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Super-potentials of positive closed currents, intersection theory and dynamics

by

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Contents

1. Introduction

Let (X, ω) be a compact Kähler manifold. It is in general quite difficult to develop a calculus on cycles of codimension $\geqslant 2$. An important approach has been introduced by Gillet–Soulé [35] who constructed appropriate potentials with tame singularities for cycles of arbitrary codimension. See also Bost–Gillet–Soulé [9], Berndtsson [7] and Polyakov–Henkin [42] for the resolution of $\partial\bar{\partial}$- and $\bar{\partial}$-equations in the projective space.

On the other hand, the calculus on positive closed currents of bidegree $(1,1)$ using potentials is very useful and quite well developed. Demailly's papers [11], [12] and book [13] contain a clear exposition of this subject. It has many applications in complex geometry and to holomorphic dynamics, see the surveys [29] and [44] for background. The recent papers [20], [18] and [14] by the authors give other applications.

Our main goal in this article is to develop a calculus on positive closed currents of bidegree (p,p). For simplicity, we restrict here to the case of the projective space $\mathbb{P}^k$. We first explain the familiar situation of currents of bidegree $(1,1)$. The reader will find in §2 some basic notions and properties of positive closed currents and of plurisubharmonic functions.

Denote by ω the standard Fubini–Study form on $\mathbb{P}^k$ normalized by $\int_{\mathbb{P}^k} \omega^k = 1$. Let S be a positive closed $(1,1)$-current on $\mathbb{P}^k$. We assume that the mass $\|S\| := \langle S, \omega^{k-1}\rangle$ is 1, that is, S is cohomologous to ω. A *quasi-potential* of S is a quasi-plurisubharmonic function u such that

$$S - \omega = dd^c u.$$

Recall that $d^c := (i/2\pi)(\bar{\partial} - \partial)$. Such a u is unique when we normalize it by $\int_{\mathbb{P}^k} u\omega^k = 0$. The correspondence $S \leftrightarrow u$ is very useful. Indeed, u has a value at every point if we allow the value $-\infty$. This makes it possible to consider the pull-back of S by dominant meromorphic maps [40] or to consider the wedge-product (intersection)

$$S \wedge S' := \omega \wedge S' + dd^c(uS')$$

when u is integrable with respect to the trace measure of a positive closed current S'.

From our point of view, the formalism in this case is as follows. Let δ_x denote the Dirac mass at x. We consider a $(k-1, k-1)$-current v, non-uniquely determined, such that $\langle v, \omega\rangle = 0$ and $dd^c v = \delta_x - \omega^k$. We then have, formally,

$$u(x) = \langle u, \delta_x\rangle = \langle u, \delta_x - \omega^k\rangle = \langle u, dd^c v\rangle = \langle dd^c u, v\rangle = \langle S - \omega, v\rangle = \langle S, v\rangle.$$

So, $\langle S, v\rangle$ is in particular independent of the choice of v. Moreover, we can extend the action of u to the convex set of probability measures $\mathscr{C}_k$. If $dd^c U_\nu = \nu - \omega^k$ with $\nu \in \mathscr{C}_k$ and $\langle U_\nu, \omega\rangle = 0$, we get

$$\langle u, \nu\rangle = \langle S, U_\nu\rangle,$$

where the value $-\infty$ is allowed. We prefer to consider that the quasi-potential is acting on $\mathscr{C}_k$. Define

$$\mathscr{U}_S(\nu) := \langle u, \nu\rangle = \langle S, U_\nu\rangle.$$

This is somehow irrelevant in this case, since Dirac masses are the extremal points of $\mathscr{C}_k$ and $\mathscr{U}_S$ is simply the affine extension of u to $\mathscr{C}_k$.

Let $\mathscr{C}_p$ denote the convex compact set of positive closed currents S of bidegree (p,p) on $\mathbb{P}^k$ and of mass 1, i.e. $\|S\|:=\langle S, \omega^{k-p}\rangle = 1$. Let U_S denote a solution to the equations

$$dd^c U_S = S - \omega^p \quad \text{and} \quad \langle U_S, \omega^{k-p+1}\rangle = 0.$$

We introduce $\mathscr{U}_S$ as a function on $\mathscr{C}_{k-p+1}$ that we will call the *super-potential* of S *of mean* 0. Suppose that R is in $\mathscr{C}_{k-p+1}$ and let $dd^c U_R = R - \omega^{k-p+1}$ with $\langle U_R, \omega^p\rangle = 0$. Then, formally,

$$\begin{aligned}\mathscr{U}_S(R) &:= \langle U_S, R\rangle = \langle U_S, R - \omega^{k-p+1}\rangle = \langle U_S, dd^c U_R\rangle \\ &= \langle dd^c U_S, U_R\rangle = \langle S - \omega^p, U_R\rangle = \langle S, U_R\rangle.\end{aligned}$$

The function $\mathscr{U}_S$ determines S. We will show that it is defined everywhere if the value $-\infty$ is allowed.

To develop the calculus, we have to consider $\mathscr{C}_p$ and $\mathscr{C}_{k-p+1}$ as infinite-dimensional spaces with special families of currents that we parametrize by the unit disc Δ in $\mathbb{C}$. We call these families *special structural discs of currents*. When $\mathscr{U}_S$ is restricted to such discs we get quasi-subharmonic functions. More precisely, if $x \mapsto R_x$ is a special structural disc of currents parametrized by $x \in \Delta$, then

$$dd^c_x \mathscr{U}_S(R_x) \geqslant -\alpha,$$

where α is a smooth $(1,1)$-form independent of S. The above definition of $\mathscr{U}_S(R)$ is valid for S or R smooth. In general, we have

$$\mathscr{U}_S(R) = \lim_{x\to 0} \mathscr{U}_S(R_x)$$

for some special discs with $R_0 = R$.

In §2, we introduce a geometry on the space $\mathscr{C}_p$, in particular the structural varieties and their curvature forms α. In §3, we establish the basic properties of super-potentials, in particular convergence theorems which make the theory useful. The main point is to extend the definition of the super-potential $\mathscr{U}_S$ from smooth forms in $\mathscr{C}_{k-p+1}$ to arbitrary currents in $\mathscr{C}_{k-p+1}$. We introduce (Definition 3.2.3) the notion of *Hartogs convergence* (or *H-convergence* for short) for currents, which is technically useful. In §4 we deal with a theory of intersection of currents. We give good conditions for the intersection of currents of arbitrary bidegrees. Two currents $R_1\in\mathscr{C}_{p_1}$ and $R_2\in\mathscr{C}_{p_2}$ are *wedgeable* if and only if a super-potential of R_1 is finite at $R_2\wedge\omega^{k-p_1-p_2+1}$. The calculus on differential forms can be extended to wedgeable currents: commutativity, associativity, convergence and continuity of wedge-product for the H-convergence. If R_2 is of bidegree $(1,1)$, then the condition means that the quasi-potentials of R_2 are integrable with respect to the trace measure of R_1. As a special case, we obtain the usual intersection of algebraic cycles. The question of developing such a theory was raised by Demailly in [11]. We give, in the last section, a satisfactory approach to the problem of pulling back a current in $\mathscr{C}_p$ by meromorphic maps. Also, in that section, we apply the theory of super-potentials to complex dynamics in higher dimension. The main applications are the following results.

As a first application, we construct Green currents of bidegree (p,p) for a large class of meromorphic maps on $\mathbb{P}^k$. This requires a good calculus using the pull-back operation. The following result holds for holomorphic maps and for Zariski generic meromorphic maps which are not holomorphic.

THEOREM 1.0.1. *Let f be an algebraically p-stable meromorphic map on $\mathbb{P}^k$ with dynamical degrees d_s, $1\leqslant s\leqslant k$. Assume that $d_{p-1}<d_p$ and that the union of the infinite fibers is of dimension $\leqslant k-p$. Then, $d_p^{-n}(f^n)^*(\omega^p)$ converge to an f^*-invariant current T which is extremal among f^*-invariant currents in $\mathscr{C}_p$.*

Note that the convergence result also holds for regular polynomial automorphisms. The current T is called the *Green current of f of bidegree (p,p)*. The convergence is still valid if we replace ω^p by a current with bounded super-potentials. The case $p=1$ was considered by the second author in [44].

Let $\mathscr{M}_d(\mathbb{P}^k)$ denote the space of dominant meromorphic self-maps of algebraic degree $d\geqslant 2$ on $\mathbb{P}^k$. Such a map can be lifted to a homogeneous polynomial self-map of $\mathbb{C}^{k+1}$ of degree d. The lift is unique up to a multiplicative constant. The space $\mathscr{M}_d(\mathbb{P}^k)$ has the structure of a Zariski dense open set in $\mathbb{P}^N$ with $N:=(k+1)(d+k)!/d!k!-1$. The space $\mathscr{H}_d(\mathbb{P}^k)$ of holomorphic self-maps of algebraic degree $d\geqslant 2$ on $\mathbb{P}^k$ is a Zariski open subset of $\mathscr{M}_d(\mathbb{P}^k)$ and $\mathscr{M}_d(\mathbb{P}^k)\setminus\mathscr{H}_d(\mathbb{P}^k)$ is an irreducible hypersurface of $\mathscr{M}_d(\mathbb{P}^k)$, see [5] and [34, p. 427].

THEOREM 1.0.2. *There is a Zariski dense open set $\mathscr{H}_d^*(\mathbb{P}^k)$ in $\mathscr{H}_d(\mathbb{P}^k)$ such that, if f is in $\mathscr{H}_d^*(\mathbb{P}^k)$ and if S is a current in $\mathscr{C}_p$, then $d^{-pn}(f^n)^*(S)$ converges to the Green current of f of bidegree (p,p) uniformly with respect to S.*

A more precise description is known for $p=1$ and $k=2$ in [31] and [27], for $p=1$ and $k\geqslant 2$ in [24] and for $p=k$ in [17] and [24] (see also [30] and [10]). Applying the previous theorem to the currents of integration on subvarieties H gives the equidistribution of

$f^{-n}(H)$ in $\mathbb{P}^k$. Another application is a rigidity theorem for polynomial automorphisms of $\mathbb{C}^k$ that we consider as birational maps on $\mathbb{P}^k$.

THEOREM 1.0.3. *Let f be a polynomial automorphism of $\mathbb{C}^k$ which is regular in the sense of* [44]. *Let I_+ denote the indeterminacy set of f at infinity and p be the integer such that $\dim I_+ = k-p-1$. Let $\mathcal{K}_+$ be the set of points $z \in \mathbb{C}^k$ with bounded orbits. Then, the Green (p,p)-current associated with f is the unique positive closed (p,p)-current of mass* 1 *with support in $\overline{\mathcal{K}}_+$.*

This result was proved by Fornæss and the second author in dimension $k=2$ [30]. Note that when $k=2$ and $p=1$, regular automorphisms are the Hénon-type automorphisms of $\mathbb{C}^2$. It is known that dynamically interesting polynomial automorphisms in $\mathbb{C}^2$ are conjugated to the regular ones [33]. Let H be an analytic subset of pure dimension $k-p$ which does not intersect the indeterminacy set I_- of f^{-1}. As a consequence of Theorem 1.0.3, we obtain that the currents of integration on $f^{-n}(H)$, properly normalized, converge to the Green (p,p)-current of f. The case $k=2$ and $p=1$ of this result was proved by Bedford and Smillie in [6].

Remark 1.0.4. The super-potential $\mathscr{U}_S$ can be extended to a function on weakly positive closed currents of bidegree $(k-p+1, k-p+1)$. For simplicity, we consider only (strongly) positive currents. We can also define super-potentials for weakly positive closed (p,p)-currents; they are functions on (strongly) positive closed currents of bidegree $(k-p+1, k-p+1)$. The super-potentials are introduced on currents of mass 1 but they can be easily extended by linearity to currents of arbitrary mass. Their domain of definition can also be extended to positive closed currents of arbitrary mass.

Other notation. Δ_r is the disc of center 0 and of radius r in $\mathbb{C}$, Δ denotes the unit disc, Δ^k the unit polydisc in $\mathbb{C}^k$ and $\Delta^* := \Delta \setminus \{0\}$. The group of automorphisms of $\mathbb{P}^k$ is a complex Lie group of dimension k^2+2k that we denote by $\mathrm{Aut}(\mathbb{P}^k) \simeq \mathrm{PGL}(k+1, \mathbb{C})$. We will work with a fixed holomorphic chart and local holomorphic coordinates y of $\mathrm{Aut}(\mathbb{P}^k)$. The automorphism with coordinates y is denoted by τ_y. Choose y so that $|y|<2$ and $y=0$ at the identity $\mathrm{id} \in \mathrm{Aut}(\mathbb{P}^k)$. In order to simplify the notation, choose a norm $|y|$ of y which is invariant under the involution $\tau \mapsto \tau^{-1}$. Fix a smooth probability measure ϱ with compact support in $\{y : |y|<1\}$. Choose ϱ radial and decreasing when $|y|$ increases. So, the involution $\tau \mapsto \tau^{-1}$ preserves ϱ. The *mass* of a positive or negative (p,p)-current S on $\mathbb{P}^k$ is defined by $\|S\| := |\langle S, \omega^{k-p} \rangle|$. Throughout the paper, S_θ, R_θ, ..., will denote the regularization of S, R, ..., defined in §2.1 below.

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2. Geometry of currents on projective spaces

In this section, we introduce some basic facts about the convex set $\mathscr{C}_p$ of positive closed (p,p)-currents of mass 1 in $\mathbb{P}^k$.

2.1. Topology and distances on the spaces of currents

Let X be a complex manifold of dimension k. Recall that a (p,p)-form Φ on X is (*strongly*) *positive* if it is positive at every point $a\in X$, that is, Φ is equal, at the point a, to a linear combination of forms with positive coefficients of the type

$$(i\varphi_1\wedge\bar\varphi_1)\wedge...\wedge(i\varphi_p\wedge\bar\varphi_p),$$

where φ_j are $(1,0)$-forms on X. Positive $(0,0)$-forms are positive functions and positive (k,k)-forms are products of volume forms with positive functions.

A (p,p)-form Φ is *weakly positive* if $\Phi\wedge\Psi$ is a positive form of maximal bidegree for every positive $(k-p,k-p)$-form Ψ. A (p,p)-current T on X is *positive* (resp. *weakly positive*) if $T\wedge\Psi$ is a positive measure for every weakly positive (resp. positive) smooth $(k-p,k-p)$-form Ψ. Positive forms and currents are weakly positive. The notions of positivity and of weak positivity coincide only for bidegrees $(0,0)$, $(1,1)$, $(k-1,k-1)$ and (k,k). We also say that Φ and T are *negative* or *weakly negative* if $-\Phi$ and $-T$ are positive or weakly positive. For real (p,p)-currents T and T', we will write $T\geqslant T'$ and $T'\leqslant T$ when $T-T'$ is positive.

Assume that X is a compact Kähler manifold and ω_X is a Kähler form on X. If T is a positive or negative (p,p)-current, the *mass* of T on a Borel set $K\subset X$ is the mass of the *trace measure* $T\wedge\omega_X^{k-p}$ of T on K; that is,

$$\|T\|_K:=|\langle T,\omega_X^{k-p}\rangle_K|.$$

The *mass* of T means its mass $\|T\|$ on $K=X$. Assume that T is positive and closed. Then, $\|T\|$ depends only on the class of T in the Hodge cohomology group $H^{p,p}(X,\mathbb{C})$. We recall the notion of density of positive closed currents. Let x denote local coordinates in a neighbourhood of a point $a\in X$ such that $x=0$ at a, and $\beta:=dd^c|x|^2$ denote the standard Euclidean form. Let B_r denote the ball $\{x:|x|<r\}$. The *Lelong number* of T at a is defined by

$$\nu(T,a):=\lim_{r\to 0}\frac{\|T\wedge\beta^{k-p}\|_{B_r}}{\pi^{k-p}r^{2k-2p}}.$$

When r decreases to 0, the expression on the right-hand side decreases to $\nu(T,a)$, which does not depend on the choice of coordinates x [47]. The Lelong number compares the mass of the current on B_r with the Euclidean volume $\pi^{k-p}r^{2k-2p}/(k-p)!$ of a ball of radius r in $\mathbb{C}^{k-p}$. A theorem of Siu says that $\{a:\nu(T,a)\geqslant c\}$ is an analytic subset of X of dimension $\leqslant k-p$ for every $c>0$ [47].

The Kähler manifolds we consider in this paper are the projective space $\mathbb{P}^k$ and the product $\mathbb{P}^k\times\mathbb{P}^k$. Let π_1 and π_2 be the canonical projections of $\mathbb{P}^k\times\mathbb{P}^k$ onto its factors. Let ω denote the Fubini–Study form on $\mathbb{P}^k$ normalized so that $\int_{\mathbb{P}^k}\omega^k=1$, and define

$$\widetilde{\omega}:=\pi_1^*(\omega)+\pi_2^*(\omega),$$

the *canonical Kähler form* on $\mathbb{P}^k\times\mathbb{P}^k$. If T is a positive closed (p,p)-current on $\mathbb{P}^k$, one proves easily that $\nu(T,a)\leqslant\|T\|$ for every $a\in\mathbb{P}^k$.

Example 2.1.1. Let V be an analytic subset of pure dimension $k-p$ in $\mathbb{P}^k$. Lelong showed in [39] that the integration on the regular part of V defines a positive closed (p,p)-current $[V]$. The mass of $[V]$ is equal to the degree of V, i.e. the number of points in the intersection of V with a generic projective plane P of dimension p. By a theorem of Thie, the Lelong number of $[V]$ at a is the multiplicity of V at a, i.e. the multiplicity at a of $V\cap P$ for P generic passing through a. This number is also equal to the number of points, in a small neighbourhood of a, of $V\cap P'$ for P' generic close enough to P. From the definition of the Lelong number, we deduce that there are constants $c,c'>0$ such that

$$cr^{2k-2}\leqslant \operatorname{volume}(V\cap B)\leqslant c'r^{2k-2}$$

for every ball B with center in V of radius $r\leqslant 1$.

We will use the *weak topology* in $\mathscr{C}_p$, i.e. the topology induced by the weak topology of currents. Recall that a sequence $\{R_n\}_{n\geqslant 0}$ of (p,p)-currents converges weakly to a current R if $\langle R_n,\Phi\rangle\to\langle R,\Phi\rangle$ for every smooth $(k-p,k-p)$-form Φ on $\mathbb{P}^k$. Since the currents in $\mathscr{C}_p$ are positive, we obtain the same topology on $\mathscr{C}_p$ if we consider real continuous forms Φ instead of smooth forms. For this topology, $\mathscr{C}_p$ is compact.

We introduce some natural *distances* on $\mathscr{C}_p$ as follows. For $\alpha\geqslant 0$ let $[\alpha]$ denote the integer part of α. Let $\mathscr{C}^\alpha_{p,q}$ be the space of (p,q)-forms whose coefficients admit derivatives of all orders $\leqslant[\alpha]$ and these derivatives are $(\alpha-[\alpha])$-Hölder continuous. We use here the sum of $\mathscr{C}^\alpha$-norms of the coefficients for a fixed atlas. If R and R' are currents in $\mathscr{C}_p$, define

$$\operatorname{dist}_\alpha(R,R'):=\sup_{\|\Phi\|_{\mathscr{C}^\alpha}\leqslant 1}|\langle R-R',\Phi\rangle|,$$

where Φ is a smooth $(k-p,k-p)$-form on $\mathbb{P}^k$. Observe that $\mathscr{C}_p$ has finite diameter with respect to these distances, since $\langle R,\Phi\rangle$ and $\langle R',\Phi\rangle$ are bounded.

LEMMA 2.1.2. *For every* $0<\alpha<\beta<\infty$ *there is a constant* $c_{\alpha,\beta}>0$ *such that*

$$\mathrm{dist}_\beta \leqslant \mathrm{dist}_\alpha \leqslant c_{\alpha,\beta}[\mathrm{dist}_\beta]^{\alpha/\beta}.$$

In particular, a function on $\mathscr{C}_p$ *is Hölder continuous for* dist_α *if and only if it is Hölder continuous for* dist_β.

Proof. The first inequality is clear. Let $L:\mathscr{C}^\infty_{k-p,k-p}\to\mathbb{C}$ be a continuous linear form. Assume that there are constants A and B such that

$$|L(\Phi)| \leqslant A\|\Phi\|_{\mathscr{C}^0} \quad \text{and} \quad |L(\Phi)| \leqslant B\|\Phi\|_{\mathscr{C}^\beta}.$$

The theory of interpolation between Banach spaces [49] implies that

$$|L(\Phi)| \leqslant c_{\alpha,\beta}A^{1-\alpha/\beta}B^{\alpha/\beta}\|\Phi\|_{\mathscr{C}^\alpha}$$

with $c_{\alpha,\beta}$ independent of A, B and L. Applying this to $L:=R-R'$ with R and R' as above, gives the second inequality in the lemma. □

When $p=k$, $\mathscr{C}_k$ is the convex set of probability measures on $\mathbb{P}^k$ and its extremal elements are the Dirac masses. One can identify the set of extremal elements of $\mathscr{C}_k$ with $\mathbb{P}^k$. Let δ_a and δ_b denote the Dirac masses at a and b, and let $\|a-b\|$ denote the distance between a and b induced by the Fubini–Study metric.

LEMMA 2.1.3. *We have*

$$\mathrm{dist}_\alpha(\delta_a,\delta_b) \simeq \|a-b\|^{\min\{\alpha,1\}}.$$

Proof. It is enough to consider the case where a and b are close. Let $x=(x_1,...,x_k)$ be local coordinates so that a and b are close to 0. Without loss of generality, one can assume that $a=0$ and $b=(t,0,...,0)$. It is clear that

$$\mathrm{dist}_\alpha(\delta_a,\delta_b) = \sup_{\|\Phi\|_{\mathscr{C}^\alpha}\leqslant 1}|\Phi(a)-\Phi(b)| \lesssim \|a-b\|^{\min\{\alpha,1\}}.$$

Using a cut-off function, one easily constructs a function Φ with bounded $\mathscr{C}^\alpha$-norm such that, near 0, $\Phi(x)=|\mathrm{Re}(x_1)|^\alpha$ if $\alpha<1$ and $\Phi(x)=\mathrm{Re}(x_1)$ if $\alpha\geqslant 1$. Hence,

$$\mathrm{dist}_\alpha(\delta_a,\delta_b) \gtrsim |\Phi(a)-\Phi(b)| = \|a-b\|^{\min\{\alpha,1\}}.$$

This implies the lemma. □

PROPOSITION 2.1.4. *For* $\alpha>0$, *the topology induced by* dist_α *coincides with the weak topology on* $\mathscr{C}_p$. *In particular,* $\mathscr{C}_p$ *is a compact separable metric space.*

Proof. It is clear that the convergence with respect to dist_α implies the weak convergence. Conversely, if a sequence converges weakly in $\mathscr{C}_p$, then it converges uniformly on compact sets of test forms with uniform norm. By Dini's theorem, the set of test forms Φ with $\|\Phi\|_{\mathscr{C}^\alpha}\leqslant 1$ is relatively compact for the uniform convergence. The proposition follows. □

Note that, since the convex set $\mathscr{C}_p$ is a Polish space, measure theory on $\mathscr{C}_p$ is quite simple. We show in Lemma 2.1.5 and Proposition 2.1.6 below that smooth forms are dense in $\mathscr{C}_p$; see [18] for the case of arbitrary compact Kähler manifolds. Here, since $\mathbb{P}^k$ is homogeneous, one can use the group $\mathrm{Aut}(\mathbb{P}^k)$ of automorphisms of $\mathbb{P}^k$ in order to regularize currents; see also [13] and [43].

Let $h_\theta(y):=\theta y$ denote the multiplication by $\theta\in\mathbb{C}$ and for $|\theta|\leqslant 1$ define $\varrho_\theta:=(h_\theta)_*\varrho$; see the introduction for the notation. Then, ϱ_0 is the Dirac mass at the identity $\mathrm{id}\in\mathrm{Aut}(\mathbb{P}^k)$ and ϱ_θ is a smooth probability measure if $\theta\neq 0$. Moreover, for every $\alpha\geqslant 0$ there is a constant $c_\alpha>0$ such that

$$\|\varrho_\theta\|_{\mathscr{C}^\alpha}\leqslant c_\alpha|\theta|^{-2k^2-4k-\alpha},$$

where $2k^2+4k$ is the real dimension of $\mathrm{Aut}(\mathbb{P}^k)$. Define, for any positive or negative (p,p)-current R on $\mathbb{P}^k$ not necessarily closed,

$$R_\theta:=\int_{\mathrm{Aut}(\mathbb{P}^k)}(\tau_y)_*R\,d\varrho_\theta(y)=\int_{\mathrm{Aut}(\mathbb{P}^k)}(\tau_{\theta y})_*R\,d\varrho(y)=\int_{\mathrm{Aut}(\mathbb{P}^k)}(\tau_{\theta y})^*R\,d\varrho(y).$$

The last equality follows from the fact that ϱ is radial and the involution $\tau\mapsto\tau^{-1}$ preserves the norm of y.

Define $R_{\theta y}:=(\tau_{\theta y})_*R$. If R is positive and closed, then $R_{\theta y}$ and R_θ are also positive and closed. Observe that, since ϱ is radial, $R_\theta=R_{\theta'}$ when $|\theta|=|\theta'|$.

LEMMA 2.1.5. *When θ tends to 0, $R_{\theta y}$ and R_θ converge weakly to R. If the restriction of R to an open set $W\subset\mathbb{P}^k$ is a form of class $\mathscr{C}^\alpha$, then $R_{\theta y}$ and R_θ converge to R in $\mathscr{C}^\alpha(W')$ for any $W'\Subset W$.*

Proof. The convergence of $R_{\theta y}$ follows from the fact that $\tau_{\theta y}$ converges to the identity in the $\mathscr{C}^\infty$ topology. This and the definition of R_θ imply the convergence of R_θ. □

PROPOSITION 2.1.6. *If $\theta\neq 0$, then R_θ is a smooth form which depends continuously on R. Moreover, for every $\alpha\geqslant 0$ there is a constant c_α independent of R such that*

$$\|R_\theta\|_{\mathscr{C}^\alpha}\leqslant c_\alpha\|R\|\,|\theta|^{-2k^2-4k-\alpha}.$$

If K is a compact set in Δ^*, then there is a constant $c_{\alpha,K}>0$ such that for $\theta,\theta'\in K$,

$$\|R_\theta-R_{\theta'}\|_{\mathscr{C}^\alpha}\leqslant c_{\alpha,K}\|R\|\,|\theta-\theta'|.$$

Proof. We may assume that R is supported at a point a, that is, $R=\delta_a\wedge\Psi$ for some tangent $(k-p,k-p)$-vector Ψ defined at a with norm $\leqslant 1$ (here, we use Federer's notation and we consider the vector Ψ as a form with negative bidegree $(p-k,p-k)$). The general case is deduced using a disintegration of R as currents with support at a point. We have

$$R_\theta=\int_{\mathrm{Aut}(\mathbb{P}^k)}(\delta_{\tau_y(a)}\wedge(\tau_y)_*\Psi)\,d\varrho_\theta(y).$$

Hence, R_θ is smooth and depends continuously on R. The estimate on $\|R_\theta\|_{\mathscr{C}^\alpha}$ follows from the estimate on the $\mathscr{C}^\alpha$-norm of ϱ_θ. The last estimate in the proposition follows from the inequality $\|\varrho_\theta-\varrho_{\theta'}\|_{\mathscr{C}^\alpha}\lesssim|\theta-\theta'|$ on K. □

Remark 2.1.7. We call R_θ the *θ-regularization* of R. In Proposition 2.1.6 we can replace $|\theta|^{-2k^2-4k-\alpha}$ by $|\theta|^{-2k-\alpha}$ but the estimates become more technical.

Let $\mathrm{dist}(\tau,\tau')$ denote the distance between τ and τ' for a fixed smooth metric on $\mathrm{Aut}(\mathbb{P}^k)$. The following simple lemma will be useful in the next sections.

LEMMA 2.1.8. *Let K be a compact subset of* $\mathrm{Aut}(\mathbb{P}^k)$. *Let W and W_0 be open sets in $\mathbb{P}^k$ such that $\overline{W}_0\subset\tau(W)$ for every $\tau\in K$. If R is of class $\mathscr{C}^\alpha$ on W, $\alpha\geqslant 0$, then $\tau_*(R)$ is of class $\mathscr{C}^\alpha$ on W_0. Moreover, there is a constant $c>0$ such that for all $\tau,\tau'\in K$,*

$$\|\tau_*(R)\|_{\mathscr{C}^\alpha(W_0)}\leqslant c\|R\|_{\mathscr{C}^\alpha(W)}$$

and

$$\|\tau_*(R)-\tau'_*(R)\|_{\mathscr{C}^\alpha(W_0)}\leqslant c\|R\|_{\mathscr{C}^\alpha(W)}\,\mathrm{dist}(\tau,\tau')^{\min(\alpha,1)}.$$

Proof. Since $\overline{W}_0\subset\tau(W)$, it is clear that $\tau_*(R)$ is of class $\mathscr{C}^\alpha$ on W_0. For $\tau\in K$, we have $\|\tau^{-1}\|_{\mathscr{C}^{\alpha+1}}\leqslant A$, which implies the first estimate. For the second one, observe that

$$\tau_*(R)-\tau'_*(R)=\tau_*[R-\tau^*\tau'_*(R)]=\tau_*[R-(\tau^{-1}\circ\tau')_*(R)].$$

This and the inequality

$$\|\tau^{-1}\circ\tau'-\mathrm{id}\|_{\mathscr{C}^{\alpha+1}}\lesssim\mathrm{dist}(\tau,\tau')$$

imply the estimate. □

2.2. Quasi-plurisubharmonic functions and capacity

Positive closed currents of bidegree $(1,1)$ admit quasi-potentials which are quasi-plurisubharmonic functions (quasi-psh for short). The compactness properties of these functions are fundamental in the study of positive closed $(1,1)$-currents. We recall here some facts; see [13] and [21].

A *quasi-psh function* is locally the difference of a psh function and a smooth one; see [13]. The first important property we will use is the following that we state only in dimension 1. It is a direct consequence of [38, Theorem 4.4.5].

LEMMA 2.2.1. *Let $\mathscr{F}$ be a compact family in $\mathscr{L}^1_{\mathrm{loc}}(\Delta)$ of subharmonic functions on Δ. Then, for every compact subset $K\subset\Delta$, there are constants $c>0$ and $A>0$ such that*

$$\|e^{-Au}\|_{\mathscr{L}^1(K)} \leqslant c \quad \textit{for every } u\in\mathscr{F}.$$

Recall that a function $\varphi:\mathbb{P}^k\to\mathbb{R}\cup\{-\infty\}$ is quasi-psh if and only if

- φ is integrable with respect to the Lebesgue measure and $dd^c\varphi\geqslant-c\omega$ for some constant $c>0$;
- φ is strongly upper semi-continuous (strongly u.s.c. for short), that is, for any Borel subset $A\subset\mathbb{P}^k$ of full Lebesgue measure, we have $\varphi(x)=\limsup_{y\to x}\varphi(y)$ with $y\in A\setminus\{x\}$.

A set $E\subset\mathbb{P}^k$ is *pluripolar* or *completely pluripolar* if there is a quasi-psh function φ such that $E\subset\varphi^{-1}(-\infty)$ or $E=\varphi^{-1}(-\infty)$, respectively.

If φ is as above, then the $(1,1)$-current $T:=dd^c\varphi+c\omega$ is positive closed and of mass c, since it is cohomologous to $c\omega$. We say that φ is a *quasi-potential* of T; it is defined everywhere on $\mathbb{P}^k$. There is a continuous one-to-one correspondence between the positive closed $(1,1)$-currents of mass 1 and the quasi-psh functions φ satisfying $dd^c\varphi\geqslant-\omega$, normalized by $\int_{\mathbb{P}^k}\varphi\omega^k=0$ or by $\max_{\mathbb{P}^k}\varphi=0$. The following compactness property is deduced from the corresponding properties of psh functions.

PROPOSITION 2.2.2. *Let $\{\varphi_n\}_{n\geqslant0}$ be a sequence of quasi-psh functions on $\mathbb{P}^k$ with $dd^c\varphi_n\geqslant-\omega$. Assume that φ_n is bounded from above by a constant independent of n. Then, either $\{\varphi_n\}_{n\geqslant0}$ converges uniformly to $-\infty$, or there is a subsequence $\{\varphi_{n_j}\}_{j\geqslant0}$ converging, in $\mathscr{L}^p$ for $1\leqslant p<\infty$, to a quasi-psh function φ with $dd^c\varphi\geqslant-\omega$.*

The next result is a consequence of the classical Hartogs lemma for psh functions.

PROPOSITION 2.2.3. *Let φ_n and φ be quasi-psh functions on $\mathbb{P}^k$ with $dd^c\varphi_n\geqslant-\omega$ and $dd^c\varphi\geqslant-\omega$. Assume that φ_n converge in $\mathscr{L}^1$ to φ. Let $\widetilde{\varphi}$ be a continuous function on a compact subset K of $\mathbb{P}^k$ such that $\varphi<\widetilde{\varphi}$ on K. Then, $\varphi_n<\widetilde{\varphi}$ on K for n large enough. In particular, we have $\limsup_{n\to\infty}\varphi_n\leqslant\varphi$ on $\mathbb{P}^k$.*

We recall a compactness property of quasi-psh functions and also an approximation result (see also Proposition 3.1.6 below).

PROPOSITION 2.2.4. *Let $\{\varphi_n\}_{n\geqslant0}$ be a decreasing sequence of quasi-psh functions with $dd^c\varphi_n\geqslant-\omega$. Then, either φ_n converge uniformly to $-\infty$, or φ_n converge pointwise and also in $\mathscr{L}^p$, $1\leqslant p<\infty$, to a quasi-psh function φ with $dd^c\varphi\geqslant-\omega$. Moreover, for every quasi-psh function φ with $dd^c\varphi\geqslant-\omega$, there is a sequence $\{\varphi_n\}_{n\geqslant0}$ of smooth functions such that $dd^c\varphi_n\geqslant-\omega$ which decreases to φ.*

Consider now a hypersurface V of $\mathbb{P}^k$ of degree m and the positive closed $(1,1)$-current $[V]$ of integration on V which is of mass m. Let φ be a quasi-potential of $[V]$, i.e. a quasi-psh function such that $dd^c\varphi=[V]-m\omega$. Let δ be an integer such that the multiplicity of V is $\leqslant\delta$ at every point. The following lemma will be useful in the next sections.

LEMMA 2.2.5. *There is a constant $A>0$ such that*

$$\delta \log \operatorname{dist}(\cdot, V)-A \leqslant \varphi \leqslant \log \operatorname{dist}(\cdot, V)+A.$$

Proof. Let $x=(x_1, ..., x_k)=(x', x_k)$ denote the coordinates of $\mathbb{C}^k$. Let $\Pi\colon \mathbb{C}^k\to\mathbb{C}^{k-1}$ with $\Pi(x):=x'$ be the projection on the first $k-1$ factors. We can reduce the problem to the local situation where V is a hypersurface of the unit polydisc Δ^k such that the projection $\Pi\colon V\to\Delta^{k-1}$ defines a ramified covering of degree $s\leqslant\delta$. For $x'\in\Delta^{k-1}$, denote by $x_{k,1}, ..., x_{k,s}$ the last coordinates of points in $\Pi^{-1}(x')\cap V$. Here, these points are repeated according to their multiplicity. So, V is the zero set of the Weierstrass polynomial

$$P(x) := (x_k-x_{k,1}) \dots (x_k-x_{k,s}).$$

This is a holomorphic function on Δ^k. It follows that $\varphi(x)-\log|P(x)|$ is a smooth function. We only have to prove that

$$\operatorname{dist}(x, V)^s \lesssim |P(x)| \lesssim \operatorname{dist}(x, V)$$

locally in Δ^k. The first inequality follows from the definition of P. Since the derivatives of P are locally bounded, it is clear that for every a in a compact set of V we have

$$|P(x)| = |P(x)-P(a)| \lesssim |x-a|.$$

Hence, $|P(x)|\lesssim\operatorname{dist}(x, V)$. $\square$

Recall that an integrable function φ on $\mathbb{P}^k$ is said to be *dsh* if it is equal outside a pluripolar set to a difference of two quasi-psh functions [21]. We identify two dsh functions if they are equal outside a pluripolar set. The space of dsh functions is endowed with the following norm:

$$\|\varphi\|_{\mathrm{DSH}} := \|\varphi\|_{\mathscr{L}^1} + \inf \|T^+\|,$$

where $T^\pm$ are positive closed $(1,1)$-currents such that $dd^c\varphi=T^+-T^-$. The currents T^+ and T^- are cohomologous and have the same mass. Note that the notion of dsh function can be easily extended to compact Kähler manifolds. We have the following lemma.

LEMMA 2.2.6. *Let $\chi:\mathbb{R}\cup\{-\infty\}\to\mathbb{R}$ be a convex increasing function such that χ' is bounded. Then, for every dsh function φ, $\chi(\varphi)$ is dsh and*

$$\|\chi(\varphi)\|_{\mathrm{DSH}} \lesssim 1+\|\varphi\|_{\mathrm{DSH}}.$$

Proof. Up to a linear change of coordinate on $\mathbb{R}\cup\{-\infty\}$, we can assume that $\|\varphi\|_{\mathrm{DSH}}\leqslant 1$. Since $\chi(x)\lesssim 1+|x|$, $\|\chi(\varphi)\|_{\mathscr{L}^1}$ is bounded. So, it is enough to prove that $\chi(\varphi)$ is dsh and to bound $dd^c\chi(\varphi)$. We can write $\varphi=\varphi^+-\varphi^-$ outside a pluripolar set, where $\varphi^\pm$ are quasi-psh with bounded DSH-norm such that $dd^c\varphi^\pm\geqslant-\omega$. Since $\varphi^\pm$ can be approximated by decreasing sequences of smooth quasi-psh functions, it is enough to consider the case where $\varphi^\pm$ and φ are smooth. It remains to bound $dd^c\chi(\varphi)$. Since χ'' is positive, we have

$$dd^c\chi(\varphi)=\chi'(\varphi)\,dd^c\varphi+\chi''(\varphi)\,d\varphi\wedge d^c\varphi\geqslant\chi'(\varphi)\,dd^c\varphi\geqslant-\|\chi'\|_\infty T^-.$$

Because χ' is bounded, $dd^c\chi(\varphi)$ can be written as a difference of positive closed currents with bounded mass. The lemma follows. ☐

Let V_t denote the *t-neighbourhood* of V, i.e. the open set of points whose distance from V is smaller than t.

LEMMA 2.2.7. *For every $t>0$ there is a smooth function χ_t, $0\leqslant\chi_t\leqslant 1$, with compact support in $V_{A_1t^{1/\delta}}$, equal to 1 on V_t and such that $\|\chi_t\|_{\mathrm{DSH}}\leqslant A_1$, where $A_1>0$ is a constant independent of t.*

Proof. We only have to consider the case where $t\ll 1$. We will construct χ_t using Lemma 2.2.6 applied twice to the function φ in Lemma 2.2.5. Let $\chi:\mathbb{R}\cup\{-\infty\}\to[0,\infty[$ be a smooth function which is convex increasing. We choose χ such that $\chi(x)=0$ on $[-\infty,-1]$ and $\chi(x)=x$ for $x\geqslant 1$. So, we have $\max\{x,0\}\leqslant\chi\leqslant\max\{x,0\}+1$. Let φ and A be as in Lemma 2.2.5. Define

$$\phi_t:=-\chi(\varphi-\log t-A-1)\quad\text{and}\quad\chi_t:=\chi(\phi_t+1).$$

Then $\phi_t-\log t$ and χ_t are smooth. From the computation in Lemma 2.2.6, their DSH-norms are bounded uniformly with respect to t. We deduce from the properties of χ that $\chi_t\geqslant 0$, $\phi_t\leqslant 0$ and $\phi_t=0$ on V_t. It follows that $\chi_t=1$ on V_t. Outside $V_{A_1t^{1/\delta}}$ with $A_1\gg 1$, by Lemma 2.2.5, we have that $\varphi-\log t-A-1\gg 0$, hence $\phi_t=-\varphi+\log t+A+1$. We deduce that $\phi_t+1\leqslant -1$ and $\chi_t=0$ there. This implies the lemma. ☐

We recall a notion of *capacity* that we introduced in [21] which can be extended to any compact Kähler manifold; see also [3] and [45]. Let

$$\mathscr{P}:=\Big\{\varphi\text{ quasi-psh}: dd^c\varphi\geqslant-\omega\text{ and }\max_{\mathbb{P}^k}\varphi=0\Big\}.$$

For $E\subset\mathbb{P}^k$, define

$$\operatorname{cap}(E):=\inf_{\varphi\in\mathscr{P}}\exp\Big(\sup_E\varphi\Big).$$

We have $\operatorname{cap}(\mathbb{P}^k)=1$, and E is pluripolar if and only if $\operatorname{cap}(E)=0$.

Consider a quasi-potential φ of a current $T\in\mathscr{C}_1$, i.e. a quasi-psh function such that $dd^c\varphi=T-\omega$. Quasi-potentials of T differ by constants. We can associate with each point $a\in\mathbb{P}^k$ the Dirac mass δ_a at a. Define a function $\mathscr{U}$ on the extremal elements of $\mathscr{C}_k$ by

$$\mathscr{U}(\delta_a):=\varphi(a).$$

We can extend this function in a unique way to an affine function on $\mathscr{C}_k$ by setting

$$\mathscr{U}(\nu):=\int_{\mathbb{P}^k}\varphi\,d\nu\quad\text{for } \nu\in\mathscr{C}_k.$$

The upper semi-continuity of φ implies that $\mathscr{U}$ is also u.s.c. on $\mathscr{C}_k$. We say that $\mathscr{U}$ is *a super-potential* of T. Super-potentials of a given current differ by constants.

Let

$$\mathscr{P}_1:=\Big\{\mathscr{U}\text{ super-potential of a current } T\in\mathscr{C}_1:\max_{\mathscr{C}_k}\mathscr{U}=0\Big\}.$$

For each set E of probability measures in $\mathscr{C}_k$, define

$$\operatorname{cap}(E):=\inf_{\mathscr{U}\in\mathscr{P}_1}\exp\Big(\sup_{\nu\in E}\mathscr{U}(\nu)\Big).$$

It is easy to check that for a single measure ν, $\operatorname{cap}(\nu)>0$ if and only if quasi-psh functions are ν-integrable, i.e. ν is PB in the sense of [17] and [21]. A definition of super-potentials for currents of any bidegree will be given in the next section.

LEMMA 2.2.8. *Let $E'\subset\mathbb{P}^k$ be a Borel set. Let E be the set of measures $\nu\in\mathscr{C}_k$ with $\nu(E')=1$. Then*, $\operatorname{cap}(E')=\operatorname{cap}(E)$.

Proof. Since $\mathscr{U}$ is affine and u.s.c., the supremum can be taken on the set of extremal points. It follows that $\max_{\mathscr{C}_k}\mathscr{U}=0$ if and only if $\max_{\mathbb{P}^k}\varphi=0$. Moreover, we have that $\sup_E\mathscr{U}=\sup_{E'}\varphi$. It is now clear that $\operatorname{cap}(E')=\operatorname{cap}(E)$. $\square$

2.3. Green quasi-potentials of currents

Let R be a current in $\mathscr{C}_p$ with $p\geqslant 1$. If U is a $(p-1,p-1)$-current such that $dd^cU=R-\omega^p$, we say that U is a *quasi-potential* of R. The integral $\langle U,\omega^{k-p+1}\rangle$ is the *mean* of U. Such currents U exist but they are not unique. When $p=1$ the quasi-potentials of R differ by constants, when $p>1$ they differ by dd^c-closed currents which can be singular. Moreover, for $p>1$, U is not always defined at every point of $\mathbb{P}^k$. This is one of the difficulties in the study of positive closed currents of higher bidegree. We will constantly use the following result which gives potentials with good estimates.

THEOREM 2.3.1. *Let R be a current in $\mathscr{C}_p$. Then, there is a negative quasi-potential U of R, depending linearly on R, such that for every r and s with $1\leqslant r<k/(k-1)$ and $1\leqslant s<2k/(2k-1)$, one has*

$$\|U\|_{\mathscr{L}^r}\leqslant c_r \quad \textit{and} \quad \|dU\|_{\mathscr{L}^s}\leqslant c_s$$

for some positive constants c_r and c_s independent of R. Moreover, U depends continuously on R with respect to the $\mathscr{L}^r$ topology on U and the weak topology on R.

We will construct U using a kernel solving the dd^c-equation for the diagonal of $\mathbb{P}^k\times\mathbb{P}^k$. We need a negative kernel with tame singularities. In the case of arbitrary compact Kähler manifolds, this is not always possible [9]. In order to simplify the notation, consider the following general situation. Let X be a homogeneous compact Kähler manifold of dimension n and let G be a complex Lie group of dimension N acting transitively on X. The following proposition gives some precisions on a result in Bost–Gillet–Soulé [9, Proposition 6.2.3]; see also Andersson [4].

PROPOSITION 2.3.2. *Let D be a submanifold of pure dimension $n-p$ in X with $p\geqslant 1$ and Ω be a real closed (p,p)-form cohomologous to the current $[D]$. Then, there is a negative $(p-1,p-1)$-form K on X smooth outside D such that $dd^cK=[D]-\Omega$ which satisfies the following inequalities near D*:

$$\|K(\cdot)\|_\infty\lesssim -\operatorname{dist}(\cdot,D)^{2-2p}\log\operatorname{dist}(\cdot,D) \quad \textit{and} \quad \|\nabla K(\cdot)\|_\infty\lesssim \operatorname{dist}(\cdot,D)^{1-2p}.$$

Moreover, there is a negative dsh function η and a positive closed $(p-1,p-1)$-form Θ smooth outside D such that $K\geqslant\eta\Theta$, $\|\Theta(\cdot)\|_\infty\lesssim\operatorname{dist}(\cdot,D)^{2-2p}$ and $\eta-\log\operatorname{dist}(\cdot,D)$ is bounded near D.

Note that $\|\nabla K\|_\infty$ is the sum $\sum_j|\nabla K_j|$, where the K_j's are the coefficients of K for a fixed atlas of X. We first prove the following lemmas.

LEMMA 2.3.3. *There is a negative dsh function η on X smooth outside D such that $\eta-\log\operatorname{dist}(\cdot,D)$ is bounded.*

Proof. Let $\pi\colon\widetilde{X}\to X$ be the blow-up of X along D. Denote by $\widehat{D}:=\pi^{-1}(D)$ the exceptional divisor. If α is a real closed $(1,1)$-form on $\widetilde{X}$ cohomologous to $[\widehat{D}]$, there is a negative quasi-psh function $\tilde{\eta}$ such that $dd^c\tilde{\eta}=[\widehat{D}]-\alpha$. It is clear that $\tilde{\eta}$ is smooth outside $\widehat{D}$ and $\tilde{\eta}-\log\operatorname{dist}(\cdot,\widehat{D})$ is bounded. Define $\eta:=\tilde{\eta}\circ\pi^{-1}$. Hence, $\eta-\log\operatorname{dist}(\cdot,D)$ is bounded. Moreover, by a theorem of Blanchard [8], $\widetilde{X}$ is Kähler. Hence, $dd^c\tilde{\eta}$ can be written as a difference of positive closed currents. It follows that $dd^c\eta=\pi_*(dd^c\tilde{\eta})$ is also a difference of positive closed currents. We deduce that η is dsh. $\square$

Proof of Proposition 2.3.2. Let $\Gamma_D\subset G\times D\times X$ denote the graph of the map

$$G\times D\longrightarrow X,$$
$$(g,x)\longmapsto g(x).$$

Let Π_G and Π_X denote the projections of Γ_D onto G and X, respectively. Observe that Π_G defines a trivial fibration. The map Π_X also defines a fibration which is locally trivial. Indeed, we can pass from a fiber to another one using the action

$$(g,x,g(x))\longmapsto(\tau(g),x,\tau(g(x))$$

on $G\times D\times X$, of an element τ of G. So, Π_X is a submersion. The integrals that we consider below are computed on some compact subset of Γ_D.

Let z be a local coordinate on G with $|z|<1$ such that $z=0$ at the identity. Let χ be a smooth positive function with compact support in $\{z:|z|<1\}$ and equal to 1 in a neighbourhood of 0. Define $K_G:=\chi(dd^c\log|z|)^{N-1}\log|z|$. This is a negative current with support in $\{z:|z|<1\}$ and $\Omega_G:=-dd^cK_G+\delta_0$ is a smooth form. We have

$$\|K_G(\,\cdot\,)\|_\infty\lesssim-|z|^{2-2N}\log|z|\quad\text{and}\quad\|\nabla K_G(\,\cdot\,)\|_\infty\lesssim|z|^{1-2N}.$$

Observe that $\widetilde{D}:=\Pi_G^{-1}(\mathrm{id})\cap\Gamma_D$ is compact and is sent by Π_X biholomorphically to D. Therefore, locally near $\widetilde{D}$, one can find coordinates $(x_D,\varrho_D,x_G)\in\mathbb{C}^{n-p}\times\mathbb{C}^p\times\mathbb{C}^{N-p}$ such that $\widetilde{D}=\{\varrho_D=x_G=0\}$ and $\Pi_X(x_D,\varrho_D,x_G)=(x_D,\varrho_D)$. Define the negative form K by

$$K:=(\Pi_X)_*(\Pi_G^*(K_G)).$$

So, K is smooth outside D. Using the coordinates (x_D,ϱ_D,x_G) and the fact that $\Pi_G:\Gamma_D\to G$ is a trivial fibration, we obtain

$$\eta\circ\Pi_X\lesssim\log\operatorname{dist}(\,\cdot\,,\widetilde{D})\lesssim-\log|\Pi_G|.$$

This, Lemma 2.3.3 and the above estimates on K_G imply that

$$K\gtrsim\eta\ (\Pi_X)_*(\Pi_G^*(\Theta_G)),$$

where $\Theta_G:=\chi(dd^c\log|z|)^{N-1}$.

Define

$$\Theta:=(\Pi_X)_*(\Pi_G^*(\Theta_G)).$$

Using the local coordinates (x_D,ϱ_D,x_G) and the fact that

$$\|\Pi_G^*(\Theta_G)\|_\infty\lesssim\operatorname{dist}(\,\cdot\,,\widetilde{D})^{2-2N}\lesssim(|\varrho_D|^2+|x_G|^2)^{1-N}$$

on Γ_D, we obtain

$$\begin{aligned}\|\Theta(\cdot)\|_\infty &\lesssim \int_{|x_G|\leqslant 1} \frac{dx_G}{(|\varrho_D|^2+|x_G|^2)^{N-1}} \leqslant \int_{|x_G|\leqslant 1} \frac{dx_G}{|\varrho_D|^{2N-2}+|x_G|^{2N-2}}\\ &\simeq \int_0^1 \frac{x^{2N-2p-1}\,dx}{|\varrho_D|^{2N-2}+x^{2N-2}} \lesssim |\varrho_D|^{2-2p}\int_0^\infty \frac{ds}{1+s^{2N-2}} \lesssim |\varrho_D|^{2-2p}.\end{aligned}$$

So, we have the estimate $\|\Theta(\cdot)\|_\infty \lesssim \operatorname{dist}(\cdot, D)^{2-2p}$.

We then deduce the desired estimate on $\|K(\cdot)\|_\infty$. We also have, near $\widetilde{D}$,

$$\|\nabla \Pi_G^*(K_G)(\cdot)\|_\infty \lesssim \operatorname{dist}(\cdot, \widetilde{D})^{1-2N}.$$

A similar computation as above gives that $\|\nabla K(\cdot)\|_\infty \lesssim \operatorname{dist}(\cdot, D)^{1-2p}$. So, the singularities of K satisfy the estimates in the proposition. We have finally

$$\begin{aligned} dd^cK &= (\Pi_X)_*(\Pi_G^*(dd^cK_G)) = (\Pi_X)_*(\Pi_G^*(\delta_{\mathrm{id}}-\Omega_G))\\ &= (\Pi_X)_*(\Pi_G^*(\delta_{\mathrm{id}})) - (\Pi_X)_*(\Pi_G^*(\Omega_G))\\ &= [D] - (\Pi_X)_*(\Pi_G^*(\Omega_G)) =: [D]-\Omega'.\end{aligned}$$

Because Ω_G is smooth, $\Omega' := (\Pi_X)_*(\Pi_G^*(\Omega_G))$ is also smooth. Since Ω and Ω' are both cohomologous to $[D]$, there is a smooth real $(p-1,p-1)$-form U such that $dd^cU = \Omega-\Omega'$. Adding to U a positive closed form large enough allows one to assume that U is positive. Replacing K by $K-U$ gives a negative form such that $dd^cK=[D]-\Omega$ with the desired tame singularities. □

Proof of Theorem 2.3.1. We apply Proposition 2.3.2 to $X := \mathbb{P}^k\times\mathbb{P}^k$,

$$G := \operatorname{Aut}(\mathbb{P}^k)\times \operatorname{Aut}(\mathbb{P}^k)$$

and D the diagonal of X. Since $\operatorname{Aut}(\mathbb{P}^k)\simeq \mathrm{PGL}(k+1,\mathbb{C})$, we can identify $\operatorname{Aut}(\mathbb{P}^k)$ with a Zariski open set in $\mathbb{P}^{k^2+2k}$ which is the projective space associated with the space of $(k+1)\times(k+1)$ matrices. The assumptions in Proposition 2.3.2 are easily verified. Let (z,ξ) denote the homogeneous coordinates of $\mathbb{P}^k\times\mathbb{P}^k$ with $z=[z_0:\ldots:z_k]$ and $\xi := [\xi_0:\ldots:\xi_k]$. The diagonal D is given by $\{(z,\xi): z=\xi\}$. Choose

$$\Omega(z,\xi) := \sum_{j=0}^k \omega(z)^j\wedge\omega(\xi)^{k-j}.$$

This form is cohomologous to $[D]$. Using the notation from Proposition 2.3.2, we define

$$U(z) := \int_{\xi\neq z} R(\xi)\wedge K(z,\xi).$$

Observe that K is smooth outside D and that its coefficients have singularities like

$$|z-\xi|^{2-2k}\log|z-\xi|$$

near D (there is an abuse of notation: we should write $|z-\xi|^{2-2k}\log|z-\xi|$ on charts $\{(z,\xi):z_j=\xi_j=1\}$, $j=0,...,k$, which cover D). It follows that the definition of U makes sense for every current R with measure coefficients. This is a form with coefficients in $\mathscr{L}^r$. An easy way to see this, is to disintegrate R into currents with support at a point. The continuity with respect to the $\mathscr{L}^r$-norm of U and the weak topology on $\mathscr{C}_p$, and the estimate on the $\mathscr{L}^r$-norm of U are easy to check.

For the rest of the theorem, by continuity, we may assume that R is a smooth form in $\mathscr{C}_p$. Denote by π_1 and π_2 the projections of $\mathbb{P}^k\times\mathbb{P}^k$ onto its factors. Note that

$$U=(\pi_1)_*(\pi_2^*(R)\wedge K).$$

Hence, U is negative since K is negative and R is positive. As R is closed, we also have

$$dd^cU=(\pi_1)_*(\pi_2^*(R)\wedge dd^cK)=(\pi_1)_*(\pi_2^*(R)\wedge[D])-(\pi_1)_*(\pi_2^*(R)\wedge\Omega)=R-\omega^p.$$

Therefore, U is a quasi-potential of R. We also have

$$dU=(\pi_1)_*(\pi_2^*(R)\wedge dK).$$

Since dK has singularities like $|z-\xi|^{1-2k}$ near D, it is clear that $\|dU\|_{\mathscr{L}^s}$ is bounded by a constant independent of R. □

Remark 2.3.4. We call U the *Green quasi-potential* of R. By Theorem 2.3.1, the mean m of U is bounded by a constant independent of R. So, $U-m\omega^{p-1}$ is a quasi-potential of mean 0 of R. Its mass is bounded uniformly with respect to R. Note that U depends on the choice of K.

We now give some properties of Green quasi-potentials.

LEMMA 2.3.5. *Let $W'\Subset W$ be open subsets of $\mathbb{P}^k$ and R be a current in $\mathscr{C}_p$. Assume that the restriction of R to W is a bounded form. Then, there is a constant $c>0$ independent of R such that*

$$\|U\|_{\mathscr{C}^1(W')}\leqslant c(1+\|R\|_{\infty,W}).$$

Proof. Observe that the derivatives of the coefficients of K have integrable singularities of order $|z-\xi|^{1-2k}$. This and the definition of U imply the result. □

The precise estimate on the behavior of U in the following proposition will be needed for the dynamical applications. It is used several times in the proof of Theorem 5.4.4.

PROPOSITION 2.3.6. *Let V, V_t and δ be as in Lemmas 2.2.5 and 2.2.7. Let T_j, $1\leqslant j\leqslant k-p+1$, be positive closed $(1,1)$-currents on $\mathbb{P}^k$, smooth on $\mathbb{P}^k\setminus V$. Assume that the quasi-potentials of T_j are α_j-Hölder continuous with $0<\alpha_j\leqslant 1$. If U is the Green quasi-potential of a current $R\in\mathscr{C}_p$, then*

$$\left|\int_{V_t\setminus V} U\wedge T_1\wedge\ldots\wedge T_{k-p+1}\right|\leqslant ct^\beta, \quad \textit{with } \beta:=(20k^2\delta)^{-k}\alpha_1\ldots\alpha_{k-p+1},$$

where $c>0$ is a constant independent of R and of t.

We will use the notation from Theorem 2.3.1 and Proposition 2.3.2. For $M>0$, define $\eta_M:=\min\{0,M+\eta\}$. As in Lemma 2.2.6, we can show that $\|\eta_M\|_{\mathrm{DSH}}$ is bounded independently of M. We have $\eta_M-M\leqslant\eta$. Define $K_M:=-M\Theta$ and $K'_M:=\eta_M\Theta$. Then, K_M is negative closed and we have $K_M+K'_M\lesssim K$. Define also

$$U_M(z):=\int_\xi R(\xi)\wedge K_M(z,\xi) \quad \text{and} \quad U'_M(z):=\int_\xi R(\xi)\wedge K'_M(z,\xi).$$

The form U_M is negative closed of mass $\simeq M$ and $U_M+U'_M\lesssim U$. Choose $M:=t^{-\beta}$. We estimate U_M and U'_M separately. Recall that U is negative and that Θ has singularities of order $\mathrm{dist}(z,\xi)^{2-2k}$.

LEMMA 2.3.7. *We have*

$$\left|\int_{V_t} U_M\wedge\omega^{k-p+1}\right|\lesssim t.$$

Proof. We may assume that $t<\frac{1}{2}$. We do not need that R is closed. So, we may assume that R has support at a point $a\in\mathbb{P}^k$. We define U_M using the same integral formula as above. Then, the coefficients of U_M have singularities of type $M|x|^{2-2k}$, where x are local coordinates such that $x=0$ at a. The problem is local. We may assume that V is a hypersurface in a neighbourhood of the unit ball B. Since $M\leqslant t^{-1/2}$, it is sufficient to prove that

$$\int_{V_t\cap B}|x|^{2-2k}(dd^c|x|^2)^k\lesssim t^{3/2}.$$

Let A be a maximal subset of $V\cap B$ such that the distance between two points in A is $\geqslant t$. The balls of radius $2t$ with center in A cover $V\cap B$ and the ones of radius $3t$ cover $V_t\cap B$. Let A_n be the set of points $p\in A$ such that $nt\leqslant|p|<(n+1)t$ and m_n be the number of elements of A_n. Observe that the $m_0+\ldots+m_n$ balls of radius $\frac{1}{2}t$ with centers in $A_1\cup...\cup A_n$ are disjoint. They cover an open subset of $V\cap\{x:|x|\leqslant(n+2)t\}$. Using Lelong's estimate in Example 2.1.1, see also [39], gives that

$$m_0+...+m_n\lesssim n^{2k-2}.$$

Note that m_0 is 0 or 1 and the integral of $|x|^{2-2k}(dd^c|x|^2)^k$ on a ball of radius $3t$ with center in A_0 is bounded by the integral of this function on the ball of center 0 and of radius $4t$. Hence, it is of order t^2. For $n\geqslant 1$, it is clear that the integral of the considered form on a ball with center in A_n is of order $n^{2-2k}t^2$. Using the estimates on m_n and Abel's transform, one obtains

$$\begin{aligned}\int_{V_t\cap B}|x|^{2-2k}(dd^c|x|^2)^k &\lesssim t^2+\sum_{1\leqslant n\leqslant 1/t} m_n n^{2-2k}t^2\\ &\lesssim t^2+\sum_{1\leqslant n\leqslant 1/t}[n^{2k-2}-(n-1)^{2k-2}]n^{2-2k}t^2\\ &\lesssim t^2+t^2\sum_{1\leqslant n\leqslant 1/t}\frac{1}{n}.\end{aligned}$$

This implies the lemma. □

We continue the proof of Proposition 2.3.6. By continuity, it is enough to consider the case where R and U are smooth. We also have that U_M is smooth.

LEMMA 2.3.8. *For every* $0\leqslant l\leqslant k-p+1$ *we have*

$$\left|\int_{V_t} U_M\wedge T_1\wedge...\wedge T_l\wedge\omega^{k-p-l+1}\right|\lesssim t^{\beta_l},\quad \textit{where } \beta_l:=(20k^2\delta)^{-l}\alpha_1\ldots\alpha_l.$$

Proof. The proof is by induction. The previous lemma implies the case $l=0$. Assume the lemma for $l-1$. Let χ_t be as in Lemma 2.2.7. We want to prove that

$$\int(-\chi_t U_M\wedge T_1\wedge...\wedge T_l\wedge\omega^{k-p-l+1})\lesssim t^{\beta_l}.$$

Write $T_l=\omega+dd^c u$ with u negative quasi-psh of class $\mathscr{C}^{\alpha_l}$. By the induction hypothesis, since χ_t has support in $V_{A_1 t^{1/\delta}}$, we obtain

$$\int(-\chi_t U_M\wedge T_1\wedge...\wedge T_{l-1}\wedge\omega^{k-p-l+2})\lesssim t^{\delta^{-1}\beta_{l-1}}\lesssim t^{\beta_l}.$$

Therefore, we only have to prove that

$$\int(-\chi_t U_M\wedge T_1\wedge...\wedge T_{l-1}\wedge dd^c u\wedge\omega^{k-p-l+1})\lesssim t^{\beta_l}.$$

By Proposition 2.1.6 and Lemma 2.1.8, there is a smooth function u_ε such that

$$\|u_\varepsilon\|_{\mathscr{C}^2}\lesssim\varepsilon^{-2k^2-4k-2}\quad\text{and}\quad\|u-u_\varepsilon\|_\infty\lesssim\varepsilon^{\alpha_l}.$$

Using Stokes theorem we can write the left-hand side of the previous inequality as

$$\int(-\chi_t U_M\wedge T_1\wedge\ldots\wedge T_{l-1}\wedge dd^c u_\varepsilon\wedge\omega^{k-p-l+1})$$
$$+\int(-dd^c\chi_t\wedge U_M\wedge T_1\wedge\ldots\wedge T_{l-1}(u-u_\varepsilon)\wedge\omega^{k-p-l+1}).$$

By the induction hypothesis, the previous estimates on $\|u_\varepsilon\|_{\mathscr{C}^2}$ and Lemma 2.2.7, we obtain that the first term is of order at most equal to $t^{\delta^{-1}\beta_{l-1}}\varepsilon^{-2k^2-4k-2}$. If we write $dd^c\chi_t=T^+-T^-$ with $T^\pm$ positive closed of bounded mass, the second term is of order less than

$$\varepsilon^{\alpha_l}\int T^+\wedge U_M\wedge T_1\wedge\ldots\wedge T_{l-1}\wedge\omega^{k-p-l+1}+\varepsilon^{\alpha_l}\int T^-\wedge U_M\wedge T_1\wedge\ldots\wedge T_{l-1}\wedge\omega^{k-p-l+1}.$$

These integrals can be computed cohomologically. The currents $T^\pm$ have bounded mass. Since $K_M=-M\Theta$, we deduce from the definition of U_M that $-U_M$ is positive and closed of mass $M=t^{-\beta}$. Therefore, the last sum is $\lesssim t^{-\beta}\varepsilon^{\alpha_l}$.

Take $\varepsilon:=t^{\delta^{-1}(2k^2+4k+2+\alpha_l)^{-1}\beta_{l-1}}$. We have

$$1-\frac{2k^2+4k+2}{2k^2+4k+2+\alpha_l}\geqslant\frac{\alpha_l}{10k^2}.$$

Then

$$t^{\delta^{-1}\beta_{l-1}}\varepsilon^{-2k^2-4k-2}\lesssim t^{\delta^{-1}\beta_{l-1}(10k^2)^{-1}\alpha_l}\lesssim t^{\beta_l}$$

and

$$t^{-\beta}\varepsilon^{\alpha_l}\lesssim t^{-\beta}t^{(10k^2\delta)^{-1}\beta_{l-1}\alpha_l}\lesssim t^{-\beta}t^{2\beta_l}\lesssim t^{\beta_l}.$$

This implies the desired estimate. □

LEMMA 2.3.9. *We have* $\|U'_M\|\lesssim\exp\big(-\frac{1}{2}M\big)$.

Proof. We can forget that R is smooth and assume that R has support at a point a. The behavior of η implies that U'_M has support in the ball of center a and radius $\lesssim\exp(-\frac{1}{2}M)$. The coefficients of U'_M have singularities $\lesssim-|x|^{2-2k}\log|x|$ for local coordinates x with $x=0$ at a. Hence, $\|U'_M\|\lesssim\exp\big(-\frac{1}{2}M\big)$. □

The following lemma completes the proof of Proposition 2.3.6, since

$$M=t^{-\beta}\gg|\log t|.$$

Lemma 2.3.10. *For every* $0\leqslant l\leqslant k-p+1$ *we have*

$$\left|\int U'_M\wedge T_1\wedge...\wedge T_l\wedge\omega^{k-p-l+1}\right|\lesssim\exp\big(-\tfrac{1}{2}(10k^2)^{-l}\alpha_1\,...\,\alpha_l M\big).$$

Proof. The previous lemma implies the case $l=0$. Assume the lemma for $l-1$ and use the notation from the proof of Lemma 2.3.8. The integral to bound is equal to

$$\int(-U'_M\wedge T_1\wedge...\wedge T_{l-1}\wedge dd^c u_\varepsilon\wedge\omega^{k-p-l+1})$$
$$+\int_{\mathbb{P}^k\times\mathbb{P}^k}(-K'_M\wedge R(\xi)\wedge T_1(z)\wedge...\wedge T_{l-1}(z)dd^c(u(z)-u_\varepsilon(z))\wedge\omega(z)^{k-p-l+1}).$$

Choose $\varepsilon=\exp(-(10k^2)^{-l}\alpha_1\,...\,\alpha_{l-1}M)$. Using the estimate on $\|u_\varepsilon\|_{\mathscr{C}^2}$, by the induction hypothesis, the first integral is of order at most equal to

$$\exp\big(-\tfrac{1}{2}(10k^2)^{-l+1}\alpha_1\,...\,\alpha_{l-1}M\big)\varepsilon^{-2k^2-4k-2}\lesssim\exp\big(-\tfrac{1}{2}(10k^2)^{-l}\alpha_1\,...\,\alpha_l M\big).$$

The second one is equal to

$$\int_{\mathbb{P}^k\times\mathbb{P}^k}(-dd^cK'_M\wedge R(\xi)\wedge T_1(z)\wedge...\wedge T_{l-1}(z)(u(z)-u_\varepsilon(z))\wedge\omega(z)^{k-p-l+1}).$$

Since the DSH-norm of η_M in the definition of K'_M is bounded, the first term in the last integral can be bounded by a positive closed current with bounded mass. So, this integral is of order at most equal to

$$\|u-u_\varepsilon\|_\infty\lesssim\varepsilon^{\alpha_l}=\exp(-(10k^2)^{-l}\alpha_1\,...\,\alpha_{l-1}\alpha_l M).$$

This implies the result. □

We will use the following lemma in the study of deformation of currents.

Lemma 2.3.11. *Let R be a current in $\mathscr{C}_p$ and U be a quasi-potential of mean m of R. Let $R_{\theta y}=(\tau_{\theta y})_*(R)$ be defined as in §2.1. Then, there is a quasi-potential $U'_{\theta y}$ of $R_{\theta y}$ of mean m such that $U'_{\theta y}-(\tau_{\theta y})_*(U)$ is a smooth form with*

$$\|U'_{\theta y}-(\tau_{\theta y})_*(U)\|_{\mathscr{C}^2}\leqslant c(1+\|U\|)|\theta|,$$

where $c>0$ is a constant independent of R, U, θ and y.

Proof. Since $\|(\tau_{\theta y})_*(\omega^p)-\omega^p\|_{\mathscr{C}^2}\lesssim|\theta|$, there is a $(p-1,p-1)$-form $\Omega_{\theta y}$ such that $\|\Omega_{\theta y}\|_{\mathscr{C}^2}\lesssim|\theta|$ and $dd^c\Omega_{\theta y}=(\tau_{\theta y})_*(\omega^p)-\omega^p$. It is clear that the mean m'' of $\Omega_{\theta y}$ is of order $\lesssim|\theta|$. Set $U'_{\theta y}:=(\tau_{\theta y})_*(U)+\Omega_{\theta y}$. So, the mean m' of $U'_{\theta y}$ satisfies

$$\begin{aligned}|m'-m|&=\left|\int(\tau_{\theta y})_*(U)\wedge\omega^{k-p+1}+m''-\int U\wedge\omega^{k-p+1}\right|\\&\leqslant|m''|+\left|\int U\wedge[(\tau_{\theta y})^*(\omega^{k-p+1})-\omega^{k-p+1}]\right|.\end{aligned}$$

The last term is of order $\lesssim\|U\||\theta|$ since $\|(\tau_{\theta y})^*(\omega^{k-p+1})-\omega^{k-p+1}\|_\infty$ is of order $\lesssim|\theta|$. Subtracting from $U'_{\theta y}$ the form $(m'-m)\omega^{p-1}$, which is of order $\lesssim|\theta|$, gives a quasi-potential satisfying the lemma. □

2.4. Structural varieties in the spaces of currents

The notion of structural varieties of $\mathscr{C}_p$ was introduced in [22]; see also [15]. In some sense, we consider $\mathscr{C}_p$ as a space of infinite dimension admitting "complex subvarieties" of finite dimension. The emphasis is that in order to connect two closed currents we use a *closed* current in higher dimension. Holomorphic families of analytic cycles of codimension p are examples of structural varieties in $\mathscr{C}_p$. Other examples of structural varieties can be obtained by deforming a given current in $\mathscr{C}_p$ using a holomorphic family of automorphisms. The reader will find in Dujardin [26] and in [16] an application of such a deformation to the dynamics of Hénon-like maps; see also [50]. General structural varieties are more flexible, and this is crucial in our study.

Let X be a complex manifold, and $\pi_X: X\times\mathbb{P}^k\to X$ and $\pi: X\times\mathbb{P}^k\to\mathbb{P}^k$ denote the canonical projections. Consider a positive closed (p,p)-current $\mathscr{R}$ in $X\times\mathbb{P}^k$. By slicing theory [28], the slices $\langle\mathscr{R},\pi_X,x\rangle$ exist for almost every $x\in X$. Such a slice is a positive closed (p,p)-current on $\{x\}\times\mathbb{P}^k$ (following [22], we can prove that the slices exist for x outside a pluripolar set). We often identify $\langle\mathscr{R},\pi_X,x\rangle$ with a (p,p)-current R_x in $\mathbb{P}^k$.

LEMMA 2.4.1. *The mass of R_x does not depend on x.*

Proof. Set $\mathscr{R}':=\mathscr{R}\wedge\pi^*(\omega^{k-p})$. Then, $\mathscr{R}'$ is positive closed on $X\times\mathbb{P}^k$ and $(\pi_X)_*(\mathscr{R}')$ is closed of bidegree $(0,0)$ on X. Hence, it is a constant function. So, the function

$$\varphi(x):=\|\langle\mathscr{R}',\pi_X,x\rangle\|=\int_{\mathbb{P}^k}R_x\wedge\omega^{k-p}=\|R_x\|$$

is constant. The lemma follows. □

We assume that the mass of R_x is equal to 1. The map $x\mapsto R_x$ is defined almost everywhere on X with values in $\mathscr{C}_p$.

Definition 2.4.2. We say that the map $x\mapsto R_x$ or the family $\{R_x\}_{x\in X}$ defines a *structural variety* in $\mathscr{C}_p$. The positive closed $(1,1)$-current

$$\alpha_{\mathscr{R}}:=(\pi_X)_*(\mathscr{R}\wedge\pi^*(\omega^{k-p+1}))$$

on X is called the *curvature* of the structural variety, see Propositions 3.1.3 and 3.2.1 below.

Definition 2.4.3. A structural variety associated with $\mathscr{R}$ is said to be *special* if R_x exists for every $x\in X$, R_x depends continuously on x and the curvature is a smooth form.

In order to simplify the argument, we restrict to special structural varieties or discs. The most useful structural discs in this work are $\{R_\theta\}_{\theta\in\Delta}$; see the introduction and Lemma 2.5.3 below.

2.5. Deformation by automorphisms

Using the automorphisms of $\mathbb{P}^k$, we will construct some special structural discs in $\mathscr{C}_p$ that we will use later on. We first construct large structural varieties parametrized by $X=\mathrm{Aut}(\mathbb{P}^k)$.

Proposition 2.5.1. *Let R be a current in $\mathscr{C}_p$. Then, the map $h\colon \mathrm{Aut}(\mathbb{P}^k)\to\mathscr{C}_p$ with $h(\tau)=R_\tau:=\tau_*(R)$ defines a special structural variety in $\mathscr{C}_p$. Moreover, its curvature is bounded by a smooth positive $(1,1)$-form independent of R.*

Proof. For any smooth test form Φ, we have $\langle R_\tau,\Phi\rangle=\langle R,\tau^*(\Phi)\rangle$. So, clearly $\tau\mapsto R_\tau$ is continuous. Consider the holomorphic map $H\colon \mathrm{Aut}(\mathbb{P}^k)\times\mathbb{P}^k\to\mathbb{P}^k$ defined by $H(\tau,z):=\tau^{-1}(z)$. The current $\mathscr{R}:=H^*(R)$ is positive closed of bidegree (p,p). It is easy to check from the definition of slices that $R_\tau=\langle\mathscr{R},\pi_X,\tau\rangle$. Hence, h defines a continuous structural variety.

Now, we have to show that the curvature

$$\alpha_{\mathscr{R}}:=(\pi_X)_*(H^*(R)\wedge\pi^*(\omega^{k-p+1}))$$

is a smooth form. We prove this for any current R of mass $\leqslant 1$ not necessarily closed. Then, we may assume that R is supported at a point a, that is, there is a tangent $(k-p,k-p)$-vector Ψ at a of norm $\leqslant 1$ such that $R=\delta_a\wedge\Psi$ (the general case is obtained using a disintegration of R into currents of the previous type). We have

$$H^*(R)=[H^{-1}(a)]\wedge\widetilde{\Psi},$$

where $\widetilde{\Psi}$ is a $(k-p,k-p)$-vector field with support in $H^{-1}(a)$ such that $H_*(\widetilde{\Psi})=\Psi$. Because H is a submersion, we can choose $\widetilde{\Psi}$ smooth on $H^{-1}(a)$.

Since $H^{-1}(a)$ is a holomorphic graph over $\mathrm{Aut}(\mathbb{P}^k)$, the form $\alpha_{\mathscr{R}}$ defined above is the direct image of $[H^{-1}(a)]\wedge\widetilde{\Psi}\wedge\pi^*(\omega^{k-p+1})$ by π_X. So, $\alpha_{\mathscr{R}}$ is smooth. Moreover, the $\mathscr{C}^s$-norm of $\alpha_{\mathscr{R}}$ on any fixed compact subset of $\mathrm{Aut}(\mathbb{P}^k)$ is uniformly bounded for every $s\geqslant 0$. The proposition follows. □

Remark 2.5.2. If $j\colon\Delta\to\mathrm{Aut}(\mathbb{P}^k)$ is a holomorphic map, then $x\mapsto j(x)_*R$, which is equal to $h\circ j$, defines a special structural disc. We can also construct a structural disc passing through R and through the current of integration on a fixed plane of codimension p [15]. So, $\mathscr{C}_p$ is connected by structural discs.

Let R be a current in $\mathscr{C}_p$. The following lemma gives us a useful special structural disc passing through R.

LEMMA 2.5.3. *Let R_θ be the currents constructed in* §2.1. *Then, the family* $\{R_\theta\}_{\theta\in\Delta}$ *defines a special structural disc whose curvature is bounded by a smooth positive* $(1,1)$-*form α which does not depend on R.*

Proof. By Proposition 2.5.1, for $|y|<1$, the family $\{R_{\theta y}\}_{\theta\in\Delta}$ defines a special disc in $\mathscr{C}_p$. Moreover, the $\mathscr{C}^s$-norm of its curvature is bounded uniformly with respect to R and y. In particular, this curvature is bounded by a positive form α which does not depend on R and y.

Let $\mathscr{R}_y$ denotes the (p,p)-current on $\Delta\times\mathbb{P}^k$ associated with the structural disc $\{R_{\theta y}\}_{\theta\in\Delta}$ and define $\mathscr{R}:=\int\mathscr{R}_y\,d\varrho(y)$. Recall that $R_\theta=\int_y R_{\theta y}\,d\varrho(y)$. Hence, $\{R_\theta\}_{\theta\in\Delta}$ is the family of slices of $\mathscr{R}$ and it defines a structural disc in $\mathscr{C}_p$. We know that R_θ depends continuously on θ. This and the above properties of $\{R_{\theta y}\}_{\theta\in\Delta}$ imply that the curvature of $\{R_\theta\}_{\theta\in\Delta}$ is bounded by α. $\square$

3. Super-potentials of currents

Consider a current S in $\mathscr{C}_p$. We introduce a *super-potential* associated with S. It is an affine upper semi-continuous (u.s.c. for short) function $\mathscr{U}_S$ defined on $\mathscr{C}_{k-p+1}$, with values in $\mathbb{R}\cup\{-\infty\}$.

3.1. Super-potentials of currents

Assume first that S is a smooth form in $\mathscr{C}_p$. The general case will be obtained using a regularization of S. Consider an element R of $\mathscr{C}_{k-p+1}$ and fix a real number m. Define

$$\mathscr{U}_S(R):=\langle S,U_R\rangle,\quad U_R \text{ a quasi-potential of mean } m \text{ of } R. \tag{3.1}$$

LEMMA 3.1.1. *The integral $\langle S,U_R\rangle$ does not depend on the choice of U_R with a fixed mean m. It defines an affine continuous function $\mathscr{U}_S$ on $\mathscr{C}_{k-p+1}$. Moreover, if U_S is a smooth quasi-potential of S with mean m, then $\mathscr{U}_S(R)=\langle U_S,R\rangle$. In particular, we have $\mathscr{U}_S(\omega^{k-p+1})=m$.*

Proof. Let U_S be a smooth quasi-potential of S with mean m. Using Stokes formula, we obtain

$$\begin{aligned}\mathscr{U}_S(R)&=\langle S,U_R\rangle=\langle S-\omega^p,U_R\rangle+\langle\omega^p,U_R\rangle=\langle dd^cU_S,U_R\rangle+m\\&=\langle U_S,dd^cU_R\rangle+m=\langle U_S,R-\omega^{k-p+1}\rangle+m=\langle U_S,R\rangle.\end{aligned}$$

This also shows that $\mathscr{U}_S(R)$ is independent of the choice of U_R and it depends continuously on R. It is clear that $\mathscr{U}_S$ is affine. $\square$

We say that $\mathscr{U}_S$ is the *super-potential of mean* m of S. One obtains the super-potential of mean m' by adding $m'-m$ to the super-potential of mean m. We will see later that the following lemma holds also for an arbitrary current S in $\mathscr{C}_p$ smooth or not; see Corollary 3.1.7 below.

LEMMA 3.1.2. *There is a constant* $c\geqslant 0$ *independent of* S *such that if* $\mathscr{U}_S$ *is the super-potential of mean* m *of* S, *then* $\mathscr{U}_S\leqslant m+c$ *everywhere.*

Proof. Without loss of generality, we may assume that $m=0$. Let U'_R be the Green quasi-potential of R which is a negative current and let m' be the mean of U'_R. Then, $U_R:=U'_R-m'\omega^{k-p}$ is a quasi-potential of mean 0 of R. By Lemma 3.1.1, since U'_R is negative and S is positive, we have

$$\mathscr{U}_S(R)=\langle S,U_R\rangle=\langle S,U'_R\rangle-m'\leqslant -m'.$$

We have seen in Remark 2.3.4 that $|m'|$ is bounded by a constant independent of R. This implies the result. □

As we have seen in §2.5, the convex set $\mathscr{C}_{k-p+1}$ can be considered as an infinite-dimensional space admitting "complex subvarieties" of finite dimension. With this point of view, we can consider $\mathscr{U}_S$ as a quasi-psh function on $\mathscr{C}_{k-p+1}$. More precisely, we will show that the restriction of $\mathscr{U}_S$ to a special structural variety is a quasi-psh function, see Proposition 3.2.1 below.

We now extend the definition of $\mathscr{U}_S$ to an arbitrary current S in $\mathscr{C}_p$. For R smooth, define $\mathscr{U}_S(R)$ as in (3.1) with U_R smooth. Observe that $\mathscr{U}_S(R)$ depends continuously on S. We can show, as in Lemma 3.1.1, that the definition is independent of the choice of U_R. We will extend $\mathscr{U}_S$ to a function on $\mathscr{C}_{k-p+1}$ with values in $\mathbb{R}\cup\{-\infty\}$. The reader can check that for $p=1$ we will obtain the same super-potentials as introduced in §2.2.

Let $\{R_\theta\}_{\theta\in\Delta}$ be the special structural disc in $\mathscr{C}_{k-p+1}$ constructed in §2.1 and §2.5 and let α be as in Lemma 2.5.3. Recall that R_θ is smooth for $\theta\neq 0$.

LEMMA 3.1.3. *The function* $u(\theta):=\mathscr{U}_S(R_\theta)$ *defined on* Δ^* *can be extended as a quasi-subharmonic function on* Δ *such that* $dd^cu\geqslant-\alpha$.

Proof. Proposition 2.1.6 implies that u is continuous on Δ^*. Lemma 3.1.2 holds for S singular and R smooth. So, u is bounded from above. Let $\mathscr{R}$ be the $(k-p+1,k-p+1)$-current in $\Delta\times\mathbb{P}^k$ associated with $\{R_\theta\}_{\theta\in\Delta}$, and let π_Δ and π be as in §2.5. Observe that $\mathscr{R}$ is smooth on $\Delta^*\times\mathbb{P}^k$. If U_S is a quasi-potential of mean m of S, then, by the definition of $\mathscr{U}_S$, we have

$$u=(\pi_\Delta)_*(\mathscr{R}\wedge\pi^*(U_S))$$

in the sense of currents on Δ^*. It follows that

$$dd^c u = (\pi_\Delta)_*(\mathscr{R}\wedge\pi^*(dd^c U_S)) \geqslant -(\pi_\Delta)_*(\mathscr{R}\wedge\pi^*(\omega^p)) \geqslant -\alpha.$$

If v is a smooth function such that $dd^c v=\alpha$, then $u+v$ is subharmonic on Δ^*. Since u is bounded from above, $u+v$ can be extended to a subharmonic function. The lemma follows. Observe that if R is a smooth form, then $u(\theta)$ is defined and is a continuous function on Δ. It is quasi-subharmonic and satisfies $dd^c u\geqslant -\alpha$. □

Recall that S_θ is defined as in §2.1 and §2.5 for S instead of R. By Lemma 2.1.5 and Proposition 2.1.6, S_θ is smooth and converges to S when θ tends to 0.

PROPOSITION 3.1.4. *Let $\mathscr{U}_{S_\theta}$ denote the super-potential of mean m of S_θ. Then, $\mathscr{U}_{S_\theta}(R)$ converges to $u(0)$ when $\theta\to 0$. In particular, if R is a smooth form, then $\mathscr{U}_{S_\theta}(R)$ converges to $\mathscr{U}_S(R)$.*

Proof. When R is smooth, we have $u(0)=\mathscr{U}_S(R)$. So, we deduce easily the last assertion from the first one. By Lemma 3.1.3, there is a constant $A>0$ independent of R and S such that $u(\theta)+A|\theta|^2$ is subharmonic. Since this function is radial (recall here that ϱ is radial; see the introduction), it decreases to $u(0)$ when $|\theta|$ decreases to 0. Therefore, the proposition is deduced from Lemma 3.1.5 below. □

LEMMA 3.1.5. *There is a constant $c>0$ independent of R and S such that*

$$|\mathscr{U}_{S_\theta}(R)-\mathscr{U}_S(R_\theta)| = |\mathscr{U}_{S_\theta}(R)-u(\theta)| \leqslant c|\theta|$$

for $\theta\in\Delta^*$.

Proof. Since R can be approximated by smooth forms in $\mathscr{C}_{k-p+1}$, we may assume that R is smooth. Then, we may also assume that S is smooth. Indeed, the following estimates are uniform with respect to R and S. Let U_S be a smooth quasi-potential of mean m of S with bounded mass. Define $U_{\theta y}:=(\tau_{\theta y})^* U_S$. We have

$$\mathscr{U}_S(R_\theta) = \int_y \langle U_S, (\tau_{\theta y})_* R\rangle \, d\varrho(y) = \int_y \langle U_{\theta y}, R\rangle \, d\varrho(y).$$

As in Lemma 2.3.11, we show that there is a quasi-potential $U'_{\theta y}$ of mean m of $(\tau_{\theta y})^*(S)$ such that $\|U'_{\theta y}-U_{\theta y}\|_{\mathscr{C}^2}\lesssim|\theta|$. We have

$$\mathscr{U}_{S_\theta}(R) = \int_y \langle U'_{\theta y}, R\rangle \, d\varrho(y).$$

The estimate on $U'_{\theta y}-U_{\theta y}$ implies that

$$|\mathscr{U}_{S_\theta}(R)-\mathscr{U}_S(R_\theta)| = \left|\int_y \langle U'_{\theta y}-U_{\theta y}, R\rangle d\varrho(y)\right| \lesssim |\theta|.$$

The proof is complete. □

Proposition 3.1.6. *There is a sequence of smooth forms* $\{S_n\}_{n\geqslant 0}$ *in* $\mathscr{C}_p$ *with super-potentials* $\mathscr{U}_n$ *of mean* m_n *such that*

- $\mathrm{supp}(S_n)$ *converge to* $\mathrm{supp}(S)$;
- S_n *converge to* S *and* $m_n\to m$;
- $\{\mathscr{U}_n\}_{n\geqslant 0}$ *is a decreasing sequence*;

Moreover, if S_n, m_n *and* $\mathscr{U}_n$ *satisfy the last two properties, then* $\mathscr{U}_n(R)$ *converge to* $u(0)$. *In particular, if* R *is a smooth form in* $\mathscr{C}_{k-p+1}$, *then* $\mathscr{U}_n(R)$ *converge to* $\mathscr{U}_S(R)$.

Proof. Consider $S_n:=S_{\theta_n}$, where $\{\theta_n\}_{n\geqslant 0}$ is a sequence in Δ^* such that $|\theta_n|$ decrease to 0 and that $\sum_{n=0}^{\infty}|\theta_n|$ is finite. Define

$$m_n := m+A|\theta_n|^2+2c\sum_{j=n}^{\infty}|\theta_j|,$$

where c and A are the constants introduced in Lemma 3.1.5 and in the proof of Proposition 3.1.4. It is clear that $S_n\to S$, $\mathrm{supp}(S_n)\to\mathrm{supp}(S)$ and $m_n\to m$. Define

$$\mathscr{U}_n := \mathscr{U}_{S_n}+m_n-m.$$

This is the super-potential of mean m_n of S_n. Lemma 3.1.5 implies that

$$\begin{aligned}\mathscr{U}_n(R)-\mathscr{U}_{n+1}(R) &\geqslant \mathscr{U}_{S_n}(R)-\mathscr{U}_{S_{n+1}}(R)+A(|\theta_n|^2-|\theta_{n+1}|^2)+2c|\theta_n|\\ &\geqslant [u(\theta_n)+A|\theta_n|^2]-[u(\theta_{n+1})+A|\theta_{n+1}|^2].\end{aligned}$$

We have seen that $u(\theta)+A|\theta|^2$ is radial subharmonic and decreases to $u(0)$ when $|\theta|$ decreases to 0. Hence, $\{\mathscr{U}_n\}_{n\geqslant 0}$ is decreasing. This implies the first assertion of the proposition.

For the second assertion, we show that $u_n(0)$ converge to $u(0)$. Observe that, by definition, $\mathscr{U}_n$ converge to $\mathscr{U}_S$ on smooth forms R in $\mathscr{C}_{k-p+1}$. Define $u_n(\theta):=\mathscr{U}_n(R_\theta)$. Hence, u_n converge to u pointwise on Δ^*. On the other hand, Lemma 3.1.3 implies that $(u_n+A|\theta|^2)$ is a decreasing sequence of subharmonic functions for A large enough. Hence, it converges pointwise to a subharmonic function. We deduce that $u_n(0)$ converge to $u(0)$. This completes the proof. □

Corollary 3.1.7. $\mathscr{U}_S$ *can be extended in a unique way to an affine u.s.c. function on* $\mathscr{C}_{k-p+1}$ *with values in* $\mathbb{R}\cup\{-\infty\}$, *also denoted by* $\mathscr{U}_S$, *such that*

$$\mathscr{U}_S(R)=\lim_{\theta\to 0}\mathscr{U}_{S_\theta}(R)=\lim_{\theta\to 0}\mathscr{U}_S(R_\theta).$$

In particular, we have

$$\mathscr{U}_S(R)=\limsup_{R'\to R}\mathscr{U}_S(R') \quad \textit{with } R' \textit{ smooth.}$$

Moreover, if c *is the constant in Lemma* 3.1.2, *then* $\mathscr{U}_S\leqslant m+c$, *independently of* S.

Proof. Proposition 3.1.6 implies that the decreasing limit of $\mathscr{U}_{S_n}$ is an extension of $\mathscr{U}_S$. Denote also this extension by $\mathscr{U}_S$. Since $\mathscr{U}_{S_n}$ are affine and continuous, $\mathscr{U}_S$ is affine and u.s.c. with values in $\mathbb{R}\cup\{-\infty\}$. In particular, we have

$$\mathscr{U}_S(R) \geqslant \limsup_{R'\to R} \mathscr{U}_S(R') \quad \text{with } R' \text{ smooth.}$$

Proposition 3.1.6 implies also that $\mathscr{U}_S(R)=u(0)$. By Proposition 3.1.4 and Lemma 3.1.5, we have

$$\mathscr{U}_S(R) = u(0) = \lim_{\theta\to 0} u(\theta) = \lim_{\theta\to 0} \mathscr{U}_S(R_\theta) = \lim_{\theta\to 0} \mathscr{U}_{S_\theta}(R).$$

The second limit is bounded above by

$$\limsup_{R'\to R} \mathscr{U}_S(R') \quad \text{with } R' \text{ smooth.}$$

It follows that

$$\mathscr{U}_S(R) = \limsup_{R'\to R} \mathscr{U}_S(R') \text{ with } R' \text{ smooth.}$$

The uniqueness of the extension of $\mathscr{U}_S$ is clear. The inequality $\mathscr{U}_S \leqslant m+c$ is a consequence of Lemma 3.1.2. □

Definition 3.1.8. We call $\mathscr{U}_S$ the *super-potential of mean m* of S.

It is clear that if $\mathscr{U}_S$ is the super-potential of mean m of S, then the super-potential of mean m' of S is equal to $\mathscr{U}_S+m'-m$. The following result applied to $I=\varnothing$, shows that the super-potentials determine the currents.

PROPOSITION 3.1.9. *Let I be a compact subset in $\mathbb{P}^k$ with $(2k-2p)$-dimensional Hausdorff measure* 0. *Let S and S' be currents in $\mathscr{C}_p$, with super-potentials $\mathscr{U}_S$ and $\mathscr{U}_{S'}$. If $\mathscr{U}_S=\mathscr{U}_{S'}$ on smooth forms in $\mathscr{C}_{k-p+1}$ with compact support in $\mathbb{P}^k\setminus I$, then $S=S'$.*

Proof. If R is a current in $\mathscr{C}_{k-p+1}$ with compact support in $\mathbb{P}^k\setminus I$, then R_θ has compact support in $\mathbb{P}^k\setminus I$ for θ small enough. On the other hand, since R_θ is smooth, we have

$$\mathscr{U}_S(R) = \lim_{\theta\to 0} \mathscr{U}_S(R_\theta) = \lim_{\theta\to 0} \mathscr{U}_{S'}(R_\theta) = \mathscr{U}_{S'}(R).$$

Hence, $\mathscr{U}_S=\mathscr{U}_{S'}$ on every current R with compact support in $\mathbb{P}^k\setminus I$. The hypothesis on the Hausdorff measure of I implies that a generic projective subspace P of dimension $p-1$ does not intersect I. We can write ω^{k-p+1} as an average of currents $[P]$. Since $\mathscr{U}_S=\mathscr{U}_{S'}$ at $[P]$ and since $\mathscr{U}_S$ and $\mathscr{U}_{S'}$ are affine, they are equal at ω^{k-p+1}. Hence, $\mathscr{U}_S$ and $\mathscr{U}_{S'}$ have the same mean. We may assume that this mean is 0.

If K is compact in $\mathbb{P}^k\setminus I$, using an average of $[P]$, we may construct a smooth form R_1 in $\mathscr{C}_{k-p+1}$ with compact support in $\mathbb{P}^k\setminus I$ which is strictly positive on K. We show that $S=S'$ on K. Let Φ be a smooth $(k-p,k-p)$-form with compact support on K. If $c>0$ is a large enough constant, $cR_1+dd^c\Phi$ is a positive closed form of mass c since it is cohomologous to cR_1. We can write $cR_1+dd^c\Phi=cR_2$ with $R_2\in\mathscr{C}_{k-p+1}$. We have $\mathscr{U}_S(R_1)=\mathscr{U}_{S'}(R_1)$ and $\mathscr{U}_S(R_2)=\mathscr{U}_{S'}(R_2)$. If U_S is a quasi-potential of mean 0 of S, we have

$$\begin{aligned}\langle S,\Phi\rangle &= \langle S-\omega^p,\Phi\rangle+\langle\omega^p,\Phi\rangle = \langle dd^cU_S,\Phi\rangle+\langle\omega^p,\Phi\rangle = \langle U_S, dd^c\Phi\rangle+\langle\omega^p,\Phi\rangle\\ &= \langle U_S, cR_2-cR_1\rangle+\langle\omega^p,\Phi\rangle = c\mathscr{U}_S(R_2)-c\mathscr{U}_S(R_1)+\langle\omega^p,\Phi\rangle.\end{aligned}$$

The current S' satisfies the same identity. We deduce that $\langle S,\Phi\rangle=\langle S',\Phi\rangle$. Hence, $S=S'$ on K. It follows that $S=S'$ on $\mathbb{P}^k\setminus I$. The hypothesis on the Hausdorff measure of I implies that S and S' have no mass on I [37]. Therefore, $S=S'$ on $\mathbb{P}^k$. □

3.2. Properties of super-potentials

The following proposition extends Lemma 3.1.3. It shows that in some sense super-potentials can be considered as quasi-psh functions on $\mathscr{C}_{k-p+1}$. In particular, they inherit the compactness property of $\mathscr{C}_p$.

PROPOSITION 3.2.1. *Let $\{R_x\}_{x\in X}$ be any special structural variety in $\mathscr{C}_{k-p+1}$ and let α be the associated curvature. Then, either $\mathscr{U}_S(R_x)=-\infty$ for every $x\in X$ or $x\mapsto\mathscr{U}_S(R_x)$ is a quasi-psh function on X such that $dd^c\mathscr{U}_S(R_x)\geqslant-\alpha$.*

Proof. By Proposition 3.1.6, it is enough to consider the case where S is smooth. The proof is the same as in Lemma 3.1.3. Let $\mathscr{R}$, π_X and π be as in §2.4. Then, $x\mapsto\mathscr{U}_S(R_x)$ is continuous and we have

$$\mathscr{U}_S(R_x) = (\pi_X)_*(\mathscr{R}\wedge\pi^*(U_S)),$$

which implies that

$$dd^c\mathscr{U}_S(R_x) = (\pi_X)_*(\mathscr{R}\wedge\pi^*(dd^cU_S)) \geqslant -(\pi_X)_*(\mathscr{R}\wedge\pi^*(\omega^p)) = -\alpha.$$

This completes the proof. □

The following result is the analogue of the classical Hartogs lemma for psh functions; see also Proposition 2.2.3.

PROPOSITION 3.2.2. *Let $\{S_n\}_{n\geqslant0}$ be a sequence in $\mathscr{C}_p$ converging to a current S. Let $\mathscr{U}_{S_n}$ (resp. $\mathscr{U}_S$) be the super-potential of mean m_n (resp. m) of S_n (resp. S). Assume that m_n converge to m. Let $\mathscr{U}$ be a continuous function on a compact subset K of $\mathscr{C}_{k-p+1}$ such that $\mathscr{U}_S<\mathscr{U}$ on K. Then, for n large enough, we have $\mathscr{U}_{S_n}<\mathscr{U}$ on K. In particular, we have $\limsup_{n\to\infty}\mathscr{U}_{S_n}\leqslant\mathscr{U}_S$ on $\mathscr{C}_{k-p+1}$.*

Proof. Recall that $\mathscr{U}_S$ is u.s.c., $\mathscr{U}$ is continuous and $\mathscr{C}_{k-p+1}$ is compact. The proposition can be applied to $K=\mathscr{C}_{k-p+1}$. Assume that there are currents R_n in K such that $\mathscr{U}_{S_n}(R_n)\geqslant\mathscr{U}(R_n)$. Extracting a subsequence allows one to assume that R_n converge to a current R in K. Let $\{R_{n,\theta}\}_{\theta\in\Delta}$ be the special structural disc associated with R_n constructed as in §2.1 and §2.5. Define $u_n(\theta):=\mathscr{U}_{S_n}(R_{n,\theta})$. Proposition 3.2.1 implies that u_n is quasi-subharmonic and $dd^c u_n\geqslant-\alpha$ with α as in Lemma 2.5.3. The first assertion of Proposition 2.1.6 implies that u_n converge pointwise to $u(\theta):=\mathscr{U}_S(R_\theta)$ on Δ^*. It follows from the Hartogs lemma for subharmonic functions that

$$\mathscr{U}_S(R)=u(0)\geqslant\limsup_{n\to\infty}u_n(0)=\limsup_{n\to\infty}\mathscr{U}_{S_n}(R_n)\geqslant\mathscr{U}(R).$$

This is a contradiction. The proof of the first assertion is complete. Taking $K=\{R\}$ and $\mathscr{U}(R)=\mathscr{U}_S(R)+\varepsilon$ gives the second assertion. □

Definition 3.2.3. Let S_n, S, $\mathscr{U}_{S_n}$, $\mathscr{U}_S$, m_n and m be as in Proposition 3.2.2. If $\mathscr{U}_{S_n}\geqslant\mathscr{U}_S$ for every n, then we say that S_n *converge to* S *in the Hartogs sense*, or S_n *H-converge to* S for short. If a current S' in $\mathscr{C}_p$ admits a super-potential $\mathscr{U}_{S'}$ such that $\mathscr{U}_{S'}\geqslant\mathscr{U}_S$, we say that S' is *more H-regular than* S or simply S' *is more diffuse than* S.

Remarks 3.2.4. By Lemma 3.2.5 below, the property that $\mathscr{U}_{S_n}$ converge pointwise to $\mathscr{U}_S$ implies that $m_n\to m$ and $S_n\to S$. If S_n H-converge to S as in Definition 3.2.3 then, by Proposition 3.2.2, we have that $\mathscr{U}_{S_n}\to\mathscr{U}_S$ pointwise. If $\mathscr{U}_{S_n}$ decrease to $\mathscr{U}_S$, then S_n H-converge to S; see also Corollary 3.2.7 below. We have seen in Proposition 3.1.6 that S_θ H-converge to S when $\theta\to 0$.

LEMMA 3.2.5. *Let $\{S_n\}_{n\geqslant 0}$ be a sequence in $\mathscr{C}_p$ and $\mathscr{U}_{S_n}$ be super-potentials of mean m_n of S_n. Assume that $\mathscr{U}_{S_n}$ converge to a finite function $\mathscr{U}$ on smooth forms in $\mathscr{C}_{k-p+1}$. Then, m_n converge to a constant m, S_n converge to a current S and $\mathscr{U}$ is equal to the super-potential of mean m of S on smooth forms in $\mathscr{C}_{k-p+1}$.*

Proof. We have $m_n=\mathscr{U}_{S_n}(\omega^{k-p+1})$. Hence, m_n converge to $m:=\mathscr{U}(\omega^{k-p+1})$. Let S and S' be limit currents of $\{S_n\}_{n\geqslant 0}$. From the definition of super-potential, we deduce that the super-potentials of mean m of S and of S' are equal to $\mathscr{U}$ on smooth forms in $\mathscr{C}_{k-p+1}$. By Proposition 3.1.9, $S=S'$. Hence, $\{S_n\}_{n\geqslant 0}$ is convergent. □

We now give a compactness property of super-potentials.

PROPOSITION 3.2.6. *Let $\mathscr{U}_{S_n}$ be a super-potential of a current S_n in $\mathscr{C}_p$. Assume that $\{\mathscr{U}_{S_n}\}_{n\geqslant 0}$ is bounded from above and does not converge uniformly to $-\infty$. Then, there is an increasing sequence $\{n_j\}_{j\geqslant 0}$ of integers such that S_{n_j} converge to a current S and $\mathscr{U}_{S_{n_j}}$ converge on smooth forms in $\mathscr{C}_{k-p+1}$ to a super-potential $\mathscr{U}_S$ of S. Moreover,*

$$\limsup_{j\to\infty}\mathscr{U}_{S_{n_j}}\leqslant\mathscr{U}_S.$$

Proof. By the last assertion in Corollary 3.1.7, since $\{\mathscr{U}_{S_n}\}_{n\geqslant 0}$ is bounded from above and does not converge to $-\infty$, their means m_n are bounded from above uniformly with respect to n and do not converge to $-\infty$. Extracting a subsequence allows one to assume that S_n converge to a current S and m_n converge to a finite value m. So, we may assume that $m_n=m=0$. Let $\mathscr{U}_S$ denote the super-potential of mean 0 of S. By the definition of $\mathscr{U}_S(R)$ for R smooth, we have that $\mathscr{U}_{S_n}(R)\to\mathscr{U}_S(R)$. The inequality $\limsup_{j\to\infty}\mathscr{U}_{S_{n_j}}\leqslant\mathscr{U}_S$ is a consequence of Proposition 3.2.2. □

COROLLARY 3.2.7. *Let $\mathscr{U}_{S_n}$ be super-potentials of mean m_n of S_n. Assume that $\mathscr{U}_{S_n}$ decrease to a function $\mathscr{U}$ which is not identically $-\infty$. Then, S_n converge to a current S, m_n converge to a constant m and $\mathscr{U}$ is the super-potential of mean m of S.*

Proof. By Lemma 3.2.5, S_n converge to a current S and m_n converge to a constant m. Define $u(\theta):=\mathscr{U}(R_\theta)$ and $u_n(\theta):=\mathscr{U}_{S_n}(R_\theta)$. As in Proposition 3.1.6, the functions u_n are quasi-subharmonic and decrease to u. Hence, u is quasi-subharmonic. On the other hand, since R_θ is smooth for $\theta\neq 0$, we have that $u(\theta)=\mathscr{U}_S(R_\theta)$ for $\theta\neq 0$, where $\mathscr{U}_S$ is the super-potential of mean m of S. The function $\theta\mapsto\mathscr{U}_S(R_\theta)$ is also quasi-subharmonic on Δ. So, we necessarily have $\mathscr{U}_S(R)=u(0)=\mathscr{U}(R)$. This holds for every R in $\mathscr{C}_{k-p+1}$. Therefore, $\mathscr{U}$ is the super-potential of mean m of S. □

COROLLARY 3.2.8. *Let $\mathscr{U}_S$ and $\mathscr{U}_R$ be super-potentials of the same mean m of S and R, respectively. Then, $\mathscr{U}_S(R)=\mathscr{U}_R(S)$.*

Proof. We have seen in the proof of Lemma 3.1.1 that the corollary holds for smooth S. Let S_n be smooth forms as in Proposition 3.1.6. The upper semi-continuity implies that

$$\mathscr{U}_S(R)=\lim_{n\to\infty}\mathscr{U}_{S_n}(R)=\lim_{n\to\infty}\mathscr{U}_R(S_n)\leqslant\mathscr{U}_R(S).$$

In the same way, we prove that $\mathscr{U}_R(S)\leqslant\mathscr{U}_S(R)$. □

LEMMA 3.2.9. *Let S and S' be currents in $\mathscr{C}_p$, and let $\mathscr{U}_S$ and $\mathscr{U}_{S'}$ be their super-potentials of mean m. Assume that there is a positive $(p-1,p-1)$-current U such that $dd^cU=S'-S$. Then, $\mathscr{U}_{S'}+\|U\|\geqslant\mathscr{U}_S$. In particular, if S has bounded super-potentials, then S' has bounded super-potentials. If $\mathscr{U}_R$ is a super-potential of a current $R\in\mathscr{C}_{k-p+1}$, then $\mathscr{U}_R(S')+\|U\|\geqslant\mathscr{U}_R(S)$.*

Proof. Let U_S be a quasi-potential of mean m of S. Then, U_S+U is a quasi-potential of mean $m+\|U\|$ of S'. For R smooth, we have

$$\mathscr{U}_{S'}(R)+\|U\|=\langle U_S+U,R\rangle\geqslant\langle U_S,R\rangle=\mathscr{U}_S(R).$$

Then, Corollaries 3.1.7 and 3.2.8 imply the result. □

We have the following important result which can be considered as a version of Lemma 2.2.1 for super-potentials. We can apply it to $K=W=\mathbb{P}^k$.

PROPOSITION 3.2.10. *Let $W\subset\mathbb{P}^k$ be an open set and $K\subset W$ be a compact set. Let S be a current in $\mathscr{C}_p$ with support in K and R be a current in $\mathscr{C}_{k-p+1}$. Assume that the restriction of R to W is a bounded form. Then, the super-potential $\mathscr{U}_S$ of mean 0 of S satisfies*

$$|\mathscr{U}_S(R)|\leqslant c(1+\log^+\|R\|_{\infty,W}),$$

where $c>0$ is a constant independent of S and R, and $\log^+:=\max\{0,\log\}$.

Proof. Recall that $u(\theta):=\mathscr{U}_S(R_\theta)$ is a quasi-subharmonic function on Δ such that $dd^cu\geqslant-\alpha$. By Proposition 2.1.6, the family of these functions u for $(S,R)\in\mathscr{C}_p\times\mathscr{C}_{k-p+1}$ is compact. So, Lemma 2.2.1 implies that $\|e^{-Au}\|_{\mathscr{L}^1(\Delta_{1/2})}\leqslant c$ for some positive constants c and A.

Suppose that the estimate in the lemma is not valid. Recall that $\mathscr{U}_S$ is bounded from above by a constant independent of S. Then, for $\varepsilon>0$ arbitrarily small, there is an R such that $M:=\log\|R\|_{\infty,W}\gg0$ and $\mathscr{U}_S(R)\leqslant-2M/\varepsilon$. It follows that $u(0)=\mathscr{U}_S(R)\leqslant-2M/\varepsilon$. We will show that $u(\theta)\leqslant-M/\varepsilon$ on a disc of radius e^{-M}, which contradicts the above estimate on e^{-Au} for ε small enough.

Let U be the Green quasi-potential of R and let m be its mean. The mass of U is bounded by a constant independent of R. By Lemma 2.3.11, there is a quasi-potential $U'_{\theta y}$ of $R_{\theta y}$ of mean m such that

$$\|U'_{\theta y}-(\tau_{\theta y})_*(U)\|_\infty\lesssim|\theta|.$$

We deduce that

$$|\mathscr{U}_S(R_\theta)-\mathscr{U}_S(R)|=\left|\int_y\langle S,U'_{\theta y}-U\rangle\,d\varrho(y)\right|\lesssim|\theta|+\left|\int_y\langle S,(\tau_{\theta y})_*(U)-U\rangle\,d\varrho(y)\right|.$$

Because θ is small, $\tau_{\theta y}^{-1}(K)\subset W'$ for some fixed open set $W'\Subset W$. Since $\tau_{\theta y}$ is close to the identity, using Lemma 2.3.5, we obtain

$$\|(\tau_{\theta y})_*(U)-U\|_{\infty,K}\lesssim|\theta|\,\|U\|_{\mathscr{C}^1(W')}\lesssim|\theta|e^M.$$

Therefore,

$$|u(\theta)-u(0)|=|\mathscr{U}_S(R_\theta)-\mathscr{U}_S(R)|\lesssim|\theta|e^M.$$

This implies the above claim and completes the proof. $\square$

3.3. Currents with regular super-potentials

The PB or PC currents are introduced in [17], [19] and [21] in the study of holomorphic dynamical systems. They correspond to currents with bounded or continuous super-potentials. We first recall the definition of the space $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$ of dsh currents. A real $(k-p,k-p)$-current Φ of finite mass is *dsh* if there are positive closed currents $R^\pm$ of bidegree $(k-p+1,k-p+1)$ such that(1) $dd^c\Phi = R^+ - R^-$. Define

$$\|\Phi\|_{\mathrm{DSH}} := \|\Phi\| + \min \|R^\pm\|$$

with $R^\pm$ as above. We consider a *weak topology* on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$. A sequence $\{\Phi_n\}_{n\geqslant 0}$ converges to Φ in $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$ if $\Phi_n \to \Phi$ in the sense of currents and $\|\Phi_n\|_{\mathrm{DSH}}$ is uniformly bounded. A positive closed (p,p)-current S is said to be PB if there is a constant $c>0$ such that

$$|\langle S, \Phi\rangle| \leqslant c\|\Phi\|_{\mathrm{DSH}}$$

for smooth real forms Φ of bidegree $(k-p,k-p)$. We say that S is PC if it can be extended to a linear form on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$ which is continuous with respect to the weak topology on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$.

PROPOSITION 3.3.1. *If a super-potential $\mathscr{U}_S$ of S is finite everywhere, then it is bounded. A current S is* PB *if and only if the super-potentials of S are bounded. A current S is* PC *if and only if the super-potentials of S are continuous.*

Proof. Subtracting a constant from $\mathscr{U}_S$, we may assume $\mathscr{U}_S \leqslant 0$. Assume that $\mathscr{U}_S$ is unbounded. Then, there are currents R_n such that $\mathscr{U}_S(R_n) \leqslant -2^n$. Set $R := \sum_{n=0}^\infty 2^{-n} R_n$. Since $\mathscr{U}_S$ is affine and negative, we have that $\mathscr{U}_S(R) \leqslant \sum_{n=0}^N 2^{-n}\mathscr{U}_S(R_n)$ for every N. Hence, $\mathscr{U}_S(R) = -\infty$. This is a contradiction. So, $\mathscr{U}_S$ is bounded. Note that this property is false for quasi-psh functions on $\mathbb{P}^k$.

Assume that the super-potential $\mathscr{U}_S$ of mean 0 of S satisfies $|\mathscr{U}_S| < M$ for some constant $M>0$. Consider a real smooth form Φ of bidegree $(k-p,k-p)$ and a constant $A \geqslant \|\Phi\|_{\mathrm{DSH}}$. We will prove that $|\langle S,\Phi\rangle| \leqslant A(1+2C+2M)$ with $C>0$ independent of S. This implies that S is PB. Since we can approximate S in the Hartogs sense by smooth forms, it is enough to prove this inequality for smooth S. Write $dd^c\Phi = A(R^+ - R^-)$ with $\|R^\pm\| = 1$. By Remark 2.3.4, there are quasi-potentials $U^\pm$ of mean 0 of $R^\pm$ such that $\|U^\pm\|_{\mathrm{DSH}} \leqslant C$, where $C>0$ is a constant. Define $\Psi := \Phi - AU^+ + AU^-$. Then $dd^c\Psi = 0$ and

$$\|\Psi\| \leqslant \|\Phi\| + A\|U^+\| + A\|U^-\| \leqslant A(1+2C).$$

(1) It is also useful to consider the space generated by such currents Φ which are negative. This is necessary in order to defined the pull-back of dsh currents by holomorphic maps.

As $dd^c\Psi=0$ and since S is cohomologous to ω^p, we have

$$|\langle S,\Psi\rangle| = |\langle \omega^p,\Psi\rangle| \leqslant A(1+2C).$$

It follows that

$$\begin{aligned}|\langle S,\Phi\rangle| &\leqslant |\langle S,\Psi\rangle| + A|\langle S,U^+\rangle| + A|\langle S,U^-\rangle|\\ &= |\langle S,\Psi\rangle| + A|\mathscr{U}_S(R^+)| + A|\mathscr{U}_S(R^-)| \leqslant A(1+2C+2M).\end{aligned}$$

Hence, S is PB.

Conversely, if S is PB, we show that $\mathscr{U}_S$ is bounded. Consider a smooth form R in $\mathscr{C}_{k-p+1}$. Let U_R be a quasi-potential of R of mean 0 such that $\|U_R\|_{\mathrm{DSH}}\leqslant C$. We have $\mathscr{U}_S(R)=\langle S,U_R\rangle$. Since S is PB, $\mathscr{U}_S(R)$ is bounded by a constant independent of R. This implies that $\mathscr{U}_S$ is bounded.

It is clear that if S is PC, $\langle S,U_R\rangle$ for smooth R can be extended to a continuous function on $\mathscr{C}_{k-p+1}$. Indeed, we can choose U_R depending continuously on R with respect to the weak topology in $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$; see Theorem 2.3.1 and Remark 2.3.4. This implies that $\mathscr{U}_S$ is continuous. Conversely, if $\mathscr{U}_S$ is continuous, we show that S is PC. If Φ and $R^\pm$ are smooth as above, we obtain

$$\langle S,\Phi\rangle = \langle \omega^p,\Psi\rangle + A\mathscr{U}_S(R^+) - A\mathscr{U}_S(R^-).$$

The right-hand side depends on Ψ and on $AR^+-AR^-=dd^c\Phi$ but not on the choice of A and $R^\pm$. Hence, since Ψ and $dd^c\Phi$ depend continuously on Φ, we can extend S to a continuous linear form on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$. The continuity is with respect to the weak topology on $\mathrm{DSH}^{k-p}(\mathbb{P}^k)$. This completes the proof. □

LEMMA 3.3.2. *If S is a form of class $\mathscr{L}^s$ with $s>k$, then S has continuous super-potentials.*

Proof. Let r be the positive number such that $1/r+1/s=1$. Then, $r<k/(k-1)$. The Green quasi-potential U_R of R is a form of class $\mathscr{L}^r$. Moreover, with respect to the $\mathscr{L}^r$ topology, it depends continuously on R, see Theorem 2.3.1. The mean m_R of U_R depends continuously on R. On the other hand, the super-potential of mean 0 of S satisfies

$$\mathscr{U}_S(R) = \langle S,U_R\rangle - m_R$$

for smooth R. The right-hand side is defined for every R and depends continuously on R. Therefore, $\mathscr{U}_R$ is continuous. □

Remark 3.3.3. U_R is in the Sobolev space $W^{1,r}$ with $r<2k/(2k-1)$. So, we can assume that $S\in W^{-1,s}$ with $1/r+1/s=1$, and still $\mathscr{U}_S$ is continuous.

Proposition 3.3.4. *Let S and S' be currents in $\mathscr{C}_p$ such that $S'\leqslant cS$ for some positive constant c. If S has bounded super-potentials, then S' has bounded super-potentials. If S has continuous super-potentials, then S' has continuous super-potentials.*

Proof. Write $S=\lambda S'+(1-\lambda)S''$ with $0<\lambda\leqslant 1$ and S'' being a current in $\mathscr{C}_p$. Let $\mathscr{U}_S$, $\mathscr{U}_{S'}$ and $\mathscr{U}_{S''}$ denote the super-potentials of mean 0 of S, S' and S''. By the definition of super-potentials, we have $\lambda\mathscr{U}_{S'}+(1-\lambda)\mathscr{U}_{S''}=\mathscr{U}_S$ on smooth forms R. Corollary 3.1.7 implies that this equality holds for every R. Since $\mathscr{U}_{S''}$ is bounded from above, if $\mathscr{U}_S$ is bounded, it is clear that $\mathscr{U}_{S'}$ is bounded. If $\mathscr{U}_S$ is continuous, as $\mathscr{U}_{S'}$ and $\mathscr{U}_{S''}$ are u.s.c., they are continuous. $\square$

Proposition 3.3.5. *Let S be a current with bounded super-potentials. Then, S has no mass on pluripolar sets of $\mathbb{P}^k$. In particular, S does not give mass to proper analytic subsets of $\mathbb{P}^k$.*

Proof. Assume that S has bounded super-potentials. Let $E\subset\mathbb{P}^k$ be a pluripolar set and u be a quasi-psh function such that $dd^c u\geqslant-\omega$ and $E\subset\{z:u(z)=-\infty\}$. Define $R:=(dd^c u+\omega)\wedge\omega^{k-p}$. This is a current in $\mathscr{C}_{k-p+1}$.

Let $\{u_n\}_{n\geqslant 0}$ be a sequence of smooth functions decreasing to u and such that $dd^c u_n\geqslant-\omega$. Define $R_n:=(dd^c u_n+\omega)\wedge\omega^{k-p}$. Observe that $u_n\omega^{k-p}$ are quasi-potentials of mean $m_n:=\int u_n\omega^k$ of R_n. If $\mathscr{U}_S$ is the super-potential of mean $m:=\int u\omega^k$ of S, then $\langle S,u_n\omega^{k-p}\rangle$ decrease to $\mathscr{U}_S(R)$. Hence, $\mathscr{U}_S(R)=\langle S,u\omega^{k-p}\rangle$. Since S has bounded super-potentials, $\langle S,u\omega^{k-p}\rangle$ is finite. It follows that S has no mass on $\{z:u(z)=-\infty\}$. $\square$

Proposition 3.3.6. *Assume that S admits a super-potential which is α-Hölder continuous with respect to the distance dist_1 on $\mathscr{C}_{k-p+1}$ for some exponent $\alpha\leqslant 1$. Let σ_S denote the trace measure of S. There is a constant $c>0$ such that if B_r is a ball of radius r, then $\sigma_S(B_r)\leqslant cr^{2k-2p+\alpha}$. In particular, S has no mass on Borel subsets of $\mathbb{P}^k$ with Hausdorff dimension less than $2(k-p)+\alpha$.*

Using Lemma 2.1.2, we deduce analogous results for a general distance $\operatorname{dist}_\beta$ on $\mathscr{C}_{k-p+1}$. Note that the last assertion in the proposition is deduced from the first one and some classical arguments. In order to prove the first assertion, it is enough to consider r small. So, we may assume that B_r is a ball of center 0 in an affine chart $\mathbb{C}^k\subset\mathbb{P}^k$. It is sufficient to show that $\int_{\Delta_r^k} S\wedge\omega^{k-p}\lesssim r^{2k-2p+\alpha}$. Let z denote the canonical coordinates in $\mathbb{C}^k$.

Lemma 3.3.7. *There are positive constants A and c independent of r, a positive $(k-p,k-p)$-current Φ and two currents $R^\pm$ in $\mathscr{C}_{k-p+1}$ such that $\Phi\geqslant(dd^c|z|^2)^{k-p}$ on Δ_r^k, $\|\Phi\|\leqslant Ar^{2k-2p+2}$, $dd^c\Phi=cr^{2k-2p}(R^+-R^-)$ and $\operatorname{dist}_1(R^+,R^-)\leqslant Ar$.*

Proof. Observe that $(dd^c|z|^2)^{k-p}$ is a combination of the forms

$$(idz_{j_1}\wedge d\bar z_{j_1})\wedge...\wedge(idz_{j_{k-p}}\wedge d\bar z_{j_{k-p}}).$$

Without loss of generality, one only has to construct Φ and $R^{\pm}$ satisfying the last three properties in the lemma and the inequality

$$\Phi\geqslant(idz_1\wedge d\bar z_1)\wedge...\wedge(idz_{k-p}\wedge d\bar z_{k-p})$$

on Δ_r^k. Taking a combination of such currents gives currents satisfying the lemma.

Let χ be a smooth cut-off function with compact support in Δ_2^k, equal to 1 on Δ_1^k. Let $v(z_{k-p+1})$ be a smooth function with support in $\{z_{k-p+1}:|z_{k-p+1}|<2r\}$ such that $0\leqslant v\leqslant 1$, $\|v\|_{\mathscr{C}^1}\lesssim r^{-1}$, $\|v\|_{\mathscr{C}^2}\lesssim r^{-2}$ and $v=1$ on $\{z_{k-p+1}:|z_{k-p+1}|\leqslant r\}$. Let $\pi:\mathbb{C}^k\to\mathbb{C}^{k-p}$ and $\pi':\mathbb{C}^k\to\mathbb{C}^{k-p+1}$ denote the canonical projections on the first factors of $\mathbb{C}^k$. Consider the restriction Θ of $idz_1\wedge d\bar z_1\wedge...\wedge idz_{k-p}\wedge d\bar z_{k-p}$ to Δ_r^{k-p} and define

$$\Phi:=v(z_{k-p+1})\chi(z)\pi^*(\Theta).$$

Then, Φ satisfies the desired lower estimate on Δ_r^k. We have to check the last three properties in the lemma.

Since π can be extended to a rational map from $\mathbb{P}^k$ to $\mathbb{P}^{k-p}$, $\pi^*(\Theta)$ can be extended to a positive closed current on $\mathbb{P}^k$ of mass $\|\Theta\|\simeq r^{2k-2p}$. Moreover, Cauchy–Schwarz's inequality implies that

$$-dd^c[v(z_{k-p+1})\chi(z)]\lesssim r^{-2}idz_{k-p+1}\wedge d\bar z_{k-p+1}+\omega.$$

Denote by Θ' the restriction of $idz_1\wedge d\bar z_1\wedge...\wedge idz_{k-p+1}\wedge d\bar z_{k-p+1}$ to $\Delta_r^{k-p}\times\Delta_{2r}$ and let

$$\Omega^-:=\lambda(\pi')^*(r^{-2}\Theta')+\lambda\omega\wedge\pi^*(\Theta)$$

with $\lambda>0$ large enough independent of r. Then, $\Omega^+:=\Omega^-+dd^c\Phi$ is positive and closed. We have $dd^c\Phi=\Omega^+-\Omega^-$. The currents $\Omega^{\pm}$ can be extended to positive closed currents on $\mathbb{P}^k$. They have the same mass since they are cohomologous. This mass is of order r^{2k-2p} and we denote it by cr^{2k-2p}. We obtain

$$dd^c\Phi=cr^{2k-2p}(R^+-R^-)$$

with $R^{\pm}:=c^{-1}r^{2p-2k}\Omega^{\pm}$. The currents R^+ and R^- are in $\mathscr{C}_{k-p+1}$. We want to bound $\operatorname{dist}_1(R^+,R^-)$. For any test form Ψ with $\|\Psi\|_{\mathscr{C}^1}\leqslant 1$, we have

$$|\langle R^+-R^-,\Psi\rangle|\simeq r^{2p-2k}|\langle dd^c\Phi,\Psi\rangle|=r^{2p-2k}|\langle d^c\Phi,d\Psi\rangle|\lesssim r^{2p-2k}\|d^c\Phi\|.$$

On the other hand, we deduce from the definition of Φ that

$$\|d^c\Phi\|\lesssim r^{2k-2p}\|d^cv\|_{\Delta_{2r}}\lesssim r^{2k-2p+1}.$$

This implies the result. □

End of the proof of Proposition 3.3.6. Let $\mathscr{U}_S$ be a super-potential of S. Since $\mathscr{U}_S$ is α-Hölder continuous, we deduce from the previous lemma that

$$\begin{aligned}\int_{\Delta_r^k} S\wedge\omega^{k-p} \leqslant \langle S,\Phi\rangle = \langle \omega^p,\Phi\rangle + \langle dd^c U_S,\Phi\rangle &= \langle \omega^p,\Phi\rangle + \langle U_S, dd^c\Phi\rangle \\ &\lesssim \langle \omega^p,\Phi\rangle + r^{2k-2p}(\mathscr{U}_S(R^+)-\mathscr{U}_S(R^-)) \lesssim r^{2k-2p+\alpha}.\end{aligned}$$

This is the required estimate. □

3.4. Capacity of currents and super-polar sets

We will define a notion of capacity for Borel subsets E of $\mathscr{C}_{k-p+1}$. This capacity does not describe how "big" the set E is, but rather how singular the currents in E are. The definition mimics the notion of capacity that we introduced in [21] for compact Kähler manifolds. Let

$$\mathscr{P}_p := \Big\{ \mathscr{U}_S \text{ super-potential of } S\in\mathscr{C}_p : \max_{\mathscr{C}_{k-p+1}} \mathscr{U}_S = 0 \Big\}.$$

Definition 3.4.1. We define the *capacity* of E to be the following quantity:

$$\operatorname{cap}(E) := \inf_{\mathscr{U}\in\mathscr{P}_p} \exp\Big(\sup_{R\in E} \mathscr{U}(R) \Big).$$

It is clear that the capacity is increasing as a set function. Propositions 3.1.6 and 3.2.2 imply that, when E is compact, in the previous definition we obtain the same capacity if we only consider super-potentials of smooth forms. We also have $\operatorname{cap}(\mathscr{C}_{k-p+1})=1$ and it follows that the set of smooth forms in $\mathscr{C}_{k-p+1}$ has capacity 1. Dense subsets of smooth forms in $\mathscr{C}_{k-p+1}$ also have capacity 1. So, there is a countable subset of $\mathscr{C}_{k-p+1}$ with capacity 1.

Definition 3.4.2. We say that E is *super-polar* or *completely super-polar* in $\mathscr{C}_{k-p+1}$ if there is a super-potential $\mathscr{U}_S$ of a current S in $\mathscr{C}_p$ such that

$$E\subset\{R:\mathscr{U}_S(R)=-\infty\} \quad \text{or} \quad E=\{R:\mathscr{U}_S(R)=-\infty\},$$

respectively.

Let $\widehat{E}$ be the *barycentric hull* of E, i.e. the set of currents $\int R\,d\nu(R)$, where ν is a probability measure on $\mathscr{C}_{k-p+1}$ such that $\nu(E)=1$. Denote by $\widetilde{E}$ the set of currents $cR+(1-c)R'$ with $R\in\widehat{E}$, $R'\in\mathscr{C}_{k-p+1}$ and $0<c\leqslant 1$. Then, $\widetilde{E}$ and $\widehat{E}$ are convex.

PROPOSITION 3.4.3. *The following properties are equivalent*:

(1) E *is super-polar in* $\mathscr{C}_{k-p+1}$;

(2) $\widehat{E}$ *is super-polar in* $\mathscr{C}_{k-p+1}$;

(3) $\widetilde{E}$ *is super-polar in* $\mathscr{C}_{k-p+1}$;

(4) $\operatorname{cap}(E)=0$.

Moreover, a countable union of super-polar sets is super-polar, completely super-polar sets are convex and $\operatorname{cap}(E)=\operatorname{cap}(\widehat{E})$.

Proof. Since every function $\mathscr{U}$ in $\mathscr{P}_p$ is affine and negative, if $\mathscr{U}$ is equal to $-\infty$ on E, it is also equal to $-\infty$ on $\widehat{E}$ and $\widetilde{E}$. Therefore, the first three properties are equivalent. We also deduce that if E is completely super-polar, then E is convex and $E=\widetilde{E}$. Moreover, for any $\mathscr{U}$ we have $\sup_E \mathscr{U}=\sup_{\widehat{E}} \mathscr{U}$. This implies that $\operatorname{cap}(E)=\operatorname{cap}(\widehat{E})$.

It is clear that if E is super-polar, then $\operatorname{cap}(E)=0$. Assume that $\operatorname{cap}(E)=0$. We show that E is super-polar. There are super-potentials $\mathscr{U}_{S_n}$ of S_n such that $\max \mathscr{U}_{S_n}=0$ and $\mathscr{U}_{S_n}\leqslant -2^n$ on E. Corollary 3.1.7 implies that the means of $\mathscr{U}_{S_n}$ are bounded. This and Corollary 3.2.7 imply that $\mathscr{U}=\sum_{n=1}^\infty 2^{-n}\mathscr{U}_{S_n}$ is a super-potential of $\sum_{n=1}^\infty 2^{-n}S_n$. It is equal to $-\infty$ on E. Hence, E is super-polar. A similar argument implies that a countable union of super-polar sets is super-polar. □

PROPOSITION 3.4.4. *Let* $E\subset\mathscr{C}_{k-p+1}$ *be a compact set. Then, E has positive capacity if and only if its barycentric hull contains a current with bounded super-potentials. Moreover, there is a current R in the barycentric hull $\widehat{E}$ of E such that its super-potential of mean* 0 *satisfies*

$$\mathscr{U}_R \geqslant \log \operatorname{cap}(E) \quad \textit{on } \mathscr{C}_p.$$

Proof. If R is a current with bounded super-potentials, then, by symmetry, $\mathscr{U}(R)\neq -\infty$ for every $\mathscr{U}\in\mathscr{P}_p$. Proposition 3.4.3 implies that $\{R\}$ is not super-polar. Hence, if $\widehat{E}$ contains a current with bounded super-potentials, $\widehat{E}$ has positive capacity. Proposition 3.4.3 also implies that E has positive capacity. Now, assume that E has positive capacity. We show that $\widehat{E}$ contains a current with bounded super-potentials. In what follows, the symbol $\mathscr{U}$ denotes a super-potential of mean 0. We have

$$\inf_{S\in\mathscr{C}_p}\sup_{R\in\widehat{E}}\mathscr{U}_S(R)\geqslant M:=\log\operatorname{cap}(E).$$

The function $\mathscr{U}_S(R)$ is affine in both variables R and S. Hence, for every convex compact set $\mathscr{C}$ of continuous forms in $\mathscr{C}_p$, the minimax theorem [46] implies that

$$\sup_{R\in\widehat{E}}\inf_{S\in\mathscr{C}}\mathscr{U}_S(R)=\inf_{S\in\mathscr{C}}\sup_{R\in\widehat{E}}\mathscr{U}_S(R)\geqslant M.$$

Consider an increasing sequence of compact sets $\{\mathscr{C}^j\}$ and define

$$E_j := \{R\in \widehat{E} : \mathscr{U}_S(R) \geqslant M-1/j \text{ for every } S\in \mathscr{C}^j\}.$$

So, $\{E_j\}$ is a decreasing sequence of compact sets. Take an element R in the intersection of E_j. If $\mathscr{C}^j$ are chosen so that their union is dense in $\mathscr{C}_p$, then $\mathscr{U}_R(S)=\mathscr{U}_S(R)\geqslant M$ for every $S\in\mathscr{C}_p$. This completes the proof. □

Consider the set of the super-potentials $\mathscr{U}$ of mean 0 of currents in $\mathscr{C}_p$ and define $c_{k,p}:=\sup_{S\in\mathscr{C}_p} \max \mathscr{U}_S$. Corollary 3.1.7 implies that this constant is finite.

Corollary 3.4.5. *For every current R in $\mathscr{C}_{k-p+1}$, if $\mathscr{U}_R$ is the super-potential of mean 0 of R, then*

$$\log \operatorname{cap}(R) \geqslant \inf_{\mathscr{C}_p} -c_{k,p}+\mathscr{U}_R.$$

Proof. Let $\mathscr{U}_S$ be the super-potential of mean 0 of S. By the definition of capacity and of $c_{k,p}$, we have

$$\log \operatorname{cap}(R) \geqslant \Big[\inf_{S\in\mathscr{C}_p} \mathscr{U}_S(R)-c_{k,p}\Big].$$

Corollary 3.2.8 implies the result. □

Corollary 3.4.6. *For every $r>k$, there is a constant $c>0$ such that if R is a form in $\mathscr{C}_{k-p+1}$ with coefficients in $\mathscr{L}^r$, then*

$$\log \operatorname{cap}(R) \geqslant -c_{k,p}-c\|R\|_{\mathscr{L}^r}.$$

Proof. Let s be the positive number such that $1/r+1/s=1$. Then, $s<k/(k-1)$. Let U_S be the Green quasi-potential of S. This is a negative form with $\mathscr{L}^s$ norm bounded uniformly with respect to S. Hence,

$$\mathscr{U}_R(S) \geqslant \langle U_S, R\rangle \geqslant -c\|R\|_{\mathscr{L}^r}$$

for some constant $c>0$. We obtain the result from Corollary 3.4.5. □

The following result is a consequence of Proposition 3.2.10.

Corollary 3.4.7. *There are constants $c>0$ and $\lambda>0$ such that for every bounded form R in $\mathscr{C}_{k-p+1}$,*

$$\operatorname{cap}(R) \geqslant c\|R\|_\infty^{-\lambda}.$$

4. Theory of intersection of currents

In this section, we develop the theory of intersection for positive closed currents of arbitrary bidegree. The method can be extended to currents on compact Kähler manifolds or in some local situation; see also [22]. Here, for simplicity, we only consider currents in the projective space.

4.1. Some universal super-functions

Let p be an integer with $1\leqslant p\leqslant k$. Define a universal function $\mathscr{U}_p$ on $\mathscr{C}_p\times\mathscr{C}_{k-p+1}$ by

$$\mathscr{U}_p(S,R):=\mathscr{U}_S(R)=\mathscr{U}_R(S),$$

where $\mathscr{U}_S$ and $\mathscr{U}_R$ are super-potentials of mean 0 of S and R; see Corollary 3.2.8. We have seen that, when S is fixed, $\mathscr{U}_p$ is quasi-psh on special varieties of $\mathscr{C}_{k-p+1}$, and when R is fixed, it is quasi-psh on special varieties of $\mathscr{C}_p$.

LEMMA 4.1.1. *The function $\mathscr{U}_p$ is u.s.c. on $\mathscr{C}_p\times\mathscr{C}_{k-p+1}$.*

Proof. Let S_n be currents in $\mathscr{C}_p$ converging to S and R_n be currents in $\mathscr{C}_{k-p+1}$ converging to R. Let $\mathscr{U}_{S_n}$ denote the super-potential of mean 0 of S_n. Choose $\mathscr{U}$ continuous with $\mathscr{U}_S<\mathscr{U}$. By Proposition 3.2.2, for n large enough, $\mathscr{U}_{S_n}<\mathscr{U}$ and hence $\mathscr{U}_{S_n}(R_n)<\mathscr{U}(R_n)$. We then get

$$\limsup_{n\to\infty}\mathscr{U}_{S_n}(R_n)\leqslant\mathscr{U}(R).$$

Since $\mathscr{U}$ is arbitrary, we deduce that

$$\limsup_{n\to\infty}\mathscr{U}_{S_n}(R_n)\leqslant\mathscr{U}_S(R).$$

This proves the lemma. □

LEMMA 4.1.2. *Let S' and R' be currents in $\mathscr{C}_p$ and $\mathscr{C}_{k-p+1}$, and let $\mathscr{U}_{S'}$ and $\mathscr{U}_{R'}$ be their super-potentials of mean 0. Assume that there are constants a and b such that $\mathscr{U}_{S'}+a\geqslant\mathscr{U}_S$ and $\mathscr{U}_{R'}+b\geqslant\mathscr{U}_R$. Then, $\mathscr{U}_p(S',R')\geqslant\mathscr{U}_p(S,R)-a-b$.*

Proof. We have

$$\mathscr{U}_p(S,R')=\mathscr{U}_{R'}(S)\geqslant\mathscr{U}_R(S)-b=\mathscr{U}_p(S,R)-b$$

and

$$\mathscr{U}_p(S',R')=\mathscr{U}_{S'}(R')\geqslant\mathscr{U}_S(R')-a=\mathscr{U}_p(S,R')-a.$$

This implies the result. □

Lemma 4.1.3. *Let $\{S_n\}_{n\geqslant 0}$ and $\{R_n\}_{n\geqslant 0}$ be sequences of currents in $\mathscr{C}_p$ and $\mathscr{C}_{k-p+1}$ H-converging to S and R, respectively. Then, $\mathscr{U}_p(S_n, R_n)$ converge to $\mathscr{U}_p(S, R)$. Moreover, if $\mathscr{U}_p(S, R)$ is finite, then $\mathscr{U}_p(S_n, R_n)$ is finite for every n.*

Proof. Let $\mathscr{U}_{S_n}$ and $\mathscr{U}_{R_n}$ be the super-potentials of mean 0 of S_n and R_n, respectively. The H-convergence implies the existence of constants a_n and b_n with limit 0, such that $\mathscr{U}_{S_n}+a_n\geqslant\mathscr{U}_S$ and $\mathscr{U}_{R_n}+b_n\geqslant\mathscr{U}_R$. It follows from Lemma 4.1.1 that

$$\limsup_{n\to\infty}\mathscr{U}_p(S_n, R_n)\leqslant\mathscr{U}_p(S, R).$$

It is sufficient to prove that

$$\mathscr{U}_p(S_n, R_n)\geqslant\mathscr{U}_p(S, R)-a_n-b_n.$$

This is a consequence of Lemma 4.1.2. □

4.2. Intersection of currents

Let p_j, $1\leqslant j\leqslant l$, be positive integers such that $p_1+...+p_l\leqslant k$. Let R_j be currents in $\mathscr{C}_{p_j}$ with $1\leqslant j\leqslant l$. We want to define the wedge-product $R_1\wedge...\wedge R_l$, as a current. In general, one cannot define this product in a consistent way; for example, when R_1 and R_2 are currents of integration on the same projective line of $\mathbb{P}^2$. We will define the intersection of the R_j's when they satisfy a quite natural condition. Consider first the case of two currents, i.e. $l=2$.

Proposition 4.2.1. *The following conditions are equivalent and are symmetric with respect to R_1 and R_2:*

(1) *$\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega)$ is finite for at least one smooth form Ω in $\mathscr{C}_{k-p_1-p_2+1}$;*

(2) *$\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega)$ is finite for every smooth form Ω in $\mathscr{C}_{k-p_1-p_2+1}$;*

(3) *there are sequences $\{R_{j,n}\}_{n\geqslant 0}$ in $\mathscr{C}_{p_j}$ converging to R_j, and a smooth form Ω in $\mathscr{C}_{k-p_1-p_2+1}$ such that $\mathscr{U}_{p_1}(R_{1,n}, R_{2,n}\wedge\Omega)$ is bounded.*

Proof. It is clear that the second condition implies the third one: we can choose $R_{j,n}=R_j$; and the third condition implies the first one because $\mathscr{U}_{p_1}$ is u.s.c. Assume the first condition. We show that $\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega')$ is finite for every smooth form Ω' in $\mathscr{C}_{k-p_1-p_2+1}$. Write $\Omega'-\Omega=dd^cU$ with U smooth. Adding a large positive closed form to U, we may assume that U is positive. If V is a quasi-potential of $R_2\wedge\Omega$, then the quasi-potential $V+R_2\wedge U$ of $R_2\wedge\Omega'$ is larger than V. Lemmas 3.2.9 and 4.1.2 imply that $\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega')$ is finite. Therefore, the three previous conditions are equivalent.

It remains to prove that the first condition is symmetric. We may assume that $\Omega=\omega^{k-p_1-p_2+1}$. Consider the case where R_1 is smooth. If U_2 is a quasi-potential of mean 0 of R_2, then $U_2\wedge\Omega$ is a quasi-potential of mean 0 of $R_2\wedge\Omega$. We have

$$\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega) = \langle R_1, U_2\wedge\Omega\rangle = \langle U_2, R_1\wedge\Omega\rangle = \mathscr{U}_{p_2}(R_2, R_1\wedge\Omega).$$

Suppose now that R_1 is arbitrary. Let $R_{1,\theta}$ be the smooth forms constructed in §2.1, starting with the current R_1. We have

$$\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega) = \lim_{\theta\to 0} \mathscr{U}_{p_1}(R_{1,\theta}, R_2\wedge\Omega) = \lim_{\theta\to 0} \mathscr{U}_{p_2}(R_2, R_{1,\theta}\wedge\Omega) \leqslant \mathscr{U}_{p_2}(R_2, R_1\wedge\Omega),$$

since $\mathscr{U}_{p_2}$ is u.s.c. In the same way, we obtain $\mathscr{U}_{p_2}(R_2, R_1\wedge\Omega)\leqslant\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega)$. Hence, $\mathscr{U}_{p_2}(R_2, R_1\wedge\Omega)=\mathscr{U}_{p_1}(R_1, R_2\wedge\Omega)$. This implies the symmetry of the first condition in the proposition. □

Definition 4.2.2. We say that R_1 and R_2 are *wedgeable* if they satisfy the conditions in Proposition 4.2.1.

Note that for R_1 fixed, the set of R_2 such that R_1 and R_2 are not wedgeable is a super-polar set in $\mathscr{C}_{p_2}$. Indeed, this is the set of R_2 such that $\mathscr{U}(R_2)=-\infty$, where $\mathscr{U}$ is a super-potential of $R_1\wedge\omega^{k-p_1-p_2+1}$. So, R_1 is wedgeable for every R_2 if and only if $R_1\wedge\omega^{k-p_1-p_2+1}$ has bounded super-potentials.

PROPOSITION 4.2.3. *Let R_j and R_j' be currents in $\mathscr{C}_{p_j}$, $j=1,2$. Assume that R_1 and R_2 are wedgeable. Then, R_1' and R_2' are wedgeable in the following cases*:

(1) *R_j' is more diffuse than R_j for $j=1,2$;*

(2) *there is a constant $c>0$ such that $R_j'\leqslant cR_j$ for $j=1,2$.*

Proof. The first assertion is a consequence of Lemma 4.1.2. For the second one, it is enough to show that R_1 and R_2' are wedgeable. Then, in the same way, R_1' and R_2' are wedgeable. Write $R_2=\lambda R_2'+(1-\lambda)R_2''$ with $0<\lambda\leqslant 1$ and $R_2''\in\mathscr{C}_{p_2}$. From the fact that $\mathscr{U}_{p_1}$ is affine, we obtain that

$$\begin{aligned}\lambda\mathscr{U}_{p_1}&(R_1, R_2'\wedge\omega^{k-p_1-p_2+1})\\ &= \mathscr{U}_{p_1}(R_1, R_2\wedge\omega^{k-p_1-p_2+1}) - (1-\lambda)\mathscr{U}_{p_1}(R_1, R_2''\wedge\omega^{k-p_1-p_2+1}) \neq -\infty,\end{aligned}$$

since $\mathscr{U}_{p_1}(R_1, R_2\wedge\omega^{k-p_1-p_2+1})\neq-\infty$ and $\mathscr{U}_{p_1}$ is bounded from above. This proves the property. □

Assume that R_1 and R_2 are wedgeable. We define the wedge-product (or the intersection) $R_1\wedge R_2$. This will be a current of bidegree (p_1+p_2, p_1+p_2). For every smooth real form Φ of bidegree $(k-p_1-p_2, k-p_1-p_2)$, write $dd^c\Phi=c(\Omega^+-\Omega^-)$, where $\Omega^\pm$ are smooth forms in $\mathscr{C}_{k-p_1-p_2+1}$ and c is a positive constant. First, consider the case where R_1 or R_2 is smooth. So, $R_1\wedge R_2$ is defined. Let U_1 be a quasi-potential of mean 0 of R_1. Choose U_1 smooth if R_1 is smooth. We have

$$\begin{aligned}\langle R_1\wedge R_2,\Phi\rangle &= \langle \omega^{p_1}\wedge R_2,\Phi\rangle+\langle (R_1-\omega^{p_1})\wedge R_2,\Phi\rangle\\ &= \langle R_2,\omega^{p_1}\wedge\Phi\rangle+\langle dd^c(U_1\wedge R_2),\Phi\rangle\\ &= \langle R_2,\omega^{p_1}\wedge\Phi\rangle+\langle U_1\wedge R_2,dd^c\Phi\rangle\\ &= \langle R_2,\omega^{p_1}\wedge\Phi\rangle+c\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^+)-c\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^-).\end{aligned}$$

We deduce that the last expression is independent of the choice of c and $\Omega^\pm$. This formally justifies the following formula for wedgeable R_1 and R_2. Define

$$\langle R_1\wedge R_2,\Phi\rangle := \langle R_2,\omega^{p_1}\wedge\Phi\rangle+c\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^+)-c\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^-). \tag{4.1}$$

The following theorem justifies our definition.

THEOREM 4.2.4. *Assume that R_1 and R_2 are wedgeable. Then, the right-hand side of* (4.1) *is independent of the choice of c and $\Omega^\pm$, and depends linearly on Φ. Moreover, $R_1\wedge R_2$ defines a positive closed (p_1+p_2,p_1+p_2)-current of mass* 1 *with support in* $\operatorname{supp}(R_1)\cap\operatorname{supp}(R_2)$ *which depends linearly on each R_j and is symmetric with respect to the variables.*

Proof. First, observe that the linear dependence of Φ and of R_j are easily deduced from the properties of $\mathscr{U}_{p_1}$. Write $dd^c\Phi=\tilde c(\widetilde\Omega^+-\widetilde\Omega^-)$ with $\tilde c\geqslant 0$ and $\widetilde\Omega^\pm$ smooth in $\mathscr{C}_{k-p_1-p_2+1}$. We have

$$c\Omega^+-c\Omega^-=\tilde c\widetilde\Omega^+-\tilde c\widetilde\Omega^-.$$

Since $\mathscr{U}_{p_1}$ is affine on each variable, we have

$$c\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^+)-c\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^-)=\tilde c\mathscr{U}_{p_1}(R_1,R_2\wedge\widetilde\Omega^+)-\tilde c\mathscr{U}_{p_1}(R_1,R_2\wedge\widetilde\Omega^-).$$

So, the right-hand side of (4.1) does not change if we replace c by $\tilde c$ and $\Omega^\pm$ by $\widetilde\Omega^\pm$.

Let $R_{j,\theta}$ be the currents constructed in §2.1 starting with the currents R_j; they are smooth for $\theta\neq 0$. Lemma 4.1.3 implies that $\mathscr{U}_{p_1}(R_{1,\theta_1},R_{2,\theta_2}\wedge\Omega^\pm)$ converge to

$$\mathscr{U}_{p_1}(R_1,R_2\wedge\Omega^\pm)$$

when $\theta_j\to 0$; see also Remarks 3.2.4. It follows that when $\theta_j\to 0$ and $(\theta_1,\theta_2)\neq(0,0)$, the currents $R_{1,\theta_1}\wedge R_{2,\theta_2}$ converge to $R_1\wedge R_2$. Hence, $R_1\wedge R_2$ is a positive closed current of mass 1. Since $\operatorname{supp}(R_{j,\theta})\to\operatorname{supp}(R_j)$, $R_1\wedge R_2$ has support in $\operatorname{supp}(R_1)\cap\operatorname{supp}(R_2)$. We also have that $R_{1,\theta_1}\wedge R_{2,\theta_2}=R_{2,\theta_2}\wedge R_{1,\theta_1}$, hence $R_1\wedge R_2=R_2\wedge R_1$. $\square$

Lemma 4.2.5. *Let R_j and R'_j be currents in $\mathscr{C}_{p_j}$. Assume that R_1 and R_2 are wedgeable. If R'_j is more diffuse than R_j for $j=1,2$, then $R'_1\wedge R'_2$ is more diffuse than $R_1\wedge R_2$.*

Proof. By Proposition 4.2.3, R'_1 and R_2 are wedgeable. Theorem 4.2.4 shows that $R_1\wedge R_2$, $R'_1\wedge R_2$, $R_1\wedge R'_2$ and $R'_1\wedge R'_2$ are well defined. We show that $R'_1\wedge R_2$ is more diffuse than $R_1\wedge R_2$. In the same way, we will get that $R'_1\wedge R'_2$ is more diffuse than $R'_1\wedge R_2$, which will complete the proof.

The symbols U and $\mathscr{U}$ below denote quasi-potentials and super-potentials of mean 0. By hypothesis, there is a constant a such that $\mathscr{U}_{R'_1}+a\geqslant\mathscr{U}_{R_1}$. Consider a smooth form R in $\mathscr{C}_{k-p_1-p_2+1}$ and choose U_R smooth. Since $dd^cU_R=R-\omega^{k-p_1-p_2+1}$, we deduce from (4.1) that

$$\mathscr{U}_{R'_1\wedge R_2}(R)=\langle R'_1\wedge R_2,U_R\rangle=\langle R_2,\omega^{p_1}\wedge U_R\rangle+\mathscr{U}_{R'_1}(R_2\wedge R)-\mathscr{U}_{R'_1}(R_2\wedge\omega^{k-p_1-p_2+1}).$$

The same identity for $R_1\wedge R_2$ and the inequality $\mathscr{U}_{R'_1}+a\geqslant\mathscr{U}_{R_1}$ imply that

$$\mathscr{U}_{R'_1\wedge R_2}(R)-\mathscr{U}_{R_1\wedge R_2}(R)\geqslant -a-\mathscr{U}_{R'_1}(R_2\wedge\omega^{k-p_1-p_2+1})+\mathscr{U}_{R_1}(R_2\wedge\omega^{k-p_1-p_2+1}).$$

The last expression is finite and independent of R. Hence, using the regularization R_θ of R for an arbitrary R in $\mathscr{C}_{k-p_1-p_2+1}$, we deduce that $\mathscr{U}_{R'_1\wedge R_2}-\mathscr{U}_{R_1\wedge R_2}$ is bounded below by a constant. So, $R'_1\wedge R_2$ is more diffuse than $R_1\wedge R_2$. $\square$

The following continuity result shows that the wedge-product is the right extension to currents of the wedge-product of smooth forms.

Proposition 4.2.6. *Let R_1 and R_2 be wedgeable currents as above and let $R_{j,n}$ be currents in $\mathscr{C}_{p_j}$ H-converging to R_j, $j=1,2$. Then $R_{1,n}$ and $R_{2,n}$ are wedgeable and $R_{1,n}\wedge R_{2,n}$ H-converge to $R_1\wedge R_2$.*

Proof. Let $\mathscr{U}_{j,n}$ and $\mathscr{U}_j$ denote the super-potentials of mean 0 of $R_{j,n}$ and R_j. Let $a_{j,n}$ be constants converging to 0 such that $\mathscr{U}_{j,n}+a_{j,n}\geqslant\mathscr{U}_j$. Define

$$\varepsilon_n:=\mathscr{U}_{1,n}(R_2\wedge\omega^{k-p_1-p_2+1})-\mathscr{U}_1(R_2\wedge\omega^{k-p_1-p_2+1}).$$

We have $\varepsilon_n\geqslant -a_{1,n}$. Since $\mathscr{U}_1(R_2\wedge\omega^{k-p_1-p_2+1})$ is finite, Proposition 3.2.2 implies that $\limsup_{n\to\infty}\varepsilon_n\leqslant 0$. So, $\varepsilon_n\to 0$. Define

$$K:=\{R_{1,1},R_{1,2},...\}\cup\{R_1\}$$

and

$$\delta_n:=\sup_{S\in K}|\mathscr{U}_{2,n}(S\wedge\omega^{k-p_1-p_2+1})-\mathscr{U}_2(S\wedge\omega^{k-p_1-p_2+1})|.$$

We first show that $\delta_n \to 0$. As $\mathscr{U}_{2,n} - \mathscr{U}_2 \geqslant -a_{2,n}$, it is enough to prove that

$$\limsup_{n\to\infty} \delta'_n \leqslant 0,$$

where

$$\delta'_n := \sup_{S\in K} (\mathscr{U}_{2,n}(S\wedge\omega^{k-p_1-p_2+1}) - \mathscr{U}_2(S\wedge\omega^{k-p_1-p_2+1})).$$

Because $R_{1,n}\to R_1$, K is compact. Since $\mathscr{U}_{1,m}\to\mathscr{U}_1$ pointwise, we have

$$\begin{aligned}\mathscr{U}_2(R_{1,m}\wedge\omega^{k-p_1-p_2+1}) &= \mathscr{U}_{1,m}(R_2\wedge\omega^{k-p_1-p_2+1}) \\ &\to \mathscr{U}_1(R_2\wedge\omega^{k-p_1-p_2+1}) = \mathscr{U}_2(R_1\wedge\omega^{k-p_1-p_2+1}).\end{aligned}$$

So $\mathscr{U}_2$, restricted to K, is continuous. Proposition 3.2.2 applied to $\mathscr{U}_2|_K + \varepsilon$, implies that $\limsup_{n\to\infty} \delta'_n \leqslant 0$. Therefore, $\delta_n \to 0$.

Proposition 4.2.3 implies that $R_{1,n}$ and $R_{2,n}$ are wedgeable, and $R_{1,n}$ and R_2 are wedgeable. Let $\mathscr{U}_n$, $\mathscr{U}'_n$ and $\mathscr{U}$ denote the super-potentials of mean 0 of $R_{1,n}\wedge R_{2,n}$, $R_{1,n}\wedge R_2$ and $R_1\wedge R_2$. We obtain as in Lemma 4.2.5 for smooth R that $\mathscr{U}_n(R)$ and $\mathscr{U}'_n(R)$ converge to $\mathscr{U}(R)$. Moreover,

$$\mathscr{U}'_n(R) - \mathscr{U}(R) \geqslant -|a_{1,n}| - |\varepsilon_n|$$

and

$$\mathscr{U}_n(R) - \mathscr{U}'_n(R) \geqslant -|a_{2,n}| - \delta_n.$$

Hence,

$$\mathscr{U}_n(R) \geqslant \mathscr{U}(R) - |a_{1,n}| - |a_{2,n}| - |\varepsilon_n| - \delta_n$$

for smooth R. Using the approximation of R by R_θ, we deduce this inequality for arbitrary R. The super-potentials $\mathscr{U}_n + |a_{1,n}| + |a_{2,n}| + |\varepsilon_n| + \delta_n$ are larger than $\mathscr{U}$ and converge to $\mathscr{U}$. Hence, the sequence $R_{1,n}\wedge R_{2,n}$ H-converges to $R_1\wedge R_2$. $\square$

LEMMA 4.2.7. *Let R_1 and R_2 be currents in $\mathscr{C}_{p_j}$. Then, for $\tau\in\mathrm{Aut}(\mathbb{P}^k)$ outside some pluripolar set, R_1 and $\tau_*(R_2)$ are wedgeable. Moreover, if R_1 and R_2 are wedgeable, then $R_1\wedge\tau_*(R_2)$ converge to $R_1\wedge R_2$ when $\tau\to\mathrm{id}$ in the fine topology on $\mathrm{Aut}(\mathbb{P}^k)$, i.e. the coarsest topology for which quasi-psh functions are continuous.*

Proof. Let $\mathscr{U}_{R_1}$ be a super-potential of R_1. Recall that $\mathscr{U}_{R_1}$ is an affine function which is finite on smooth forms R in $\mathscr{C}_{k-p_1+1}$. On the other hand, using an average of $\tau_*(R_2)\wedge\omega^{k-p_1-p_2+1}$ we can obtain a smooth form R in $\mathscr{C}_{k-p_1+1}$. Therefore, the function $\tau\mapsto\mathscr{U}_{R_1}(\tau_*(R_2)\wedge\omega^{k-p_1-p_2+1})$ is not identically $-\infty$. So, it is a quasi-psh function on $\mathrm{Aut}(\mathbb{P}^k)$ and is finite outside a pluripolar set. Hence, R_1 and $\tau_*(R_2)$ are wedgeable for τ outside this pluripolar set.

Assume now that R_1 and R_2 are wedgeable. Let Φ be a real smooth form of bidegree $(k-p_1-p_2, k-p_1-p_2)$. By (4.1), $\langle R_1\wedge\tau_*(R_2),\Phi\rangle$ can be written as a difference of quasi-psh functions on $\mathrm{Aut}(\mathbb{P}^k)$. Hence, in the fine topology on $\mathrm{Aut}(\mathbb{P}^k)$, $\langle R_1\wedge\tau_*(R_2),\Phi\rangle$ converge to $R_1\wedge R_2$ when $\tau\to\mathrm{id}$. The lemma follows. □

In order to define the wedge-product of several currents, we need the following result.

LEMMA 4.2.8. *Assume that R_1 and R_2 are wedgeable, and that $R_1\wedge R_2$ and R_3 are wedgeable. Then, R_2 and R_3 are wedgeable, and R_1 and $R_2\wedge R_3$ are wedgeable. Moreover, we have*

$$(R_1\wedge R_2)\wedge R_3 = R_1\wedge(R_2\wedge R_3).$$

Proof. We use the symbols U and $\mathscr{U}$ for quasi-potentials and super-potentials of mean 0. Since ω^{p_1} is more diffuse than R_1, by Lemma 4.2.5, $\omega^{p_1}\wedge R_2$ is more diffuse than $R_1\wedge R_2$. Proposition 4.2.3 implies that $\omega^{p_1}\wedge R_2$ and R_3 are wedgeable. Hence, $\mathscr{U}_{R_3}(\omega^{k-p_2-p_3+1}\wedge R_2)$ is finite. It follows that R_2 and R_3 are wedgeable.

We show that R_1 and $R_2\wedge R_3$ are wedgeable. By Proposition 4.2.6 and Remark 3.2.4, $R_{2,\theta}\wedge R_{3,\theta}$ H-converge to $R_2\wedge R_3$. Using Lemma 4.1.3, for $p=p_1+p_2+p_3$, we obtain

$$\begin{aligned}
&\mathscr{U}_{R_1}(R_2\wedge R_3\wedge\omega^{k-p+1})\\
&\quad=\lim_{\theta\to0}\mathscr{U}_{R_1}(R_{2,\theta}\wedge R_{3,\theta}\wedge\omega^{k-p+1})\\
&\quad=\lim_{\theta\to0}\langle U_{R_1}, R_{2,\theta}\wedge R_{3,\theta}\wedge\omega^{k-p+1}\rangle\\
&\quad=\lim_{\theta\to0}\langle R_{3,\theta}, U_{R_1}\wedge R_{2,\theta}\wedge\omega^{k-p+1}\rangle\\
&\quad=\lim_{\theta\to0}\mathscr{U}_{R_{3,\theta}}(R_1\wedge R_{2,\theta}\wedge\omega^{k-p+1})+\langle\omega^{p_3}, U_{R_1}\wedge R_{2,\theta}\wedge\omega^{k-p+1}\rangle-\mathscr{U}_{R_3}(R_2\wedge\omega^{k-p_2-p_3+1})\\
&\quad=\mathscr{U}_{R_3}(R_1\wedge R_2\wedge\omega^{k-p+1})+\mathscr{U}_{R_1}(R_2\wedge\omega^{k-p_1-p_2+1})-\mathscr{U}_{R_3}(R_2\wedge\omega^{k-p_2-p_3+1}).
\end{aligned}$$

The last sum is finite. Hence, by Proposition 4.2.1, R_1 and $R_2\wedge R_3$ are wedgeable.

We now prove the identity in the lemma. Proposition 4.2.6 and Remarks 3.2.4 imply that $R_{1,\theta}\wedge(R_{2,\theta}\wedge R_{3,\theta})$ converge to $R_1\wedge(R_2\wedge R_3)$ and $(R_{1,\theta}\wedge R_{2,\theta})\wedge R_{3,\theta}$ converge to $(R_1\wedge R_2)\wedge R_3$. For $\theta\neq0$, since $R_{j,\theta}$ are smooth, we have

$$(R_{1,\theta}\wedge R_{2,\theta})\wedge R_{3,\theta} = R_{1,\theta}\wedge(R_{2,\theta}\wedge R_{3,\theta}).$$

Letting $\theta\to0$ gives the result. □

Definition 4.2.9. We say that $R_1, ..., R_l$ are *wedgeable* if $R_1\wedge...\wedge R_m$ and R_{m+1} are wedgeable for $m=1, ..., l-1$.

Lemma 4.2.8 implies that this property and the wedge-product $R_1\wedge...\wedge R_l$ are symmetric with respect to R_j. The wedge-product is a positive closed current of mass 1. Applying inductively Proposition 4.2.6 gives the following result.

THEOREM 4.2.10. *Let* $\{R_{j,n}\}_{n\geqslant 0}$ *be sequences of currents in* $\mathscr{C}_{p_j}$ *H-converging to* R_j, $j=1,...,l$. *Assume that* $R_1,...,R_l$ *are wedgeable. Then*, $R_{1,n},...,R_{l,n}$ *are wedgeable and* $R_{1,n}\wedge...\wedge R_{l,n}$ *converge to* $R_1\wedge...\wedge R_l$ *in the Hartogs sense.*

Definition 4.2.11. Let S and R be wedgeable currents in $\mathscr{C}_p$ and $\mathscr{C}_{k-p}$ respectively. Let a be a point in $\mathbb{P}^k$. We let $\nu_R(S,a)$ denote the mass of $S\wedge R$ at a and we refer to it as the *Lelong number of* S *at* a *relative to* R.

This notion is related to the directional Lelong numbers of S developed in [12] and [13]. Consider a classical example.

Example 4.2.12. Let S be a current in $\mathscr{C}_1$ and u be a quasi-potential of S. We have $S=\omega+dd^c u$. If R is the current of integration on a projective line D which is not contained in $\{z:u(z)=-\infty\}$, then S and $[D]$ are wedgeable and $\nu_{[D]}(S,a)$ exists for every a. It is equal to the mass of $S\wedge[D]=dd^c(u[D])+\omega\wedge[D]$ at a, i.e. to the mass of $dd^c(u[D])$ at a.

We will see in Proposition 4.3.4 below that if R is locally bounded in a neighbourhood of a hypersurface, then $\nu_R(S,a)$ exists for every S. For the classical case, when R is locally bounded outside a; see [13].

4.3. Intersection with currents with regular potentials

In this section, we will give sufficient conditions for currents to be wedgeable.

PROPOSITION 4.3.1. *Let* R_j *be currents in* $\mathscr{C}_{p_j}$ *with* $1\leqslant j\leqslant l$. *Assume that* R_j *have bounded super-potentials for* $1\leqslant j\leqslant l-1$. *Then*, $R_1,...,R_l$ *are wedgeable. If moreover* R_l *has bounded super-potentials*, *then* $R_1\wedge...\wedge R_l$ *has bounded super-potentials.*

Proof. Consider $R_j':=\omega^{p_j}$. Their super-potentials of mean 0 vanish identically. It is clear that $R_1',...,R_{l-1}',R_l$ are wedgeable. Since R_j have bounded super-potentials, they are more diffuse than R_j'. Proposition 4.2.3 implies that $R_1,...,R_l$ are wedgeable.

Assume that the super-potentials of R_l are bounded. Then, R_l are more diffuse than R_l'. Lemma 4.2.5 implies that $R_1\wedge...\wedge R_l$ is more diffuse than $R_1'\wedge...\wedge R_l'$. It follows that $R_1\wedge...\wedge R_l$ has bounded super-potentials. □

PROPOSITION 4.3.2. *Let* R_j *be currents in* $\mathscr{C}_{p_j}$, $1\leqslant j\leqslant l$. *Assume that* R_j *have continuous super-potentials for* $1\leqslant j\leqslant l-1$. *Then*, $R_1\wedge...\wedge R_l$ *depends continuously on* R_l. *If moreover* R_l *has continuous super-potentials*, *then* $R_1\wedge...\wedge R_l$ *has continuous super-potentials.*

Proof. We only have to consider the case where $l=2$. Since R_1 has continuous super-potentials, it follows from (4.1) that $R_1\wedge R_2$ depends continuously on R_2. Assume that R_2 also has continuous super-potentials. Let $\mathscr{U}_{R_1\wedge R_2}$ and $\mathscr{U}_{R_j}$ denote the super-potentials of mean 0 of $R_1\wedge R_2$ and R_j, respectively. Applying (4.1) to a smooth quasi-potential U_R of mean 0 of a smooth form R in $\mathscr{C}_{k-p_1-p_2+1}$ gives

$$\mathscr{U}_{R_1\wedge R_2}(R)=\langle R_1\wedge R_2, U_R\rangle=\mathscr{U}_{R_2}(\omega^{p_1}\wedge R)+\mathscr{U}_{R_1}(R_2\wedge R)-\mathscr{U}_{R_1}(R_2\wedge\omega^{k-p_1-p_2+1}).$$

Since $\mathscr{U}_{R_j}$ are continuous and $R_2\wedge R$ depends continuously on R, the last expression can be extended continuously to R in $\mathscr{C}_{k-p_1-p_2+1}$. Hence, $R_1\wedge R_2$ has continuous super-potentials. □

Definition 4.3.3. A compact subset K of $\mathbb{P}^k$ is $(p+1)$-*pseudoconvex* if there is a current in $\mathscr{C}_{k-p}$ with compact support in $\mathbb{P}^k\setminus K$; see also [32].

Observe that one can approximate the previous current by smooth elements of $\mathscr{C}_{k-p}$ with compact support in $\mathbb{P}^k\setminus K$. So, there is a smooth positive closed $(k-p,k-p)$-form Θ with compact support in $\mathbb{P}^k\setminus K$. If the $2(k-p)$-dimensional Hausdorff measure of K vanishes, then K is $(p+1)$-pseudoconvex. Indeed, generic projective planes of dimension p do not intersect K. In particular, analytic sets of pure codimension p are p-pseudoconvex.

To explain the terminology, observe that we may assume that Θ has mass 1 and there is a smooth $(k-p-1,k-p-1)$-form Φ such that $dd^c\Phi=-\Theta+\omega^{k-p}$. So, $dd^c\Phi$ is strictly positive on K. Adding a large positive closed form to Φ allows one to assume that Φ is positive on $\mathbb{P}^k$; compare with Definition 5.2.1 for $X=\mathbb{P}^k$.

PROPOSITION 4.3.4. *Let R_j be currents in $\mathscr{C}_{p_j}$, $j=1,2$. Assume that R_j are locally bounded forms on open sets $W_j\subset\mathbb{P}^k$ such that $\mathbb{P}^k\setminus(W_1\cup W_2)$ is (p_1+p_2)-pseudoconvex. Then, R_1 and R_2 are wedgeable.*

Proof. Let Θ be a smooth form in $\mathscr{C}_{k-p_1-p_2+1}$ with compact support in $W_1\cup W_2$. Fix open sets $W_j'\Subset W_j$ such that $\operatorname{supp}(\Theta)\subset W_1'\cup W_2'$. Reducing W_j if necessary, we may assume that R_j are bounded on W_j. Proposition 4.2.1 implies that it suffices to show that

$$\mathscr{U}_{p_1}(R_1,R_2\wedge\Theta)\geqslant -A(1+\|R_1\|_{\infty,W_1}+\|R_2\|_{\infty,W_2}),$$

where $A>0$ is independent of R_j. This estimate is uniform with respect to R_j, we can then use a regularization and assume that R_j are smooth.

Let U_j denote the Green quasi-potentials of R_j and let m_j denote their means. Lemma 2.3.5 implies that

$$\|U_j\|_{\mathscr{C}^1(W_j')}\leqslant c(1+\|R_j\|_{\infty,W_j})\quad\text{and}\quad |m_j|\leqslant c$$

for $c>0$ independent of R_j. Let χ_j be positive smooth functions with compact support in W'_j such that $\chi_1+\chi_2=1$ on $\operatorname{supp}(\Theta)$. We have

$$\mathscr{U}_{p_1}(R_1, R_2\wedge\Theta) = \langle U_1, R_2\wedge\Theta\rangle - m_1 = \langle \chi_2 U_1, R_2\wedge\Theta\rangle + \langle \chi_1 U_1, R_2\wedge\Theta\rangle - m_1.$$

Since $\chi_1 U_1$ is bounded, we only have to estimate the first integral. By Stokes formula, it is equal to the sum of $\langle \chi_2 U_1, \omega^{p_2}\wedge\Theta\rangle$, which is bounded, and of the integral

$$\begin{aligned}\langle \chi_2 U_1, dd^c U_2\wedge\Theta\rangle &= \langle \chi_2 dd^c U_1, U_2\wedge\Theta\rangle + \langle d\chi_2\wedge d^c U_1, U_2\wedge\Theta\rangle \\ &\quad - \langle d^c\chi_2\wedge dU_1, U_2\wedge\Theta\rangle + \langle U_1\wedge dd^c\chi_2, U_2\wedge\Theta\rangle \\ &= \langle \chi_2 R_1, U_2\wedge\Theta\rangle - \langle \chi_2\omega^{p_1}, U_2\wedge\Theta\rangle - \langle d\chi_1\wedge d^c U_1, U_2\wedge\Theta\rangle \\ &\quad + \langle d^c\chi_1\wedge dU_1, U_2\wedge\Theta\rangle - \langle U_1\wedge dd^c\chi_1, U_2\wedge\Theta\rangle.\end{aligned}$$

We used that $d\chi_2=-d\chi_1$ and $dd^c\chi_2=-dd^c\chi_1$ on $\operatorname{supp}(\Theta)$. It is clear that the last sum is of order at most equal to $1+\|R_1\|_{\infty,W_1}+\|R_2\|_{\infty,W_2}$. Indeed, we have $\|U_j\|\leqslant c$ and each integral is over a domain where we can use the estimates on $\|U_j\|_{\mathscr{C}^1(W'_j)}$. □

Remark 4.3.5. It is enough to assume that R_j are in $\mathscr{L}^s_{\mathrm{loc}}(W_j)$ with $s>2k$.

We deduce from Proposition 4.3.4 and Lemma 2.3.5 the following results.

COROLLARY 4.3.6. *Let R_j be currents in $\mathscr{C}_{p_j}$, $j=1,...,l$. Assume, for $j=2,...,l$, that the intersection of the supports of $R_1,...,R_j$ is $(p_1+...+p_j)$-pseudoconvex. Then, $R_1,...,R_l$ are wedgeable.*

COROLLARY 4.3.7. *Let V_j be analytic subsets of pure codimension p_j in $\mathbb{P}^k$, $1\leqslant j\leqslant l$. Assume that their intersection is of pure codimension $p_1+...+p_l$. Let I_n denote the components of $V_1\cap...\cap V_l$ and m_n their multiplicities. Then, the currents of integration on V_j are wedgeable and we have*

$$[V_1]\wedge...\wedge[V_l] = \sum_n m_n[I_n].$$

Proof. It is clear that $V_1\cap...\cap V_j$ is of pure codimension $p_1+...+p_j$. Hence, it is $(p_1+...+p_j)$-pseudoconvex. By Corollary 4.3.6, $V_1,...,V_l$ are wedgeable and $[V_1]\wedge...\wedge[V_l]$ has support in $V_1\cap...\cap V_l$, which is of pure codimension $p_1+...+p_l$. Then $[V_1]\wedge...\wedge[V_l]$ is a combination of $[I_n]$. For the identity in the corollary, by induction, it is enough to prove it for $l=2$. Since $\sum_n m_n[I_n]$ depends continuously on V_1 and V_2, Lemma 4.2.7 implies that it is enough to prove the corollary for V_1 and $\tau(V_2)$, where τ is a generic automorphism close enough to the identity. So, we may assume that $m_n=1$ for all n. Hence, for a generic point a in $V_1\cap V_2$, a belongs to the regular parts of V_1 and V_2, and

V_1 and V_2 intersect transversally at a. It is enough to prove that $[V_1]\wedge[V_2]=[V_1\cap V_2]$ in a neighbourhood of a. In this neighbourhood, the θ-regularization $[V_2]_\theta$ of $[V_2]$ is an average of currents of integration on manifolds $\tau(V_2)$, where τ is an automorphism close to the identity. Observe that $\tau(V_2)$ is close to V_2 and intersects V_1 transversally on a manifold close to $V_1\cap V_2$. Hence, $[V_1]\wedge[V_2]_\theta$ is an average of $[V_1\cap\tau(V_2)]$. When θ tends to 0, this mean converges to $[V_1\cap V_2]$. On the other hand, we have seen in Proposition 4.2.6 that $[V_1]\wedge[V_2]_\theta$ converge to $[V_1]\wedge[V_2]$. Therefore, $[V_1]\wedge[V_2]=[V_1\cap V_2]$. The corollary follows. □

4.4. Intersection with $(1,1)$-currents

Consider now the case where $p_2=...=p_l=1$. For $2\leqslant j\leqslant l$, there is a quasi-psh function u_j on $\mathbb{P}^k$ such that

$$dd^c u_j = R_j-\omega.$$

We have the following lemma.

LEMMA 4.4.1. *The currents $R_1,...,R_l$ are wedgeable if and only if, for all $2\leqslant j\leqslant l$, u_j is integrable with respect to the trace measure of $R_1\wedge...\wedge R_{j-1}$. In particular, the last condition is symmetric with respect to $R_2,...,R_l$.*

Proof. It is enough to consider the case $l=2$. We may assume that u_2 is of mean 0. Let $u_{2,\theta}$ be the quasi-potential of mean 0 of $R_{2,\theta}$. Since $R_{2,\theta}$ H-converge to R_2, there are constants a_θ converging to 0 such that $u_{2,\theta}+a_\theta\geqslant u_2$, and $u_{2,\theta}$ converge pointwise to u_2. If $\mathscr{U}_{R_1}$ is the super-potential of mean 0 of R_1, then

$$\mathscr{U}_{R_1}(R_2\wedge\omega^{k-p_1}) = \lim_{\theta\to 0}\mathscr{U}_{R_1}(R_{2,\theta}\wedge\omega^{k-p_1}) = \lim_{\theta\to 0}\langle R_1, u_{2,\theta}\omega^{k-p_1}\rangle = \langle R_1, u_2\omega^{k-p_1}\rangle.$$

Therefore, $\mathscr{U}_{R_1}(R_2\wedge\omega^{k-p_1})$ is finite if and only if u_2 is integrable with respect to the trace measure $R_1\wedge\omega^{k-p_1}$ of R_1. This implies the lemma. □

If R_2 has a quasi-potential integrable with respect to R_1, it is classical to define the wedge-product $R_1\wedge R_2$ by

$$R_1\wedge R_2 := dd^c(u_2R_1)+\omega\wedge R_1.$$

One defines $R_1\wedge...\wedge R_l$ by induction.

LEMMA 4.4.2. *The previous definition coincides with the definition given in* §4.2.

Proof. Proposition 4.2.6 implies that $R_1\wedge R_{2,\theta}$ converge to $R_1\wedge R_2$ when $\theta\to 0$. Since $R_{2,\theta}$ is smooth, we have

$$R_1\wedge R_{2,\theta} = R_1\wedge(dd^c u_{2,\theta}+\omega) = dd^c(u_{2,\theta}R_1)+\omega\wedge R_1.$$

It is clear that the last expression converge to $dd^c(u_2R_1)+\omega\wedge R_1$. □

5. Complex dynamics in higher dimension

Super-potentials allow us to construct and study invariant currents in complex dynamics. We will give here some applications of this new notion.

5.1. Pull-back of currents by meromorphic maps

The results in this section hold for meromorphic correspondences, in particular for the inverse of a dominant meromorphic map. For simplicity, we only consider meromorphic maps on $\mathbb{P}^k$. Recall that a meromorphic map $f:\mathbb{P}^k\to\mathbb{P}^k$ is holomorphic outside an analytic subset I of codimension $\geqslant 2$ in $\mathbb{P}^k$. Let Γ denote the closure of the graph of the restriction of f to $\mathbb{P}^k\setminus I$. This is an irreducible analytic set of dimension k in $\mathbb{P}^k\times\mathbb{P}^k$.

Let π_1 and π_2 denote the canonical projections of $\mathbb{P}^k\times\mathbb{P}^k$ on the factors. The *indeterminacy locus* I of f is the set of points $z\in\mathbb{P}^k$ such that $\dim\pi_1^{-1}(z)\cap\Gamma\geqslant 1$. We assume that f is *dominant*, that is, $\pi_2(\Gamma)=\mathbb{P}^k$. The *second indeterminacy set* of f is the set I' of points $z\in\mathbb{P}^k$ such that $\dim\pi_2^{-1}(z)\cap\Gamma\geqslant 1$. Its codimension is also at least equal to 2. If A is a subset of $\mathbb{P}^k$, define

$$f(A):=\pi_2(\pi_1^{-1}(A)\cap\Gamma)\quad\text{and}\quad f^{-1}(A):=\pi_1(\pi_2^{-1}(A)\cap\Gamma).$$

Define formally for a current S on $\mathbb{P}^k$, not necessarily positive or closed, the pull-back $f^*(S)$ by

$$f^*(S):=(\pi_1)_*(\pi_2^*(S)\wedge[\Gamma]),\tag{5.1}$$

where $[\Gamma]$ is the current of integration of Γ. This makes sense if the wedge-product $\pi_2^*(S)\wedge[\Gamma]$ is well defined, in particular, when S is smooth. Note that when S is smooth $f^*(S)$ is an $\mathscr{L}^1$ form. Consider now the case of positive closed currents. We need some preliminary results.

LEMMA 5.1.1. *Let S be a current in $\mathscr{C}_p$. Assume that the restriction of S to a neighbourhood of I' is a smooth form. Then, formula* (5.1) *defines a positive closed (p,p)-current. Moreover, the mass λ_p of $f^*(S)$ does not depend on S.*

Proof. Since $\pi_2|_\Gamma$ is a finite map outside $\pi_2^{-1}(I')\cap\Gamma$, the current $\pi_2^*(S)\wedge[\Gamma]$ is well defined there, and depends continuously on S; see [23]. So, if S is smooth in a neighbourhood of I', $\pi_2^*(S)\wedge[\Gamma]$ is well defined in a neighbourhood of $\pi_2^{-1}(I')\cap\Gamma$, hence, $f^*(S)$ is well defined and is positive. Let U be the Green quasi-potential of S. This is a negative form such that $S-\omega^p=dd^cU$. By [23], $\pi_2^*(U)\wedge[\Gamma]$ is well defined outside $\pi_2^{-1}(I')$. Lemma 2.3.5 implies that U is continuous in a neighbourhood of I'. Hence, as for S, we obtain that $f^*(U)$ is well defined. We have $f^*(S)-f^*(\omega^p)=dd^cf^*(U)$. It follows that $f^*(S)$ and $f^*(\omega^p)$ are cohomologous. Therefore, they have the same mass. □

The operator f_* is formally defined by

$$f_*(R):=(\pi_2)_*(\pi_1^*(R)\wedge[\Gamma]). \tag{5.2}$$

LEMMA 5.1.2. *Let R be a current in $\mathscr{C}_{k-p+1}$ which is smooth in a neighbourhood of I. Then, the formula* (5.2) *defines a positive closed $(k-p+1,k-p+1)$-current. Moreover, the mass of $f_*(R)$ does not depend on R and is equal to λ_{p-1}.*

Proof. We obtain the first part as in Lemma 5.1.1. Since $f_*(\omega^{k-p+1})$ and $f^*(\omega^{p-1})$ have $\mathscr{L}^1$ coefficients, we also have

$$\|f_*(R)\|=\|f_*(\omega^{k-p+1})\|=\int f_*(\omega^{k-p+1})\wedge\omega^{p-1}=\int \omega^{k-p+1}\wedge f^*(\omega^{p-1})=\lambda_{p-1},$$

which proves the last assertion in the lemma. $\square$

In order to define $f^*(S)$, we need to define $\pi_2^*(S)\wedge[\Gamma]$. For this purpose, we can introduce the notion of super-potential in $\mathbb{P}^k\times\mathbb{P}^k$ and study the intersection of currents there. We avoid this here. We call λ_p the *intermediate degree of order* p of f. Let, for simplicity, $L:=\lambda_p^{-1}f^*$ and $\Lambda:=\lambda_{p-1}^{-1}f_*$. With this normalization, for $S\in\mathscr{C}_p$ and $R\in\mathscr{C}_{k-p+1}$, the currents $L(S)$ and $\Lambda(R)$ have mass 1 when they are well defined.

LEMMA 5.1.3. *Let S be a smooth form in $\mathscr{C}_p$ and $\mathscr{U}_S$ be a super-potential of S. If $\mathscr{U}_{L(\omega^p)}$ is a super-potential of $L(\omega^p)$, then $\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}$ is equal to a super-potential of $L(S)$ on the currents $R\in\mathscr{C}_{k-p+1}$ which are smooth on a neighbourhood of I.*

Proof. We may assume that $\mathscr{U}_S$ and $\mathscr{U}_{L(\omega^p)}$ are of mean 0. Let $\mathscr{U}_{L(S)}$ be the super-potential of mean 0 of $L(S)$. Let U_S be a smooth quasi-potential of mean 0 of S and U_R be a quasi-potential of mean 0 of R which is smooth in a neighbourhood of I. Since $L(S)$ and $L(\omega^p)$ are smooth outside I, the following computation holds

$$\begin{aligned}
\mathscr{U}_{L(S)}(R) &= \langle L(S),U_R\rangle\\
&= \lambda_p^{-1}\langle S, f_*(U_R)\rangle\\
&= \lambda_p^{-1}\langle S-\omega^p, f_*(U_R)\rangle+\lambda_p^{-1}\langle \omega^p, f_*(U_R)\rangle\\
&= \lambda_p^{-1}\langle dd^cU_S, f_*(U_R)\rangle+\lambda_p^{-1}\langle f^*(\omega^p), U_R\rangle\\
&= \lambda_p^{-1}\langle U_S, f_*(dd^cU_R)\rangle+\mathscr{U}_{L(\omega^p)}(R)\\
&= \lambda_p^{-1}\langle U_S, f_*(R)\rangle-\lambda_p^{-1}\langle U_S, f_*(\omega^{k-p+1})\rangle+\mathscr{U}_{L(\omega^p)}(R)\\
&= \lambda_p^{-1}\lambda_{p-1}\mathscr{U}_S(\Lambda(R))-\lambda_p^{-1}\langle U_S, f_*(\omega^{k-p+1})\rangle+\mathscr{U}_{L(\omega^p)}(R).
\end{aligned}$$

This implies the result, since the second term in the last line is independent of R. $\square$

Definition 5.1.4. We say that a current S in $\mathscr{C}_p$ is *f^*-admissible* if there is a current R_0 in $\mathscr{C}_{k-p+1}$ which is smooth on a neighbourhood of I, such that the super-potentials of S are finite at $\Lambda(R_0)$.

LEMMA 5.1.5. *Let S be an f^*-admissible current in $\mathscr{C}_p$. Then, the super-potentials of S are finite at $\Lambda(R)$ for every smooth R in $\mathscr{C}_{k-p+1}$. In particular, if $S'\in\mathscr{C}_p$ is such that $S'\leqslant cS$ for some positive constant c, or if S' is more diffuse than S, then S' is also f^*-admissible.*

Proof. Since R admits a smooth quasi-potential, we can find a positive current U such that $dd^cU=R-R_0$ and U is smooth in a neighbourhood of I. We have $dd^c\Lambda(U)=\Lambda(R)-\Lambda(R_0)$ and, by Lemma 3.2.9,

$$\mathscr{U}_S(\Lambda(R))\geqslant\mathscr{U}_S(\Lambda(R_0))-\|\Lambda(U)\|.$$

This implies the first assertion. When $S'\leqslant cS$, as in Proposition 3.3.4, we obtain

$$\mathscr{U}_{S'}(\Lambda(R_0))>-\infty.$$

This also holds when S' is more diffuse than S. Hence, S' is f^*-admissible. □

LEMMA 5.1.6. *Let S be an f^*-admissible current in $\mathscr{C}_p$. Let S_n be smooth forms in $\mathscr{C}_p$ H-converging to S. Then, $f^*(S_n)$ H-converge to a positive closed (p,p)-current of mass λ_p which does not depend on the choice of S_n.*

Proof. Let $\mathscr{U}_{S_n}$ and $\mathscr{U}_S$ be super-potentials of mean 0 of S_n and S. Let c_n be constants converging to 0 such that $\mathscr{U}_{S_n}+c_n\geqslant\mathscr{U}_S$. Recall that $\mathscr{U}_{S_n}$ converge pointwise to $\mathscr{U}_S$. If R is smooth in a neighbourhood of I, we have

$$\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_{S_n}(\Lambda(R))+\mathscr{U}_{L(\omega^p)}(R)\to\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_S(\Lambda(R))+\mathscr{U}_{L(\omega^p)}(R).$$

Lemma 5.1.5 implies that the last sum is not identically $-\infty$.

Lemmas 5.1.3 and 3.2.5 imply that $L(S_n)$ converge to a positive closed current S' of bidegree (p,p). Lemma 5.1.1 implies that the mass of S' is 1. Moreover,

$$\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_{S_n}\circ\Lambda+\mathscr{U}_{L(\omega^p)}\quad(\text{resp. }\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)})$$

is equal on smooth forms R to some super-potential of $L(S_n)$ (resp. of S'). Denote by $\mathscr{U}_{L(S_n)}$ and $\mathscr{U}_{S'}$ these super-potentials. We have $\mathscr{U}_{L(S_n)}+\lambda_p^{-1}\lambda_{p-1}c_n\geqslant\mathscr{U}_{S'}$ on smooth forms R. Corollary 3.1.7 implies that this inequality holds for every R. Therefore, $L(S_n)\to S'$ in the Hartogs sense.

Finally, observe that if S_n' are smooth forms in $\mathscr{C}_p$ H-converging to S, then S_1, S_1', S_2, S_2', ... H-converge also to S. It follows that $L(S_1)$, $L(S_1')$, $L(S_2)$, $L(S_2')$, ... converge. We deduce that the limit S' does not depend on the choice of S_n. We can also obtain the result using the fact that $\mathscr{U}_{S'}(R)$ does not depend on the choice of S_n. □

Definition 5.1.7. Let S and S_n be as in Lemma 5.1.6. The limit of $f^*(S_n)$ is denoted by $f^*(S)$ and is called the *pull-back of S under f*. We say that S is *invariant under f^** or *f^*-invariant* if S is f^*-admissible and $f^*(S)=\lambda_p S$.

The following result extends Lemmas 5.1.3 and 5.1.6 when S and S_n are not necessarily smooth.

PROPOSITION 5.1.8. *Let S be an f^*-admissible current in $\mathscr{C}_p$. Let $\mathscr{U}_S$ and $\mathscr{U}_{L(\omega^p)}$ be super-potentials of S and $L(\omega^p)$. Let S_n be currents in $\mathscr{C}_p$ H-converging to S. Then, S_n are f^*-admissible and $f^*(S_n)$ H-converge towards $f^*(S)$. Moreover,*

$$\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}$$

is equal to a super-potential of $L(S)$ for $R\in\mathscr{C}_{k-p+1}$, smooth in a neighbourhood of I.

Proof. If $\mathscr{U}_{S_n}$ are super-potentials of mean 0 of S_n, there are constants c_n converging to 0 such that $\mathscr{U}_{S_n}+c_n\geqslant\mathscr{U}_S$. The last assertion in the proposition was already obtained in the proof of Lemma 5.1.6. Let $\mathscr{U}_{L(S)}$ denote the super-potential of $L(S)$ which is equal to $\lambda_p^{-1}\lambda_{p-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}$ for smooth R in $\mathscr{C}_{k-p+1}$. Let $\mathscr{U}_{L(S_n)}$ denote the analogous super-potentials of $L(S_n)$. Since $\mathscr{U}_{S_n}\to\mathscr{U}_S$ pointwise, $\mathscr{U}_{L(S_n)}\to\mathscr{U}_{L(S)}$ on smooth forms in $\mathscr{C}_{k-p+1}$. As in Lemma 5.1.6, we obtain $\mathscr{U}_{L(S_n)}+\lambda_p^{-1}\lambda_{p-1}c_n\geqslant\mathscr{U}_{L(S)}$, and this implies that $L(S_n)$ H-converge towards $L(S)$. $\square$

In the same way, we have the following.

Definition 5.1.9. We say that a current R in $\mathscr{C}_{k-p+1}$ is *f_*-admissible* if the super-potentials of R are finite at $L(S_0)$ for at least one current S_0 in $\mathscr{C}_p$ which is smooth in a neighbourhood of I' (or equivalently, for every S_0 smooth in $\mathscr{C}_p$).

If $R'\in\mathscr{C}_{k-p+1}$ is such that $R'\leqslant cR$ for some positive constant c, or R' is more diffuse than R, then R' is also f_*-admissible. We easily get the following lemma.

LEMMA 5.1.10. *Let R be an f_*-admissible current in $\mathscr{C}_{k-p+1}$. Let R_n be smooth forms in $\mathscr{C}_{k-p+1}$ H-converging to R. Then, R_n are f_*-admissible and $f_*(R_n)$ H-converge to a positive closed $(k-p+1,k-p+1)$-current of mass λ_{p-1} which does not depend on the choice of R_n.*

Definition 5.1.11. Let R and R_n be as in Lemma 5.1.10. The limit of $f_*(R_n)$ is denoted by $f_*(R)$ and is called the *push-forward of R under f*. We say that R is *invariant under f_** or *f_*-invariant* if R is f_*-admissible and $f_*(R)=\lambda_{p-1}R$.

PROPOSITION 5.1.12. *Let R be an f_*-admissible current in $\mathscr{C}_{k-p+1}$. Let $\mathscr{U}_R$ and $\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ be super-potentials of R and $\Lambda(\omega^{k-p+1})$. Let R_n be f_*-admissible currents in $\mathscr{C}_{k-p+1}$ H-converging to R. Then, $f_*(R_n)$ H-converge to $f_*(R)$. Moreover,*

$$\lambda_p\lambda_{p-1}^{-1}\mathscr{U}_R\circ L+\mathscr{U}_{\Lambda(\omega^{k-p+1})}$$

is equal to a super-potential of $\Lambda(R)$ on $S\in\mathscr{C}_p$, smooth in a neighbourhood of I'.

Note that if an analytic subset H of pure dimension in $\mathbb{P}^k$, of a given degree, is generic in the Zariski sense, then $[H]$ is f^*- and f_*-admissible. One can check that $f^*[H]$ and $f_*[H]$ depend continuously on H.

5.2. Pull-back by maps with small singularities

In this section we will give sufficient conditions, easy to check, in order to define the pull-back and push-forward operators. We need some preliminary results. In what follows, X is a complex manifold of dimension k and ω_X is a Hermitian form on X.

Definition 5.2.1. A compact subset K of X is *weakly p-pseudoconvex* if there is a positive smooth $(k-p,k-p)$-form Φ on X such that $dd^c\Phi$ is strictly positive on K.

Note that, using a cut-off function, we may assume that Φ has compact support in X. It follows from the discussion after Definition 4.3.3 that p-pseudoconvex sets in $\mathbb{P}^k$ are weakly p-pseudoconvex.

LEMMA 5.2.2. *If the $(2k-2p+1)$-dimensional Hausdorff measure of K is zero, then K is weakly p-pseudoconvex.*

Proof. Consider a point a in K. We construct a positive smooth $(k-p,k-p)$-form Φ_a such that $dd^c\Phi_a$ is positive on K and strictly positive at a. Since K is compact, there is a finite sum Φ of such forms satisfying Definition 5.2.1. Consider local coordinates $z=(z_1,...,z_k)$ such that $z=0$ at a. Define $z':=(z_1,...,z_{k-p})$ and $z'':=(z_{k-p+1},...,z_k)$. The hypothesis on the measure of K allows us to choose z so that K does not intersect the set $\{z:|z'|\leqslant 1$ and $1-\varepsilon\leqslant|z''|\leqslant 1\}$, where $\varepsilon>0$ is a constant. Let Θ be a positive $(k-p,k-p)$-form with compact support in the unit ball $\{z':|z'|<1\}$ of $\mathbb{C}^{k-p}$, strictly positive at 0. Let φ be a positive function with compact support in the unit ball of $\mathbb{C}^p$ such that $\varphi=|z''|^2$ for $|z''|\leqslant 1-\varepsilon$. Let π denote the projection $z\mapsto z'$ and define $\Psi_a:=\varphi(z'')\pi^*(\Theta)$. It is clear that Ψ_a is positive with compact support in X and $dd^c\Psi_a\geqslant 0$ on K. Nevertheless, $dd^c\Psi_a$ is not strictly positive at 0, but it does not vanish at 0. Observe that if τ is a linear automorphism of $\mathbb{C}^k$ close enough to the identity, then $\tau^*(\Psi_a)$ satisfies the same properties as Ψ_a does. Taking a finite sum of $\tau^*(\Psi_a)$ gives a form Φ_a which is strictly positive at 0. □

The following result is a version of Oka's inequality; see [32].

Proposition 5.2.3. *Let K be a weakly p-pseudoconvex compact subset of X. Let T be a positive (p,p)-current on X, not necessarily closed. Then, for every negative $(p-1,p-1)$-current U on X with $dd^cU \geqslant -T$, we have*

$$\|U\|_X \leqslant c(1+\|U\|_{X\setminus K}),$$

where $c>0$ is a constant independent of U.

Proof. Since $\|U\|_X = \|U\|_{X\setminus K} + \|U\|_K$, we only have to bound the mass of U on K. Let Φ be as in Definition 5.2.1 with compact support. Without loss of generality, we may assume that $dd^c\Phi \geqslant \omega_X^{k-p+1}$ on K. We have, for some positive constant c',

$$\begin{aligned}\|U\|_K &= -\int_K U\wedge\omega_X^{k-p+1} \leqslant -\int_K U\wedge dd^c\Phi = \int_{X\setminus K} U\wedge dd^c\Phi - \int_X U\wedge dd^c\Phi \\ &\leqslant c'\|U\|_{X\setminus K} - \int_X dd^cU\wedge\Phi \leqslant c'\|U\|_{X\setminus K} + \int_X T\wedge\Phi.\end{aligned}$$

This implies the result, since T is fixed. □

Let $\widetilde{\Sigma}'$ denote the analytic subset of the points x in Γ such that π_2 restricted to Γ is not locally finite at x. Define $\Sigma' := \pi_1(\widetilde{\Sigma}')$. We have $\widetilde{\Sigma}' \subset \pi_2^{-1}(I')\cap\Gamma$ and $\Sigma'\subset f^{-1}(I')$. The following proposition gives a sufficient condition in order to define the pull-back of a (p,p)-current, see also Lemma 5.2.7 below. The result can be applied to a generic meromorphic map in $\mathbb{P}^k$; see Proposition 5.3.6 below. Note that the hypothesis is satisfied for $p=1$, and in this case the result is due to Méo [40].

Proposition 5.2.4. *Assume that* $\dim\Sigma' \leqslant k-p$. *Then, every positive closed (p,p)-current S is f^*-admissible. Moreover, the pull-back operator $S\mapsto f^*(S)$ is continuous with respect to the weak topology on currents.*

Proof. Let S_n be smooth forms in $\mathscr{C}_p$ converging to S. Let $\mathscr{U}_{S_n}$ denote the super-potentials of mean 0 of S_n. It is sufficient to prove that, for R smooth in $\mathscr{C}_{k-p+1}$, $\mathscr{U}_{S_n}(\Lambda(R))$ converge to a finite number. Propositions 5.1.8 and 3.2.2 will imply that S is f^*-admissible. The convergence implies also that the limit does not depend on the choice of S_n (see the last argument in Lemma 5.1.6) and that f^* is continuous.

Let U_{S_n} denote the Green quasi-potentials of S_n which are smooth negative forms such that $dd^cU_{S_n} \geqslant -\omega^p$. These forms converge in $\mathscr{L}^1$ to the Green quasi-potential U_S of S. Hence, the means c_{S_n} of U_{S_n} converge to the mean c_S of U_S. Since U_{S_n} and R are smooth, we have

$$\mathscr{U}_{S_n}(\Lambda(R)) = \langle U_{S_n}, \Lambda(R)\rangle - c_{S_n} = \lambda_{p-1}^{-1}\langle f^*(U_{S_n}), R\rangle - c_{S_n}.$$

So, it is enough to prove that $f^*(U_{S_n})$ converge in the sense of currents.

The restriction of π_2 to $\Gamma\setminus\widetilde{\Sigma}'$ is a finite map. Under this hypothesis, it was proved in [23] that $\pi_2^*(U_{S_n})\wedge[\Gamma]$ converge in $\mathbb{P}^k\times\mathbb{P}^k$ outside $\widetilde{\Sigma}'$. It follows that $f^*(U_{S_n})$ converge outside Σ'. Hence, the mass of $f^*(U_{S_n})$ outside a small neighbourhood V of Σ' is bounded uniformly with respect to n. By Lemma 5.2.2, Σ' is weakly p-pseudoconvex in $\mathbb{P}^k$. Hence, since V is small, $\overline{V}$ is also p-pseudoconvex. Using the fact that $dd^c f^*(U_{S_n})\geqslant -f^*(\omega^p)$, Proposition 5.2.3 gives

$$\|f^*(U_{S_n})\|\leqslant c(1+\|f^*(U_{S_n})\|_{\mathbb{P}^k\setminus V})$$

with $c>0$ independent of S_n. Therefore, the mass of $f^*(U_{S_n})$ is bounded uniformly with respect to n. We can extract convergent subsequences from $f^*(U_{S_n})$. In order to prove the convergence of $f^*(U_{S_n})$ in $\mathbb{P}^k$, it remains to check that the limit values U of $f^*(U_{S_n})$ have no mass on Σ'.

Let W be a small open set in $\mathbb{P}^k$. Write $f^*(\omega^p)=dd^c\Phi$ with Φ negative on W. So, Φ and $U':=U+\Phi$ are negative currents with dd^c positive. Since the currents U and Φ are of bidimension $(k-p+1,k-p+1)$ and $\dim\Sigma'\leqslant k-p$, it follows from a result of Alessandrini–Bassanelli [2, Theorem 5.10] that Φ and U' have no mass on Σ'. This implies the result. □

Remark 5.2.5. Assume that $\dim\Sigma'\leqslant k-p$. The previous proof gives a definition of $f^*(U_S)$ which depends continuously on U_S. The definition can be extended to negative currents U such that dd^cU is bounded below by a negative closed current of bounded mass. We still have that $f^*(U)$ depends continuously on U.

PROPOSITION 5.2.6. *Under the hypothesis of Proposition* 5.2.4, *if R is a current in $\mathscr{C}_{k-p+1}$ with bounded* (*resp. continuous*) *super-potentials*, *then R is f_*-admissible and $\Lambda(R)$ is a current in $\mathscr{C}_{k-p+1}$ with bounded* (*resp. continuous*) *super-potentials.*

Proof. Assume that the super-potentials of R are bounded. It is clear that R is f_*-admissible. Proposition 5.1.12 implies that $\Lambda(R)$ admits a super-potential equal to $\lambda_p\lambda_{p-1}^{-1}\mathscr{U}_R\circ L+\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ on smooth $S\in\mathscr{C}_p$. The first term is bounded. By Proposition 5.2.4, it can be extended to a continuous function on $\mathscr{C}_p$ if R has continuous super-potentials. So, it is sufficient to prove that the super-potential $\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ of mean 0 of $\Lambda(\omega^{k-p+1})$ is continuous. Let U_S be the Green quasi-potential of S and c_S be its mean. Recall that $U_S-c_S\omega^{p-1}$ is a quasi-potential of mean 0 of S and c_S depends continuously on S. For smooth S, we have

$$\mathscr{U}_{\Lambda(\omega^{k-p+1})}(S)=\langle U_S-c_S\omega^{p-1},\Lambda(\omega^{k-p+1})\rangle=\lambda_{p-1}^{-1}\langle f^*(U_S)-c_Sf^*(\omega^{p-1}),\omega^{k-p+1}\rangle.$$

By Remark 5.2.5, the left-hand side can be extended continuously to S in $\mathscr{C}_p$. So, $\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ is continuous. □

If $g: \mathbb{P}^k \to \mathbb{P}^k$ is a dominant meromorphic map, the composition $g \circ f$ is well defined on a Zariski dense open set. We extend it as a meromorphic map by compactifying the graph. The *iterate of order* n of f is the map $f^n := f \circ ... \circ f$ (n times). The inverse of f^n is denoted by f^{-n}. It should be distinguished from $f^{-1} \circ ... \circ f^{-1}$. Define I_n, I'_n and Σ'_n as above for f^n instead of f. The following lemma will be useful in our dynamical study.

Lemma 5.2.7. *The following conditions are equivalent*:

(1) $\dim \Sigma' \leqslant k-p$;

(2) $\dim f^{-1}(A) \leqslant k-p$ *for every analytic subset* A *of* $\mathbb{P}^k$ *with* $\dim A \leqslant k-p$;

(3) $\dim \Sigma'_n \leqslant k-p$ *for every* $n \geqslant 1$.

Proof. It is easy to check that (1) implies (2) and (3) implies (1). Suppose now that (2) holds. We prove that (1) is satisfied. If not, we can find an irreductible analytic subset A of I', of minimal dimension, such that $\dim \pi_1(\pi_2^{-1}(A) \cap \widetilde{\Sigma}') > k-p$. The second condition in the lemma implies that $\dim A > k-p$. Let $\tilde{A}$ be an irreducible component of $\pi_2^{-1}(A) \cap \widetilde{\Sigma}'$ such that $A' := \pi_1(\tilde{A})$ has dimension $> k-p$. By definition of $\widetilde{\Sigma}'$, we have $\dim \tilde{A} \geqslant \dim A + 1 \geqslant k-p+2$.

Choose a dense Zariski open set Ω of $\tilde{A}$ such that $\pi_1: \Omega \to A'$ and $\pi_2: \Omega \to A$ locally are submersions. Denote these maps by τ_1 and τ_2. If H is a hypersurface of A then $\widetilde{H} := \tau_2^{-1}(H)$ is a hypersurface of Ω. It has dimension $\geqslant k-p+1$. The minimality of $\dim A$ implies that $\dim \tau_1(\widetilde{H}) \leqslant k-p < \dim \widetilde{H}$. Hence, the fibers of τ_1 are of positive dimension. Moreover, $\tau_1(\widetilde{H})$ has positive codimension in A'. Therefore, since $\widetilde{H}$ is a hypersurface in $\tilde{A}$, it should be a union of components of the fibers of τ_1. This holds for every H. Hence, the fibers of τ_2, which can be obtained as intersections of such $\widetilde{H}$, are unions of components of the fibers of τ_1. The intersection of a fiber of τ_1 and a fiber of τ_2 contains at most one point. We deduce that τ_1 is locally finite, which is a contradiction.

Now, assume that (1) and (2) hold. It remains to check that $\dim \Sigma'_n \leqslant k-p$ for $n \geqslant 2$. Using (2) inductively, we get that $f^{-1} \circ ... \circ f^{-1}(\Sigma')$ has dimension $\leqslant k-p$. Observe that Σ'_n is the union of the components of dimension $\geqslant 1$ in the fibers $f^{-n}(x)$. So, Σ'_n is contained in the union of $f^{-1} \circ ... \circ f^{-1}(\Sigma')$. This gives the result. □

5.3. Green super-functions for algebraically stable maps

Consider a dominant meromorphic map f on $\mathbb{P}^k$ of algebraic degree $d \geqslant 2$ and the associated sets I, I', I_n, I'_n, Σ' and Σ'_n as in §5.1 and §5.2. Some results in this section can be easily extended to the case of correspondences, in particular to f^{-1} instead of f. Let λ_p denote the intermediate degree of order p of f and $\lambda_p(f^n)$ the intermediate degree of order p of f^n. Note that $\lambda_1(f) = d$. We have the following elementary lemma; see [18] and [20] for a more general context.

LEMMA 5.3.1. *The sequence of intermediate degrees* $\lambda_p(f^n)$ *is sub-multiplicative, i.e.* $\lambda_p(f^{m+n})\leqslant\lambda_p(f^m)\lambda_p(f^n)$. *We also have* $\lambda_{p+q}(f^n)\leqslant\lambda_p(f^n)\lambda_q(f^n)$ *and* $\lambda_p(f^n)\leqslant d^{pn}$.

Proof. Observe that $(f^{m+n})^*(\omega^p)$ has no mass on analytic sets. Let S_j be smooth positive closed forms of mass $\lambda_p(f^n)$ converging locally uniformly to $(f^n)^*(\omega^p)$ on a Zariski open set. Then, the currents $(f^m)^*(S_j)$ are of mass $\lambda_p(f^m)\lambda_p(f^n)$ and converge to $(f^{m+n})^*(\omega^p)$ on a Zariski open set. If S is a limit of $(f^m)^*(S_j)$ in $\mathbb{P}^k$, it is of mass $\lambda_p(f^m)\lambda_p(f^n)$ and it satisfies $S\geqslant(f^{m+n})^*(\omega^p)$. Hence, $\|S\|\geqslant\|(f^{m+n})^*(\omega^p)\|$. The first inequality in the lemma follows.

In the same way, we approximate $(f^n)^*(\omega^p)$ and $(f^n)^*(\omega^q)$ locally uniformly on a suitable Zariski open set by smooth forms S_j and S_j'. If S is a limit current of $S_j\wedge S_j'$ in $\mathbb{P}^k$, it satisfies $S\geqslant(f^n)^*(\omega^{p+q})$. This implies that $\lambda_{p+q}(f^n)\leqslant\lambda_p(f^n)\lambda_q(f^n)$. For $p=1$, the first assertion in the lemma implies that $\lambda_1(f^p)\leqslant d^p$. Applying the second inequality inductively for $q=1$ gives $\lambda_p(f^n)\leqslant d^{pn}$. □

The previous lemma implies that the limit

$$d_p := \lim_{n\to\infty}\lambda_p(f^n)^{1/n} = \inf_n \lambda_p(f^n)^{1/n}$$

exists. It is called the *dynamical degree of order* p of f. We have $d_p\leqslant d^p$ for every p. The last dynamical degree d_k is also called the *topological degree* of f. It is equal to the number of points in a generic fiber of f, and we have $\lambda_k(f^n)=d_k^n$. In general, $\lambda_p(f^n)$ is the degree of $f^{-n}(H)$, where H is a generic projective plane of codimension p. So, $\lambda_p(f^n)$ is an integer. A result by Gromov [36, Theorem 1.6] implies that $p\mapsto\log\lambda_p(f^n)$ is concave in p. It follows that $p\mapsto\log d_p$ is also concave in p. If f is holomorphic, we have $d_p=\lambda_p=d^p$. If f is not holomorphic, it is easy to prove that $d_k<d^k$. Indeed, if a is the intersection of generic hyperplanes $H_1,...,H_k$, then $f^{-1}(a)\subset f^{-1}(H_1)\cap...\cap f^{-1}(H_k)\setminus I$. By Bézout's theorem, the last set has cardinal $\leqslant d^k-1$ since all the hypersurfaces $f^{-1}(H_j)$ contain I.

Definition 5.3.2. We say that f is *algebraically* p*-stable* if $\lambda_p(f^n)=\lambda_p^n$ for every $n\geqslant1$.

For such a map we have $d_p=\lambda_p$. For $p=1$, the algebraic 1-stability coincides with the notion introduced by Fornæss and the second author [44], i.e. no hypersurface is sent by f^n to I; see also [41] and Lemma 5.3.4 below.

LEMMA 5.3.3. *Assume that* $\dim\Sigma'\leqslant k-p$. *Then,* f *is algebraically* p*-stable if and only if* $(f^*)^n=(f^n)^*$ *on* $\mathscr{C}_p$.

Proof. Recall that, by Proposition 5.2.4 and Lemma 5.2.7, $(f^n)^*$ is well defined and is continuous on $\mathscr{C}_p$. If $(f^*)^n=(f^n)^*$ on $\mathscr{C}_p$, it is clear that

$$\lambda_p(f^n)=\|(f^n)^*(\omega^p)\|=\|(f^*)^n(\omega^p)\|=\lambda_p^n.$$

Hence, f is algebraically p-stable. Conversely, by continuity, it is enough to prove the identity $(f^*)^n = (f^n)^*$ on smooth forms S in $\mathscr{C}_p$. Observe that $(f^*)^n(S) = (f^n)^*(S)$ on a Zariski dense open set V such that V, $f(V)$, ..., $f^{n-1}(V)$ do not intersect I. As we observed after definition (5.1), since S is smooth, $(f^n)^*(S)$ has no mass on analytic sets. So, $(f^*)^n(S) \geqslant (f^n)^*(S)$. When f is algebraically p-stable, $(f^*)^n(S)$ and $(f^n)^*(S)$ have mass λ_p^n and $\lambda_p(f^n)$, which are equal. It follows that $(f^*)^n(S) = (f^n)^*(S)$. □

LEMMA 5.3.4. *Assume that* $\dim \Sigma' \leqslant k-p$. *For every analytic subset* A_0 *of* $\mathbb{P}^k$ *of dimension* $k-p$, *define by induction* $A_n := f(A_{n-1} \setminus I)$, *and assume that* A_n *is not contained in* I *for every* $n \geqslant 0$. *Then*, f *is algebraically* l-*stable for* $l \leqslant p$.

Proof. It is enough to show that $(f^*)^n(\omega^l) = (f^n)^*(\omega^l)$. We have seen that the identity holds outside $A := I \cup f^{-1}(I) \cup ... \cup (f^{-1})^n(I)$ and that $(f^*)^n(\omega^l) \geqslant (f^n)^*(\omega^l)$. The hypothesis implies that A is of dimension $< k-p$. Hence, $(f^*)^n(\omega^l)$ has no mass on A because $(f^*)^n(\omega^l)$ is of bidimension $(k-l, k-l)$. This completes the proof. □

PROPOSITION 5.3.5. *If* $\dim \Sigma' < k-p$, *then* f *is algebraically* l-*stable for* $l \leqslant p$. *In particular*, *if* f *is finite*, *i.e.* $I' = \varnothing$, *then* f *is algebraically* p-*stable for every* p.

Proof. When $\dim \Sigma' < k-p$, Proposition 5.2.6, applied to $l+1$ instead of p, implies that $(f_*)^n(\omega^{k-l})$ is well defined and has no mass on analytic sets. We deduce, as in Lemma 5.3.4, that $(f_*)^n(\omega^{k-l}) = (f^n)_*(\omega^{k-l})$ and that f is algebraically l-stable. □

Let f be a finite map. We have $f^{-n} = f^{-1} \circ ... \circ f^{-1}$, n times, therefore,

$$I_n = I \cup f^{-1}(I) \cup ... \cup f^{-n+1}(I).$$

So, the dimension of I_n is independent of n. It is not difficult to prove that $d_p = d^p$ for $p < k - \dim I$. Indeed, for such p, we have $f^*(\omega^p) = f^*(\omega) \wedge ... \wedge f^*(\omega)$, p times. The following proposition implies that generic maps in $\mathscr{M}_d(\mathbb{P}^k) \setminus \mathscr{H}_d(\mathbb{P}^k)$ are algebraically p-stable.

PROPOSITION 5.3.6. *The family of finite meromorphic maps of algebraic degree* $d \geqslant 2$ *on* $\mathbb{P}^k$, *whose dynamical degrees* d_s *satisfy* $d_1 < ... < d_k$, *contains a Zariski dense open set of* $\mathscr{M}_d(\mathbb{P}^k) \setminus \mathscr{H}_d(\mathbb{P}^k)$.

Proof. Let for simplicity $\mathscr{M} := \mathscr{M}_d(\mathbb{P}^k) \setminus \mathscr{H}_d(\mathbb{P}^k)$ and recall that this is an irreducible hypersurface of $\mathscr{M}_d(\mathbb{P}^k)$ [34]. We can easily check, using the coefficients of f, that the set $\mathscr{M}'$ of maps f in $\mathscr{M}$ which are finite and of (maximal) topological degree $d^k - 1$ is a Zariski open set in $\mathscr{M}$. For such a map, we have $d_{k-1} \leqslant d^{k-1} < d_k$, and since $p \mapsto \log d_p$ is concave, we obtain $d_1 < ... < d_k$. It remains to check that $\mathscr{M}'$ is not empty.

Consider the map defined on homogeneous coordinates by

$$f[z_0:\ldots:z_k]:=[z_0^{d-1}z_1:z_0^{d-1}z_2-z_1^d:\ldots:z_0^{d-1}z_k-z_{k-1}^d:z_0^{d-1}z_1-z_k^d].$$

The indeterminacy set is the common zero set of the components of f. So, I contains only the point $[1:0:\ldots:0]$. The map f is not holomorphic, hence $d_k\leqslant d^k-1$. On the other hand, if t is a root of order d^k-1 of the unity, $[1:t:t^d:\ldots:t^{d^{k-1}}]$ is sent to I by f. Hence, $d_k=d^k-1$. We show that f is *finite*, i.e. I' is empty. If not, there is $(a_0,\ldots,a_k)\neq 0$ in $\mathbb{C}^{k+1}$ such that the equations

$$z_0^{d-1}z_1=a_0,\quad z_0^{d-1}z_2-z_1^d=a_1,\quad \ldots,\quad z_0^{d-1}z_1-z_k^d=a_k$$

define an algebraic set of positive dimension. Consider a sequence of solutions $z^{(n)}=(z_0^{(n)},\ldots,z_k^{(n)})$ such that $|z^{(n)}|$ tend to infinity and that $z_j^{(n)}/|z^{(n)}|$ converge to some values x_j. We have $|x|=1$ and

$$x_0^{d-1}x_1=0,\quad x_0^{d-1}x_2-x_1^d=0,\quad \ldots,\quad x_0^{d-1}x_1-x_k^d=0.$$

Hence, $|x_0|=1$ and $x_1=\ldots=x_k=0$. Therefore, we may assume that $z_0^{(n)}$ tends to infinity and is strictly large than the other $z_j^{(n)}$. Extracting a subsequence allows one to assume that for some index $m\geqslant 1$, $z_m^{(n)}$ is the largest coordinate between $z_1^{(n)},\ldots,z_k^{(n)}$. The equation $z_0^{d-1}z_m-z_{m-1}^d=a_m$ implies that $z_m^{(n)}\to 0$. Hence, $z_j^{(n)}\to 0$ for every $j\geqslant 1$. On the other hand, we deduce from the considered equations that $z_k^d=a_0-a_k$. So, $a_k=a_0$ and $z_k^{(n)}=0$. Using the given equations and the fact that $z_j^{(n)}\to 0$, we obtain inductively that $z_j^{(n)}=0$ for $j\geqslant 1$ and then $a_j=0$ for every $j\geqslant 0$. This is a contradiction. □

THEOREM 5.3.7. *Let $f:\mathbb{P}^k\to\mathbb{P}^k$ be an algebraically p-stable meromorphic map of dynamical degrees d_s and let Σ' be as above. Assume that $\dim\Sigma'\leqslant k-p$ and $d_{p-1}<d_p$. Let S_n be currents in $\mathscr{C}_p$ and let $\mathscr{U}_{S_n}$ be super-potentials of S_n such that*

$$\|\mathscr{U}_{S_n}\|_\infty=o(d_{p-1}^{-n}d_p^n).$$

Then, $d_p^{-n}(f^n)^*(S_n)$ H-converge to an f^*-invariant current T in $\mathscr{C}_p$ which does not depend on S_n.

We call T the *Green (p,p)-current* associated with f. Set, for simplicity, $L:=d_p^{-1}f^*$ and $\Lambda:=d_{p-1}^{-1}f_*$. Proposition 5.3.5 implies that f is algebraically $(p-1)$-stable. Hence, $\lambda_{p-1}=d_{p-1}<d_p$. We have seen that $L:\mathscr{C}_p\to\mathscr{C}_p$ is continuous and $L^n=d_p^{-n}(f^n)^*$ on $\mathscr{C}_p$. It follows that the convex set of f^*-invariant currents S in $\mathscr{C}_p$ is not empty. Indeed, it contains all the limit values of the Cesàro means

$$\frac{1}{N}\sum_{j=0}^{N-1}L^j(\omega^p).$$

Let $\mathscr{C}^b_{k-p+1}$ denote the set of currents R in $\mathscr{C}_{k-p+1}$ with bounded super-potentials. By Proposition 5.2.6, the operator $\Lambda: \mathscr{C}^b_{k-p+1} \to \mathscr{C}^b_{k-p+1}$ is well defined. Consider a current S in $\mathscr{C}_p$, a super-potential $\mathscr{U}_S$ of S and a negative super-potential $\mathscr{U}_{L(\omega^p)}$ of $L(\omega^p)$.

LEMMA 5.3.8. *The current $L(S)$ admits a super-potential which is equal to*

$$d_{p-1}d_p^{-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}$$

on $\mathscr{C}^b_{k-p+1}$. If S_0 is an f^*-invariant current in $\mathscr{C}_p$, then it admits a super-potential $\mathscr{U}_{S_0}$ satisfying $\mathscr{U}_{S_0}=d_{p-1}d_p^{-1}\mathscr{U}_{S_0}\circ\Lambda+\mathscr{U}_{L(\omega^p)}$ on $\mathscr{C}^b_{k-p+1}$.

Proof. We prove the first assertion. By Proposition 5.1.8, we may assume that S is smooth. Moreover, there is a super-potential $\mathscr{U}_{L(S)}$ of $L(S)$ which is equal to $d_{p-1}d_p^{-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}$ on smooth forms in $\mathscr{C}_{k-p+1}$. Consider a current R in $\mathscr{C}^b_{k-p+1}$ and smooth forms R_n in $\mathscr{C}_{k-p+1}$ H-converging to R. We have $\mathscr{U}_{L(S)}(R_n)\to\mathscr{U}_{L(S)}(R)$ and $\mathscr{U}_{L(\omega^p)}(R_n)\to\mathscr{U}_{L(\omega^p)}(R)$. By Proposition 5.1.12, $\Lambda(R_n)\to\Lambda(R)$. Since $\mathscr{U}_S$ is continuous, we deduce that $\mathscr{U}_S(\Lambda(R_n))\to\mathscr{U}_S(\Lambda(R))$. Therefore, $\mathscr{U}_{L(S)}=d_{p-1}d_p^{-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}$ at R.

For the second assertion, if $\mathscr{U}$ is a super-potential of S_0, since $L(S_0)=S_0$, the first assertion implies that $\mathscr{U}=d_{p-1}d_p^{-1}\mathscr{U}\circ\Lambda+\mathscr{U}_{L(\omega^p)}+c$ on $\mathscr{C}^b_{k-p+1}$, where c is a constant. The super-potential $\mathscr{U}_{S_0}:=\mathscr{U}-cd_p(d_p-d_{p-1})^{-1}$ satisfies the lemma. Here we use the property that $d_p\neq d_{p-1}$. □

Proof of Theorem 5.3.7. Replacing $\mathscr{U}_{S_n}$ by $\mathscr{U}_{S_n}+\|\mathscr{U}_{S_n}\|_\infty$ allows one to assume that $\mathscr{U}_{S_n}$ are positive. We apply inductively Lemma 5.3.8 for $S=L^j(S_n)$. We obtain that $L^n(S_n)$ admits a super-potential $\mathscr{U}_{L^n(S_n)}$ satisfying

$$\mathscr{U}_{L^n(S_n)}=d_{p-1}^n d_p^{-n}\mathscr{U}_{S_n}\circ\Lambda^n+\sum_{j=0}^{n-1}d_{p-1}^j d_p^{-j}\mathscr{U}_{L(\omega^p)}\circ\Lambda^j$$

on $\mathscr{C}^b_{k-p+1}$. By hypothesis, the first term converges to 0. Since $\mathscr{U}_{L(\omega^p)}$ is negative, the second term decreases to

$$\mathscr{U}:=\sum_{j=0}^{\infty}d_{p-1}^j d_p^{-j}\mathscr{U}_{L(\omega^p)}\circ\Lambda^j.$$

Hence, $\mathscr{U}_{L^n(S_n)}$ converge pointwise in $\mathscr{C}^b_{k-p+1}$ to $\mathscr{U}$. We show that $\mathscr{U}$ is not identically $-\infty$. Let S_0 be an f^*-invariant current in $\mathscr{C}_p$ and $\mathscr{U}_{S_0}$ be a super-potential as in Lemma 5.3.8. We have

$$\mathscr{U}_{S_0}=d_{p-1}d_p^{-1}\mathscr{U}_{S_0}\circ\Lambda+\mathscr{U}_{L(\omega^p)}$$

on $\mathscr{C}^b_{k-p+1}$. Iterating this identity gives

$$\mathscr{U}_{S_0}=d_{p-1}^n d_p^{-n}\mathscr{U}_{S_0}\circ\Lambda^n+\sum_{j=0}^{n-1}d_{p-1}^j d_p^{-j}\mathscr{U}_{L(\omega^p)}\circ\Lambda^j.$$

Since $\mathscr{U}_{S_0}$ is bounded from above and since $d_{p-1}<d_p$, letting $n\to\infty$ gives $\mathscr{U}\geqslant\mathscr{U}_{S_0}$. So, $\mathscr{U}$ is not identically $-\infty$.

We deduce from Propositions 3.1.9 and 3.2.6 that $L^n(S_n)$ converge to a current T which admits a super-potential equal to $\mathscr{U}$ on $\mathscr{C}^b_{k-p+1}$. The fact that $\mathscr{U}$ does not depend on S_n implies that T is also independent of S_n. Because $\mathscr{U}_{S_n}$ are positive, the convergence is in the Hartogs sense. We have

$$L(T)=L\Big(\lim_{n\to\infty} L^n(S_n)\Big)=\lim_{n\to\infty} L^{n+1}(S_n)=T.$$

Hence, T is f^*-invariant. □

THEOREM 5.3.9. *Let f be as in Theorem 5.3.7. Then, the Green (p,p)-current T of f is the most diffuse current in $\mathscr{C}_p$ which is f^*-invariant. In particular, T is extremal in the convex set of f^*-invariant currents in $\mathscr{C}_p$.*

Proof. We have seen in the proof of Theorem 5.3.7 that T admits a super-potential $\mathscr{U}_T$ which is equal to $\mathscr{U}$ on $\mathscr{C}^b_{k-p+1}$. It follows that

$$\mathscr{U}_T=d_{p-1}d_p^{-1}\mathscr{U}_T\circ\Lambda+\mathscr{U}_{L(\omega^p)}$$

on $\mathscr{C}^b_{k-p+1}$. It is clear that $\mathscr{U}_T$ is the unique super-potential of T satisfying this identity. Let S_0 and $\mathscr{U}_{S_0}$ be as above. We have seen that $\mathscr{U}_T\geqslant\mathscr{U}_{S_0}$ on $\mathscr{C}^b_{k-p+1}$. By Corollary 3.1.7, this inequality holds on $\mathscr{C}_{k-p+1}$. Hence, T is the most diffuse current in $\mathscr{C}_p$ which is f^*-invariant.

We now prove that T is extremal among f^*-invariant currents in $\mathscr{C}_p$. Assume that $T=\frac{1}{2}(T_1+T_2)$ with T_j in $\mathscr{C}_p$ invariant under f^*. By Lemma 5.3.8, the T_j admit super-potentials $\mathscr{U}_{T_j}$ such that

$$\mathscr{U}_{T_j}=d_{p-1}d_p^{-1}\mathscr{U}_{T_j}\circ\Lambda+\mathscr{U}_{L(\omega^p)}$$

on $\mathscr{C}^b_{k-p+1}$. This and the uniqueness of $\mathscr{U}_T$ imply that $\mathscr{U}_T=\frac{1}{2}(\mathscr{U}_{T_1}+\mathscr{U}_{T_2})$. On the other hand, we have $\mathscr{U}_T\geqslant\mathscr{U}_{T_j}$. Hence, $\mathscr{U}_T=\mathscr{U}_{T_j}$ and $T_j=T$. This completes the proof. □

THEOREM 5.3.10. *Let $f:\mathbb{P}^k\to\mathbb{P}^k$ be a dominant meromorphic map of dynamical degrees d_s and Σ' be defined as above. Assume that $\dim\Sigma'\leqslant k-p$ and that $d_p<d_{p-1}$. Let R_n be currents in $\mathscr{C}_{k-p+1}$ and $\mathscr{U}_{R_n}$ be super-potentials of R_n such that*

$$\|\mathscr{U}_{R_n}\|_\infty=o((d_p+\varepsilon)^{-n}d_{p-1}^n)$$

for some constant $\varepsilon>0$. Then, $d_{p-1}^{-n}(f^n)_*(R_n)$ H-converge to an f_*-invariant current T' in $\mathscr{C}_{k-p+1}$ which does not depend on R_n and has continuous super-potentials.

Proof. Proposition 5.3.5 implies that f is algebraically $(p-1)$-stable. It follows that $\lambda_{p-1}=d_{p-1}$. By Proposition 5.2.6, the operator $\Lambda:\mathscr{C}^b_{k-p+1}\to\mathscr{C}^b_{k-p+1}$ is well defined. By Proposition 5.2.4, $L:\mathscr{C}_p\to\mathscr{C}_p$ is well defined and continuous, but we do not necessarily have $L^n=d_p^{-n}(f^n)^*$. Replacing f by an iterate f^N, allows one to assume that $\lambda_p<d_{p-1}$ and $\|\mathscr{U}_{R_n}\|_\infty=o(\lambda_p^{-n}d^n_{p-1})$. We may also assume that $\mathscr{U}_{R_n}$ are positive. Let $\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ be a negative super-potential of $\Lambda(\omega^{k-p+1})$. By Proposition 5.2.6, $\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ is continuous. Proposition 5.1.12 implies that $\Lambda^n(R_n)$ admits a super-potential which equals

$$\lambda_p^n d_{p-1}^{-n}\mathscr{U}_{R_n}\circ L^n+\sum_{j=0}^{n-1}\lambda_p^j d_{p-1}^{-j}\mathscr{U}_{\Lambda(\omega^{k-p+1})}\circ L^j$$

on smooth forms in $\mathscr{C}_p$. Letting $n\to\infty$, the first term tends to 0, the second term decreases to a continuous function on $\mathscr{C}_p$, since $\mathscr{U}_{\Lambda(\omega^{k-p+1})}$ and L are continuous and $\lambda_p<d_{p-1}$. This function does not depend on R_n. We deduce that $\Lambda^n(R_n)$ converge to a current T' which is independent of R_n. The convergence is in the Hartogs sense because $\mathscr{U}_{R_n}$ are positive. Moreover, T' admits a super-potential $\mathscr{U}_{T'}$ such that

$$\mathscr{U}_{T'}:=\sum_{j=0}^{\infty}\lambda_p^j d_{p-1}^{-j}\mathscr{U}_{\Lambda(\omega^{k-p+1})}\circ L^j$$

on smooth forms in $\mathscr{C}_p$. We have seen that the right-hand side defines a continuous function on $\mathscr{C}_p$. Hence, $\mathscr{U}_{T'}$ is continuous and the last identity holds on $\mathscr{C}_p$. It follows from the convergence of $\Lambda^n(R_n)$ that T' is f_*-invariant. □

THEOREM 5.3.11. *Let f and T' be as in Theorem* 5.3.10. *Then*, *T' is the only f_*-invariant current in $\mathscr{C}_{k-p+1}$ which has bounded super-potentials. Moreover*, *it is extremal in the convex set of f_*-invariant currents in $\mathscr{C}_{k-p+1}$.*

Proof. Let R be a current in $\mathscr{C}_{k-p+1}$ with bounded super-potentials. Theorem 5.3.10 implies that $\Lambda^n(R)\to T'$. So, if R is f_*-invariant, then $R=T'$. This implies the first assertion. We deduce from this and Proposition 3.3.4 the extremality of T'. □

5.4. Equidistribution problem for endomorphisms

Consider a holomorphic map $f:\mathbb{P}^k\to\mathbb{P}^k$ of algebraic degree $d\geqslant 2$. Recall that f^* acts continuously on positive closed currents of any bidegree [23], [40]; see also §5.1 and §5.2. It is well known that $d^{-n}(f^n)^*(\omega)$ converge to a positive closed $(1,1)$-current T with Hölder continuous quasi-potentials. One deduces from the intersection theory of currents that $d^{-pn}(f^n)^*(\omega^p)$ converge to T^p; see [29] and [44] for the first stages of the theory. The current T^p is the *Green current of order p* and its super-potentials are the

Green super-functions of order p of f. In the following result, we give a new construction and new properties of T^p.

THEOREM 5.4.1. *Let* $f\colon \mathbb{P}^k\to\mathbb{P}^k$ *be a holomorphic map of algebraic degree* $d\geqslant 2$. *Then, the Green super-potentials of* f *are Hölder continuous. Moreover,* T^p *is extremal in the convex set of* f^**-invariant currents* S *in* $\mathscr{C}_p$. *If* S_n *are currents in* $\mathscr{C}_p$ *of super-potentials* $\mathscr{U}_{S_n}$ *such that* $\|\mathscr{U}_{S_n}\|_\infty = o(d^n)$, *then* $d^{-pn}(f^n)^*(S_n)$ *H-converge to* T^p.

We will see that the proof also gives that $(f,R)\mapsto\mathscr{U}_{T^p}(R)$ is locally Hölder continuous on $\mathscr{H}_d(\mathbb{P}^k)\times\mathscr{C}_{k-p+1}$. The following lemma is a special case of [19, Proposition 2.4]. For the reader's convenience, we here give the proof.

LEMMA 5.4.2. *Let* K *be a metric space with finite diameter and* $\Lambda\colon K\to K$ *be a Lipschitz map*: $\|\Lambda(a)-\Lambda(b)\|\leqslant A\|a-b\|$ *with* $A>0$. *Let* $\mathscr{U}$ *be an* α*-Hölder continuous function on* K. *Then,* $\sum_{n=0}^\infty d^{-n}\mathscr{U}\circ\Lambda^n$ *converges pointwise to a function which is* β*-Hölder continuous on* K *for every* β *such that* $\beta<\alpha$ *and* $\beta\leqslant\log d/\log A$.

Proof. Here, $\|a-b\|$ denotes the distance between two points a and b in K. Since K has finite diameter (it is enough to assume that $\mathscr{U}$ is bounded), it is sufficient to consider $\|a-b\|\ll 1$. By hypothesis, there is a constant $A'>0$ such that $|\mathscr{U}(a)-\mathscr{U}(b)|\leqslant A'\|a-b\|^\alpha$. Define $A'':=\|\mathscr{U}\|_\infty$. Since K has finite diameter, A'' is finite. If N is an integer, we have

$$\begin{aligned}
\Big|\sum_{n=0}^\infty d^{-n}\mathscr{U}\circ\Lambda^n(a)-\sum_{n=0}^\infty d^{-n}\mathscr{U}\circ\Lambda^n(b)\Big| & \\
&\leqslant \sum_{n=0}^N d^{-n}|\mathscr{U}\circ\Lambda^n(a)-\mathscr{U}\circ\Lambda^n(b)| + \sum_{n=N+1}^\infty d^{-n}|\mathscr{U}\circ\Lambda^n(a)-\mathscr{U}\circ\Lambda^n(b)| \\
&\leqslant A'\sum_{n=0}^N d^{-n}\|\Lambda^n(a)-\Lambda^n(b)\|^\alpha + 2A''\sum_{n=N+1}^\infty d^{-n} \\
&\lesssim \|a-b\|^\alpha\sum_{n=0}^N d^{-n}A^{n\alpha}+d^{-N}.
\end{aligned}$$

If $A^\alpha\leqslant d$, the last sum is of order at most equal to $N\|a-b\|^\alpha+d^{-N}$. For a given β, $0<\beta<\alpha$, choose $N\simeq-\beta\log\|a-b\|/\log d$. So, the last expression is $\lesssim\|a-b\|^\beta$. In this case, the function is β-Hölder continuous for every $0<\beta<\alpha$. When $A^\alpha>d$, the sum is $\lesssim d^{-N}A^{N\alpha}\|a-b\|^\alpha+d^{-N}$. If $N\simeq-\log\|a-b\|/\log A$, the last expression is $\lesssim\|a-b\|^\beta$ with $\beta:=\log d/\log A$. Therefore, the function is β-Hölder continuous. $\square$

Define $L:=d^{-p}f^*$ and $\Lambda:=d^{-p+1}f_*$. Recall that $L\colon\mathscr{C}_p\to\mathscr{C}_p$ and $\Lambda\colon\mathscr{C}_{k-p+1}\to\mathscr{C}_{k-p+1}$ are well defined and continuous.

LEMMA 5.4.3. *The operator* Λ *is Lipschitz with respect to the distance* dist_α *on* $\mathscr{C}_{k-p+1}$ *for* $\alpha>0$.

Proof. If Φ is a $\mathscr{C}^\alpha$ test $(p-1,p-1)$-form such that $\|\Phi\|_{\mathscr{C}^\alpha}\leqslant 1$, it is clear that $\|f^*(\Phi)\|_{\mathscr{C}^\alpha}\leqslant c_\alpha$ for a constant $c_\alpha>0$ independent of Φ. If R and R' are currents in $\mathscr{C}_{k-p+1}$, we have

$$|\langle\Lambda(R)-\Lambda(R'),\Phi\rangle| = |\langle R-R', d^{-p+1}f^*(\Phi)\rangle| \leqslant c_\alpha\,\mathrm{dist}_\alpha(R,R').$$

The lemma follows. Observe that the estimates are locally uniform with respect to $f\in\mathscr{H}_d(\mathbb{P}^k)$. □

Proof of Theorem 5.4.1. Theorems 5.3.7 and 5.3.9 imply that $L^n(S_n)$ H-converge to a current T_p which does not depend on S_n and is extremal among f^*-invariant currents in $\mathscr{C}_p$. For $S_n=\omega^p$ and $\mathscr{U}_{S_n}=0$, the computation in those theorems shows that T_p admits a super-potential $\mathscr{U}_{T_p}$ satisfying

$$\mathscr{U}_{T_p}=\sum_{j=0}^{\infty} d^{-j}\,\mathscr{U}_{L(\omega^p)}\circ\Lambda^j$$

on smooth forms in $\mathscr{C}_{k-p+1}$. Since $L(\omega^p)$ is smooth, $\mathscr{U}_{L(\omega^p)}$ is Lipschitz. By Lemmas 5.4.2 and 5.4.3, the latter sum defines a Hölder continuous function on $\mathscr{C}_{k-p+1}$. It follows that the last identity holds everywhere on $\mathscr{C}_{k-p+1}$. So, T_p has Hölder continuous super-potentials.

Let T denote the first Green current of f. So, T is the limit of $d^{-n}(f^n)^*(\omega)$ in the Hartogs sense. By Theorem 4.2.10, $d^{-pn}(f^n)^*(\omega^p)$ converge to T^p. Hence, $T_p=T^p$. □

Here is one of our main applications of super-potentials.

THEOREM 5.4.4. *There is a Zariski dense open set* $\mathscr{H}_d^*(\mathbb{P}^k)$ *in* $\mathscr{H}_d(\mathbb{P}^k)$ *such that, if* f *is in* $\mathscr{H}_d^*(\mathbb{P}^k)$, *then* $d^{-pn}(f^n)^*(S)\to T^p$ *uniformly with respect to* $S\in\mathscr{C}_p$. *In particular, for* f *in* $\mathscr{H}_d^*(\mathbb{P}^k)$, T^p *is the unique current in* $\mathscr{C}_p$ *which is* f^**-invariant.*

The open set $\mathscr{H}_d^*(\mathbb{P}^k)$ is given by the following lemma.

LEMMA 5.4.5. *There is a Zariski dense open set* $\mathscr{H}_d^*(\mathbb{P}^k)$ *in* $\mathscr{H}_d(\mathbb{P}^k)$ *and an integer* $N\geqslant 1$ *such that, if* f *is in* $\mathscr{H}_d^*(\mathbb{P}^k)$ *and if* δ *denotes the maximal multiplicity of* f^N *at a point in* $\mathbb{P}^k$, *then* $(20k^2\delta)^{8k}<d^N$.

Proof. Fix an N large enough. Observe that the set $\mathscr{H}_d^*(\mathbb{P}^k)$ of f satisfying the previous inequality is a Zariski open set in $\mathscr{H}_d(\mathbb{P}^k)$. We only have to construct such a map f in order to obtain the density of $\mathscr{H}_d^*(\mathbb{P}^k)$. Choose a rational map $h\colon\mathbb{P}^1\to\mathbb{P}^1$ of degree d whose critical points are simple and have disjoint infinite orbits. Observe that the multiplicity of h^N at every point is at most equal to 2. We construct the map f using an idea of Ueda. Let σ_k denote the group of permutations of $\{1, ..., k\}$. It acts in a canonical way on $\mathbb{P}^1\times...\times\mathbb{P}^1$, k times. Using the symmetric functions on $(x_1, ..., x_k)\in\mathbb{P}^1\times...\times\mathbb{P}^1$, one shows that $\mathbb{P}^1\times...\times\mathbb{P}^1$ divided by σ_k is isomorphic to $\mathbb{P}^k$. Let $\pi:\mathbb{P}^1\times...\times\mathbb{P}^1\to\mathbb{P}^k$ denote the canonical map. If $\hat{f}$ is the endomorphism of $\mathbb{P}^1\times...\times\mathbb{P}^1$, k times, defined by $\hat{f}(x_1, ..., x_k):=(h(x_1), ..., h(x_k))$, then there is a holomorphic map $f:\mathbb{P}^k\to\mathbb{P}^k$ of algebraic degree d such that $f\circ\pi=\pi\circ\hat{f}$. We also have $f^N\circ\pi=\pi\circ\hat{f}^N$. Consider a point x in $\mathbb{P}^k$ and a point $\hat{x}$ in $\pi^{-1}(x)$. The multiplicity of $\hat{f}^N$ at $\hat{x}$ is at most equal to 2^k. It follows that the multiplicity of f^N at x is at most equal to $2^k k!$, since π has degree $k!$. Therefore, f satisfies the desired inequality if N is large enough. □

Replacing f by f^N, one may assume that f satisfies the lemma for $N=1$. Let δ be the maximal multiplicity of f at a point in $\mathbb{P}^k$. We introduce some notation. We call *dynamical super-potential* of S the function $\mathscr{V}_S$ defined by

$$\mathscr{V}_S := \mathscr{U}_S - \mathscr{U}_{T^p} - c_S, \quad \text{where} \quad c_S := \mathscr{U}_S(T^{k-p+1}) - \mathscr{U}_{T^p}(T^{k-p+1}),$$

and $\mathscr{U}_S$ and $\mathscr{U}_{T^p}$ are the super-potentials of mean 0 of S and T^p. We also call *dynamical Green quasi-potential* of S the form

$$V_S := U_S - U_{T^p} - (m_S - m_{T^p} + c_S)\omega^{p-1},$$

where U_S and U_{T^p} are the Green quasi-potentials of S and T^p, and m_S and m_{T^p} their means.

LEMMA 5.4.6. *We have* $\mathscr{V}_S(T^{k-p+1})=0$, $\mathscr{V}_S(R)=\langle V_S, R\rangle$ *for smooth* R *in* $\mathscr{C}_{k-p+1}$, *and* $\mathscr{V}_{L(S)}=d^{-1}\mathscr{V}_S\circ\Lambda$ *on* $\mathscr{C}_{k-p+1}$. *Moreover*, $\mathscr{U}_S-\mathscr{V}_S$ *is bounded by a constant independent of* S.

Proof. It is clear that $\mathscr{V}_S(T^{k-p+1})=0$. Since T^{k-p+1} has bounded super-potentials, c_S is bounded by a constant independent of S. Hence, as $\mathscr{U}_{T^p}$ is bounded, $\mathscr{U}_S-\mathscr{V}_S$ is bounded by a constant independent of S. For smooth R, we have

$$\langle V_S, R\rangle = (\langle U_S, R\rangle - m_S) - (\langle U_{T^p}, R\rangle - m_{T^p}) - c_S = \mathscr{U}_S(R) - \mathscr{U}_{T^p}(R) - c_S = \mathscr{V}_S(R).$$

It remains to prove that $\mathscr{V}_{L(S)}=d^{-1}\mathscr{V}_S\circ\Lambda$. Since $\Lambda(T^{k-p+1})=T^{k-p+1}$, we have $\mathscr{V}_{L(S)}=d^{-1}\mathscr{V}_S\circ\Lambda=0$ at T^{k-p+1}. Hence, we only have to show that $\mathscr{V}_{L(S)}-d^{-1}\mathscr{V}_S\circ\Lambda$ is constant. By Proposition 5.1.8, we have

$$\mathscr{U}_{L(S)} = d^{-1}\mathscr{U}_S\circ\Lambda + \mathscr{U}_{L(\omega^p)} + \text{const}$$

and, since $L(T^p)=T^p$, this implies that

$$\mathscr{U}_{T^p}=d^{-1}\mathscr{U}_{T^p}\circ\Lambda+\mathscr{U}_{L(\omega^p)}+\text{const}.$$

It follows that

$$\mathscr{V}_{L(S)}=d^{-1}\mathscr{U}_S\circ\Lambda-d^{-1}\mathscr{U}_{T^p}\circ\Lambda+\text{const}.$$

So, $\mathscr{V}_{L(S)}-d^{-1}\mathscr{V}_S\circ\Lambda$ is constant. □

LEMMA 5.4.7. *Let W_ε be the ε-neighbourhood of the set P of critical values of f and W_ε^c be the complement of W_ε with $0<\varepsilon\ll 1$. There is a constant $c>0$ independent of ε such that, for smooth R in $\mathscr{C}_{k-p+1}$ and for $0<\varepsilon'\ll\varepsilon$, we have*

$$\|\Lambda(R)_{\varepsilon'}-\Lambda(R)\|_{\infty,W_\varepsilon^c}\leqslant c\|R\|_{\mathscr{C}^1}\varepsilon^{-5k}\varepsilon',$$

where $\Lambda(R)_{\varepsilon'}$ is the ε'-regularization of $\Lambda(R)$; see Remark 2.1.7 for the terminology.

Proof. Let B_ε be the ball of radius ε centered at a given point a of W_ε^c. Since B_ε does not intersect P, f admits d^k inverse branches on B_ε. More precisely, there are d^k injective holomorphic maps $g_j: B_\varepsilon\to\mathbb{P}^k$ such that $f\circ g_j=\text{id}$ on B_ε. Observe that, since f is finite, when the diameter of a ball B tends to 0, the connected components of $f^{-1}(B)$ tend to single points. So, $g_j(B_\varepsilon)$ have small size. Using Cauchy's integral, it is easy to check that all the derivatives of order n of g_j on $B_{\varepsilon/2}$ are $\lesssim\varepsilon^{-n}$. On B_ε, we have

$$\Lambda(R)=d^{-p+1}\sum_{j=1}^{d^k}g_j^*(R).$$

For fixed local real coordinates $(x_1,...,x_{2k})$, R is a combination with smooth coefficients of $dx_{j_1}\wedge...\wedge dx_{j_{2k-2p+2}}$. Hence, the estimate on the derivatives of g_j implies that

$$\|g_j^*(R)\|_{\mathscr{C}^1(B_{\varepsilon/2})}\lesssim\|R\|_{\mathscr{C}^1}\varepsilon^{-2k+2p-3}\lesssim\|R\|_{\mathscr{C}^1}\varepsilon^{-5k}.$$

It follows that

$$\|\Lambda(R)\|_{\mathscr{C}^1(W_{\varepsilon/2}^c)}\lesssim\|R\|_{\mathscr{C}^1}\varepsilon^{-5k}.$$

Let τ be an automorphism of $\mathbb{P}^k$ close enough to the identity. Lemma 2.1.8 implies that

$$\|\tau_*(\Lambda(R))-\Lambda(R)\|_{\infty,W_\varepsilon^c}\lesssim\|R\|_{\mathscr{C}^1}\varepsilon^{-5k}\operatorname{dist}(\tau,\text{id}).$$

We then deduce the desired estimate from the definition of $\Lambda(R)_{\varepsilon'}$. □

Lemma 5.4.8. *The quasi-potentials of $f_*(\omega)$ are δ^{-1}-Hölder continuous.*

Proof. Let B be a small ball in $\mathbb{P}^k$. The inverse image $f^{-1}(B)$ of B is a union of small open sets. Hence, there is a smooth psh function u on $f^{-1}(B)$ such that $\omega = dd^c u$ there. Define the function v on B by

$$v(z) := \sum_{w \in f^{-1}(z)} u(w),$$

where the points in $f^{-1}(z)$ are repeated according to their multiplicity. It is clear that v is continuous and $dd^c v = f_*(\omega)$. We only have to show that v is δ^{-1}-Hölder continuous. Recall that the multiplicity of f at every point is $\leqslant \delta$. By Łojasiewicz's inequality [24, Lemma 4.3], we can write, for z and z' in B,

$$f^{-1}(z) = \{w_1, ..., w_{d^k}\} \quad \text{and} \quad f^{-1}(z') = \{w'_1, ..., w'_{d^k}\},$$

so that $\mathrm{dist}_{\mathrm{FS}}(w_j, w'_j) \lesssim \mathrm{dist}_{\mathrm{FS}}(z, z')^{\delta^{-1}}$. Hence,

$$|v(z) - v(z')| \leqslant d^k \|u\|_{\mathscr{C}^1} \max \mathrm{dist}_{\mathrm{FS}}(w_j, w'_j) \lesssim \mathrm{dist}_{\mathrm{FS}}(z, z')^{\delta^{-1}}.$$

This implies the lemma. □

Lemma 5.4.9. *Let P denote the set of critical values of f as above. If R is smooth, then $\mathscr{V}_S(\Lambda(R)) = \langle V_S, \Lambda(R) \rangle_{\mathbb{P}^k \setminus P}$.*

Proof. Observe that $\Lambda(R)$ is smooth outside P. We will show that

$$\mathscr{U}_S(\Lambda(R)) = \langle U_S, \Lambda(R) \rangle_{\mathbb{P}^k \setminus P} - m_S.$$

This and the same identity for T^p imply the result. Since $R \leqslant c\omega^{k-p+1}$ for a constant $c > 0$, we have

$$\Lambda(R) \leqslant cd^{1-p} f_*(\omega^{k-p+1}) \leqslant cd^{1-p} [f_*(\omega)]^{k-p+1}.$$

Lemma 5.4.8 and Proposition 2.3.6 imply that, when $\theta \to 0$, $\langle U_{S_\theta}, \Lambda(R) \rangle_{\mathbb{P}^k \setminus P}$ converge to $\langle U_S, \Lambda(R) \rangle_{\mathbb{P}^k \setminus P}$. So, it is enough to consider the case where S is smooth. In this case, U_S is smooth. Since $\Lambda(R)$ has no mass on P, we have

$$\langle U_S, \Lambda(R) \rangle_{\mathbb{P}^k \setminus P} - m_S = \langle U_S, \Lambda(R) \rangle - m_S = \mathscr{U}_S(\Lambda(R)).$$

This completes the proof. □

PROPOSITION 5.4.10. *For every smooth form* R *in* $\mathscr{C}_{k-p+1}$, $d^{-4n/5}\mathscr{V}_S(\Lambda^n(R))$ *converge to* 0 *uniformly with respect to* S. *In particular*, *we have* $|\log\operatorname{cap}(\Lambda^n(R))|=o(d^{4n/5})$.

Fix an integer n large enough and define $\varepsilon:=d^{-n}$. In what follows, the symbols $\lesssim$ and $\gtrsim$ mean inequalities up to multiplicative constants which are independent of n and j. Observe that we may assume S to be smooth. Define $\varepsilon_j:=\varepsilon^{(20k^2\delta)^{6kj}}$ for $0\leqslant j\leqslant n$. The main point here is that $\varepsilon_j/\varepsilon_{j-1}$ has to be small. Define also by induction $R_0:=R$ and $R_j:=\Lambda(R_{j-1})_{\varepsilon_j}$, the ε_j-regularization of $\Lambda(R_{j-1})$; see Remark 2.1.7 for the terminology. Let V_j be the Green dynamical quasi-potentials of $L^j(S)$. They are forms with bounded mass.

LEMMA 5.4.11. *We have* $d^{-j}|\mathscr{V}_S(R_j)|\lesssim(-\log\varepsilon)d^{-j/4}$.

Proof. By Proposition 2.1.6, we have

$$\|R_j\|_\infty\lesssim\varepsilon_j^{-2k^2-4k}\lesssim\varepsilon_j^{-4k^2}.$$

Hence, Proposition 3.2.10 applied to $K=\mathbb{P}^k$ implies that

$$d^{-j}|\mathscr{V}_S(R_j)|\lesssim d^{-j}(-\log\varepsilon_j)=d^{-j}(-\log\varepsilon)(20k^2\delta)^{6kj}.$$

Lemma 5.4.5 implies the result. Recall that we suppose $N=1$. □

LEMMA 5.4.12. *We have* $\langle V_{n-j},\Lambda(R_{j-1})-R_j\rangle_{\mathbb{P}^k\setminus P}\gtrsim-\varepsilon^j$.

Proof. An analogous inequality for $\pm U_{T^p}$ instead of V_{n-j} is easily deduced from the Hölder continuity of the Green super-functions, since $\operatorname{dist}_1(\Lambda(R_{j-1}),R_j)\lesssim\varepsilon_j$. Observe also that $V'_{n-j}:=V_{n-j}+U_{T^p}-c\omega^p$ is negative for some universal constant $c>0$. Since $\Lambda(R_{j-1})$ and R_j have the same mass, we also have

$$\langle V'_{n-j},\Lambda(R_{j-1})-R_j\rangle_{\mathbb{P}^k\setminus P}=\langle V_{n-j}+U_{T^p},\Lambda(R_{j-1})-R_j\rangle_{\mathbb{P}^k\setminus P}.$$

Proposition 2.1.6 implies that

$$\|R_{j-1}\|_{\mathscr{C}^1}\lesssim\varepsilon_{j-1}^{-2k^2-4k-1}\lesssim\varepsilon_{j-1}^{-5k^2}.$$

Let W_j denote the $\varepsilon_j^{(10k)^{-1}}$-neighbourhood of P and W_j^c its complement. We obtain from Lemma 5.4.7 applied to $R:=R_{j-1}$ that

$$\|\Lambda(R_{j-1})-R_j\|_{\infty,W_j^c}\lesssim\|R_{j-1}\|_{\mathscr{C}^1}[\varepsilon_j^{(10k)^{-1}}]^{-5k}\varepsilon_j\lesssim\varepsilon_{j-1}^{-5k^2}\varepsilon_j^{1/2}\lesssim\varepsilon^j.$$

As V'_{n-j} has bounded mass, we deduce that

$$|\langle V'_{n-j},\Lambda(R_{j-1})-R_j\rangle_{W_j^c}|\lesssim\varepsilon^j.$$

It remains to prove that

$$\langle V'_{n-j}, \Lambda(R_{j-1}) - R_j \rangle_{W_j \setminus P} \geqslant -\varepsilon^j.$$

Since V'_{n-j} is negative and R_j is positive, it is enough to bound the integral

$$\langle V'_{n-j}, \Lambda(R_{j-1}) \rangle_{W_j \setminus P}.$$

By Proposition 2.1.6, we have

$$R_{j-1} \lesssim \|R_{j-1}\|_\infty \omega^{k-p+1} \lesssim \varepsilon_{j-1}^{-4k^2} \omega^{k-p+1}.$$

It follows that

$$\Lambda(R_{j-1}) \lesssim \varepsilon_{j-1}^{-4k^2} f_*(\omega^{k-p+1}) \lesssim \varepsilon_{j-1}^{-4k^2} [f_*(\omega)]^{k-p+1}.$$

Lemma 5.4.8 and Proposition 2.3.6 then imply that

$$|\langle V'_{n-j}, \Lambda(R_{j-1}) \rangle_{W_j \setminus P}| \lesssim \varepsilon_{j-1}^{-4k^2} \varepsilon_j^{(10k)^{-1}(20k^2\delta)^{-k}\delta^{-k}} \lesssim \varepsilon_{j-1}^{-(20k^2\delta)^{2k}} \varepsilon_j^{(20k^2\delta)^{-3k}} \lesssim \varepsilon^j.$$

This completes the proof. □

End of the proof of Proposition 5.4.10. Since $\mathscr{V}_S$ is bounded from above by a constant independent of S, we only have to bound $\mathscr{V}_S(\Lambda^n(R))$ from below. By Lemmas 5.4.6 and 5.4.9, since $R_0 = R$ and the R_j are smooth, we have

$$\begin{aligned} d^{-n}\mathscr{V}_S(\Lambda^n(R)) &= d^{-1}\mathscr{V}_{L^{n-1}(S)}(\Lambda(R_0)) \\ &= d^{-1}\langle V_{n-1}, \Lambda(R_0) - R_1 \rangle_{\mathbb{P}^k \setminus P} + d^{-1}\langle V_{n-1}, R_1 \rangle \\ &= d^{-1}\langle V_{n-1}, \Lambda(R_0) - R_1 \rangle_{\mathbb{P}^k \setminus P} + d^{-1}\mathscr{V}_{L^{n-1}(S)}(R_1) \\ &= d^{-1}\langle V_{n-1}, \Lambda(R_0) - R_1 \rangle_{\mathbb{P}^k \setminus P} + d^{-2}\mathscr{V}_{L^{n-2}(S)}(\Lambda(R_1)). \end{aligned}$$

By induction, we obtain

$$\begin{aligned} d^{-n}\mathscr{V}_S(\Lambda^n(R)) &= d^{-1}\langle V_{n-1}, \Lambda(R_0) - R_1 \rangle_{\mathbb{P}^k \setminus P} \\ &\quad + \ldots + d^{-n}\langle V_0, \Lambda(R_{n-1}) - R_n \rangle_{\mathbb{P}^k \setminus P} + d^{-n}\mathscr{V}_S(R_n). \end{aligned}$$

It follows from Lemmas 5.4.11 and 5.4.12 that

$$d^{-n}\mathscr{V}_S(\Lambda^n(R)) \gtrsim -d^{-1}\varepsilon - \ldots - d^{-n}\varepsilon^n - d^{-n/4}(-\log\varepsilon) \gtrsim -\varepsilon - d^{-n/4}(-\log\varepsilon).$$

Since $\varepsilon = d^{-n}$, we get the result. □

End of the proof of Theorem 5.4.4. Consider a current S in $\mathscr{C}_p$ and a smooth form R in $\mathscr{C}_{k-p+1}$. We want to prove that $L^n(S)$ converge to T^p uniformly with respect to S. By Propositions 3.2.6 and 3.1.9, it is enough to show that $\mathscr{V}_{L^n(S)}(R)$ converge to 0 uniformly with respect to S. By Lemma 5.4.6, we have that

$$\mathscr{V}_{L^n(S)}(R) = d^{-n}\mathscr{V}_S(\Lambda^n(R)).$$

Proposition 5.4.10 implies the result. □

PROPOSITION 5.4.13. *Assume that f is in $\mathscr{H}_d^*(\mathbb{P}^k)$. For any $\alpha>0$, there are constants $c>0$ and $\lambda>1$ such that if S is in $\mathscr{C}_p$ and Φ is a test $(k-p,k-p)$-form of class $\mathscr{C}^\alpha$, then*

$$|\langle d^{-pn}(f^n)^*(S)-T^p,\Phi\rangle| \leqslant c\lambda^{-n}\|\Phi\|_{\mathscr{C}^\alpha}.$$

In particular, if φ is a $\mathscr{C}^\alpha$ function such that $\langle T^k,\varphi\rangle=0$, then

$$\|d^{-kn}(f^n)_*(\varphi)\|_\infty \leqslant c\lambda^{-n}\|\varphi\|_{\mathscr{C}^\alpha}.$$

Proof. We prove the first assertion. Using the theory of interpolation as in Lemma 2.1.2, we only have to prove the case $\alpha=3$. Assume that Φ has a bounded $\mathscr{C}^3$-norm. Multiplying Φ by a constant allows one to assume that $dd^c\Phi=R^+-R^-$, where $R^\pm$ are $\mathscr{C}^1$ forms in $\mathscr{C}_{k-p+1}$ with bounded $\mathscr{C}^1$-norm. A straighforward computation as above gives

$$\langle d^{-pn}(f^n)^*(S)-T^p,\Phi\rangle = d^{-n}\mathscr{V}_S(\Lambda^n(R^+))-d^{-n}\mathscr{V}_S(\Lambda^n(R^-)).$$

The estimates we obtained above give

$$d^{-n}\mathscr{V}_S(\Lambda^n(R^\pm)) \gtrsim -nd^{-n/4}.$$

On the other hand, since $\mathscr{V}_S$ is bounded from above uniformly with respect to S, we have

$$d^{-n}\mathscr{V}_S(\Lambda^n(R^\pm)) \lesssim d^{-n}.$$

So, it is enough to take a λ smaller than $d^{1/4}$.

For the second assertion, if δ_a is the Dirac mass at a, then

$$\langle d^{-kn}(f^n)^*(\delta_a),\varphi\rangle = \langle \delta_a, d^{-kn}(f^n)_*(\varphi)\rangle = d^{-kn}(f^n)_*(\varphi)(a).$$

Since $\langle T^k,\varphi\rangle=0$, we deduce from the first assertion that

$$|d^{-kn}(f^n)_*(\varphi)(a)| \leqslant c\lambda^{-n}\|\varphi\|_{\mathscr{C}^\alpha}.$$

This completes the proof. □

Note that, for $\alpha\leqslant 2$, we can take λ to be any constant smaller than $d^{\alpha/2}$ if we replace $\mathscr{H}_d^*(\mathbb{P}^k)$ by a suitable Zariski open set depending on λ. In dimension 1, Drasin–Okuyama proved in [25] that the second assertion holds for every f if a is a point on the Julia set, i.e. on the support of the equilibrium measure.

5.5. Equidistribution problem for automorphisms

In this section, we consider the class of regular polynomial automorphisms introduced by the second author in [44]. Let f be a polynomial automorphism of $\mathbb{C}^k$. We extend f to a birational map on $\mathbb{P}^k$ that we still denote by f. Let I_+ and I_- be the indeterminacy sets of f and f^{-1}, respectively. With the notation of §5.1, we have $I=I_+$ and $I'=I_-$. They are analytic subsets of codimension $\geqslant 2$ in $\mathbb{P}^k$. The map f is said to be *regular* if $I_+\cap I_-=\varnothing$. We summarize here some properties of f, which are deduced from the above assumption [44].

The indeterminacy sets $I_\pm$ are irreducible and there is an integer p such that

$$\dim I_+ = k-p-1 \quad \text{and} \quad \dim I_- = p-1.$$

They are contained in the hyperplane at infinity L_∞. We also have $f(L_\infty\setminus I_+)=f(I_-)=I_-$ and $f^{-1}(L_\infty\setminus I_-)=f^{-1}(I_+)=I_+$. If $d_\pm$ denote the algebraic degrees of $f^\pm$, then $d_+^p=d_-^{k-p}$. Denote by $\mathcal{K}_+$ (resp. $\mathcal{K}_-$) the set of points z in $\mathbb{C}^k$ such that the forward orbit $\{f^n(z)\}_{n\geqslant 0}$ (resp. the backward orbit $\{f^{-n}(z)\}_{n\geqslant 0}$) is bounded in $\mathbb{C}^k$. They are closed subsets in $\mathbb{C}^k$ and $\bar{\mathcal{K}}_\pm=\mathcal{K}_\pm\cup I_\pm$. Moreover, I_- is attracting for f and $\mathbb{P}^k\setminus\bar{\mathcal{K}}_+$ is the attracting basin; I_+ is attracting for f^{-1} and $\mathbb{P}^k\setminus\bar{\mathcal{K}}_-$ is the attracting basin.

The positive closed $(1,1)$-currents $d_\pm^{-n}(f^{\pm n})^*(\omega)$ converge to the Green $(1,1)$-currents $T_\pm$ associated with $f^{\pm 1}$. These currents have Hölder continuous quasi-potentials outside $I_\pm$ and satisfy $f^*(T_+)=d_+T_+$ and $f_*(T_-)=d_-T_-$. The self-intersections T_+^p and T_-^{k-p} are positive closed currents of mass 1 with support in the boundaries of $\bar{\mathcal{K}}_+$ and $\bar{\mathcal{K}}_-$, respectively. The probability measure $\mu:=T_+^p\wedge T_-^{k-p}$ is supported in the boundary of $\mathcal{K}:=\mathcal{K}_+\cap\mathcal{K}_-$. The current T_+^s, $1\leqslant s\leqslant p$, is the *Green current of order* s of f and its super-potentials are called *Green super-potentials of order* s of f.

Let $\mathscr{C}_{k-s+1}(W)$ denote the set of currents in $\mathscr{C}_{k-s+1}$ with compact support in an open set W. We assume that W is a neighbourhood of I_- such that $\overline{W}\cap I_+=\varnothing$. Since $\dim I_-=p-1$, $\mathscr{C}_{k-s+1}(W)$ is not empty for $s\leqslant p$. If $\mathscr{U}$ is a function on $\mathscr{C}_{k-s+1}(W)$, define

$$\|\mathscr{U}\|_{\infty,W} := \sup_{R\in\mathscr{C}_{k-s+1}(W)} |\mathscr{U}(R)|.$$

In the following result, we give a new construction of the currents T_+^s and T_-^s. Note that we cannot apply the results of §5.3 here, since $\Sigma'=L_\infty$. Indeed, we apply f^* only to currents without mass on L_∞.

THEOREM 5.5.1. *Let f and W be as above. Then, the Green super-potentials of order s of f, $1\leqslant s\leqslant p$, are Hölder continuous on $\mathscr{C}_{k-s+1}(W)$. Let S_n be currents in $\mathscr{C}_s$ and $\mathscr{U}_{S_n}$ be super-potentials of S_n such that $\|\mathscr{U}_{S_n}\|_{\infty,W}=o(d_+^n)$ for an open set W which contains $\bar{\mathcal{K}}_-$. Then, $d_+^{-sn}(f^n)^*(S_n)\to T_+^s$.*

It is shown in [44] that the current $f^*(\omega^s)$ is of mass d_+^s for $1\leqslant s\leqslant p$; see also §5.1. It follows that $f_*(\omega^{k-s})$ is of mass d_+^s. Define $L_s:=d_+^{-s}f^*$ and $\Lambda_s:=d_+^{-s+1}f_*$. Assume that the super-potentials of S are finite on $\mathscr{C}_{k-s+1}(W)$. Then, S is f^*-admissible, because $\Lambda_s(R)$ belongs to $\mathscr{C}_{k-s+1}(W)$ when $\operatorname{supp}(R)$ is close enough to I_-. By Lemma 5.1.6 and Proposition 5.1.8, the current $f^*(S)$ is well defined and is of mass d_+^s. Consider a super-potential $\mathscr{U}_{L_s(\omega^s)}$ of $L_s(\omega^s)$. Since $L_s(\omega^s)$ is smooth on W, it is easy to check that $\mathscr{U}_{L_s(\omega^s)}$ is Lipschitz on $\mathscr{C}_{k-s+1}(W)$. We first prove the following result.

Proposition 5.5.2. *Let S_n be currents in $\mathscr{C}_s$ and $\mathscr{U}_{S_n}$ be super-potentials of S_n with $\|\mathscr{U}_{S_n}\|_{\infty,W}=o(d_+^n)$. If S is a limit value of $d_+^{-sn}(f^n)^*(S_n)$, then S admits a super-potential which is equal on $\mathscr{C}_{k-s+1}(\mathbb{P}^k\setminus\bar{\mathcal{K}}_+)$ to $\sum_{n=0}^\infty d_+^{-n}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^n$. Moreover, this equality holds on $\mathscr{C}_{k-s+1}(\mathbb{P}^k\setminus I_+)$ when W contains $\bar{\mathcal{K}}_-$.*

Proof. Reducing W allows one to assume that $f(W)\Subset W$. If W contains $\bar{\mathcal{K}}_-$, we can keep this property. Fix an open set W_0 relatively compact in $\mathbb{P}^k\setminus\bar{\mathcal{K}}_+$ which contains I_-. If W contains $\bar{\mathcal{K}}_-$, we can take W_0 relatively compact in $\mathbb{P}^k\setminus I_+$. Observe that $f^{-m}(W)$ contains W_0 for m large enough. So, replacing S_n by $d_+^{-sm}(f^m)^*(S_{n+m})$ and W by some open set of $f^{-m}(W)$ allows one to assume that $W_0\Subset W$.

By Proposition 5.1.8, there is a super-potential of $L_s(S_n)$ which is equal to

$$d_+^{-1}\mathscr{U}_{S_n}\circ\Lambda_s+\mathscr{U}_{L_s(\omega^s)}$$

on $\mathscr{C}_{k-s+1}(W)$. We apply again this proposition to $L_s(S_n)$. There is a super-potential of $L_s^2(S_n)$ which is equal to

$$d_+^{-2}\mathscr{U}_{S_n}\circ\Lambda_s^2+\mathscr{U}_{L_s(\omega^s)}+d_+^{-1}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s$$

on $\mathscr{C}_{k-s+1}(W)$. By induction, $L_s^n(S_n)$ admits a super-potential $\mathscr{U}_{L_s^n(S_n)}$ equal to

$$d_+^{-n}\mathscr{U}_{S_n}\circ\Lambda_s^n+\mathscr{U}_{L_s(\omega^s)}+d_+^{-1}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s+...+d_+^{-n+1}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^{n-1}$$

on $\mathscr{C}_{k-s+1}(W)$. By hypothesis, the first term tends to 0. Hence, $\mathscr{U}_{L_s^n(S_n)}$ converge to $\sum_{n=0}^\infty d_+^{-n}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^n$ on $\mathscr{C}_{k-s+1}(W)$. This sum converges since $\mathscr{U}_{L_s(\omega^s)}$ is Lipschitz on $\mathscr{C}_{k-s+1}(W)$.

By Proposition 3.2.6, it remains to show that $\mathscr{U}_{L_s^n(S_n)}$ are bounded from above uniformly with respect to n. For this purpose, it is enough to show that the means $\mathscr{U}_{L_s^n(S_n)}(\omega^{k-s+1})$ of $\mathscr{U}_{L_s^n(S_n)}$ are bounded from above uniformly with respect to n. If R_0 is a smooth form in $\mathscr{C}_{k-s+1}(W_0)$, then we have

$$\mathscr{U}_{L_s^n(S_n)}(R_0)=d_+^{-n}\mathscr{U}_{S_n}(\Lambda_s^n(R_0))+\mathscr{U}_{L_s(\omega^s)}(R_0)+...+d_+^{-n+1}\mathscr{U}_{L_s(\omega^s)}(\Lambda_s^{n-1}(R_0)).$$

This sum is bounded from above. On the other hand, R_0 admits a positive quasi-potential, since it is smooth. Lemma 3.2.9 implies the result. $\square$

End of the proof of Theorem 5.5.1. Since W contains $\overline{\mathcal{K}}_-$, by Proposition 5.5.2, any cluster point of $L_s^n(S_n)$ has a super-potential which is equal to $\sum_{n=0}^\infty d_+^{-n}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^n$ on $\mathscr{C}_{k-s+1}(\mathbb{P}^k\setminus I_+)$. Proposition 3.1.9 implies that there is only one cluster point for the sequence $L_s^n(S_n)$, hence $L_s^n(S_n)$ converge to a current T_s. This current does not depend on S_n, since it admits a super-potential independent of S_n. For $S_n=\omega^s$, we obtain that T_s is the Green current of order s of f. It admits a super-potential $\mathscr{U}_{T_s}$ equal to $\sum_{n=0}^\infty d_+^{-n}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^n$ on $\mathscr{C}_{k-s+1}(\mathbb{P}^k\setminus I_+)$. Lemma 5.4.2 implies that this function is Hölder continuous on $\mathscr{C}_{k-s+1}(W)$.

Let $T_+:=T_1$. We next want to prove that $T_s=T_+^s$. For this purpose, it is sufficient to show that T_s and T_l are wedgeable and $T_s\wedge T_l=T_{s+l}$ when $s+l\leqslant p$. Since $s+l\leqslant p$, there is a smooth form $\Omega\in\mathscr{C}_{k-s-l+1}$ with compact support in $\mathbb{P}^k\setminus I_+$. Hence, $\Omega\wedge T_l$ has compact support in $\mathbb{P}^k\setminus I_+$ and the super-potentials of T_s are finite at $\Omega\wedge T_l$. It follows that T_s and T_l are wedgeable.

The computation in Proposition 5.5.2 implies that $L_s^n(\omega^s)$ admits a super-potential $\mathscr{U}_{L_s^n(\omega^s)}$ which is equal to $\sum_{j=0}^{n-1} d_+^{-j}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^j$ on $\mathscr{C}_{k-s+1}(\mathbb{P}^k\setminus I_+)$. Fix a real smooth test form Φ of bidegree $(k-s-l,k-s-l)$ with compact support in $\mathbb{P}^k\setminus I_+$. As in Proposition 3.1.9, write $dd^c\Phi=c(\Omega^+-\Omega^-)$ with $c>0$ and $\Omega^\pm$ in $\mathscr{C}_{k-s-l+1}(\mathbb{P}^k\setminus I_+)$. The sequence $\Omega^\pm\wedge L_l^n(\omega^l)$ converges to $\Omega^\pm\wedge T_l$. Since these currents have supports in a fixed compact subset of $\mathbb{P}^k\setminus I_+$, the values of $\mathscr{U}_{L_s^n(\omega^s)}$ at $\Omega^\pm\wedge L_l^n(\omega^l)$ converge to the value of $\mathscr{U}_{T_s}$ at $\Omega^\pm\wedge T_l$. The formula (4.1) implies that $L_s^n(\omega^s)\wedge L_l^n(\omega^l)$ converge to $T_s\wedge T_l$. On the other hand, $L_{s+l}^n(\omega^{s+l})$ and $L_s^n(\omega^s)\wedge L_l^n(\omega^l)$ are smooth forms which are equal outside I_+. They have no mass on I_+ because $\dim I_+<k-s-l$. Hence, these currents are equal. Therefore, letting $n\to\infty$ gives $T_{s+l}=T_s\wedge T_l$, and in particular $T_s=T_+^s$. □

THEOREM 5.5.3. *The Green current T_+^s is the most diffuse f^*-invariant current in $\mathscr{C}_s$. In particular, it is extremal in the convex set of f^*-invariant currents in $\mathscr{C}_s$.*

Proof. It follows from the convergence in Theorem 5.5.1 that T_+^s is f^*-invariant. Let T be an f^*-invariant current in $\mathscr{C}_s$ and $\mathscr{U}_T$ be a super-potential of T. Proposition 5.1.8 implies that $L_s(T)$ admits a super-potential $\mathscr{U}$ which is equal to $d_+^{-1}\mathscr{U}_T\circ\Lambda_s+\mathscr{U}_{L_s(\omega^s)}$ on smooth R in $\mathscr{C}_{k-s+1}$. Since $L_s(T)=T$, there is a constant c such that $\mathscr{U}=\mathscr{U}_T+c$. Subtracting an appropriate constant from $\mathscr{U}_T$ gives another super-potential that we still denote by $\mathscr{U}_T$, such that

$$\mathscr{U}_T=d_+^{-1}\mathscr{U}_T\circ\Lambda_s+\mathscr{U}_{L_s(\omega^s)}$$

on R in $\mathscr{C}_{k-s+1}$ which is smooth in a neighbourhood of I_+. The condition on R is invariant under Λ_s. So, iterating the above identity gives

$$\mathscr{U}_T=d_+^{-n}\mathscr{U}_T\circ\Lambda_s^n+\sum_{j=0}^{n-1} d_+^{-j}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^j.$$

Since $\mathscr{U}_T$ is bounded from above, letting $n\to\infty$, we obtain

$$\mathscr{U}_T \leqslant \sum_{j=0}^{\infty} d_+^{-j}\mathscr{U}_{L_s(\omega^s)}\circ\Lambda_s^j = \mathscr{U}_{T_+^s}.$$

This identity holds on smooth forms R in $\mathscr{C}_{k-s+1}$. Hence, T_+^s is more diffuse than T.

We now prove that T_+^s is extremal among f^*-invariant currents. Assume that $T_+^s = \frac{1}{2}(T+T')$ with T and T' in $\mathscr{C}_s$ invariant by f^*. Let $\mathscr{U}_T$ be as above. Let $\mathscr{U}_{T'}$ be the analogous super-potential of T'. It is the unique super-potential which satisfies

$$\mathscr{U}_{T'} = d_+^{-1}\mathscr{U}_{T'}\circ\Lambda_s + \mathscr{U}_{L_s(\omega^s)}$$

on smooth forms in $\mathscr{C}_{k-s+1}$. Observe that $\frac{1}{2}(\mathscr{U}_T+\mathscr{U}_{T'})$ is a super-potential of T_+^s satisfying the same property. It follows that

$$\tfrac{1}{2}(\mathscr{U}_T+\mathscr{U}_{T'}) = \mathscr{U}_{T_+^s}.$$

We deduce from the inequalities $\mathscr{U}_T\leqslant\mathscr{U}_{T_+^s}$ and $\mathscr{U}_{T'}\leqslant\mathscr{U}_{T_+^s}$ that $\mathscr{U}_T$ and $\mathscr{U}_{T'}$ are equal to $\mathscr{U}_{T_+^s}$. Hence, $T=T'=T_+^s$. This implies the result. □

In the case of bidegree (p,p), we have the following stronger result which is another main application of the super-potentials. It was proved by Fornæss and the second author in the case of dimension 2, [30].

THEOREM 5.5.4. *The current T_+^p is the unique positive closed current of bidegree (p,p) of mass* 1 *supported in $\bar{\mathcal{K}}_+$. The current T_-^{k-p} is the unique positive closed current of bidegree $(k-p,k-p)$ of mass* 1 *supported in $\bar{\mathcal{K}}_-$.*

In what follows, we only consider currents S in $\mathscr{C}_p$ with support in $\bar{\mathcal{K}}_+$. By Proposition 3.2.10, their super-potentials of mean 0 are bounded on $\mathscr{C}_{k-p+1}(W)$ uniformly with respect to S when $W\Subset\mathbb{P}^k\setminus\bar{\mathcal{K}}_+$. In particular, they are bounded at the current $R_\infty:=(\deg I_-)^{-1}[I_-]$. We call *dynamical super-potential of* S the function $\mathscr{V}_S$ defined by

$$\mathscr{V}_S := \mathscr{U}_S - \mathscr{U}_{T_+^p} - c_S, \quad \text{where } c_S := \mathscr{U}_S(R_\infty) - \mathscr{U}_{T_+^p}(R_\infty),$$

and $\mathscr{U}_S$ and $\mathscr{U}_{T_+^p}$ are the super-potentials of mean 0 of S and T_+^p. We also call the *dynamical Green quasi-potential of* S the form

$$V_S := U_S - U_{T_+^p} - (m_S - m_{T_+^p} + c_S)\omega^{p-1},$$

where U_S and $U_{T_+^p}$ are the Green quasi-potentials of S and T_+^p, and m_S and $m_{T_+^p}$ are their means. Denote, for simplicity, $L:=L_p$ and $\Lambda:=\Lambda_p$.

LEMMA 5.5.5. *Let $W\Subset\mathbb{P}^k\setminus I_+$ be an open set. Then, $\mathscr{V}_S(R_\infty)=0$, $\mathscr{V}_S(R)=\langle V_S,R\rangle$ for smooth R in $\mathscr{C}_{k-p+1}(W)$ and $\mathscr{V}_{L(S)}=d_+^{-1}\mathscr{V}_S\circ\Lambda$ on $\mathscr{C}_{k-p+1}(W)$. Moreover, $\mathscr{U}_S-\mathscr{V}_S$ is bounded on $\mathscr{C}_{k-p+1}(W)$ by a constant independent of S.*

Proof. It is clear that $\mathscr{V}_S(R_\infty)=0$. Recall that m_S, $m_{T_+^p}$ and c_S are bounded. Since $\mathscr{U}_{T_+^p}$ is continuous on $\mathscr{C}_{k-p+1}(W)$, $\mathscr{U}_S-\mathscr{V}_S$ is bounded on $\mathscr{C}_{k-p+1}(W)$ by a constant independent of S. We also have, for smooth R in $\mathscr{C}_{k-p+1}(W)$,

$$\langle V_S,R\rangle=(\langle U_S,R\rangle-m_S)-(\langle U_{T_+^p},R\rangle-m_{T_+^p})-c_S=\mathscr{U}_S(R)-\mathscr{U}_{T_+^p}(R)-c_S=\mathscr{V}_S(R).$$

It remains to prove that $\mathscr{V}_{L(S)}=d_+^{-1}\mathscr{V}_S\circ\Lambda$ on $\mathscr{C}_{k-p+1}(W)$. Observe that, since I_- is irreducible, $\Lambda(R_\infty)=R_\infty$. We deduce that $\mathscr{V}_{L(S)}=d_+^{-1}\mathscr{V}_S\circ\Lambda=0$ at R_∞. Hence, we only have to show that $\mathscr{V}_{L(S)}-d_+^{-1}\mathscr{V}_S\circ\Lambda$ is constant. By Proposition 5.1.8 (see also Proposition 5.5.2), we have

$$\mathscr{U}_{L(S)}=d_+^{-1}\mathscr{U}_S\circ\Lambda+\mathscr{U}_{L(\omega^p)}+\text{const},$$

and since $L(T_+^p)=T_+^p$, this implies that

$$\mathscr{U}_{T_+^p}=d_+^{-1}\mathscr{U}_{T_+^p}\circ\Lambda+\mathscr{U}_{L(\omega^p)}+\text{const}.$$

It follows that

$$\mathscr{V}_{L(S)}=d_+^{-1}\mathscr{U}_S\circ\Lambda-d_+^{-1}\mathscr{U}_{T_+^p}\circ\Lambda+\text{const}.$$

It is clear that $\mathscr{V}_{L(S)}-d_+^{-1}\mathscr{V}_S\circ\Lambda$ is constant. □

Proof of Theorem 5.5.4. Consider a current S in $\mathscr{C}_p(\mathbb{P}^k)$ with support in $\bar{\mathcal{K}}_+$. Define $S_n:=d_+^{pn}(f^n)_*(S)$ on $\mathbb{C}^k$. These currents are positive closed with support in $\bar{\mathcal{K}}_+$. Since $\bar{\mathcal{K}}_+=\mathcal{K}_+\cup I_+$, S_n are defined on $\mathbb{P}^k\setminus I_+$. As $\dim I_+<k-p$, S_n can be extended to positive closed currents on $\mathbb{P}^k$ without mass on I_+ [37]. We also denote this extension by S_n. Since f^n is an automorphism in $\mathbb{C}^k$, we have $(f^n)^*(S_n)=d_+^{pn}S$ on $\mathbb{C}^k$. The equality holds in $\mathbb{P}^k$ because the currents have supports in $\bar{\mathcal{K}}_+$ and hence, have no mass at infinity. So, necessarily, S_n have mass 1. Let $\mathscr{V}_{S_n}$ and $\mathscr{V}_S$ denote the dynamical super-potentials of S_n and S, respectively. We want to prove that $S=T_+^p$. According to Proposition 3.1.9, it is enough to show that $\mathscr{V}_S=0$ on $\mathscr{C}_{k-p+1}(W)$ for any W disjoint from I_+.

We have $L^n(S_n)=S$, hence Lemma 5.5.5 implies that $\mathscr{V}_S=d_+^{-n}\mathscr{V}_{S_n}\circ\Lambda^n$. Since $\mathscr{V}_{S_n}$ is bounded from above on $\mathscr{C}_{k-p+1}(W)$ by a constant independent of n, the last identity implies that $\mathscr{V}_S\leqslant 0$ on $\mathscr{C}_{k-p+1}(W)$. If $\mathscr{V}_S\neq 0$ on $\mathscr{C}_{k-p+1}(W)$, there is a smooth form R in $\mathscr{C}_{k-p+1}(W)$ such that $\mathscr{V}_S(R)<0$. It follows that $\mathscr{V}_{S_n}(\Lambda^n(R))\lesssim -d_+^n$. Let W'' be a neighbourhood of $\bar{\mathcal{K}}_+$, disjoint from I_-, such that $f^{-1}(W'')\subset W''$. Hence, $\|Df^{-1}\|$

is bounded on W'' by some constant M. It follows that $\|\Lambda^n(R)\|_{\infty,W''}\lesssim M^{3kn}$. The inequality $\mathscr{V}_{S_n}(\Lambda^n(R))\lesssim -d_+^n$ contradicts Proposition 3.2.10, which gives

$$|\mathscr{V}_{S_n}(\Lambda^n(R))|\lesssim 1+\log M^{3kn}.$$

So, $\mathscr{V}_S=0$ on $\mathscr{C}_{k-p+1}(W)$ and this completes the proof. □

The following result holds for currents of integration on generic varieties of dimension $k-p$ in $\mathbb{P}^k$.

COROLLARY 5.5.6. *Let S be a current in $\mathscr{C}_p$ such that* $\mathrm{supp}(S)\cap I_-=\varnothing$. *Then, $d_+^{-pn}(f^n)^*(S)$ converge to T_+^p.*

Proof. Let W be a neighbourhood of I_- such that $f(W)\Subset W$ and $W\cap\mathrm{supp}(S)=\varnothing$. Hence, $f^{-n}(W)\subset f^{-n-1}(W)$ and $d_+^{-pn}(f^n)^*(S)$ has support in $\mathbb{P}^k\setminus f^{-n}(W)$. It follows that the limit values of $d_+^{-pn}(f^n)^*(S)$ are supported in the complement of $\bigcup_{n\geqslant 0} f^{-n}(W)$, which is contained in $\overline{\mathcal{K}}_+$. By Theorem 5.5.4, the only limit value is T_+^p. Following the proof of that theorem, it is not difficult to obtain here a speed of convergence. □

Remark 5.5.7. In [48], de Thélin proved that the measure μ is hyperbolic. It admits $k-p$ strictly negative and p strictly positive Lyapounov exponents. Pesin's theory implies that if a point a is generic with respect to μ, then it admits a stable manifold of dimension $k-p$ and an unstable manifold of dimension p. If $p=k-1$ and if $\tau:\mathbb{C}\to\overline{\mathcal{K}}_+$ is an entire curve, using the Ahlfors construction [1], we obtain positive closed $(k-1,k-1)$-currents with support in $\overline{\tau(\mathbb{C})}$. Indeed, Ahlfors inequality implies the existence of $r_n\to\infty$ such that the currents of integration on $\tau(\Delta_{r_n})$, properly normalized, converge to a positive closed current of mass 1. Theorem 5.5.4 implies that this current is equal to T_+^{k-1}. Hence $\overline{\tau(\mathbb{C})}$ contains the support of T_+^{k-1}. This result holds for generic stable manifolds of μ.

Remark 5.5.8. For $1\leqslant s\leqslant p$, if S is a current in $\mathscr{C}_s$ with super-potentials bounded on $\mathscr{C}_{k-s+1}(W)$ for some small neighbourhood W of I_-, then we can prove in the same way that $d_+^{-sn}(f^n)^*(S)$ converge to T_+^s. The proof follows the same lines as in Theorem 5.5.4. We should choose W'' large enough, in particular we have $W''\cup W=\mathbb{P}^k$. In order to apply Proposition 3.2.10, we write R as a combination of a current in $\mathscr{C}_{k-p+1}(W)$ and a smooth form with bounded $\mathscr{C}^0$-norm.

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