpha\,, \\
&\delta A^*_{[mn]} = \frac{1}{2}(\partial_m C_n - \partial_n C_m)\,,
\end{aligned}
\tag{3.9}
$$

where C_m and C_α are two arbitrary superfields.

For what concerns the ghost-number three BRST cohomology, we point out that the only solution is a constant scalar field for non-zero momentum cohomology. One expects that there is no cohomology at ghost number beyond three. The easiest way to show this is to first determine the zero-momentum cohomology because the latter contains the case $k^m \neq 0$ as a special case, while it is easier to compute. So we now turn to the zero-momentum cohomology.

3.2. Zero momentum cohomology and tree level measure

We denote here with Ψ the string field without any restriction on the ghost number. Direct evaluation of $\{Q, \Psi\} = 0$ with $Q = \lambda^\alpha \partial_\alpha + \frac{i}{2} b_m \lambda\gamma^m\lambda$ shows that the most general solution for the cohomology at zero momentum is given by[9]

$$
\begin{aligned}
&\Psi = c\,U^0 + a_m U^{(1)m} + a^{*,m} U^{(2)}_m + c^* U^{(3)}\,, \\
&U^{(0)} = 1\,, \\
&U^{(1),m} = \frac{i}{2}\lambda^\alpha \gamma^m_{\alpha\beta}\theta^\beta - c^m\,, \\
&U^{(2)}_m = \epsilon_{mnr}\left(\frac{1}{4}\lambda\gamma^n\theta\,\lambda\gamma^r\theta - i\lambda\gamma^n\theta\, c^r + c^n\, c^r\right)\,, \\
&U^{(3)} = \epsilon_{mnr}\left(\frac{3}{2}\lambda\gamma^m\theta\,\lambda\gamma^n\theta\, c^r - 3i\lambda\gamma^m\theta\, c^n\, c^r + c^m\, c^n\, c^r\right)\,.
\end{aligned}
\tag{3.10}
$$

[9] For example, at ghost number one the equations to solve are $\partial_\alpha A_m = 0$ and $\partial_{(\alpha} A_{\beta)} - \frac{i}{2}\gamma^m_{\alpha\beta} A_m = 0$, which yield $U^{(1)} = \lambda^\alpha A_\alpha - c^m A_m = \lambda^\alpha\left(\frac{i}{2}\gamma^m_{\alpha\beta} a_m \theta^\beta\right) - c^m a_m$ with constant a_m.

To show that Q annihilates $U^{(3)}$ we used the identity $\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2$, and $[\gamma^m, \gamma^n] = 2i\epsilon^{mnr}\gamma_r$. The only remaining degree of freedom is the gauge vector a_m, which cannot be removed at zero momentum by a gauge transformation $\delta\Psi = \{Q, \Omega\}$. On the contrary, the gaugino can be removed by a gauge transformation at zero momentum. The constant fields c and c^* are the ghost and its antifield, and they are gauge invariant.

The path integral measure can be decomposed into two factors $d\mu = d\mu_0 d\widetilde{\mu}$, where $d\mu_0$ is the measure on zero modes and $d\widetilde{\mu}$ is the measure on non-zero modes. The latter is chosen as usual in the conventional way, the free measure weighted with classical action evalutated on non-zero modes. (In conformal field theory this part of the measure is obtained by performing all the possible OPE's among the insertion of vertex operators). Since the action for the zero modes vanishes we should choose a different path to define it.

The path integral measure for zero modes is defined in the following way

$$\int d\mu_0 U^{(3)} = 1 \tag{3.11}$$

where $U^{(3)}$ is an element of the cohomology $H^3(Q)$. This follows the usual requirement that the integral measure is the Poincaré dual to the top form here represented by the highest non-vanishing element of the BRST cohomology. The main property of this measure is $\int d\mu_0 Q\Lambda = 0$ for any Λ^2. The total ghost number of $d\mu_0$ is -3. In the following sections, we will show that this number is associated to the anomalies of $U(1)$ currents.

According to this definition, it is easy to see that

$$\langle U^{(0)} U^{(3)} \rangle = \int d\mu_0 U^{(0)} U^{(3)} = constant\,, \qquad \langle U^{(1)} U^{(2)} \rangle = \int d\mu_0 U^{(1)} U^{(2)} = constant\,. \tag{3.12}$$

The cohomology U^3 in (3.10) is composed by three independent monomials and we have to establish the integration rules for each of them. Thus we need

$$\begin{aligned}
&\langle c^m \lambda^\alpha \lambda^\beta \rangle = \epsilon^{mpq} \gamma_p^{\alpha\alpha'} \gamma_q^{\beta\beta'} \partial_{\alpha'} \partial_{\beta'}\,, \\
&\langle c^m c^n \lambda^\alpha \rangle = u\, \epsilon^{mnq} \gamma_q^{\alpha\alpha'} \partial_{\alpha'}\,, \\
&\langle c^m c^n c^r \rangle = w\, \epsilon^{mnr}\,.
\end{aligned} \tag{3.13}$$

where u and w are arbitrary constants. The path integral measure associated with the correlators (3.13) is given by

$$\begin{aligned}
&\langle F(x, \theta, \lambda, c) \rangle = \int d\mu_0 F(x, \theta, \lambda, c)\,, \\
&d\mu_0 = \Big(\epsilon^m{}_{nr} c^n c^r \gamma_m^{\alpha\beta} \partial_{\lambda^\alpha} \partial_{\lambda^\beta} - u(\theta \not{\partial}_\lambda) + w\, \theta^2 \Big) \delta^2(\lambda)\, d^3x\, d^2\theta\, d^2\lambda\, d^3c
\end{aligned} \tag{3.14}$$

where ∂_λ are derivatives which act on the delta function $\delta^2(\lambda)$. The presence of delta functions is due to bosonic ghost fields and we have to reabsorb their zero modes to have a non-vanishing and non-divergent contribution. In sec. 4, it is illustrated how this measure is related to the usual bosonic measure of Chern-Simons theory c^3. To fix the free parameter u and w, we adopt the following strategy: we check which term provides a supersymmetric amplitude and we check the gauge invariance.

▷ Supersymmetry.

Notice that the last term of the functional measure gives the bosonic Chern-Simons action which is not supersymmetric. In fact, as is clear from (3.14) the last term does not provide a supersymmetric measure. The function $F(x, \theta, \lambda, c)$ is a polynomial combination of superfields and transforms as $\delta_\epsilon F(x, \theta, \lambda, c) = \epsilon^\alpha D_\alpha F(x, \theta, \lambda, c)$ under supersymmetry. Integrating by parts, one has

$$\langle \delta_\epsilon F(x, \theta, \lambda, c) \rangle = \epsilon^\alpha \int d^3x\, d^2\theta\, d^2\lambda\, d^3c\; (D_\alpha \mu_0)\; F(x, \theta, \lambda, c) = 0\,. \tag{3.15}$$

where $d\mu_0 = \mu_0 d^3x\, d^2\theta\, d^2\lambda\, d^3c$. Since $(D_\alpha\mu_0) \neq 0$, we set $u = w = 0$.

▷ Gauge invariance.

Another essential property for the measure is gauge invariance. Namely, the integration of a BRST trivial vertex should vanish: $\langle \{Q, \Omega\} \rangle = \int d\mu_0 \{Q, \Omega\} = 0$. In fact, inserting the vertex $Q(\lambda^\alpha\lambda^\beta\Omega_{(\alpha\beta)} + \lambda^\alpha c^m \Omega_{\alpha m} + c^m c^n \Omega_{[mn]})$ and selecting only terms of the form $\lambda^\alpha\lambda^\beta c^m$ because $u = w = 0$,

$$\Big\langle -\lambda^\alpha\lambda^\beta c^n \partial_n \Omega_{(\alpha\beta)} + \lambda^\alpha c^m \lambda^\beta D_\beta \Omega_{\alpha m} + i\, \lambda^\alpha \lambda^\beta c^n \gamma^{m\alpha\beta} \Omega_{[mn]} \Big\rangle = \tag{3.16}$$

$$= \int d^3x d^2\theta \gamma^{n\alpha\beta} \Big(\partial_n \Omega_{(\alpha\beta)} + D_{(\beta}\Omega_{\alpha)n} + \gamma^m_{\alpha\beta}\Omega_{[mn]} \Big) = 0\,.$$

The first term vanishes because it is a total derivative, the second because it is a total spinorial derivative and the last one because $\mathrm{tr}(\gamma^m\gamma^n) = 2\eta^{mn}$ and $\Omega_{[mn]}$ is antisymmetric.[10] This is not unexpected since, as will be shown in sec. 4, the delta functions apper in the PCO which are BRST closed.

---

[10] The measure (3.14) can be rewritten in the following form

$$\langle F \rangle = \int d^3x\, d^2\theta\, d^2\lambda\, d^3c \int d^2w\, \epsilon_{\alpha\delta} (\not{\partial} w)^\alpha (\not{\partial} w)^\delta e^{w_\alpha \lambda^\alpha} F \tag{3.17}$$

where we introduced the integral representation for the delta function. The last line in the above

3.3. The action

It is convenient to use a Witten-like string field theory to derive the action. Moreover, this derivation leads to a BV action which contains the classical action plus all the fields and antifields needed to implement the symmetries of the model. For that we need a supersymmetric BV measure ω_{BV} (see app. A for further details) which reads

$$\omega_{BV} = \langle \Psi, \Psi \rangle = \int d^3x \int d^2\theta\, \gamma^{m\alpha\beta} \Big(\delta A^*_{(\alpha\beta)} \delta A_m + \delta A^*_{\alpha m} \delta A_\beta + \delta C^*_{\alpha\beta m}\, \delta C \Big) \,. \qquad (3.19)$$

The BV action for the fields and antifields which appear in the most general string field

$$\begin{aligned} \Psi = {} & C + \lambda^\alpha A_\alpha + c^m A_m + \lambda^\alpha \lambda^\beta A^*_{\alpha\beta} + \lambda^\alpha c^m A^*_{\alpha m} + c^m c^n A^*_{[mn]} \\ & + \lambda^\alpha \lambda^\beta \lambda^\gamma C^*_{\alpha\beta\gamma} + \lambda^\alpha \lambda^\beta c^m C^*_{(\alpha\beta)m} + \lambda^\alpha c^m c^n C^*_{\alpha[mn]} + c^m c^n c^r C^*_{[mnr]} \,. \end{aligned} \qquad (3.20)$$

should couple the fields A_α and A_m to the corresponding antifields $A^*_{\alpha\beta}$, $A^*_{\alpha m}$ and A^*_{mn}, and the ghost C to the corresponding antifields $C^*_{\alpha\beta\gamma}, C^*_{\alpha\beta m}, C^*_{\alpha mn}$ and C^*_{mnr}. (From the zero momentum cohomology, we know that there is only one non-vanishing state corresponding to a scalar superfield $C^* = \gamma^{m\alpha\beta} C^*_{(\alpha\beta)m}$.)

The string action S, which satisfies the master equation, is obtained from

$$S = \frac{1}{2} \langle \Psi, Q\, \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi\Psi \rangle \,, \qquad (3.21)$$

where Q is the BRST charge given (2.8) and the generic state is described in (3.20). Again the product in the interaction term is given by $\langle \Psi_1 \Psi_2 \Psi_3 \rangle = \int \mu(\lambda, c, \theta) \mathrm{tr}\, (\Psi_1 \Psi_2 \Psi_3)$. This vanishes in the abelian case. In the non-abelian case, the string field Ψ carries an index in the adjoint representation of the gauge group and $\mathrm{tr}\, (\Psi_1 \Psi_2 \Psi_3) = f_{abc} \Psi^a_1 \Psi^b_2 \Psi^c_3$. The inner product by given by

$$\langle \Psi_1, \Psi_2 \rangle = \int \mu(x, \lambda, c, \theta)\, (\Psi_1 \Psi_2) \qquad (3.22)$$

where the product $(\Psi_1 \Psi_2)$ is the usual superspace product of superfields. Notice that in the non-abelian case, the superfield product should include the trace over the internal

---

equation can again be written as an exponent by introducing a new fermionic zero mode p_α

$$\langle F \rangle = \int d^3x d^3c\, d^2\theta\, d^2p\, d^2\lambda\, d^2w\, e^{(p_\alpha \gamma^{\alpha\beta}_m\, w_\beta\, c^m + w_\alpha \lambda^\alpha)} F \,. \qquad (3.18)$$

A similar analysis has been pursued in [63] where we first point out the necessity of delta functions in the path integral of λ^α.

gauge group: $\mathrm{tr}\,(\Psi_1\Psi_2)$. The fields $C^*_{\alpha\beta\gamma},\ldots,C^*_{[mnr]}$ are the antighost fields for the ghost C.

In string field theory the master equation is defined by $(S,S)=\int\mu\frac{\delta S}{\delta\Psi}\frac{\delta S}{\delta\Psi}$. Substitution of S in (3.21) yields $\int\mu(Q\Psi+\Psi^2)(Q\Psi+\Psi^2)$. This vanishes because $\int\mu Q\Psi Q\Psi=\int Q(\Psi Q\Psi)=0$, while $\int\mu(Q\Psi)\Psi^2=\frac{1}{3}\int Q(\Psi^3)$, and finally $\Psi^4=0$ because Ψ is anticommuting. Subsitution of the measure yields the more familiar result:

$$\{S,S\}_{BV}=\int d^3x\int d^2\theta\,\gamma^{m\alpha\beta}\left(\frac{\partial_l S}{\partial A^*_{(\alpha\beta)}}\frac{\partial_r S}{\partial A_m}+\frac{\partial_l S}{\partial A^*_{\alpha m}}\frac{\partial_r S}{\partial A_\beta}+\frac{\partial_l S}{\partial C^*_{(\alpha\beta)m}}\frac{\partial_r S}{\partial C}\right)=0\,. \tag{3.23}$$

Inserting the expression of Ψ and of the BRST charge, from (3.21) using the measure μ given in (3.15) one obtains the supersymmetric invariant action

$$\begin{aligned}S&=\int d^3x\int d^2\theta\gamma^{m\alpha\beta}\mathrm{tr}X_{m\alpha\beta}\,,\\ X_{MNR}&=\frac{1}{2}A_{[M}D_NA_{R\}}+\frac{1}{3}A_{[M}A_NA_{R\}}+A^*_{[MN}D_{R\}}C+C^*_{[MNR\}}C^2\end{aligned} \tag{3.24}$$

where the index M refers to both the vector index m and the spinorial index α, and $D_M=\{\partial_m,D_\alpha\}$. To check that (3.24) reproduces exactly the Chern-Simons action, we can compute the kinetic terms, neglecting both the contributions coming from the non-abelian terms and from the antifields. The form of the action has also been described in [34].

Assuming the decomposition

$$A_m=a_m+\theta^\beta\hat{\xi}_{m\beta}+(\theta^\alpha\epsilon_{\alpha\beta}\theta^\beta)\xi_m\,,\qquad A_\alpha=\chi_\alpha+\theta^\beta\hat{\chi}_{\alpha\beta}+\theta^2\psi_\alpha\,. \tag{3.25}$$

we have for the first term in (3.24)[11]

$$\begin{aligned}&\Big\langle c^m\lambda^\alpha\lambda^\beta\left(-\frac{1}{2}\gamma^n_{\alpha\beta}A_mA_n+A_mD_\beta A_\alpha-A_\alpha D_\beta A_m+A_\alpha\partial_mA_\beta\right)\Big\rangle=\\ &\qquad\int d^3x\,\epsilon^{mpq}\Big[-2\epsilon_{pq}{}^n(\xi_ma_n+\xi_na_m)-(\hat{\xi}_m\,\gamma_p\,\gamma^n\,\gamma_q\,\hat{\xi}_n)-4\,\xi_m\mathrm{tr}(\gamma_q\hat{\chi}\gamma_p)\\ &\qquad-2a_m\mathrm{tr}(\gamma_q\partial_p\hat{\chi})+a_m\mathrm{tr}(\partial_s\hat{\chi}\,\gamma_p\gamma^s\gamma_q)-2(\hat{\xi}_m\gamma_q\partial_p\chi)\\ &\qquad+4(\hat{\xi}_m\gamma_p\gamma_q\psi)+(\hat{\xi}_m\gamma_p\gamma^s\gamma_q\partial_s\chi)-4(\psi\gamma_q\gamma_p\partial_m\chi)\\ &\qquad-\mathrm{tr}(\gamma_q\hat{\chi})\partial_m\mathrm{tr}(\gamma_p\hat{\chi})+\mathrm{tr}(\hat{\chi}\gamma_p\partial_m\hat{\chi}\gamma_q)\Big]\,,\end{aligned} \tag{3.26}$$

[11] We use the conventions: $\mathrm{tr}(\gamma^m\gamma^n\gamma^p)=-2\epsilon^{mnp}$, $\epsilon^{mnp}\epsilon_{mqr}=\delta^n_q\delta^p_r-\delta^n_r\delta^p_q$, $\epsilon^{mnp}\epsilon_{mnq}=2!\,\delta^p_q$ and $\epsilon^{mnp}\epsilon_{mnp}=3!$. As always, one has $\mathrm{tr}(\gamma^m)=0$, $(\gamma^m\gamma^n\gamma^r)_{\alpha\beta}=\gamma^m_{\alpha\beta}\eta^{nr}-\gamma^n_{\alpha\beta}\eta^{mr}+\gamma^r_{\alpha\beta}\eta^{mn}+\epsilon^{mnr}\epsilon_{\alpha\beta}$. The tensor $\epsilon_{\alpha\beta}$ is normalized as $\epsilon_{12}=1$. Notice that $\theta^\alpha\theta^\beta=-\frac{1}{2}\epsilon^{\alpha\beta}\theta^2$.

The action is invariant under the gauge transformations

$$
\begin{aligned}
&\delta\Psi = \lambda^\alpha \delta A_\alpha + c^m \delta A_m = Q\left(\Omega + \theta^\alpha \omega_\alpha + \theta^2 \eta\right) , \\
&\delta\chi_\alpha = \omega_\alpha , \qquad \delta\hat{\chi}_{\alpha\beta} = \frac{1}{2}\gamma^m_{\alpha\beta}\partial_m \Omega + 2\epsilon_{\alpha\beta}\eta , \\
&\delta\psi_\alpha = -\frac{1}{4}\gamma^m_{\alpha\beta}\epsilon^{\beta\gamma}\partial_m \omega_\gamma , \qquad \delta a_m = \partial_m \Omega , \\
&\delta\hat{\xi}_{m\beta} = \partial_m \omega_\beta , \qquad \delta\xi_m = \partial_m \eta .
\end{aligned}
\tag{3.27}
$$

Moreover, since the gauge transformations of χ_α and of the scalar part of $\hat{\chi}_{\alpha\beta} = \frac{1}{2}\gamma^m_{\alpha\beta}\chi_m + \epsilon_{\alpha\beta}\chi$, namely χ, are pure shifts one can remove these fields from the action by setting them to zero. Hence, one gets

$$
\begin{aligned}
S = \int d^3x\, \epsilon^{mpq}\Big[&- 2\epsilon_{pq}{}^n(\xi_m a_n + \xi_n a_m) - (\hat{\xi}_m\, \gamma_p\, \gamma^n\, \gamma_q\, \hat{\xi}_n) - 4\,\xi_m \mathrm{tr}(\gamma_q \hat{\chi} \gamma_p) \\
&- 2a_m \mathrm{tr}(\gamma_q \partial_p \hat{\chi}) + a_m \mathrm{tr}(\partial_s \hat{\chi}\, \gamma_p \gamma^s \gamma_q) + 4(\hat{\xi}_m \gamma_p \gamma_q \psi) \\
&- \mathrm{tr}(\gamma_q \hat{\chi})\partial_m \mathrm{tr}(\gamma_p \hat{\chi}) + \mathrm{tr}(\hat{\chi}\gamma_p \partial_m \hat{\chi}\gamma_q)
\end{aligned}
\tag{3.28}
$$

Eliminating the auxiliary fields ξ_m and $\hat{\xi}_{m\beta}$, one finds the super-Chern-Simons action

$$
S = -6 \int d^3x \Big(\epsilon^{mpq} a_m \partial_p a_q + 3\psi^2\Big) . \tag{3.29}
$$

This proves the fact that the measure chosen in the present derivation is supersymmetric and gauge invariant.

4. Supergeometry, Picture Changing Operators and BRST Cohomology

The discussion in the previous section concerning the construction of a measure for the zero-modes integrals can be better understood when considered in the broader context of the theory of differential forms and integration on supermanifolds. This framework permits a clear discussion about PCO and supergeometry for topological theories with supermanifolds as target space. This subject has a vast literature and we can direct the reader to the book [16] for a summary of results. In order to discuss our application, we discuss the integration on supermanifolds by means of pseudoforms and the Baranov-Schwarz transformations, then we construct the superforms for our topological model, we derive the Cartan calculus on the superforms, and finally we construct the PCO. This technique could be applied to the RNS superstrings, where the superforms are represented by distributions of the superghosts associated with the worldsheet local supersymmetry, and part of the analysis was performed in [27].

4.1. Integration on Supermanifolds: Pseudoforms and Densities

One of the main problem in the superform differential analysis is the construction of a consistent theory of integration. This has been deeply analized by Schwarz, Voronov and collaborators (for a more extensive discussion see [30]), where the construction of singular superforms and their relation with the usual integration on manifold have been worked out. Here we consider some aspects of their formulation. We will not dwell on the issue of the precise definition of a supermanifold, which can be found in several places (a good general reference is [16]).

As soon as the problem was posed, it became clear that the theory cannot be developed very far without departing from the analogy with the usual theory for bosonic manifolds. In particular, the main problem is that the straightforward generalization of a differential form does not have the right properties for integration on submanifolds. Several different solutions have been proposed, using various generalizations of differential forms.

The easiest ones to describe and work with are the *pseudodifferential forms* (or simply pseudoforms) of Bernstein-Leites [15] . Given a supermanifold $X^{M|N}$ (the exponent denotes as usual the bosonic/fermionic dimension), a pseudoform is a distribution on the cotangent space T^*X. In practice then it is a generalized function $\omega(z, dz)$ of the coordinates $(z^A) = (x^m, \theta^\alpha)$ and their differentials $(dx^m, d\theta^\alpha)$. It can be integrated on T^*X if it decreases sufficiently fast at infinity in the bosonic directions along the fibres. On the other hand the integration along a submanifold cannot always be defined, because in general it would involve a product of distributions (or said differently: unlike a function, a distribution cannot always be restricted to a subspace). Nevertheless, since they are easier to manipulate, we will work mainly with the complex of pseudoforms, denoted by $\Omega^*(X) = \sum_{m,n} \Omega^{m|n}(X)$ (where the range of the summation will be clarified later), after establishing their relation with two other types of objects, called *densities* [31] .

The densities are constructed to be integrated over submanifolds, and so they come with a (bi)grading corresponding to the superdimension of the submanifold. We define the *$m|n$-densities* $\mathcal{D}^{m|n}(X)$ to be functions $A(v_1, \dots v_m, w_1, \dots w_n)$ of m even and n odd vector fields. At each point of X, the arguments of A span an $m|n$ dimensional subspace of the tangent space TX. Writing a matrix R having the vectors (v_i, w_j) as columns, this subspace is just the image of R, seen as a linear map

$$R : \mathbf{R}^{m|n} \to TX \, .$$

So, without any ambiguity, we write the density A as $A(R)$. Under a linear change of basis $L \in GL(m|n)$, a density should transform as $A(RL) = A(R)\,\mathrm{Ber}\,L$, where Ber is the berezinian of a supermatrix defined as follows

$$Ber(L) = \frac{Det(L_1 - L_3 L_4^{-1} L_2)}{Det L_4}$$

where L_i are the four blocks of a supermatrix. Recalling the properties of the Berezinian one can see that densities are homogeneous of degree -1 in the odd vectors, and therefore will necessarily be singular if the odd vectors are not linearly independent (*i.e.* if R does not have maximal rank in the fermionic subspace). Given a submanifold $Y \subset X$ of dimension $m|n$, in terms of a parametrization $z^A = f^A(\zeta^K)$ where the index K runs over bosonic and fermonic indices, it is possible to define the integral over Y of a density:

$$\int_Y d\zeta^K A\Big(f(\zeta), \frac{\partial f^A}{\partial \zeta^K}\Big)\,.$$

This is well-defined thanks to the transformation properties of A. Notice that a density of degree equal to the dimension of the manifold is the same as a volume form on the manifold. In the notations we are using, this is defined by $A(R) = \mathrm{Ber}\,R$. Of course, other volume forms can be obtained by multiplication with a function $f(z)$, and the integral over the whole space will be non-vanishing only when $f = g(x)\theta^1 \dots \theta^N$. Then it is natural to single this one out as a preferred volume form vol_X.

There are also D-*densities* $\tilde{\mathcal{D}}^{m|n}$, defined as functions of $M - m|N - n$ cotangent vectors, or of an operator

$$S : TX \to \mathbf{R}^{M-m|N-n}.$$

Such an operator has an $m|n$-dimensional kernel which has to be thought of as the tangent space of a submanifold. If we require the transformation property $B(TS) = B(S)\,\mathrm{Ber}\,T$, for $T \in GL(M - m|N - n)$, then the following integral is well defined:

$$\int_X B\Big(z, \frac{\partial \Phi^K}{\partial z^A}\Big) \prod_K \delta(\Phi^K(z)) dz^A\,.$$

Thus D-densities can be integrated over a submanifold defined in terms of equations $\{\Phi^K(z) = 0, K = 1, \dots m|n\}$.

Clearly the concepts of densities and D-densities are equivalent, and there is a canonical isomorphism $\mathcal{D}^{m|n} \simeq \tilde{\mathcal{D}}^{M-m|N-n}$ given by $B(z, S) = A(z, S^\dagger)$. However, we can do more if X as a metric: we can define an Hodge duality. First of all, note that if R, S have maximal rank, we can assume, after a change of basis, that they are isometries on the image and kernel respectively. Then is sufficient to define a density on such isometries and extend it to a general operator using the covariance properties. Given R, there is an orthogonal decomposition $TX = V \oplus W$, with $V = \operatorname{Im} R$. Then $W \simeq \mathbf{R}^{M-m|N-n}$, and one defines S as the orthogonal projection over W. Then setting $B(S) = A(R)$, we define a 1-1 correspondence $\mathcal{D}^{m|n} \simeq \tilde{\mathcal{D}}^{m|n}$. Combining this with the natural isomorphism given above, we see that

$$\mathcal{D}^{m|n} \simeq \mathcal{D}^{M-m|N-n} .$$

This is what we expect from Hodge duality. We need however one more step: the classical formula which defines Hodge dual in the bosonic case is

$$\alpha \wedge *\beta = (\alpha, \beta)\, vol_X .$$

We would like this to hold with the preferred choice of the volume form that has non-vanishing integral. This uniquely defines the * operation in our case.

The relation between densities and pseudoforms on a supermanifold X is expressed by the Baranov-Schwarz transformation. It is given as a collection of maps

$$\lambda^{m|n} : \Omega^* \to \mathcal{D}^{m|n} , \tag{4.1}$$

$$\omega(z^A, dz^A) \mapsto [\lambda^{m|n}\omega](z^A, dz^A) = A(z, R) \equiv \int_{\mathbf{R}^{m|n}} D(dt^F)\, \omega(z^A, dt^F R_F^A) .$$

To avoid confusion, we notice that here $D(dt)$ denotes Berezin integration over the variable dt. A pseudoform does not have a grading in principle. But if its image under the transformation is zero except for one $\lambda^{m|n}$ then it can be said to be homogeneous of degree $m|n$. We can then formulate Hodge duality for homogeneous pseudoforms. To see how this works explicitly, we consider the simplest possible case, namely $X = \mathbf{R}^{1|1}$, with coordinate (x, θ). A vector on $TX \simeq X$ will be written as $v = (v^1, v^{\bar{1}})$. Take the pseudoform $\omega = dx$. Its transform in $\mathcal{D}^{1|0}$, a function of one even vector, is given by:

$$A(v) = \left[\lambda^{1|0}\omega\right](v) = \int D(dt)\, dt\, v^1 = v^1 .$$

We find now the dual form in $\mathcal{D}^{0|1}$, a function of an odd vector $w = (w^1, \tilde{w}^{\bar{1}})$ with w^1 anticommuting and $\tilde{w}^{\bar{1}}$ commuting:

$$\tilde{A}(w) = \tilde{A}\begin{pmatrix} w^1 \\ w^{\bar{1}} \end{pmatrix} = \frac{1}{w^{\bar{1}}}\tilde{A}\begin{pmatrix} w^1/w^{\bar{1}} \\ 1 \end{pmatrix} = \frac{1}{w^{\bar{1}}}B(\frac{w^1}{w^{\bar{1}}}, 1) = \frac{1}{w^{\bar{1}}}A\begin{pmatrix} 1 \\ w^1/w^{\bar{1}} \end{pmatrix} = \frac{1}{w^{\bar{1}}} . \tag{4.2}$$

In the first equality we used the linearity of the $\tilde{A}$. Then we use the properties of the BS transformations. It is easy to see that this is the Baranov-Schwarz transform of $\delta(d\theta)$ (in fact $[\lambda^{1|0}\omega](v) = \int D(dt)\delta(\tilde{w}^{\bar{1}}dt) = \int D(\tilde{w}^{\bar{1}}dt)\delta(\tilde{w}^{\bar{1}}dt)/\tilde{w}^{\bar{1}} = \frac{1}{\tilde{w}^{\bar{1}}}$.)

If we start instead with $\omega = d\theta$,

$$A(v) = \lambda^{1|0}\omega(v) = \int D(dt)\, dt\, v^{\bar{1}} = v^{\bar{1}}\,.$$

The computation of the dual form is the same as in (4.2) , except at the last step when A now picks up the fermionic component of the argument, so

$$\tilde{A}(w) = \frac{w^1}{(w^{\bar{1}})^2}\,.$$

This is the BS-transform of $\omega = dx\delta'(d\theta)$, in fact

$$\lambda^{0|1}\omega(w) = \int D(dt) w^1\, dt\, \delta'(w^{\bar{1}}dt) = \frac{w^1}{(w^{\bar{1}})^2}\,. \tag{4.3}$$

In conclusion, we find the action of $*$ in $\mathbb{R}^{1|1}$ to be

$$\begin{aligned} *(dx) &= \theta\,\delta(d\theta) \\ *(d\theta) &= dx\,\theta\,\delta'(d\theta)\,. \end{aligned} \tag{4.4}$$

Applying again the BS transform, it is easy to show that $*^2 = 1$.

We come now to the case of interest to us, $X = \mathbf{R}^{3|2}$. We will compute the dual of the forms that appear in the measure (3.14) . We use now notations compatible with those of the previous section, which amounts to making the identifications $dx^m \to c^m$, $d\theta^\alpha \to \lambda^\alpha$. Let us start with the last term, $\omega = \theta^2\delta^2(\lambda)$ and we consider a vector of $T^{3|2}X$ denoted by (w^m, w^α). Its BS transform in $\mathcal{D}^{0|2}$ is $A(w^\alpha) = \det^{-1}(w^\alpha_\beta)$. The dual form $A \in \mathcal{D}^{3|0}$ is

$$\tilde{A}(v^m) = \tilde{A}\begin{pmatrix} v^m_n \\ v^m_\alpha \end{pmatrix} = \det(v^m_n)\tilde{A}\begin{pmatrix} 1 \\ v^m_\alpha (v^{-1})^n_m \end{pmatrix} = \det(v^m_n)A(w_\alpha) = \det(v^m_n)\,, \tag{4.5}$$

where w_α are odd vectors with $w^\alpha_\beta = \delta^\alpha_\beta$, and $w^\alpha_m = v^\alpha_n(v^{-1})^n_m$. Then $A(w_\alpha) = 1$ from which the last equality follows. Clearly $\tilde{A}$ is the BS transform of the pseudoform $\epsilon_{mnp}c^m c^n c^p$. The other terms are less easy. We will explicitly compute only the first one, $\omega = \epsilon_{mnr}c^m c^n \gamma^{r\alpha\beta}\frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\lambda^\beta}\delta^2(\lambda)$. Its transform in $\mathcal{D}^{0|2}$ is

$$A(w_\alpha) = \det{}^{-1}(w^\beta_\alpha)\epsilon_{mnr}w^m_\rho(w^{-1})^\rho_\alpha\gamma^{r\alpha\beta}(w^{-1})^\sigma_\beta w^n_\sigma\,.$$

In computing the dual, we have the same relation as before between v_m and w_α, and using the last expression in (4.5) we get

$$\tilde{A}(v_m) = \det(v_i^j)\epsilon_{mnp}(v^{-1})^m_r(v^{-1})^n_s v^r_\alpha \gamma^{p\alpha\beta} v^s_\beta = v^r_\alpha \gamma^{p\alpha\beta} v^s_\beta \epsilon_{rst} v^t_p \,. \tag{4.6}$$

Noting that the expression is linear in the bosonic part and bilinear in the fermionic part of v_m, we can conclude that this is the BS transform of a pseudoform which is schematically $c\lambda\lambda$. A careful computation shows that $*\omega = \epsilon_{mnr} c^m (\theta\gamma^n\lambda)(\theta\gamma^r\lambda)$.

We can now understand the origin of the measure (3.14) : its terms are precisely the duals of the terms generated from $\omega_0 = \epsilon_{mnp}\hat{c}^m\hat{c}^n\hat{c}^p$, by the unitary transformation (2.10), which acts on the fields as follows:

$$\begin{aligned}
\hat{p}_\alpha &= p_\alpha + \frac{i}{2} b_m (\gamma^m\lambda)_\alpha \,, \\
\hat{c}_m &= c_m + \frac{i}{2}(\lambda\gamma_m\theta) \,, \\
\hat{w}_\alpha &= w_\alpha - \frac{i}{2} b_m (\gamma^m\theta)_\alpha \,.
\end{aligned} \tag{4.7}$$

In the next section we study in detail the complex of pseudoforms and its cohomology, and we will show that ω_0 is the unique element of the cohomology of Q' in degree $3|0$.

4.2. Superforms

We have learnt that we can use the BS transform of superforms to define a meaningful integration on supermanifold. So, in the following we will consider only the pseudoforms and we describe them. Afterwards we discuss the BRST complexes of these forms and we show how the PCO operators play a role in the present framework.

In the case of the superspace $\mathbf{R}^{3|2}$, the complex of superforms contains the following spaces:

$$\begin{aligned}
\Omega^{r|0} &\neq 0 \,, \qquad \text{for} \quad r \geq 0 \,, \\
\Omega^{r|1} &\neq 0 \,, \qquad \text{for} \quad r \in \mathbf{Z} \,, \\
\Omega^{r|2} &\neq 0 \,, \qquad \text{for} \quad r \leq 3 \,,
\end{aligned} \tag{4.8}$$

and the spaces are empty for other values of $r|s$. Here s counts the number of delta functions in the pseudoform. Again, instead of using the superform notation dx^m and $d\theta^\alpha$, we replace them with the ghosts c^m and λ^α which have the same statistics.

In order to write covariant expressions involving $\delta(\lambda)$, we have two possibilities: $\delta^2(\lambda)$ and $\delta(v_\alpha\lambda^\alpha)$ where v_α is a spinor. In general, v_α is not constant and to define an expression with a single delta function, we have to choose a direction in the spinorial space (represented by the real spinor v_α), but we should cover the complete space. So, we introduce two of such spinors $v^{(i)}_\beta$ where $i = 1, 2$ labels the elements of a basis of the vector space and we define the two delta functions as $\delta(v^{(i)}_\beta \lambda^\beta)$. In addition, the basis is orthonormal $v^{(i)}_\gamma \epsilon^{\gamma\delta} v^{(i)}_\delta = \epsilon^{ij}$. This reduces the number of indipendent real components of $v^{(i)}_\beta$ to three and they belong to $SL(2, \mathbf{R})$. We want to reduce further the number of independent components to two, therefore we factorize the compact subgroup $U(1)$. We assume therefore that the coordinates $v^{(i)}_\beta$ belong to the space $SL(2, \mathbf{R})/U(1)$ which is not compact. They essentially form a set of harmonic coordinates and therefore in order to render the computation covariant at the end we integrated over the coset.[12] One interesting choice is $v^{(i)}_\beta(P) = \gamma^{i\gamma}_m \epsilon_{\gamma j} P^m/\sqrt{P^2}$ which has $\det(v^{(i)}_\beta) = 1$ and where we factorize the $SO(2)$ rotation in the $x - y$ plane. The vectors $v^{(i)}$ can be viewed as gauge fixing parameters needed to fix the zero mode gauge (the action is clearly invariant under any transformations of the zero modes) and, by integrating over them, the Lorentz covariance is reestablished [43].

Explicitly, the $\Omega^{r|0}$ forms have the structure

$$\Omega^{r|0} = f(x, \theta)(c)^l (\lambda)^{r-l}\,, \qquad 0 \le l \le 3\,, \qquad r \ge 0\,. \tag{4.9}$$

for they have no delta functions (the vertex operator $U^{(1|0)} = \lambda^\alpha A_\alpha - c^m A_m$ discussed in Sec. 3.2 belongs to this space); for one delta function vertex operators

$$\Omega^{r|1} = g(x, \theta)(c)^l (v_\perp \cdot \lambda)^k \delta^{(k+l-r)}(v \cdot \lambda)\,, \qquad 0 \le l \le 3\,, \qquad k \ge 0\,, \qquad r \ge 0\,. \tag{4.10}$$

for the 1-picture forms, where $v_\perp$ is the orthogonal direction to v. Here, we used

$$\delta^{(n)}(\lambda^\beta) = \partial_{\lambda^{\alpha_1}} \dots \partial_{\lambda^{\alpha_n}} \delta(\lambda^\beta)\,, \qquad n > 0\,, \qquad \text{and} \qquad \delta^{(0)}(\lambda^\beta) = \delta(\lambda^\beta)\,. \tag{4.11}$$

The indices of the commuting ghosts λ^α in front of the Dirac delta functions and of argument of the latter are different. For instance, $\lambda^\alpha \delta^{(0)}(\lambda^\beta) = \epsilon^{\alpha\beta} \lambda^\alpha \delta^{(0)}(\lambda_\alpha)$ in such a way that it does not vanish. We also use the rule $\lambda^\alpha \delta^{(n)}(\lambda^\alpha) = (-)\delta^{(n-1)}(\lambda^\alpha)$ where α is not summed.

[12] Be aware that the integration over non-compact coordinates belonging a coset can be done by the Haar measure, but some behavior at infinity has to be assumed for the space of functions of the coordinates $v^{(i)}_\beta$.

Finally, the two-delta function forms are given by

$$\Omega^{r|2} = h(x,\theta)(c)^l \Big[\delta^2(\lambda)\Big]^{(l-r)}, \quad 0 \le l \le 3\,, \quad r \le 3\,. \tag{4.12}$$

The superscript over the delta function means $l-r$ derivative of the delta whit the conventions $\partial_{\lambda^\alpha}\delta^2(\lambda) = \epsilon_{\beta\gamma}\Big(\partial_{\lambda^\alpha}\delta(\lambda^\beta)\Big)\delta(\lambda^\gamma)$. The functions f, g and h are superfields. In the specific example of superforms that we are considering, we take into account only the sector with $\lambda^\alpha, \theta^\alpha, x^m, c^m$ as variable and they span the manifold $\mathcal{M} = \mathbf{R}^{3|2} \oplus T^*\mathbf{R}^{3|2}$.

Notice that forms on $\mathcal{M}$ have only positive pictures. Later we will enlarge the space and have negative pictures as well.

Let us discuss the de Rham cohomology in the space of superforms. We consider first the simplest example of $R^{1|1}$, with coordinates (x,θ), and differential $d = c\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial\theta}$. The space is the product of the ordinary real line and the 1-dimensional odd superspace $R^{0|1}$, and the differential splits in a sum of operators acting independently on the two factors, so we can compute the cohomology separately in each factor. For $R^{1|0}$, this is the usual de Rham cohomology, but we have to pay some attention to the behavior at infinity. If we work in the complex of forms which decay at infinity, then the cohomolgy is generated by [13] $\{1,\, c\delta(x)\}$. We will have to consider also the zero-momentum cohomology, defined in the complex of x-independent forms. Then the generators are $\{1,\, c\}$. In the odd direction, the forms can have zero or one delta function. Without any delta function, the most general form is $\alpha = (a + b\theta)\,\lambda^n \in \Omega^{n|0}$. It is closed iff $b = 0$, whilst the term proportional to a is exact since $\lambda^n = d(\theta\,\lambda^{n-1})$, so the only cohomology class is $1 \in H^{0|0}$. At picture 1, the most general form is $\alpha = \sum_{n\ge 0}(a_n + b_n\theta)\delta^{(n)}(\lambda) \in \oplus_{n\ge 0}\Omega^{-n|1}$. It is closed iff $b_n = 0$, $n \ge 1$; moreover $\delta^{(n)}(\lambda) = d(\theta\delta^{(n+1)}(\lambda)$. So the only generator is $\theta\delta(\lambda) \in H^{0|1}$. The total cohomology is obtained by tensoring the ones for the two factors, and so it is

$$H^*(R^{1|1}) = \{1,\, c,\, \theta\delta(\lambda),\, c\,\theta\delta(\lambda)\}. \tag{4.13}$$

It is interesting to see that $H^{*|0}$ and $H^{*|1}$ are isomorphic spaces. We will see that the isomorphism can be given explicitly by the picture changing operators. It is now easy to generalise from $\mathbf{R}^{1|1}$ to $\mathbf{R}^{3|2}$. We can again use a Künneth-type argument to see that the cohomology at the various pictures is given by:

$$\begin{aligned}
H^{*|0}(\mathbf{R}^{3|2}, d) =& \{1, c^m, c^m c^n, \epsilon_{mnp}c^m c^n c^p\}\,,\\
H^{*|1}(\mathbf{R}^{3|2}, d) =& \{1, c^m, c^m c^n, \epsilon_{mnp}c^m c^n c^p\} \otimes \{v^{(i)}\cdot\theta\,\delta(v^{(i)}\cdot\lambda)\}\,,\\
H^{*|2}(\mathbf{R}^{3|2}, d) =& \{1, c^m, c^m c^n, \epsilon_{mnp}c^m c^n c^p\} \otimes \{\theta^2\delta^2(\lambda)\}\,.
\end{aligned} \tag{4.14}$$

[13] We denote by $\{\ldots\}$ the span of the elements inside the brackets.

We are ultimately interested in the cohomology of the complex (4.8) with the differential given by the BRST operator as in (2.8) . We can observe that at zero momentum the two differentials coincide, and moreover, as previously noticed, there is a unitary transformation (2.10) that brings the BRST operator into the de Rham one. But it can be convenient to work out directly the BRST cohomology in order to have manifestly supersymmetric expressions. For instance, the most general vertex operator at ghost number +1 has the form

$$U^{1|2} = c^m A_m \delta^2(\lambda) + c^m c^n A_{nm}{}^{\alpha} \partial_{\lambda^\alpha} \delta^2(\lambda) + c^m c^n c^p A_{pnm}{}^{\alpha\beta} \partial_{\lambda^\alpha} \partial_{\lambda^\beta} \delta^2(\lambda) \,. \tag{4.15}$$

One can check that for $A_{nm}{}^{\alpha} = A_{pnm}{}^{\alpha} = 0$ and $F_{mn} = \partial_{[m} A_{n]} = 0$, this vertex is in the BRST cohomology. However, it will be easy to see that vertex can be obtained by acting on it with suitable differential operators which map the complesses of pseudoforms transversally. Notice that both Q and d map $\Omega^{r|s} \to \Omega^{r+1|s}$, and neither of them changes the number of delta functions.

In fact, as will become clear later, to define the integration measure for multiloops, we have to enlarge the complex (4.8) to include forms that do not have a direct geometrical interpretation as form living in the target space. Once we make the identification of the differentials with the ghosts of the conformal field theory, we have to take into account the presence of antighosts. We are then led to consider the full space $\mathcal{M} \oplus \hat{\mathcal{M}}$ where $\hat{\mathcal{M}}$ is the variety described by the conjugate momenta P_m and p_α and the vector fields w_α and b_m belonging to $T_*(\mathbf{R}^{3|2} \oplus T^*\mathbf{R}^{3|2})$. The variables $(x^m, \theta^\alpha, c^m, \lambda^\alpha)$ parametrize $\mathbf{R}^{3|2} \oplus T^*\mathbf{R}^{3|2}$, while $(P_m, d_\alpha, b_m, w_\alpha)$ parametrize the space $T_*(\mathbf{R}^{3|2} \oplus T^*\mathbf{R}^{3|2})$. Vectors in the latter space can be identified with the differential operators $(\partial_m, D_\alpha, \iota_m, \iota_\alpha)$ with

$$w_\alpha \equiv \partial_{\lambda^\alpha} = \frac{\partial}{\partial(d\theta^\alpha)} = \iota_{v_{(\alpha)}} \,, \qquad b_m \equiv \partial_{c^m} = \frac{\partial}{\partial(dx^m)} = \iota_{v_{(m)}} \,, \tag{4.16}$$

where $v_{(\alpha)} = v^\beta_{(\alpha)} D_\beta + v^m_{(\alpha)} \partial_m$, with $v^\beta_{(\alpha)} = \delta^\beta_\alpha$ and $v^\beta_m = 0$, while $v_{(m)} = v^\beta_{(m)} D_\beta + v^n_{(m)} \partial_n$, with $v^\beta_{(m)} = 0$ and $v^n_m = \delta^n_m$. Furthermore, we have

$$d_\alpha + b_m(\gamma^m \lambda)_\alpha = \Big[Q, w_\alpha\Big] \equiv \Big[d, \iota_{v_{(\alpha)}}\Big] = \mathcal{L}_{v_{(\alpha)}} \,, \tag{4.17}$$

$$\mathcal{L}_{v_{(\alpha)}} = D_\alpha + (\lambda\gamma^m)_\alpha \iota_{v_{(m)}} \,,$$

$$P_m = \Big\{Q, b_m\Big\} \equiv \Big\{d, \iota_{v_{(m)}}\Big\} = \mathcal{L}_{v_{(m)}} \,,$$

$$\mathcal{L}_{v_{(m)}} = \partial_m \, .$$

The differential operator $\iota_{v_{(\alpha)}}$ (in the following we denote this operator as ι_α) is commuting in contrast to the usual interior multiplication provided by $\iota_{v_{(m)}}$ (it will be denoted by ι_m) which is anticommuting. Therefore, as for the superforms λ^α, we can consider the superforms of negative picture of type $\delta^2(w) = \delta^2(\iota_\alpha)$ and in general we can consider forms like these:

$$\Omega^{r|-2} = m(x,\theta)(c)^l(b)^r \left[\delta^2(w)\right]^{l-r-s} \tag{4.18}$$

For instance, we see that the space $\Omega^{0|0}$ of forms with vanishing picture will contain also an element $\delta^2(\lambda)\delta^2(w)$.

4.3. Cartan Calculus and PCO

Given an odd vector field $\tilde{v} = v^\alpha(x,\theta)D_\alpha + v^m(x,\theta)\partial_m \equiv: v^\alpha(x,\theta)d_\alpha + v^m(x,\theta)P_m$:[14] where v^α and v^m are, respectively, commuting and anticommuting functions of x, θ, λ and c, we define

$$\iota_{\tilde{v}} = v^\alpha \partial_{\lambda^\alpha} + (v^m + v^\alpha \gamma^m_{\alpha\beta}\theta^\beta)\partial_{c^m} \equiv v^\alpha w_\alpha + (v^m + v^\alpha \gamma^m_{\alpha\beta}\theta^\beta) b_m \, , \tag{4.19}$$

$$\mathcal{L}_{\tilde{v}} = [d, \iota_{\tilde{v}}] \equiv \left[Q, v^\alpha w_\alpha + (v^m + v^\alpha \gamma^m_{\alpha\beta}\theta^\beta) b_m\right] ,$$

where $\iota_{\tilde{v}}$ is a commuting differential operators action on the space of pseudoforms $\Psi^{r|s}$. For an even vector v, the usual rules apply. The first expression in (4.19) is evaluated on functions which depend only on λ^α and c^m and they are independent of their derivatives. Notice that the operator $\iota_{\tilde{v}}$ reduce the form degree and equivalently it reduces the the ghost number. However, both the operations Q and $\iota_{\tilde{v}}$ do not change the number of the delta functions. Since Q is anticommuting and $\iota_{\tilde{v}}$ is commuting, we used the commutator to define the Lie derivative.

The Cartan algebra is respected:

$$Q^2 = 0 \, , \qquad \{Q, \mathcal{L}_{\tilde{v}}\} = 0 \, , \qquad \{\mathcal{L}_{\tilde{v}}, \iota_{\tilde{u}}\} = \iota_{\{\tilde{v},\tilde{u}\}} \, , \quad [\iota_v, \iota_{\tilde{u}}] = 0 \tag{4.20}$$

where $\tilde{u}$ is another odd vector.

[14] In the operator-formalism expressions in $\tilde{v}$ and $\iota_{\tilde{v}}$, we used the normal ordering in the case that the component of v^α and v^m depend upon x and θ.

Indeed $\mathcal{L}_{\tilde{v}}$ is an anticommuting differential operator and has the explicit expression

$$\mathcal{L}_{\tilde{v}} = - : v^\alpha d_\alpha - v^m P_m + \Big[\lambda^\alpha D_\alpha v^\beta + c^m \partial_m v^\beta\Big] w_\beta : \tag{4.21}$$
$$+ : \Big[\lambda^\alpha D_\alpha v^n + c^m \partial_m v^n + \lambda^\alpha D_\alpha v^\beta \gamma^m_{\beta\gamma} \theta^\gamma + 2(\lambda\gamma^m v)\Big] b_n : \,.$$

Since the differential $\iota_{\tilde{v}}$ is commuting it can be used to define a formal Dirac delta function, its derivatives and the Heaviside function

$$\delta(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt e^{it\iota_{\tilde{v}}} \,, \qquad \delta^{(n)}(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt\, t^n e^{it\iota_{\tilde{v}}} \,, \qquad \Theta(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt \frac{1}{t} e^{it\iota_{\tilde{v}}} \,, \tag{4.22}$$

which are well-defined operations on the space of pseudoforms $\Omega^{r|s}$. The above expressions can be rewritten in terms of w_α and b_m by substituting the interior differentials with the combination of ghosts and we have

$$\delta(\iota_{\tilde{v}}) = \delta\Big(v^\alpha w_\alpha\Big) + i\delta'\Big(v^\alpha w_\alpha\Big) v^m b_m - \frac{1}{2}\delta''\Big(v^\alpha w_\alpha\Big) v^m v^n b_m b_n - \frac{i}{6}\delta'''\Big(v^\alpha w_\alpha\Big) v^3 b^3 \,, \tag{4.23}$$

where $v^3 = \epsilon_{mnr} v^m v^n v^r$ and $b^3 = \epsilon_{mnr} b^m b^n b^r$. A fundamental property used in the next sections is

$$\iota_{\tilde{v}} \delta(\iota_{\tilde{v}}) = 0 \,. \tag{4.24}$$

It follows directly by the integral definition (4.22). For an even vector field v we identify $\delta(\iota_v) = \iota_v$ and equation (4.24) is obvious.

Finally, we can define the Picture Changing Operator as follows

$$Z_{\tilde{v}} = \Big[Q, \Theta(\iota_{\tilde{v}})\Big] = \delta(\iota_{\tilde{v}})\mathcal{L}_{\tilde{v}} + \delta'(\iota_{\tilde{v}})\iota_{\tilde{v}^2} \,, \tag{4.25}$$

where

$$\begin{aligned} \tilde{v}^2 &= \left(v^\beta D_\beta v^\alpha + v^m \partial_m v^\alpha\right) d_\alpha + \left(v^\beta D_\beta v^n + v^m \partial_m v^n\right) P_n \,, \\ \iota_{\tilde{v}^2} &= \left(v^\beta D_\beta v^\alpha + v^m \partial_m v^\alpha\right) w_\alpha + \left(v^\beta D_\beta v^n + v^m \partial_m v^n\right) b_n \,, \end{aligned} \tag{4.26}$$

The general form of those operators is rather complicate, however one can choose the most convenient vector $\tilde{v}$ in order to simplify those operators. The amplitudes will not depend on the choice of the odd vector $\tilde{v}$ since

$$\delta_{\tilde{v}^\alpha} Z_{\tilde{v}} = \Big[Q, (\delta_{\tilde{v}^\alpha} \iota_{\tilde{v}}) \delta(\iota_{\tilde{v}})\Big] \,. \tag{4.27}$$

Notice that, as opposed to $Z_{\tilde{v}}$, its variation is a BRST variation of a pseudoform. Therefore, by inserting the PCO in the amplitudes we are guaranteed that the latter are independent of the choice of the gauge parameters $\tilde{v}$. In the same way, the dependence of the PCO upon the time (the worldline coordinate) is also a BRST variation of a delta function which is an element of the space of the pseudoforms. This implies that the PCO are independent of the position on the worldline.

The odd vector can be field-dependent or field-independent. For example, let us choose $\tilde{v}_1^\alpha = constant$ or a ghost-number one combination $\tilde{v}_2^\alpha = B_{mn}(\gamma^{mn}\lambda)^\alpha$ where B_{mn} is constant, we have

$$Z_{\tilde{v}_1} = \tilde{v}_1^\alpha (d + b^p \gamma_p \lambda)_\alpha \delta(\tilde{v}_1^\alpha w_\alpha)\,, \tag{4.28}$$

$$Z_{\tilde{v}_2} = B_{mn}\lambda\gamma^{mn}(d + b^p\gamma_p\lambda)\delta(B_{mn}\lambda\gamma^{mn}w)\,,$$

where it can easily be seen that in the second operator, the generator of the Lorentz transformations $N^{mn} = \lambda\gamma^{mn}w$ appears. The gauge parameters B_{mn} can be chosen in such a way that the no normal ordering is necessary to define $Z_{\tilde{v}_2}$. These PCO have been discussed and used in [43] and in [24].

Acting on the space of superforms on $\mathcal{M}$, the PCO $Z_{\tilde{v}_1}$ becomes

$$Z_{\tilde{v}_1}\Omega^{(p|q)} = \Big[\tilde{v}^\alpha (D + \lambda\gamma^m \partial_{c^m})_\alpha \delta(\tilde{v}^\alpha \partial_{\lambda^\alpha})\Big]\Omega^{(p|q)} \to \Omega^{(p|q-1)}\,. \tag{4.29}$$

The delta function $\delta(\tilde{v}^\alpha\partial_{\lambda^\alpha})$ reduces the number of delta functions $\delta(\lambda^\alpha)$ present in the pseudoform since

$$\delta(\tilde{v}^\alpha\partial_{\lambda^\alpha})\delta(f_\alpha\lambda^\alpha) = \int_{-\infty}^{\infty} dt\, e^{it\tilde{v}^\alpha\partial_{\lambda^\alpha}}\, \delta(f_\alpha\lambda^\alpha) = \int_{-\infty}^{\infty} dt\, \delta\Big(f_\alpha(\lambda^\alpha + itv^\alpha)\Big) = -i\frac{1}{f_\alpha v^\alpha}\,.$$

where f_α is a λ-independent parameter.[15] Notice that the operator $\delta(\tilde{v}^\alpha\partial_{\lambda^\alpha})$ is ill-defined on pseudoforms which carry a different picture ($v^\alpha f_\alpha = 0$). In that case, we define it to

[15] In conformal field theory

$$\delta(\tilde{v}^\alpha w_\alpha(y))\delta(f_\alpha\lambda^\alpha(z)) = \int_0^{\infty} dt\, e^{it\tilde{v}^\alpha w_\alpha(y)}\, \delta(f_\alpha\lambda^\alpha(z)) =$$

$$\int_0^{\infty} dt\, \delta\Big(f_\alpha(\lambda^\alpha + \frac{it}{(y-z)}v^\alpha)\Big) = -i\frac{(y-z)}{f_\alpha v^\alpha}\,.$$

So, if $\Omega^{(0|1)} = i(f_\alpha\theta^a)\delta(f_\alpha\lambda^\alpha)$, then we have that

$$\lim_{y\to z}\Big(Z_{v_1}(y)\Omega^{(0|1)}(z)\Big) = \lim_{y\to z}\Big[\tilde{v}_1^\alpha(d + b^m\gamma_m\lambda)_\alpha(y)\delta(\tilde{v}_1^\alpha w_\alpha(y))\Omega^{(0|1)}(z)\Big] = 1\,.$$

So, the vertex $\Omega^{(0|1)} = i(f_\alpha\theta^a)\delta(f_\alpha\lambda^\alpha)$, which is BRST invariant, is the inverse picture changing operator. Notice that the gauge parameters $\tilde{v}^\alpha$ and f_α cancel out.

vanish. So, effectively the PCO reduce the number of delta functions in the vertex operator belonging to $\Omega^{(p|q)}$.

The PCO $Z_{\tilde{v}}$ are designed to reabsorb the zero modes of w_α in the path integral measure. However, the number of zero modes is $n \times g$ where n is the Grassmann dimension of the superspace $\mathbf{R}^{(m|n)}$ and g is the genus of the Riemann surface. Therefore, we need to stack a corresponding number of PCO. This can be done by choosing a basis of abelian differential ω_i^α where $\alpha = 1, \ldots, n$ and $i = 1, \ldots, g$ defined such that

$$\oint_{a_i} \omega_j^\alpha = \delta_{ij} v^\alpha \tag{4.30}$$

where a_i is the a-cycle on the Riemann surface and v^α is a constant (or depending on the zero modes θ^α, λ^α and x^m). Using this basis, we can finally construct the operator

$$\prod_{i=1}^{g} \prod_{\alpha_i=1}^{n} Z_{\omega_i^\alpha} \tag{4.31}$$

which maps the cohomology H onto the cohomology group a different number of pictures. Also in the present case we conjecture that the number of zero modes for w_α and b_m is equal to number of differential forms on a genus g Riemann surface. However, in the particle limit the antighost fields loose their conformal weight. We assume however that the number of zero modes remains invariant. (An analysis of higher loop expansion for superparticle model was performed in [25]). The problem to derive this number of zero modes has to be ascribed to the non-manifold nature of particle graph. Those graphs are embedded into string theory graph and the conformal weights of the fields play an important role.

Let us now consider the odd form of $\Omega^{(1|0)}$: $\tilde{f} = f_\alpha \lambda^\alpha + f_m c^m$ where f_α and f_m are commuting and anticommuting, respectively. These components can also be field dependent and the total parity of the odd form is 0. We define the exterior product as usual:

$$e_{\tilde{f}} : \Omega^{(n|p)} \to \Omega^{(n+1|p)}\,, \qquad e_{\tilde{f}} \omega = \tilde{f} \wedge \omega\,. \tag{4.32}$$

This operation, together with $\iota_{\tilde{v}}$ for an odd vector $\tilde{v}$, satisfies the following Clifford algebra:

$$\Big[e_{\tilde{f}}, e_{\tilde{g}}\Big] = 0\,, \qquad \Big[e_{\tilde{f}}, \iota_{\tilde{v}}\Big] = \tilde{f}(\tilde{v}) \equiv f_\alpha v^\alpha + f_m v^m\,. \tag{4.33}$$

Again, since the operation $e_{\tilde{f}}$ is an even operation, we can define Dirac delta functions of this operator. In particular we can construct an inverse operation to $Z_{\tilde{v}}$ as follows:

given an anticommuting superfield $f = f(x, \theta)$ on the supermanifold, its differential $df = \{Q, f\} \equiv \tilde{f} \in \Omega^{(1|0)}$ is an odd differential form and we define the inverse PCO with

$$Y_f = f(x, \theta)\delta(e_{\{Q,f\}})\,. \tag{4.34}$$

It is obvious that Y_f is BRST invariant, in fact action with Q on Y_f, one has $\{Q, Y_f\} = \{Q, f\}\delta(e_{\{Q,f\}}) = 0$ because of the property of delta functions. In addition, any change of the function f is BRST exact

$$\delta Y_f = \delta f \delta(e_{\{Q,f\}}) - f\{Q, \delta f\}\delta'(e_{\{Q,f\}}) = \Big[Q, f\delta f \delta(e_{\{Q,f\}})\Big]\,. \tag{4.35}$$

It is therefore convenient to choose very simple representation for Y_f, such as, for example those related to the choices $f = f_\alpha \theta^\alpha$ and with $f_\alpha = constant$ or field dependent $f_\alpha = w_\beta C^\beta{}_\alpha$ (with $C^\beta_\alpha = constant$)

$$Y_{f_1} = f_\alpha \theta^\alpha \delta(e_{f_\alpha \lambda^\alpha})\,. \tag{4.36}$$

$$Y_{f_2} = \Big(w_\beta C^\beta{}_\alpha \theta^\alpha\Big)\delta(e_{[(d+b^p\gamma_p\lambda)_\beta C^\beta{}_\alpha \theta^\beta + w_\beta C^\beta{}_\alpha \lambda^\alpha]})\,.$$

In the following, we will just write $\tilde{f}$ instead of $e_{\tilde{f}}$ in the delta function since there is no ambiguity.

Given a set of anticommuting functions f_i where $i = 1, \dots, n$ where n is the odd dimension of the superspace $\mathbf{R}^{m|n}$ and n is the maximal number of delta functions of a given vertex, we can define the PCO

$$Y_{\{f_i\}_i} = \prod_{i=1}^{n} f_i \delta(\{Q, f_i\}) \tag{4.37}$$

which has the same properties of Y_f, but this maps the cohomology $H^{(m|0)}(Q)$ into $H^{(m|n)}(Q)$, the cohomology group with the maximal number of delta functions.

4.4. Changing the Number of Delta Functions

To show how the PCO work, we consider several examples. First, we can go back again to the simplest case of $\mathbf{R}^{1|1}$ with the de Rham differential. We observed after (4.13) that there is a 1-1 correspondence between $H^{*|0}$ and $H^{*|1}$. We can now exhibit the isomorphism as a PCO. In this case there is only one constant odd vector field, namely $\tilde{v} = \frac{\partial}{\partial \theta}$, and the corresponding picture-lowering operator is $Z = \delta(\iota_{\tilde{v}})\mathcal{L}_{\tilde{v}}$ and acts as $\theta\delta(\lambda) \mapsto 1$. The inverse operation is evidently $Y = \theta\delta(\lambda)$, which corresponds to (4.34) with $f = \theta$. The two operators Z and Y_f are in this case inverses on the whole complex $\Omega^{*|*}$. A different choice of f would give an equivalent Y, which would in general fail to be an exact inverse to Z, but would still be an inverse in the cohomology.

We consider the vertex operator $U^{(1|0)} = \lambda^\alpha A_\alpha - c^m A_m$ described in sec. 3.1, and we can act with the PCO

$$Y_{\{f^{(\beta)}\}} = \prod_{\beta=1}^{2} (f_\alpha^{(\beta)}\theta^\alpha)\delta(f_\alpha^{(\beta)}\lambda^\alpha) = \theta^2\delta^2(\lambda^\alpha)$$

(where we normalize the function f such that $f_\alpha^{(\gamma)} f_\beta^{(\delta)}\epsilon^{\alpha\beta} = \epsilon^{\gamma\delta}$) to have

$$U^{(1|2)} = Y_{\{f^{(\beta)}\}}U^{(1|0)} = \theta^2\delta^2(\lambda)(\lambda^\alpha A_\alpha - c^m A_m) = -c^m(A'_m)\delta^2(\lambda) \tag{4.38}$$

where $A'_m = A_m\theta^2 = a_m(x)\theta^2$. It is easy to check that $U^{(1|2)}$ is in the cohomology $H^{(1|2)}(Q)$ if a_m is a flat connection. Since $U^{(1|0)}$ was in the cohomology $H^{(1|0)}(Q)$, one can easily show that the new vertex operator $U^{(1|2)}$ depends upon the choice of the parameters f_α in the PCO only through BRST exact terms which do not affect the amplitudes. We can also compute the transformed vertex operator $U^{(2|2)}$ as follows

$$U^{(2|2)} = Y_{\{f^{(\beta)}\}}U^{(2|0)} = \theta^2\delta^2(\lambda)(\lambda^\alpha\lambda^\beta A^*_{\alpha\beta} - c^m\lambda^\beta A^*_{m\beta}c^m c^p A^*_{[mp]}) \tag{4.39}$$

$$= -c^m c^p (A^*_{[mp]}\theta^2)\delta^2(\lambda)$$

which is again in the cohomology $H^{(2|2)}$.

In the same way, we can consider the vertex operator in the zero momentum cohomology $U^{(3|0)}$ given in (3.10) and we can construct the vertex $U^{(3|2)}$ with 2 delta functions as follows

$$U^{(3|2)} = Y_{\{f^{(\beta)}\}}U^{(3|0)} = c^m c^n c^r \epsilon_{mnr}\theta^2\delta^2(\lambda) \tag{4.40}$$

which is the top form of $\mathbf{R}^{(3|2)}$ and it can be integrated on this space. Again, the dependence on $f_\alpha^{(\beta)}$ is only through BRST exact terms and $U^{(3|2)}$ is representative of the cohomology group $H^{(3|2)}(Q)$ at zero momentum.

It is also very interesting to use the PCO $Z_{\tilde{v}}$ to lower the number of delta functions in the vertex operators. For that purpose, we consider the simplest PCO $Z_{\tilde{v}} = \tilde{v}^\alpha(d +$

$b_m\gamma^m\lambda)_\alpha\delta(\tilde{v}^\alpha w_\alpha)$ where $\tilde{v}^\alpha$ is a constant parameter and we act on the vertex operator $U^{(1|2)}$ derived above (4.38). We have

$$Z_{\tilde{v}}U^{(1|2)} = \tilde{v}^\alpha(d + b_m\gamma^m\lambda)_\alpha\delta(\tilde{v}^\alpha w_\alpha)\Big[c^m\tilde{A}_m\delta(f_\alpha\lambda^\alpha)\delta(f_\perp\lambda^\alpha)\Big] \tag{4.41}$$

$$= \tilde{v}^\alpha(D + \lambda\gamma^m\partial_{c^m})_\alpha\delta(\tilde{v}^\alpha\partial_{\lambda_\alpha})\Big[c^m\tilde{A}_m\delta(f_\alpha\lambda^\alpha)\delta(f_\perp\lambda^\alpha)\Big]$$

$$= \Big(c^m\tilde{v}^\alpha D_\alpha\tilde{A}_m + (\lambda\gamma^m\tilde{v})\tilde{A}_m\Big)\delta(f_{\perp,\alpha}\lambda^\alpha)$$

where we choose to decompose the product $\delta^2(\lambda)$ along two orthogonal spinors $F^i_\alpha = (f_\alpha, f_{\perp,\alpha})$ such that $\epsilon^{\alpha\beta}F^i_\alpha F^j_\beta = \epsilon^{ij}$. In addition, we also choose $\tilde{v}^\alpha f_\alpha = 1$ which is not essential and will not change our conclusions. This however implies, as seen above, that $\delta(\tilde{v}^\alpha w_\alpha)\delta(f_\alpha\lambda^\alpha) = 1$. [16]

We can repeat the operation with an orthogonal PCO $Z_{\tilde{v}^\alpha_\perp} = \tilde{v}^\alpha_\perp(d+b_m\gamma^m\lambda)_\alpha\delta(\tilde{v}^\alpha_\perp w_\alpha)$. This gives

$$U^{(1|0)} = Z_{\tilde{v}_\perp}Z_{\tilde{v}}U^{(1|2)} = \tilde{v}^\alpha\tilde{v}^\beta_\perp\Big[c^m D_\alpha D_\beta\tilde{A}_m + (\gamma^m\lambda)_{(\alpha}D_{\beta)}\tilde{A}_m\Big] \tag{4.42}$$

Inserting the definition of $\tilde{A}_m$, one gets the original vertex operator (3.1) $U(1|0) = \lambda^\alpha A_\alpha + c^m A_m$ with

$$A_\alpha(x,\theta) = \Big((\gamma^m\theta)_\alpha + (\tilde{v}\gamma^m\tilde{v}_\perp)\theta_\alpha\Big)a_m(x)\,, \tag{4.43}$$

$$A_m(x,\theta) = a_m(x) + \theta^2(\tilde{v}\gamma^p\tilde{v}_\perp)\partial_m a_p(x)\,.$$

These superfields differ from the orignal solution found in sec. 3.1 by a choice of gauge. Indeed, it can be shown that the vertex operator satisfied the Wess-Zumino-like gauge condition

$$\Big(\theta^\alpha - (\tilde{v}\gamma^m\tilde{v}_\perp)(\gamma_m\theta)^\alpha\Big)A_\alpha = 0.$$

This is a Lorentz-transformed Wess-Zumino-like gauge $\theta^\alpha N_\alpha{}^\beta A_\beta = 0$ where $N_\alpha{}^\beta$ is the infinitesimal Lorentz transformation whose parameter is $\omega_{mn} = e_{mnp}(\tilde{v}\gamma^p\tilde{v}_\perp)$. This is not surprising since to define the PCO we had to choose a gauge given by the constant parameters $\tilde{v}^\alpha$ and $\tilde{v}^\alpha_\perp$.

[16] To justify the above manipulations is better to use CFT techniques and we have $U^{(1|1)} = \lim_{x\to y}\,:Z_{\tilde{v}}(x)U^{(1|2)}(y):$. The normal ordering removes all poles from the contraction of the PCO with vertex operator and the limit removes the zeros of the form $(z-w)^n$ leaving only the finite and non vanishing parts.

4.5. Genus g differential forms and anomalies

With the geometric theory of integration at our disposal, we only need one more ingredient before we can proceed to present the computation of the correlators. As we have already remarked, at tree-level there is a perfect identification of the superforms on the target manifold with the vertex operators that depend only on the zero modes of the scalar fields, in our case $x^m, \theta^\alpha, \lambda^a$ and c^m. We can denote this as tree-level forms. As soon as we want to consider higher-genus Riemann surfaces, we have to take into account the presence of the conjugate momenta, resp. P_m, p_α, w_α and b_m; these are fields of conformal weight 1 and have g zero modes on a surface Σ of genus g. Making a choice of a symplectic base $a_i, b_i, i = 1, \dots, g$ for the first homology group of Σ, the zero-modes can be conveniently described as the integrals $\oint_{a_i} w_{\alpha z}(z)dz$ (and similarly for the other fields).

Since in the particle limit the Riemann surface degenerates into a graph and one cannot define a notion of conformal weight, and even the counting of zero modes becomes ill-defined, we have to consider this issue of higher-loops in the full string theory. Once we have written the prescriptions, they can be used also for the particle.

It is then convenient to use a conformal field theory formalism. We can introduce conserved currents that assign quantum numbers to the ghosts, and the number of insertions that are necessary to absorb the zero modes are counted by the anomaly in their OPE with the stress energy tensor. In the usual topological sigma model there is only one such current, associated to the ghost number. In our case we will need another one associated to the picture. There is a certain freedom in defining these currents, simply because any linear combination of them would also be conserved. We will make the following choice:

$$J_{gh} = c^m b_m - \lambda^\alpha w_\alpha \,, \qquad J_{pi} = -\Theta(w_\alpha)\partial\Theta(\lambda^\alpha) \,, \tag{4.44}$$

These expressions need some clarification. In order to compute their anomalies and their assignment of charges to the various fields, it is more convenient to use a bosonized form for the commuting ghosts, even though this breaks the manifest Lorentz invariance. Then we can use the following dictionary (see e.g. [66]): for each pair of conjugate fields (λ, w) (in our case there are 2 such pairs),

$$\begin{aligned} \lambda = \eta e^{\phi} \,, &\quad w = \partial\xi e^{-\phi} \,, \\ \delta(\lambda) = e^{-\phi} \,, &\quad \delta(w) = e^{\phi} \,, \\ \partial\Theta(\lambda) = \partial\lambda\delta(\lambda) = \eta \,, &\quad \Theta(w) = \xi \\ \lambda w = \partial\phi \,. & \end{aligned} \tag{4.45}$$

Here ξ, η form as usual an anticommuting system of spin $(0,1)$, and ϕ is a chiral boson with background charge $-1/2$, so that the exponentials in the first line of (4.45) have weights -1 and 0 respectively.

Then one can see easily that λ, w have ghost number ± 1 and picture ∓ 1, while $\delta(\lambda), \delta(w)$ have ghost number ∓ 1 and no picture. This definition of the ghost current is in agreement with the requirement that its anomaly should count the number of zero modes of the fields charged under it. In fact, an insertion of $\delta(\lambda)$ can be used to reabsorb a zero mode of λ, and so it carries the opposite ghost number. Using the bosonization rules (4.45) it is straightforward to compute the anomalies of the currents:

$$c_{gh} = -3 + 2 = -1\,, \quad c_{pi} = -2\,. \tag{4.46}$$

This result will be important when we discuss the prescription for the computation of amplitudes. At tree level, an operator that saturates the anomaly is $c\lambda^2\theta^2\delta^2(\lambda)$, and we have already seen that with the measure given by (3.14) , this operator has a non-vanishing vev. So the selection rules given by the anomalies are consistent with the tree level prescription we have found, and moreover they enable us to extend the prescription to higher loops. One has to recall that the anomaly is proportional to the Euler character of the Riemann surface, and at genus g it is given by $c(1-g)$, where c is the coefficient computed by the OPE with the stress-energy tensor.

We note also that – as in the RNS formalism – the BRST charge Q is filtered with the respect to this number as follows $Q = Q_0 + Q_{-1} + Q_{-2}$ where $Q_0 = c_m P^m, Q_{-1} = \lambda^\alpha d_\alpha$ and $Q_{-2} = \frac{1}{2} b_m \lambda\gamma^m\lambda$. In the same way the vertex operators are filtered according to this number.

Whereas for tree-level forms we can identify the measure with an element of the zero-momentum cohomology of top degree, which usually turns out to be unique, this is not the case for higher loop forms. For instance, let us consider the case of 1-loop forms. By using the unitary transformation (2.10) , the BRST operator becomes simply $Q = \lambda^\alpha \hat{p}_\alpha - \hat{c}^m P_m$ and we are effectively computing the de Rham cohomology in the space of 1-loop forms on $\mathcal{M}$. This can be written as

$$\prod_{m=1}^{3} \{1,\, \hat{c}^m\} \otimes \prod_{n=1}^{3} \{1,\, b^n\} \otimes \prod_{\alpha=1}^{2} \{1,\, \theta^\alpha \delta(\lambda^\alpha)\} \otimes \prod_{\beta=1}^{2} \{1,\, \hat{p}_\beta\, \delta(\hat{w}_\beta)\}\,. \tag{4.47}$$

We are particularly interested in the space $H^{0|0}$ that should be relevant for the 1-loop integration measure, since all the anomalies vanish at genus 1. But it is clear from (4.47) that this space is not 1-dimensional. There is however a distinguished element, which contains the zero modes of all the fields, and is the only class $\omega \in H^{3|0}(x,\hat{c}) \otimes H^{-3|0}(b,P) \otimes H^{0|2}(\theta,\lambda) \otimes H^{0|-2}(\hat{p},\hat{w})$. Explicitly, $\omega = c^3\, b^3\, \theta^2\, p^2\, \delta^2(\lambda)\, \delta^2(w)$. Note that ω has the same form in term of the transformed and the original fields. This cohomology class is the natural volume form on $\mathcal{M} \oplus \hat{\mathcal{M}}$, and determines a pairing $<,>$: $H^{p|q} \times H^{-p|-q} \to \mathbf{C}$. The space $H^{0|0}$ is dual to itself under this pairing, and $< 1, \omega >= 1$. It is natural to consider the factors in ω depending on the ghosts as related to the measure for the tree-level fields; then the antighost introduce additional zero modes that make the total anomaly vanish and require the introduction of the PCO in the correlators. Analogous statements hold for higher loops, as will be illustrated in the next section.

5. Multiloop multilegged amplitudes

In this section, we clarify completely the prescription for multiloop computation in the present model. The prescription is obtained by mapping the Riemann surface into graphs. We separately discuss the tree level (which has been discussed in pevious sections), one loop and finally for $g > 1$. This is needed in order to fix the isometries of tree level and one loop.

5.1. $g = 0$, N-point functions

We start to give the prescription of tree level multilegged amplitude. In the present situation, there is no zero modes for the conjugated momenta d_α, w_α and b_m. It is easy to see that, based on the previous considerations, the amplitude is given by

$$\mathcal{A}^N_{g=0} \sim \int \Big(d^3x_0 d^3P \hat{\mathcal{D}}x\Big) d^2\theta d^2\lambda d^3c\Big((\not{\epsilon}\partial_\lambda)\cdot(\not{\epsilon}\partial_\lambda)\delta^2(\lambda)\Big)\times \tag{5.1}$$

$$\times \prod_{i=1}^{3} U^{(1)}_i \prod_{j=4}^{N} \int d\tau_j V^{(0)}_j(\tau_j)\,,$$

where x_0 is the center of mass and $\mathcal{D}x$ is the measure for non zero modes and

$$\langle x^m(\tau_1)x^n(\tau_2)\rangle_{g=0} = \int \hat{\mathcal{D}}x\Big(x^m(\tau_1)x^n(\tau_2)\Big) = \eta^{mn}|\tau_1-\tau_2|\,. \tag{5.2}$$

The insertion of 3 vertex operators $U^{(1)}$ with ghost number one is needed in order to fix the isometries at tree level (in string theory this corresponds to fixing the Möbious group $SL(2,\mathbf{R})$ of the disk). In the case of the particle limit, one of the vertex $U^{(1)}$ is situated at the beginning of the worldline ($T = -\infty$) one at the other end and the third in the middle for a generic T (this analysis can be found in [67,68]). The insertion of integrated operators is needed to have N-point amplitudes and they are given by $\int d\tau V^{(0)}(\tau) = \int d\tau \dot{x}^m(\tau) A_m(x(\tau))$. The form of the connection A_m depends on the background chosen. Example of these amplitudes are already given in the previous sections.

Regarding the saturation of the anomalies, we can easily see that if we consider the combination $[\mathcal{D}c\mathcal{D}\lambda]_{(-1|2)} = d^2\lambda d^3 c (\not{c}\partial_\lambda)\cdot(\not{c}\partial_\lambda)$ as part of the measure (which defines the vacuum of the Hilbert space), it has ghost anomaly equal to -1 which is the ghost charge of the vacuum while it has picture number equal to 2 which is again what dictates the picture number anomaly. On the other side the insertions, namely $\prod_{i=1}^{3} U_i^{(1)}$ and $\delta^2(\lambda)$, have ghost number $+1$ which saturate the ghost anomaly of the vacuum and out to the three vertex operators $U_i^{(1)}$ we have to select the picture -2 part. That leads to a non-vanishing amplitude.[17]

5.2. $g = 1$, N-point functions

The next step is to consider one loop amplitudes. Here the presence of zero modes for the conjugated variables d_α, w_α and b_m is important. For one-loop one has to integrate over the modulus of the torus T. To this modulus, there is a corresponding antighost field denoted B_T. The one-loop N-point function is given by

$$\mathcal{A}^N_{g=1} \sim \int_0^1 dT \int \Big(d^3x_0 d^3 P \hat{\mathcal{D}}x\Big) d^2\theta d^2 d d^2\lambda d^2 w d^3 c d^3 b \Big((\not{\theta}\partial_\lambda)\cdot(\not{\theta}\partial_\lambda)\delta^2(\lambda)\Big)\times \tag{5.3}$$
$$\times B_T \prod_{k=2}^{N}\int_0^T d\tau_k B_k \prod_{i=1}^{2} Z_{v_i} U^{(1)}(T) \prod_{j=2}^{N} U_j^{(1)}(\tau_j)\,,$$

where $B = b_n \dot{x}^n$ and

$$Z_{v_i} = \{Q, \Theta(v_i^\alpha w_\alpha)\} = v_i^\alpha \Big(d - b_n\gamma^n\lambda\Big)_\alpha \delta(v_i^\alpha w_\alpha)\,.$$

The parameters τ_j are the Schwinger parameters. The counting of zero modes goes as follows:

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[17] In pure spinor approach the measure for the zero modes of open superstrings is given by $[\mathcal{D}\lambda]_{(8|3)} = d\lambda^{\alpha_1}\wedge\ldots\wedge d\lambda^{\alpha_{11}}\epsilon_{\alpha_1\ldots\alpha_{16}}(\gamma_m\gamma_n\gamma_r\gamma^{mnr})^{[\alpha_{12}\ldots\alpha_{16}](\beta_1\ldots\beta_3)}\frac{\partial}{\partial\lambda^{\beta_1}}\cdots\frac{\partial}{\partial\lambda^{\beta_3}}$. This measure has ghost number $+8$ as prescribed by the ghost anomaly $TJ \sim -8/(z-w)^3 + \ldots$ (see [43] for further details) and has picture $+3$, and this is prescribed by a similar operator as in the previous section.

1) there are 3 zero modes for c^m and 3 zero modes for b_m. Notice that b_m behave like abelian differentials of string theory, (we have to recall here that the 1-loop graph is indeed a manifold and the abelian differentials on it are well defined. There is one modulus for the torus for each vertex insertion. For higher loops the counting of moduli cannot be done in the same way since higher loop graphs are not manifolds.

2) there are 2 zero modes for θ^α and 2 zero modes of d_α. They are fermionic and they have to be saturated in order to get a non-vanishing amplitude,

3) there are 2 zero modes of λ^α and 2 zero modes for w_α. They are bosonic and the delta functions $\delta^2(\lambda)$ and $\delta^2(w)$ have to be inserted. The first delta functions are inserted by means of the picture lowering operator $\Big((\not{\epsilon}\partial_\lambda) \cdot (\not{\epsilon}\partial_\lambda)\delta^2(\lambda)\Big)$ – which is BRST invariant[18] – there are two derivatives on delta functions which select those products of vertices that contain at least two powers of λ^α. The second type of delta functions $\delta^2(w)$ is contained in the picture changing operators Z_v. They depend on the gauge parameters v_i^m where $i = 1, 2$ and they are chosen in such a way that no normal ordering is needed for the expression of Z_v. This dependence is BRST exact

$$\delta_{v_i^\alpha} Z_v = \{Q, \delta_i^j w_\alpha \delta(v_j^\alpha w_\alpha)\}\,.$$

4) The total ghost number of the amplitude as it can be checked by

$$-2_{\delta^2(\lambda)} - (N+1)_{(B_k)} + 2_{(Z_v)} + (N+1)_{(U^{(1)})} = 0$$

where in bracket we inserted the sources of the ghost charge.

5) We also have to take into account that the picture should be saturated. Indeed we have

$$-2_{(\delta^2(\lambda))} + 2_{(Z_v)} = 0\,.$$

As an example, we consider one-loop, three point function.

$$\mathcal{A}^3_{g=1} \sim \int_0^\infty \frac{dT}{T} \int \Big(d^3x_0 d^3P\hat{\mathcal{D}}x\Big) d^2\theta d^2 d d^2\lambda d^2 w d^3 c d^3 b \Big((\not{\epsilon}\partial_\lambda)\cdot(\not{\epsilon}\partial_\lambda)\delta^2(\lambda)\Big)\times \tag{5.4}$$

$$(b_m\dot{x}^m)(T)\int_0^T d\tau_1 (b_m\dot{x}^m)(\tau_1)\int_0^T d\tau_2 (b_m\dot{x}^m)(\tau_2)\times$$

[18] The BRST variation of c^m produces $\lambda\gamma^m\lambda$, and therefore using the two derivatives appearing on the delta function, it is easy to see that the BRST variation of this monomial is zero.

$$
\begin{aligned}
&v_1^\alpha\Big(d-b_m(\gamma^m\lambda)\Big)_\alpha v_2^\beta\Big(d-b_m(\gamma^m\lambda)\Big)_\beta\delta(v_1^\alpha w_\alpha)\delta(v_2^\beta w_\beta)\times\\
&(\lambda^\alpha A_\alpha-c^m A_m)(\lambda^\alpha A_\alpha-c^m A_m)(\lambda^\alpha A_\alpha-c^m A_m)\,.
\end{aligned}
$$

It is easy to check that indeed all the zero modes are saturated and therefore this amplitude does not vanish. To evaluate it, one needs to compute all the integration on zero modes and finally the contraction of x's is performed with the propagator

$$
\langle x^m(\tau_1)x^n(\tau_2)\rangle_{g=1}=\eta^{mn}\Big(|\tau_1-\tau_2|-\frac{(\tau_1-\tau_2)^2}{T}\Big)\,. \tag{5.5}
$$

To define the partition function at one loop, one has to insert powers of the ghost current $J=b_m c^m+w_\alpha\lambda^\alpha$, but it vanishes as it can be seen by just counting the zero modes. This is equivalent to the vanishing of the partition function for Green-Schwarz superstring in the light-cone gauge on the flat background.

5.3. $g>1$, N-point functions

In this last section, we consider the multiloop when $g>1$ and we do not have to take into account any isometry of the graph. We have to notice that by counting the internal lines I of a graph with V internal vertices at g loops, one has $g-1=I-V$. Since the vertices are trivalent $3V=2I+N$ where N is the external insertions, we have that $I=3(g-1)+N$ which gives the number of internal lines to whom we assign a modulus each.

By respecting the ghost number and zero modes saturation, we propose the prescription

$$
\mathcal{A}_g^N\sim\sum_{h=1}^{3g-3+N}\int dm_h\int d^3x_0 d^3P\hat{\mathcal{D}}xd^2\theta d^2\lambda d^3c\prod_{i=1}^{2g}d^2w_id^2d_i\prod_{j=1}^{3g}d^3b_j\times \tag{5.6}
$$

$$
\times\Big(((\not{\epsilon}\partial_\lambda)\cdot(\not{\epsilon}\partial_\lambda)\delta^2(\lambda)\Big)\prod_{k=1}^{3(g-1)+N}\int d\tau_k\mu_k(\tau_k)B_k\prod_{l=1}^{2g+N}Z_{v_l}\prod_{m=1}^{N}U_j^{(1)}(\tau_j)=,
$$

$$
\sim\sum_{h=1}^{3g-3+N}\int dm_h\Big\langle\Big(((\not{\epsilon}\partial_\lambda)\cdot(\not{\epsilon}\partial_\lambda)\delta^2(\lambda)\Big)\prod_{k=1}^{3(g-1)+N}\int d\tau_k\mu_k(\tau_k)B_k\prod_{l=1}^{2g+N}Z_{v_l}\prod_{m=1}^{N}U_j^{(1)}(\tau_j)\Big\rangle\,.
$$

In the present case, we notice that we add the contribution from the $3g$ zero modes of b_m and $2g$ zero modes of d_α and of w_α. They are saturated by the presence of B's and by the picture changing operators. The counting of ghost number and picture number goes as follows

$$
-2_{(\delta^2(\lambda))}-(3(g-1)+N)_{(B_k)}+(2g+N)_{(Z_v)}+N_{(U^{(1)})}=-(g-1)+N
$$

$$-2_{(\delta^2(\lambda))} + (2g+N)_{(Z_v)} = 2(g-1) + N \,.$$

The last expression counts the number of delta functions for λ and w and the number is exaclty matched by the number of integration variables $d^2\lambda$ and $d^2 w$.

We notice therefore that in the case of a given number of possible pictures, let us say, n different pictures, the counting of the insertions Z_v and of the delta functions $\delta(\lambda)$ should be changed to n, namely the insertions should be replaced by

$$H(c, \partial_\lambda)\delta^n(\lambda) \prod_{l=1}^{n\,g+N} Z_{v_l}$$

which conpensate the picture charge of the vacuum $n(g-1)+N$. [19]

It is clear from this analysis that several amplitudes are vanishing. However, we have to recall that if the computation is done in topological field theory there some terms coming from the curvature terms of the action. This phenomena is illustrated in [69].

5.4. Derivation of (5.6)

In order to derive the formulat (5.6) it is necessary to couple the theory to an extended topological system on the worldsheet (or on the worldline).

There are essentially two type of zero modes relevant for our derivation: the zero modes and moduli of the Riemann surface and the supermoduli associated to commuting ghosts λ^α, w_α. The first type of zero modes are parametrized by suitable combinations of the ghost fields (c^m, b_m), while the second by the ghosts $(\lambda^\alpha, w_\alpha)$. In order to derive the formula (5.6), we introduce four new type of ghost fields (b', c'), (β', γ') (with conformal spin (2,-1) and with fermionic and bosonic statistics, respectively) and (β''_i, γ''^i), (ξ''_i, η''^i) where $i = 1, 2$ (this number depends on the number of independent pictures) with conformal (1,0) with bosonic and fermionic statistics. This set of fields resembles the set of ghost

[19] For example in the pure spinor approach, the number of independent components is computed by solving the constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$, where λ^α is a Weyl spinor in 10d and $\gamma^m_{\alpha\beta}$ are symmetric real matrices of $Spin(10)$. This yields 11 components for λ^α and the insertion is given by

$$\Big(\theta^{\alpha_1} \dots \theta^{\alpha_{11}} \epsilon_{\alpha_1 \dots \alpha_{16}} \gamma_m^{\alpha_{12}\beta_1} \gamma_n^{\alpha_{13}\beta_2} \gamma_p^{\alpha_{14}\beta_3} \gamma^{mnp,\alpha_{15}\alpha_{16}} \partial_{\lambda_{\beta_1}} \partial_{\lambda_{\beta_2}} \partial_{\lambda_{\beta_3}} \delta^{11}(\lambda)\Big) \prod_{l=1}^{11\,g+N} Z_{v_l}$$

where the term in the bracket absorbs the zero modes of λ^α and the PCO Z_v absorbs the zero modes of w_α. The total number of delta functions absorbed is $11(g-1)+N$.

of a fermionic string with an extended supersymmetry, whose conformal central charge vanishes since these new fields always appear in quartets. On the other side, the anomaly of the ghost currents

$$J_{gh} = b'c' + 2\beta'\gamma' \,, \qquad J_{pi} = \beta''_\alpha \gamma''^\alpha + 2\xi''_\alpha \eta''^\alpha$$

are -3 and $-n$, respectively, where n is the number of independent components of λ^α.

Let us first derive the insertions of the $B = (b_m P^m)$ of the formula (5.6). They are related to the moduli of the Riemann surface and therefore it is useful to introduce a new BRST charge defined as follows

$$Q_{new} = c'P^2 + \gamma'(b' - b_m P^m) + Q \tag{5.7}$$

which is clearly nilpotent (since $\{Q, b_m P^m\} = P^2$) and has the properties

$$Q_{new} b' = P^2 \,, \quad Q_{new}\beta' = b' + b_m P^m \,, \quad Q_{new} c' = \gamma' \,, \quad Q_{new}\gamma' = 0 \,.$$

This new BRST operator contains the old BRST charge Q (which implements the topological symmetry) but also the BRST symmetry of the diffeomerphisms with the term $c'P^2$. The additional terms are needed to render the BRST operator nilpotent. The term $\gamma' b'$ is needed in order to generate a topological symmetry of the system c', b', γ', β' such that they do not enter the cohomology.

In terms of c', b', γ' and β', we know what are the correct insertions for the moduli of a Riemann surface together with the supermoduli associated to γ' and β' (here we consider $g > 1$), namely one has to insert the following BRST invariant combinations to reabsorb the zero modes of b' and β'

$$\prod_i^{3(g-1)+N} \int \mu_i b'_i \{Q_{new}, \Theta(\int \mu_i \beta'_i)\} = \prod_i^{3(g-1)+N} \int \mu_i b'_i (b'_i + (b_m P^m)_i) \delta(\beta'_i) \tag{5.8}$$

where μ_i are the Beltrami differentials. Since the fields b' and β' are decoupled, they can be integrated out leaving the insertion of $\prod_i^{3(g-1)+N} \int \mu_i (b_m P^m)_i$ which are the insertions obtained in (5.6). Notice that for genus 1, one has to reabsorb the isometries of the sphere and of the torus. This can be done by inserting the PCO $c'\delta(\gamma')$ to soak up the zero modes of c' and γ'. For genus zero, we have already lengthly discussed it in the previous sections.

The next step, we derive the insertion of the PCO Z_{v_i}. The number of v_i is equal to the number of independent components of w_α. We have to notice the following: there exist two solutions to the equation $\{Q, b^\alpha\} = \lambda^\alpha P^2$ (we refer to [43] for a complete analsys of this equation in the pure spinor framework). One solution is provided by $\lambda^\alpha b_m P^m$ and the second is given by $(\not{P} d)^\alpha$ (this is similar to the solution in [43], in [24] and in [70]) and the latter resembles the generator of the κ-symmetry of the Brink-Schwarz formulation of the superparticle.[20] Given those two solutions, we can form the BRST invariant combination

$$K^\alpha = (\not{P} d)^\alpha - \lambda^\alpha b_m P^m \,.$$

This combination is not only invariant, but it is also BRST-exact:

$$K^\alpha = \{Q, \not{P} w - b^m b^n \epsilon_{mnp}(\gamma^p \lambda)\} \,. \tag{5.9}$$

This suggests that the operator $\Xi^\alpha = \not{P} w - b^m b^n \epsilon_{mnp}(\gamma^p \lambda)$ can play the same role of $B = b_m P^m$ and its BRST transformed P^2 introduced above. In addition, we can check that the those operators form a closed algebra whose main commutation relation is given by

$$\{K^\alpha, K^\beta\} = P^m \gamma_m^{\alpha\beta} P^2 \,, \tag{5.10}$$

which is the usual relation between the κ-symmetry generators and the Virasoro constraint $P^2 \sim 0$. As a check, one can also compute the commutation relation between K^α and Ξ^β to have

$$\{K^\alpha, \Xi^\beta\} = -P^m \gamma_m^{\alpha\beta} B \,, \tag{5.11}$$

which leads to (5.10) by acting with Q on both side of the equal sign.

Notice that the main difference between the operator B and Ξ^α is the spinorial index carried by the second one. In addition, while B is anticommuting, Ξ^α is commuting. This last observation suggests that we can introduce a new BRST operator for each spinorial component of Ξ^α and K^α. It is however convenient to introduce again the gauge parameters $v^i_{(\alpha)}$ and define the BRST charge as follows

$$Q_{fin} = Q_{new} + \sum_i \Big[\xi''_i v^i_\alpha K^\alpha \delta'(v^i_\alpha \Xi^\alpha) + \gamma''_i \Big(\eta''^i + \delta(v^i_\alpha \Xi^\alpha)\Big)\Big] \tag{5.12}$$

where we converted the commuting operators $v^i_\alpha \Xi^\alpha$ into the fermionized ones $\delta(v^i_\alpha \Xi^\alpha)$. Notice that we could have also used the other fermionized operators given by $\Theta(v^i_\alpha \Xi^\alpha)$,

[20] Recently paper [71] appeared, this provides a useful link between GS formalism and pure spinor formalism (as previously done so in [72]. It would be interesting to see the relation between PCO and the coupling to topological gravity for the critical superstring [55].

but this is not a pseudoform. In addition, $\Theta(v^i_\alpha \Xi^\alpha)$ does not carry any ghost number. On the other side, $\delta(v^i_\alpha \Xi^\alpha)$ satisfies all the conditions.[21] The action of the final BRST operator Q_{fin} is given by

$$Q\beta''^i = \eta''^i + \delta(v^i_\alpha \Xi^\alpha)\,, \qquad Q\eta''^i = v^i_\alpha K^\alpha \delta'(v^i_\alpha \Xi^\alpha) \tag{5.13}$$
$$Q\xi''_i = \gamma''_i\,, \qquad Q\gamma''_i = 0\,.$$

It is easy to show that the dependence upon the gauge parameters v^α_i is always BRST trivial.

Following the previous derivation, we can derive the insertion to reabsorb the zero modes of η''^i and of β''^i

$$\prod_{k=1}^{ng+N} \oint_{A_k} \Big(\eta''^k v^k_\alpha K^\alpha\Big) \Big\{Q_{fin}, \Theta(\oint \beta''^k)\Big\} \tag{5.14}$$

Notice that at genus g there are two zero modes for η''^i and two zero modes β''^i. The integral is performed over an A cycle of the Riemann surface (again here we dropped all this detail and we naively restrict ourselves to the worldline model). It is easy to check that those insertions are BRST closed, but not BRST exact. Notice that since the symmetry generated by K^α are not worldsheet symmetry, but rather target space symmetry, we expect that the insertions (5.14) are exaclty BRST closed (not up to a total derivatives on the moduli). By computing the action of Q_{fin} on β''^k and using the fact that η''^i is nilpotent (for each i) we get the following aswer

$$\prod_{k=1}^{ng+N} \eta''^k v^k_\alpha K^\alpha \delta(\beta''^k) \delta(v^k_\alpha \Xi^\alpha) \tag{5.15}$$

then finally integrating out η''^i and β''^i, we obtain the insertion of the spacetime PCO

$$\prod_{k=1}^{ng+N} Z_{v^k}\,. \tag{5.16}$$

where a suitable odd vector has been chosen which gives $Z_{v^k} = \{Q, \Theta(v^k_\alpha \Xi^\alpha)\}$. Notice that the dependence upon v^k is entirely through BRST exact terms and they do not affect the amplitudes. Finally, $\Big\{Q_{fin}, \Theta(\oint \beta''^k)\Big\}$ is the well-known worldsheet PCO constructed by fermonizing the superghost β''^i and it is interesting to see the interplay between a worldsheet PCO and the target space PCO.

[21] The BRST charge (5.12) is very similar to the BRST operator $Q_{RNS} + \oint \eta$ proposed in [73] where $\oint \eta$ is the BRST operator which reduces the large Hilbert space to the small one.

6. Future Directions

The present analysis is a first step toward a more complete study of topological string theory on supertarget spaces. As we pointed out the theory of integration of superforms is directly related to the path integral formulation of superstrings and topological strings. Here we would like to indicate some future directions that we consider relevant for applications and developements.

i) The analysis of boundary conditions of the sigma model given in sec. 2.2 and the study of A-branes and the analysis of their properties has be explored. As is well known the presence of fermionic degrees of freedom for the superstring D-branes drastically changes the spectrum of the theory. We suspect that also in the present context the study of A-branes might reveal some interesting properties. In addition, to our knowledge, the Super-Chern-Simon action is the only supersymmetric string field theory action constructed so far which is fully super-Poincaré invariant in the target space. We hope that the present construction can be useful to construct the full-fledged superstring field theory.

ii) Together with the A model, one can study the B model. In that case some anomalies coming from the holomorphic nature of the model should be properly compensated. It would be interesting to construct the top pseudoform to be integrated in the path integral. This certainly generalizes the usual form Ω of the super Calabi-Yau

$$\Omega = \epsilon_{IJKL} Z^I dZ^J dZ^K dZ^L \epsilon_{ABCD} d\Psi^A d\Psi^B d\Psi^C d\Psi^D$$

The observables of the model should be identified with superforms and the PCO should enter in the prescription for the amplitudes. It would be very interesting to understand the implications of these modifications in some specific calculation of amplitudes.

iii) The techniques of supergeometry can be applied to pure spinor string theory where the space of differential is modfied by the introduction of the pure spinor constraints. This will change the structure of the top form, but a complete analysis is missing. Interesting steps in that direction have been already achieved by Movshev and Schwarz in [74,75]. In that context, one should be able to derive the multiloop prescription for pure spinor string theory by coupling it to topological gravity.

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8. Appendix: BV formalism

Here we view the BV formalism (see, cf. [76]) as an integral I_{BV} of the BV differential form Ω_{BV} along the Lagrangian submanifold $\mathcal{L}$ in the BV space:

$$I_{BV} = \int_{\mathcal{L}} \Omega_{BV} \tag{8.1}$$

The BV space $\mathcal{M}$ is equipped with the canonical odd symplectic form ω_{BV}. One can choose local coordinates to identify $\mathcal{M}$ with $\Pi T^* N$ where N is some (super)manifold, where the symplectic form has a canonical form

$$\omega_{BV} = \delta Z_a^+ \wedge \delta Z^a \tag{8.2}$$

where Z^a denotes the (super)coordinates on N and Z_a^+- corresponding coordinates on the cotangent fiber.

The submanifold $\mathcal{L}$ is Lagrangian with respect to the canonical form ω_{BV} (in the physical literature its generating function is called the gauge fermion).

The BV differential form Ω_{BV} is constructed out of two ingredients [76]: the BV action S and the BV measure ν: $\Omega_{BV} = \left(\nu e^{-S}\right)$.

The action S must obey the so-called BV master equation:

$$\{S, S\}_{BV} := \omega_{BV}^{-1}(\partial_l S \wedge \partial_r S) = 0 \tag{8.3}$$

One calls the coordinates Z^a the fields and Z_a^+ the anti-fields. Sometimes one distinguishes the classical part of N and the auxiliary fields used for gauge fixing. Also, the identification of the BV phase space with $\Pi T^* N$ is not unique and is not global in general, so the partition of all the fields involved on the fields and anti-fields is not unique.

The deformations of the action S that preserve (8.3) are (in the first order approximation) the functions Φ on $\mathcal{M}$ which are Q_{BV}-closed, where the differential Q_{BV} acts as $Q_{BV}\Phi = \{S, \Phi\}_{BV}$. The deformations which are Q_{BV}-exact are trivial in the sense that they could be removed by a symplectomorphism of $\mathcal{M}$ (one has to make sure that this symplectomorphism preserves ν to guarantee that the quantum theory is not sensitive to such a Q_{BV}–exact term).

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