T. Ishikawa
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SUPER CONGRUENCE FOR THE APÉRY NUMBERS

TSUNEO ISHIKAWA

§0. Introduction

Let, for any $n \geq 0$,

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad u(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 .$$

R. Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ made use of these numbers (see [10]). As a result, many properties of the Apéry numbers were found (see [1]-[9]). In particular, Beukers and Stienstra showed the interesting congruence (see [11, Theorem 13.1]).

THEOREM 1 (Beukers and Stienstra). *Let $p \geq 3$ be a prime, and write*

$$\sum_{n=1}^{\infty} \lambda_n q^n = q \prod_{n=0}^{\infty} (1 - q^{4n})^6 . \tag{1}$$

Let m, $r \in \boldsymbol{N}$, m odd, then we have

$$a\left(\frac{mp^r - 1}{2}\right) - \lambda_p a\left(\frac{mp^{r-1} - 1}{2}\right) + (-1)^{(p-1)/2} p^2 a\left(\frac{mp^{r-2} - 1}{2}\right) \equiv 0 \bmod p^r . \tag{2}$$

Moreover they conjectured that congruence (2) holds $\bmod p^{2r}$ if $p \geq 5$, and they called these congruences *super congruences* in [4] and [11].

In this paper we shall prove the conjecture for $r = 1$.

THEOREM 2. *Let $p \geq 5$ be a prime and $m \in \boldsymbol{N}$, m odd, then we have*

$$a\left(\frac{mp - 1}{2}\right) - \lambda_p a\left(\frac{m - 1}{2}\right) \equiv 0 \bmod p^2 .$$

F. Beukers informed me that L. Van Hamme proved the case of $p \equiv 1 \bmod 4$ using properties of the p-adic gamma function (see [7]). We prove the general case involving $p \equiv 3 \bmod 4$ by entirely different method. Our

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method is applicable to super congruences of other numbers such as $u(n)$ (see [8]).

§1. Congruence of $a(n)$

The numbers $a(n)$ satisfy the recurrence

$$(3)\qquad (n+1)^2a(n+1) = (11n^2+11n+3)a(n) + n^2a(n-1) \qquad n \geq 1 .$$

We know the following result. Let p be an odd prime, and $m \geq 0$, then

$$(4)\qquad a(mp) \equiv a(m) \qquad \bmod p^2 ,$$

$$(5)\qquad a(p-1) \equiv 1 \qquad \bmod p^2 .$$

By (3), (4) and (5), we have $a(p-2) \equiv -3 + 5p \bmod p^2$, $a(p+1) \equiv 9 + 15p \bmod p^2$.

PROPOSITION 1. *Let $m \geq 0$, $n \geq 0$ and $m + n = p - 1$. Then*

$$a(m) \equiv (-1)^m a(n) \qquad \bmod p .$$

Proof. We proceed by induction on m to show that $a(m) \equiv (-1)^m a(p-m-1) \bmod p$. From the above result, $a(0) \equiv a(p-1) \equiv 1 \bmod p$ and $a(1) \equiv -a(p-2) \equiv 3 \bmod p$. Let $0 < m < p-1$. From the recurrence (3),

$$\begin{aligned}
&(m+1)^2a(m+1)\\
&\quad = (11m^2+11m+3)a(m) + m^2a(m-1)\\
&\quad \equiv \{11(p-m)^2 - 11(p-m) + 3\}a(m) + (p-m)^2a(m-1)\\
&\quad \equiv \begin{cases} -\{11(p-m)^2 - 11(p-m) + 3\}a(p-m-1) + (p-m)^2a(p-m) & \text{if } m\text{: odd}\\ \{11(p-m)^2 - 11(p-m) + 3\}a(p-m-1) - (p-m)^2a(p-m) & \text{if } m\text{: even}\end{cases}\\
&\quad \equiv \begin{cases} (m+1)^2a(p-m-2) & \text{if } m\text{: odd}\\ -(m+1)^2a(p-m-2) & \text{if } m\text{: even}\end{cases} \qquad \bmod p .
\end{aligned}$$

Q.E.D.

PROPOSITION 2. *For all primes p, $n \geq 0$ and $0 \leq m \leq p-1$, we have*

$$a(np+m) \equiv a(m)a(n) \qquad \bmod p .$$

Proof. This congruence follows from the similar method of the proof of [6, Theorem 1]. Q.E.D.

§ 2. Congruence of $b(n)$

Let $b(n) = 0$ and, for any $n \geq 1$,

$$b(n) = \sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left[\frac{2}{n-k+1} + \cdots + \frac{2}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k} \right].$$

These numbers are (differential) of $a(n)$ and they take important parts in the congruence of $\bmod p^2$ as shown in [6, Theorem 4].

PROPOSITION 3. *The numbers $b(n)$ satisfy the recurrence*

$$(6) \quad (n+1)^2 b(n+1) = (11n^2 + 11n + 3) b(n) + n^2 b(n-1) - 2(n+1) a(n+1) + 11(2n+1) a(n) + 2na(n-1),$$

and for all primes $p \geq 3$, $n \geq 0$ and $0 \leq m \leq p-1$, we have

$$a(np+m) \equiv \{a(m) + pnb(m)\} a(n) \qquad \bmod p^2.$$

Proof. Let

$$B_{n,k} = (k^2 + 3(2n+1)k - 11n^2 - 9n - 2) \binom{n}{k}^2 \binom{n+k}{k} H_{n,k} + (6k - 22n - 9) \binom{n}{k}^2 \binom{n+k}{k},$$

and

$$H_{n,k} = \frac{2}{n-k+1} + \cdots + \frac{2}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k},$$

then we have

$$\begin{aligned} B_{n,k} - B_{n,k-1} = {} & (n+1)^2 \binom{n+1}{k}^2 \binom{n+1+k}{k} H_{n+1,k} \\ & - (11n^2 + 11n + 3) \binom{n}{k}^2 \binom{n+k}{k} H_{n,k} \\ & - n^2 \binom{n-1}{k}^2 \binom{n-1+k}{k} H_{n-1,k} \\ & + 2(n+1) \binom{n+1}{k}^2 \binom{n+1+k}{k} \\ & - 11(2n+1) \binom{n}{k}^2 \binom{n+k}{k} - 2n \binom{n-1}{k}^2 \binom{n-1+k}{k}. \end{aligned}$$

Taking summation from 1 to $n+1$ on k, recurrence (6) follows. The congruence can be proved in the similar method of the proof of [6, Theorem 4] by congruences (4) and (5). Q.E.D.

Proposition 4. *Let $m \geq 0$, $n \geq 0$ and $m + n = p - 1$. Then*

$$b(m) \equiv (-1)^{m-1} b(n) \quad \bmod p .$$

Proof. From the congruence (4), (5) and Proposition 3, $b(0) \equiv -b(p-1) \equiv 0 \bmod p$. And by the definition of $b(n)$, $\mathrm{ord}_p b(p) \geq 0$. Then $b(1) \equiv b(p-2) \equiv 5 \bmod p$ by the recurrence (6). By induction on m, similarly in Proposition 1, we can prove. Q.E.D.

Theorem 3. *Let $m \geq 0$, $n \geq 0$ and $m + n = p - 1$. Then*

$$a(m) \equiv (-1)^m \{a(n) - pb(n)\} \quad \bmod p^2.$$

Proof. It is clear from (4), (5) and Proposition 4 in the case of $m = 0, 1$. From the recurrence (3), (6) and the congruence

$$\begin{aligned} &(m+1)^2 a(m+1) \\ &\quad \equiv \{11(p-m)^2 - 11(p-m) + 3\} a(m) + (p-m)^2 a(m-1) \\ &\qquad - 11p\{2(p-m) - 1\} a(m) - 2p(p-m) a(m-1) \quad \bmod p^2 , \end{aligned}$$

it can be also shown by inductive method. Q.E.D.

§ 3. Congruence of $c(n)$

If $p \equiv 3 \bmod 4$, we can not obtain the congruence of $b((p-1)/2)$ from Proposition 4. Therefore we prepare the numbers $c(n)$.

Let, for all odd numbers $n \geq 1$,

$$c(n) = \sum_{k=1}^{n} \binom{n}{k}^3 (-1)^k \left[\frac{3}{n-k+1} + \cdots + \frac{3}{n}\right].$$

Let p be an odd prime. From the congruence

$$\binom{\frac{p-1}{2} + k}{k} \equiv (-1)^k \binom{\frac{p-1}{2}}{k} \quad \bmod p$$

and

$$\frac{1}{\frac{p-1}{2} + k + 1} + \cdots + \frac{1}{\frac{p-1}{2}} + \frac{1}{\frac{p+1}{2}} + \cdots + \frac{1}{\frac{p-1}{2} + k} \equiv 0 \quad \bmod p$$

where $1 \leq k \leq (p-1)/2$, we have

$$3b\left(\frac{p-1}{2}\right) \equiv c\left(\frac{p-1}{2}\right) \quad \bmod p \quad \text{if } p \equiv 3 \bmod 4 .$$

PROPOSITION 5. *The numbers $c(n)$ satisfy the recurrence*

$$n^2c(n) = -3\{9(n-1)^2 - 1\}c(n-2) \tag{7}$$

for all odd numbers $n \geq 3$.

Proof. Let

$$\begin{aligned} f_n(k) &= 2(14n^2 + n - 1) - 3(26n^2 - n - 3)k/n + 3(29n^2 - 3)k^2/n^2 \\ &\quad - 3(15n^2 + 2n - 1)k^3/n^3 + 3(3n + 1)k^4/n^3 , \\ g_n(k) &= 2(28n + 1) - 3(26n^2 + 3)k/n^2 + 18k^2/n^3 \\ &\quad + 3(15n^2 + 14n - 3)k^3/n^4 - 9(2n + 1)k^4/n^4 , \end{aligned}$$

and

$$C_{n,k} = \frac{3}{n-k+1} + \cdots + \frac{3}{n} .$$

Then we have

$$\begin{aligned} &(n+1)^2\binom{n+1}{k}^3 C_{n+1,k} + 3(9n^2 - 1)\binom{n-1}{k}^3 C_{n-1,k} \\ &\quad + 2(n+1)\binom{n+1}{k}^3 + 54n\binom{n-1}{k}^3 \\ &= f_n(k)\binom{n}{k}^3 C_{n,k} + f_n(k-1)\binom{n}{k-1}^3 C_{n,k-1} \\ &\quad + g_n(k)\binom{n}{k}^3 + g_n(k-1)\binom{n}{k-1}^3 . \end{aligned}$$

We multiply both sides by $(-1)^k$. Taking summation from 1 to $n+1$ on k,

$$\begin{aligned} &(n+1)^2c(n+1) + 3(9n^2 - 1)c(n-1) \\ &\quad + 2(n+1)\sum_{k=0}^{n+1}\binom{n+1}{k}^3(-1)^k + 54n\sum_{k=0}^{n-1}\binom{n-1}{k}^3(-1)^k = 0 . \end{aligned} \tag{8}$$

If $n \equiv 0 \bmod 2$, two latter summations are equal to 0. Q.E.D.

Remark. The numbers $c(n)$ satisfy the recurrence (8) if $n \equiv 1 \bmod 2$.

PROPOSITION 6. *Let $p \equiv 3 \bmod 4$ be a prime, we have*

$$c\left(\frac{p-1}{2}\right) \equiv 0 \quad \bmod p .$$

Proof. It is trivial if $p = 3$. If $p \equiv 7 \bmod 12$ then $(p+2)/3$ is odd. By (7), we have

$$\left(\frac{p+2}{3}\right)^2 c\left(\frac{p+2}{3}\right) + 3\left\{9\left(\frac{p-1}{3}\right)^2 - 1\right\} c\left(\frac{p-4}{3}\right) = 0 .$$

Then $c((p+2)/3) \equiv 0 \bmod p$. Hence, $c(n) \equiv 0 \bmod p$ for $(p+2)/3 \leq n \leq p-2$ and n odd. If $p \equiv 11 \bmod 12$ then $(p+4)/3$ is odd. Then it can be proved in the same way. Q.E.D.

§ 4. Proof of Theorem 2

Beukers and Stienstra showed that the generating function of $a(n)$ is a holomorphic solution of the Picard-Fuchs equation associated to the family of elliptic curves. From this argument and the ζ-function of a certain K3-surface, they proved Theorem 1 (see [2, 11]). Moreover we know that the right hand side of (1) is equal to $\eta(4z)^6$ with $q = e^{2\pi i z}$, $\operatorname{Im}(z) > 0$, where $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is Dedekind's η-function. From the Jacobi-Macdonald formula, we see

$$\lambda_p = \begin{cases} 4a^2 - 2p & \text{if } p \equiv 1 \bmod 4 \text{ and } p = a^2 + b^2 , \quad a \equiv 1 \bmod 2 \\ 0 & \text{if } p \equiv 3 \bmod 4 . \end{cases}$$

Hence if $p \equiv 1 \bmod 4$ then $\lambda_p \not\equiv 0 \bmod p$. According to Theorem 1, $m = 1$ and $r = 1$ then $a((p-1)/2) \equiv \lambda_p \not\equiv 0 \bmod p$.

Let us prove Theorem 2 using congruences of $a(n)$, $b(n)$, $c(n)$, and Theorem 1.

If $p \equiv 1 \bmod 4$ then $\frac{p-1}{2}$ is even. From Proposition 4, $b\left(\frac{p-1}{2}\right) \equiv -b\left(\frac{p-1}{2}\right) \bmod p$. Hence $b\left(\frac{p-1}{2}\right) \equiv 0 \bmod p$. Then $a\left(\frac{mp^2-1}{2}\right) \equiv a\left(\frac{mp-1}{2}\right) a\left(\frac{p-1}{2}\right) \bmod p^2$ and $a\left(\frac{mp-1}{2}\right) \equiv a\left(\frac{m-1}{2}\right) a\left(\frac{p-1}{2}\right) \bmod p^2$. Putting $r = 2$ in Theorem 1, $a\left(\frac{mp^2-1}{2}\right) \equiv \lambda_p a\left(\frac{mp-1}{2}\right) \bmod p^2$. Since $a\left(\frac{p-1}{2}\right) \not\equiv 0 \bmod p$, this is reduced to $a\left(\frac{mp-1}{2}\right) \equiv \lambda_p a\left(\frac{m-1}{2}\right) \bmod p^2$.

If $p \equiv 3 \bmod 4$ and $p \neq 3$ then

$$a\left(\frac{p-1}{2}\right) \equiv \frac{p}{2} b\left(\frac{p-1}{2}\right) \equiv \frac{p}{6} c\left(\frac{p-1}{2}\right) \qquad \bmod p^2$$

by Theorem 3. From Proposition 6, we have $a\left(\frac{p-1}{2}\right) \equiv 0 \bmod p^2$. Hence

$$a\left(\frac{mp-1}{2}\right) \equiv a\left(\frac{p-1}{2}\right)a\left(\frac{m-1}{2}\right) \equiv 0 \qquad \mathrm{mod}\, p^2 .$$

Thus we have completed the proof. Q.E.D.

§ 5. Application for other numbers

Above method is applicable to other numbers which satisfy the relation such as (2) (see [11]), and super congruence of $u(n)$ is shown in [8]. i.e.

THEOREM 4. *Let $p \geq 3$ be a prime, and write*

$$\sum_{n=1}^{\infty} \xi_n q^n = q \sum_{n=0}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4 .$$

If $u\left(\frac{p-1}{2}\right) \neq 0 \bmod p$ *then*

$$u\left(\frac{p-1}{2}\right) \equiv \xi_p \qquad \mathrm{mod}\, p^2 .$$

Moreover we cite another example in this section.

Let, for any $n \geq 0$,

$$u(n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^3 .$$

F. Beukers and J. Stientstra showed the following congruence in [11]. Let $p \geq 3$, and write

$$\sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{2n})(1-q^{4n})(1-q^{8n})^2 .$$

Then, for $m, r \in N$, m odd,

$$v\left(\frac{mp^r-1}{2}\right) - \gamma_p v\left(\frac{mp^{r-1}-1}{2}\right) + \left(\frac{-2}{p}\right) p^2 v\left(\frac{mp^{r-2}-1}{2}\right) \equiv 0 \qquad \mathrm{mod}\, p^r ,$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi-Legendre symbol.

The numbers $w(n)$ which is (differential) of $v(n)$ can be formulate to

$$w(n) = 3(-1)^n \sum_{k=1}^{n} \binom{n}{k}^3 \left[\frac{1}{n-k+1} + \cdots + \frac{1}{n}\right] .$$

And for all primes $p \geq 3$, $n \geq 0$ and $0 \leq m \leq p-1$, we have

$$v(np+m) \equiv \{v(m) + pnw(m)\} v(n) \qquad \mathrm{mod}\, p^2 .$$

Then $v\left(\frac{p-1}{2}\right)$ of $\bmod p^2$ is determined by our method if $\left(\frac{-2}{p}\right)=1$, that is

$$v\left(\frac{p-1}{2}\right) \equiv \gamma_p + \frac{p}{2} w\left(\frac{p-1}{2}\right) \qquad \bmod p^2 .$$

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Department of Mathematics
Kobe University
Rokko, Kobe 657
Japan