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SUPER-RIGIDITY FOR HOLOMORPHIC MAPPINGS BETWEEN HYPERQUADRICS WITH POSITIVE SIGNATURE

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Abstract

We study local holomorphic mappings sending a piece of a real hyperquadric in a complex space into a hyperquadric in another complex space of possibly larger dimension. We show that these mappings possess strong super-rigidity properties when the hyperquadrics have positive signatures. These results are applied in the context of holomorphic mappings between classical domains in complex projective spaces of different dimensions.

1. Introduction

In this paper, we study holomorphic mappings from a piece of a real hyperquadric with positive signature into a hyperquadric in a complex space of larger dimension. We will prove that, unlike in the case of Heisenberg hypersurfaces (i.e., hyperquadrics with 0-signature), the maps possess strong super-rigidity properties. This phenomenon is somewhat analogous to that encountered in the study of holomorphic maps between irreducible bounded symmetric domains of rank at least two (see e.g., the book of Mok [**18**] for results and extensive references on this matter). We first state our results in the context of holomorphic mappings between classical domains in complex projective spaces of different dimensions.

For $0 \le \ell < n$, denote by $\mathbb{B}^n_\ell$ the domain in $\mathbb{CP}^n$ given by

$$\mathbb{B}^n_\ell := \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : |z_0|^2 + \cdots + |z_\ell|^2 > |z_{\ell+1}|^2 + \cdots + |z_n|^2\}.$$

For $0 \le k \le m$, let $E_{(k,m)}$ denote the $m \times m$ diagonal matrix with its first k diagonal elements -1 and the rest $+1$, and define

$$U(n+1, \ell+1) = \left\{A \in \mathrm{GL}(n+1, \mathbb{C}),\ A E_{(\ell+1,n+1)} \overline{A}^t = E_{(\ell+1,n+1)}\right\}.$$

In what follows, we will regard $U(n+1, \ell+1)$ as a subgroup of the automorphism group of $\mathbb{CP}^n$ by identifying an element A in

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$U(n+1,\ell+1)$ with the holomorphic linear map $\sigma \in \mathrm{Aut}(\mathbb{CP}^n)$ defined by $\sigma([z_0,\dots,z_n]) = [z_0,\dots,z_n]A$. Then, it is clear that $U(n+1,\ell+1)$ can actually be identified with a subgroup of $\mathrm{Aut}(\mathbb{B}^n_\ell)$, acting transitively on $\partial\mathbb{B}^n_\ell$. In fact, it is well known that, with this identification, we have $U(n+1,\ell+1) = \mathrm{Aut}(\mathbb{B}^n_\ell)$ (see e.g., [**5**], Section 1) or Remark 5.1 below). We can now state our first result.

Theorem 1.1. *For $0 < \ell < n-1$, let p be a boundary point of $\mathbb{B}^n_\ell$ and let U_p be an open neighborhood of p in $\mathbb{CP}^n$ with $U_p \cap \mathbb{B}^n_\ell$ connected. Assume that F is a holomorphic map from $U_p \cap \mathbb{B}^n_\ell$ into $\mathbb{B}^N_\ell$ for $N \geq n$. Suppose that for any sequence $\{Z_j\} \subset U_p \cap \mathbb{B}^n_\ell$ with $\lim_{j\to\infty} Z_j \in \partial\mathbb{B}^n_\ell$, all the limit points of the sequence $\{F(Z_j)\}$ lie in $\partial\mathbb{B}^N_\ell$. Then, F extends to a totally geodesic embedding from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_\ell$, i.e., there exist $\sigma \in U(n+1,\ell+1)$ and $\tau \in U(N+1,\ell+1)$ such that*

$$\tau \circ F \circ \sigma([z_0,\dots,z_n]) \equiv [z_0,\dots,z_n,0,\dots,0].$$

It should be mentioned that by making use of a theorem of Siuand Ivashkovich ([**22**], [**17**]), it suffices to prove Theorem 1.1, under the additional assumption that the map F in Theorem 1.1, extends holomorphically to U_p (see Section 5 below). As an immediate application of Theorem 1.1, we have the following global super-rigidity result.

Corollary 1.2. *Any proper holomorphic map from $\mathbb{B}^n_\ell$ to $\mathbb{B}^N_\ell$ is a totally geodesic embedding whenever $0 < \ell < n-1$ and $N \geq n$.*

In Corollary 1.2, the case $\ell = 0$ has to be excluded. Indeed, we do know that super-rigidity does not hold for general holomorphic maps between $\mathbb{B}^n_0$ and $\mathbb{B}^N_0$ with $N > n > 1$, unless more restrictions are imposed. For instance, the Whitney map given by $W([z_0,z_1,z_2]) := [z_0^2, z_1z_0, z_1z_2, z_2^2]$ maps properly $\mathbb{B}^2_0$ into $\mathbb{B}^3_0$, and is not linear. Establishing various rigidity results for the case $\ell = 0$ has attracted much attention since the work of Poincaré [**21**]. Here, we mention the work in [**25**],[**10**], [**4**], [**12**], [**13**], [**7**], [**8**], [**14**], [**15**], to name a few. We refer the reader to the second author's papers [**15**], [**16**] for more references.

We remark that there are no proper holomorphic maps from $\mathbb{B}^n_\ell$ into $\mathbb{B}^N_{\ell'}$ for $\ell' < \ell$ (see Section 2 below). Also, one cannot expect, in general, that Corollary 1.2 holds for any $\ell' > \ell$. For instance, we have the following:

Example 1.3. Let $F([z_0,z_1,z_2,z_3]) = [z_0^2, \sqrt{2}z_0z_1, z_1^2, z_2^2, \sqrt{2}z_2z_3, z_3^2]$. Then, F maps properly $\mathbb{B}^3_1$ into $\mathbb{B}^5_2$, and clearly, F is not linear.

However, with the same arguments used in the proof of Theorem 1.1, we can also prove the following results:

Theorem 1.4. *Let* p *be a boundary point of* $\mathbb{B}^n_\ell$ *and let* U_p *be a neighborhood of* p *in* $\mathbb{CP}^n$ *with* $U_p \cap \mathbb{B}^n_\ell$ *connected. Assume that* F *is a holomorphic map from* $U_p \cap \mathbb{B}^n_\ell$ *into* $\mathbb{B}^{n+k}_{\ell+k}$ *for* $k \geq 0$ *and* $0 < \ell < n-1$. *Suppose that for any sequence* $\{Z_j\} \subset U_p \cap \mathbb{B}^n_\ell$ *with* $\lim_{j\to\infty} Z_j \in \partial\mathbb{B}^n_\ell$, *all the limit points of the sequence* $\{F(Z_j)\}$ *lie in* $\partial\mathbb{B}^{n+k}_{\ell+k}$. *Then,* F *extends to a totally geodesic embedding from* $\mathbb{B}^n_\ell$ *into* $\mathbb{B}^{n+k}_{\ell+k}$. *Namely, there exist* $\sigma \in U(n+1, \ell+1)$ *and* $\tau \in U(n+k+1, \ell+k+1)$ *such that* $\tau \circ F \circ \sigma([z_0, \ldots, z_n]) = [z_0, \ldots, z_n, 0, \ldots, 0]$.

Corollary 1.5. *Any proper holomorphic map from* $\mathbb{B}^n_\ell$ *to* $\mathbb{B}^{n+k}_{\ell+k}$ *is a totally geodesic embedding whenever* $0 < \ell < n-1$ *and* $k \geq 0$.

In light of Example 1.3, the target dimension $n+k$ cannot be improved in the statements of Theorem 1.4 and Corollary 1.5. Our proofs of the above results are of local nature. In fact, we will reduce the proofs to statements for local holomorphic mappings between hyperquadratics as stated earlier. Before we state our main technical result, we introduce the following notation. For $0 \leq \ell \leq n-1$, we define the generalized Siegel upper-half space

$$\mathbb{S}^n_\ell :\,= \Big\{(z,w) = (z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \quad w = u + iv, \qquad v > -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\Big\},$$

where the first sum is understood to be 0 if $\ell = 0$. The boundary of $\mathbb{S}^n_\ell$ is the standard hyperquadric

$$\mathbb{H}^n_\ell := \Big\{(z,w) = (z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \quad w = u + iv, \qquad v = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2\Big\}.$$

Here, ℓ is called the signature of $\mathbb{H}^n_\ell$. If $0 < \ell < n-1$, it is well known that any CR function defined over a connected open piece M of $\mathbb{H}^n_\ell$ extends to a holomorphic function in a neighborhood of M in $\mathbb{C}^n$ (see e.g., [**2**]). We denote by $\mathrm{Aut}_0(\mathbb{H}^n_\ell)$ the stability group of the local biholomorphisms of $\mathbb{C}^n$ preserving a piece of $\mathbb{H}^n_\ell$ near the origin and sending 0 to itself. We now can state the main technical result of the paper.

Theorem 1.6. *Let* M *be a small neighborhood of* 0 *in* $\mathbb{H}^n_\ell$ *with* $0 < \ell < n-1$. *Suppose that* $F = (f_1, \ldots, f_{N-1}, g)$ *is a holomorphic map from a neighborhood* U *of* M *in* $\mathbb{C}^n$ *into* $\mathbb{C}^N$ *with* $F(M) \subset \mathbb{H}^N_\ell$, $N \geq n$, *and* $F(0) = 0$*. Suppose either* $\ell \leq (n-1)/2$ *or* F *preserves sides in the sense that* $F(U \cap \mathbb{S}^n_\ell) \subset \overline{\mathbb{S}^N_\ell}$*. Then, the following hold:*

(i) *If* $\dfrac{\partial g}{\partial w}(0) \neq 0$*, then* F *is linear fractional. Moreover, there exists* $\tau \in \mathrm{Aut}_0(\mathbb{H}^N_\ell)$ *such that either*

$$\begin{aligned}&\tau \circ F(z_1, \ldots, z_{n-1}, w) = (z_1, \ldots, z_{n-1}, 0, \ldots, 0, w), \qquad \textit{or}\\ &\tau \circ F(z_1, \ldots, z_{n-1}, w) = (z_{\ell+1}, \ldots, z_{n-1}, z_1, \ldots, z_\ell, 0, \ldots, 0, -w),\end{aligned}$$

and the latter can only happen when $\ell = (n-1)/2$*.*

(ii) *If* $\dfrac{\partial g}{\partial w}(0) = 0$*, then* $F(U) \subset \mathbb{H}^N_\ell$*. More precisely, there is a constant* $(N-\ell-1) \times \ell$ *complex matrix* V*, with* $V\overline{V}^t = \mathrm{Id}_{N-\ell-1}$*, such that*

$$g \equiv 0, \qquad (f_1, \ldots, f_\ell) \equiv (f_{\ell+1}, \ldots, f_{N-1})V.$$

When $N = n$, Theorem 1.6, Part (i), is a classical result of Tanaka [**23**] and Chern–Moser [**5**]. For $n \leq N \leq 2n-2$, Theorem 1.6, Part (i), was first obtained in ajoint work of the second author with Ebenfelt and Zaitsev [**9**]. The proofs in [**9**] are based on a new normal space for the Chern–Moser operator and are different from those given in the present paper. It should be mentioned that the approach in [**9**] cannot be used to derive Theorem 1.6 (i) when $N \geq 2n-1$. We should mention that when $\ell > (n-1)/2$, the side preserving assumption in Theorem 1.6 is crucial for the conclusion (i) to hold as shown by the following example.

Example 1.7. Let $F(z_1, z_2, z_3, w) = (z_3, z_1^2, z_1, z_2, z_1^2, -w)$. Clearly, F embeds $\mathbb{H}^4_2$ into $\mathbb{H}^6_2$, but does not preserve sides. Although $g_w(0) \neq 0$, F is not linear fractional.

For holomorphic maps sending a piece of $\mathbb{H}^n_\ell$ in $\mathbb{H}^N_{\ell'}$ with $\ell' > \ell$, we have the following result.

Theorem 1.8. *Let* M *be a small neighborhood of* 0 *in* $\mathbb{H}^n_\ell$ *with* $0 < \ell < n-1$*. For* $k \geq 0$*, let* $F = (f_1, \ldots, f_{n+k-1}, g)$ *be a holomorphic map from a neighborhood* U *of* M *in* $\mathbb{C}^n$ *into* $\mathbb{C}^{n+k}$ *with* $F(M) \subset \mathbb{H}^{n+k}_{\ell+k}$*, and* $F(0) = 0$*. Suppose that* F *preserves sides in the sense that* $F(U \cap \mathbb{S}^n_\ell) \subset \overline{\mathbb{S}^{n+k}_{\ell+k}}$*. Then, the following hold:*

(i) *If* $\dfrac{\partial g}{\partial w}(0) \neq 0$*, then* F *is linear fractional. Moreover, there exists* $\tau \in \mathrm{Aut}_0(\mathbb{H}^{n+k}_{\ell+k})$ *such that either*

$$\tau \circ F(z_1, \ldots, z_{n-1}, w) = (z_1, \ldots, z_{n-1}, 0, \ldots, 0, w).$$

(ii) *If* $\frac{\partial g}{\partial w}(0) = 0$*, then* $F(U) \subset \mathbb{H}^{n+k}_{\ell+k}$*. More precisely, there is an* $(n-\ell-1)\times(\ell+k)$ *constant complex matrix* V*, with* $V\overline{V}^t = \mathrm{Id}_{n-\ell-1}$*, such that*

$$g \equiv 0, \qquad (f_1, \ldots, f_{\ell+k}) \equiv (f_{\ell+k+1}, \ldots, f_{n+k-1})V.$$

It can be easily checked that the non-vanishing of $(\partial g/\partial w)(0)$ in Part (i) of Theorem 1.6 (resp. Theorem 1.8) is equivalent to the transversality at 0 of the mapping F to the hypersurface $\mathbb{H}^N_\ell$ (resp. $\mathbb{H}^{n+k}_{\ell+k}$). This means, say, in the context of Theorem 1.6, that the non-vanishing condition $(\partial g/\partial w)(0) \neq 0$ is equivalent to the transversality condition $F'(0)(T_0(\mathbb{C}^n)) + T_0\mathbb{H}^N_\ell = T_0\mathbb{C}^N$. In the equidimensional case, such conditions were previously considered for local holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ sending a hypersurface into another (see e.g., [**3**]).

2. Normalization and Chern–Moser–Gauss Equation

In this section, we set up some basic notation to be used throughout the rest of the paper. We then derive a fundamental equation, called the *Chern–Moser equation* or *Gauss equation*, for embeddings in hyperquadrics. This equation geometrically reflects the curvature relations of the hyperquadrics. Let $\mathbb{H}^n_\ell \subset \mathbb{C}^n$ and $\mathbb{H}^N_{\ell'} \subset \mathbb{C}^N$ be the standard hyperquadrics defined by

$$\begin{aligned} \mathbb{H}^n_\ell &:= \left\{ (z, w = u + iv) \in \mathbb{C}^n,\ v = \operatorname{Im} w = \sum_{j=1}^{n-1} \delta_{j,\ell}|z_j|^2 \right\}; \\ \mathbb{H}^N_{\ell'} &:= \left\{ (z^*, w^* = u^* + iv^*) \in \mathbb{C}^N,\ v^* = \sum_{j=1}^{N-1} \delta_{j,\ell'}|z^*_j|^2 \right\}. \end{aligned} \tag{2.1}$$

Here and in what follows, we denote by $\delta_{j,\ell}$ the symbol which takes value -1 when $1 \leq j \leq \ell$ and 1 otherwise. For $\ell' \geq \ell$ and $N \geq n > \ell - 1$, we define

$$\mathbb{H}^N_{\ell,\ell',n} := \left\{ (z^*, w^*) \in \mathbb{C}^N,\ \operatorname{Im} w^* = \sum_{j=1}^{N-1} \delta_{j,\ell,\ell',n}|z^*_j|^2 \right\}. \tag{2.2}$$

with $\delta_{j,\ell,\ell',n} = -1$ for $j \leq \ell$ or $n \leq j \leq n + \ell' - \ell - 1$, and $\delta_{j,\ell,\ell',n} = 1$ otherwise. Notice that $\mathbb{H}^N_{\ell,\ell',n}$ is the same as $\mathbb{H}^N_{\ell'}$ for $\ell' = \ell$. When $\ell' > \ell$,

$\mathbb{H}^N_{\ell'}$ is holomorphically equivalent to $\mathbb{H}^N_{\ell,\ell',n}$ by the linear map

$$\begin{aligned}(2.3)\quad &\sigma_{\ell,\ell',n}(z^*,w^*) := \\ &(z_1^*,\ldots,z_\ell^*,z_{\ell'+1}^*,\ldots,z_{n-1}^*,z_{\ell+1}^*,\ldots,z_{\ell'}^*,z_n^*,\ldots,z_{N-1}^*,w^*).\end{aligned}$$

Write $L_j = 2i\delta_{j,\ell}\overline{z_j}\dfrac{\partial}{\partial w} + \dfrac{\partial}{\partial z_j}$ for $j = 1,\ldots,n-1$ and $T = \dfrac{\partial}{\partial u}$. Then, $\{L_1,\ldots,L_{n-1}\}$ forms a global basis for the complex tangent bundle $\mathrm{T}^{(1,0)}\mathbb{H}^n_\ell$ of$\mathbb{H}^n_\ell$, and T is a tangent vector field of $\mathbb{H}^n_\ell$ transversal to $T^{(1,0)}\mathbb{H}^n_\ell \cup T^{(0,1)}\mathbb{H}^n_\ell$. Parameterize $\mathbb{H}^n_\ell$ by $(z,\overline{z},u)$ through the map $(z,\overline{z},u) \mapsto (z, u + i\sum_{j=1}^{n-1}\delta_{j,\ell}|z_j|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non-negative integer m, a function $h(z,\overline{z},u)$ defined in a small neighborhood M of 0 in $\mathbb{H}^n_\ell$ is said to be $o_{\mathrm{wt}}(m)$, if $h(tz,t\overline{z},t^2u)/|t|^m \to 0$ uniformly for (z,u) on any compact subset of M for $t \in \mathbb{R}, t \to 0$. (In this case, we write $h = o_{\mathrm{wt}}(m)$). By convention, we write $h = o_{\mathrm{wt}}(0)$ if $h(z,\overline{z},u) \to 0$ as $(z,\overline{z},u) \to 0$. For a smooth function $h(z,\overline{z},u)$ defined in U, we denote by $h^{(k)}(z,\overline{z},u)$ the sum of terms of weighted degree k in the Taylor expansion of h at 0. We also denote by $h^{(k)}(z,\overline{z},u)$ a weighted homogeneous polynomial of weighted degree k, (even if there is no specified function h). When $h^{(k)}(z,\overline{z},u)$ extends to a weighted holomorphic polynomial of weighted degree k, we write it as $h^{(k)}(z,w)$, or $h^{(k)}(z)$ if it depends only on z. Here again, z has weight 1 and w has weight 2.

For two m-tuples $x = (x_1,\ldots,x_m), y = (y_1,\ldots,y_m)$ of complex numbers, we write

$$\langle x,y\rangle_\ell = \sum_{j=1}^m \delta_{j,\ell}x_jy_j, \quad \text{and} \quad |x|^2_\ell = \sum_{j=1}^m \delta_{j,\ell}|x_j|^2.$$

For $\ell' \geq \ell$ and $\ell - 1 \leq n \leq m$, we write $\langle x,y\rangle_{\ell,\ell',n} = \sum_{j=1}^m \delta_{j,\ell,\ell',n}x_jy_j$. When $\langle x,\overline{y}\rangle_{\ell,\ell',n} = 0$, we write $x \perp_{\ell,\ell',n} y$.

Let $F = (f_1,\ldots,f_{N-1},g) := (\widetilde{f},g)$ be a non-constant holomorphic map from an open neighborhood M of 0 in $\mathbb{H}^n_\ell$ into $\mathbb{H}^N_{\ell'}$ with $1 \leq \ell < n-1$ and $1 \leq \ell' < N-1$. For the rest of this section, we shall assume that $g_w(0) = \lambda \neq 0$.

Write $\widetilde{f} = zA + w\mathbf{a} + O(|(z,w)|^2)$ with A an $(n-1)\times(N-1)$ matrix and $\mathbf{a}$ an $(N-1)$-row vector. Applying $L^\alpha = L_1^{\alpha_1}\cdots L_{n-1}^{\alpha_{n-1}}$, where $\alpha = (\alpha_1,\ldots,\alpha_{n-1})$, to the equation on M

$$\mathrm{Im}(g) = \sum_{j=1}^{N-1}\delta_{j,\ell'}|f_j|^2, \tag{2.4}$$

and then evaluating at 0, we see that

$$g(z,0) \equiv 0. \tag{2.5}$$

Hence, we have $g = \lambda w + O(|(z,w)|^2)$. Collecting terms in (2.4) of weighted degree 2, we have

$$\mathrm{Im}(\lambda w) = \widetilde{f}^{(1)}(z) E_{(\ell', N-1)} \overline{\widetilde{f}^{(1)}(z)}^{t}, \quad w = u + i|z|_\ell^2,$$

with $E_{(\ell',N-1)}$ as defined above. Since u and z are independent variables, taking the coefficients of u and identifying the coefficients of $z_j \overline{z_k}$, we conclude that $\lambda \in \mathbb{R} \setminus \{0\}$ and

$$\lambda E_{(\ell,n-1)} = A E_{(\ell',N-1)} \overline{A}^t. \tag{2.6}$$

Counting the number of negative eigenvalues in both sides of (2.6) and using elementary linear algebra, we conclude the following:

Lemma 2.1. *Let F be a holomorphic map as above sending a neighborhood of* 0 *in $\mathbb{H}_\ell^n$ into $\mathbb{H}_{\ell'}^N$ with $F(0) = 0$ and $\lambda = g_w(0) \neq 0$. Then, necessarily $\lambda \in \mathbb{R} \setminus \{0\}$, $N \geq n$, and* Rank $(A) = n - 1$. *Moreover, one has the following:*

(a) *If $\lambda > 0$, then $\ell' \geq \ell$.*
(b) *If $\ell', \ell < (n-1)/2$, then $\lambda > 0$.*
(c) *If $\ell = (n-1)/2$, composing F on the right by the linear map*

$$\sigma_{00}(z,w) = (z_{\ell+1}, \ldots, z_{n-1}, z_1, \ldots, z_\ell, -w) \in \mathrm{Aut}(\mathbb{H}^n_{(n-1)/2}) \tag{2.7}$$

if necessary, one can assume $\lambda > 0$.

In what follows, we shall always assume that $N \geq n$ and $\ell' \geq \ell$. We now write,

$$F = (\widetilde{f}, g) = (f_1, \ldots, f_{n-1}, f_n, \ldots, f_{N-1}, g) := (f, \phi, g) = (f_1, \ldots, f_{n-1}, \phi_1, \ldots, \phi_{N-n}, g).$$

Then, $\sigma_{\ell,\ell',n} \circ F$ sends $\mathbb{H}_\ell^n$ into $\mathbb{H}^N_{\ell,\ell',n}$. For simplicity of notation, we still write, in what follows, F for $\sigma_{\ell,\ell',n} \circ F$. Hence (2.6), associated to the new F, reads as

$$\lambda E_{(\ell,n-1)} = A E_{(\ell,\ell',n,N-1)} \overline{A}^t, \tag{2.8}$$

where $E_{(\ell,\ell',n,N-1)}$ is an $(N-1) \times (N-1)$ diagonal matrix with -1 for its first ℓ diagonal elements as well as elements at position $n-1+j$, $1 \leq j \leq \ell' - \ell$, and with 1 for the remaining diagonal elements. Similarly to Lemma 2.1, we have the following.

Lemma 2.1′. *Let F be as above, mapping a neighborhood of* 0 *in $\mathbb{H}_\ell^n$ into $\mathbb{H}^N_{\ell,\ell',n}$ with $F(0) = 0$ and $\lambda = g_w(0) \neq 0$. Then, the rank of A is $n-1$ and the statements* (b) *and* (c) *of Lemma 2.1 hold.*

For the rest of this section, we shall assume $\lambda > 0$. Write

$$A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \tag{2.9}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are $(N-1)$-row complex vectors. We get, from (2.8),

$$\langle \alpha_j, \overline{\alpha_k} \rangle_{\ell,\ell',n} = \lambda \delta_{j,\ell} \delta_k^j, \quad 1 \le j, k \le n-1.$$

Here, δ_j^k is the usual Kronecker symbol. Let

$$S_{n-1} := \operatorname{span}\{\alpha_1, \ldots, \alpha_{n-1}\} \subset \mathbb{C}^{N-1},$$

and let $S_{n-1}^{\perp_{\ell,\ell',n}} \subset \mathbb{C}^{N-1}$ be the orthogonal complement of S_{n-1} with respect to the Hermitian form $\langle \cdot, \bar{\cdot} \rangle_{\ell,\ell',n}$. When $N > n$, we claim that there is a vector $\alpha_n \in S_{n-1}^{\perp_{\ell,\ell',n}}$ such that $|\alpha_n|^2_{\ell,\ell',n} = \langle \alpha_n, \overline{\alpha_n} \rangle_{\ell,\ell',n} \neq 0$. To prove this claim, we argue by contradition. If there is no such vector α_n, then for any $\alpha,\ \beta \in S_{n-1}^{\perp_{\ell,\ell',n}}$, we have

$$\begin{aligned} 0 = |\alpha + \beta|^2_{\ell,\ell',n} &= |\alpha|^2_{\ell,\ell',n} + |\beta|^2_{\ell,\ell',n} + 2 \operatorname{Re} \langle \alpha, \overline{\beta} \rangle_{\ell,\ell',n} \\ &= 2 \operatorname{Re} \langle \alpha, \overline{\beta} \rangle_{\ell,\ell',n}. \end{aligned}$$

Replacing β by $i\beta$, we conclude that $\langle \alpha, \overline{\beta} \rangle_{\ell,\ell',n} = 0$. Hence, any $\beta \in S_{n-1}^{\perp_{\ell,\ell',n}}$ is also in $(S_{n-1}^{\perp_{\ell,\ell',n}})^{\perp_{\ell,\ell',n}} = S_{n-1}$. This contradicts the non-degeneracy of the Hermitian form $\langle \cdot, \bar{\cdot} \rangle_{\ell,\ell',n}$. Hence, we can choose $\alpha_n \in S_{n-1}^{\perp_{\ell,\ell',n}}$ such that $\langle \alpha_n, \overline{\alpha_n} \rangle_{\ell,\ell',n} = \pm\lambda$. By induction, we can further find α_j, $n \le j \le N-1$, such that $\langle \alpha_j, \overline{\alpha_j} \rangle_{\ell,\ell',n} = \pm\lambda$ and $\alpha_j \perp_{\ell,\ell',n} \alpha_r$ for $1 \le r < j$. Hence, denoting by $\widetilde{A}$ the invertible $(N-1) \times (N-1)$ matrix, whose rows are the vectors $\alpha_1, \ldots, \alpha_{N-1}$, we have

$$\widetilde{A}\, E_{(\ell,\ell',n,N-1)} \overline{\widetilde{A}}^t = \lambda \widetilde{E}, \tag{2.10}$$

where $\widetilde{E}$ is an $(N-1) \times (N-1)$ diagonal matrix whose first $(n-1)$ diagonal elements are the same as those of $E_{(\ell,n-1)}$ and the remaining ones are ± 1. Since $\lambda > 0$, by comparing the number of the negative eigenvalues in (2.10) and changing the order of α_j for $j > n-1$ if necessary, we can make $\widetilde{E} = E_{(\ell,\ell',n,N-1)}$. Hence, we have the following

$$\widetilde{A}\, E_{(\ell,\ell',n,N-1)} \overline{\widetilde{A}}^t = \lambda E_{(\ell,\ell',n,N-1)}. \tag{2.10$'$}$$

Define the invertible linear map of $\mathbb{C}^N$ by

$$\tau^*_{\lambda,\tilde{A}}(z^*, w^*) := \left(z^* \widetilde{A}^{-1}, \frac{1}{\lambda} w^* \right). \tag{2.11}$$

It easily follows from (2.10$'$) that $\tau^*_{\lambda,\tilde{A}} \in \mathrm{Aut}_0(\mathbb{H}^N_{\ell,\ell',n})$. We define the first normalization of F by setting

$$F^* = (\widetilde{f^*}, g^*) := \tau^*_{\lambda,\tilde{A}} \circ F = \left(\widetilde{f}\,\widetilde{A}^{-1}, \frac{1}{\lambda} g\right).$$

Clearly, F^* maps M into $\mathbb{H}^N_{\ell,\ell',n}$ with $g^* = w + O(|(z,w)|^2)$ and $\widetilde{f^*} = (z,0) + \mathbf{a}w + O(|(z,w)|^2)$. Let $r = \mathrm{Re}(\partial^2 g^*/\partial w^2)(0)$ and let $G \in \mathrm{Aut}_0(\mathbb{H}^N_{\ell,\ell',n})$ be defined by

$$\begin{aligned} &G_{r,\mathbf{a}}(z^*,w^*) := \left(\frac{z^* - \mathbf{a}w^*}{\Delta(z^*,w^*)}, \ \frac{w^*}{\Delta(z^*,w^*)}\right), \qquad \text{with} \\ &\Delta(z^*,w^*) := 1 + 2i\langle z^*, \overline{\mathbf{a}}\rangle_{\ell,\ell',n} + (r - i\langle \mathbf{a}, \overline{\mathbf{a}}\rangle_{\ell,\ell',n})w^*. \end{aligned} \tag{2.12}$$

We then define the second normalization F^{**} of F to be the composition of F^* with G on the left. Namely, we have

$$F^{**} = (f^{**}, \phi^{**}, g^{**}) = (\widetilde{f}^{**}, g^{**}) := G_{r,\mathbf{a}} \circ F^* = G_{r,\mathbf{a}} \circ \tau_{\lambda,\tilde{A}} \circ F. \tag{2.13}$$

We then still have the basic equation on M:

$$\begin{aligned} \mathrm{Im}(g^{**}) &= \sum_{j=1}^{N-1} \delta_{j,\ell,\ell',n} f_j^{**}\overline{f_j^{**}} \\ &= \sum_{j=1}^{n-1} \delta_{j,\ell} f_j^{**}\overline{f_j^{**}} + \sum_{j=1}^{N-n} \delta_{j,\ell'-\ell}\phi_j^{**}\overline{\phi_j^{**}}. \end{aligned} \tag{2.13$'$}$$

Now as in the Heisenberg hypersurface case ([**14**], Lemma 5.2), by collecting terms of weighted degree 2, 3 and 4 in (2.13$'$), we derive the following fundamental equation, called the Chern–Moser equation or the Gauss equation for the embedding of a neighborhood of 0 in $\mathbb{H}^n_\ell$ into $\mathbb{H}^N_{\ell,\ell',n}$:

$$\begin{aligned} f^{**}(z,w) &= z + \frac{i}{2} a^{(1)}(z) w + o_{\mathrm{wt}}(3), \\ \phi^{**}(z,w) &= \phi^{**(2)}(z) + o_{\mathrm{wt}}(2), \\ g^{**}(z,w) &= w + o_{\mathrm{wt}}(4), \qquad \text{with} \\ \left\langle a^{(1)}(z), \overline{z}\right\rangle_\ell |z|^2_\ell &= -\sum_{j=1}^{\ell'-\ell} |\phi_j^{**(2)}(z)|^2 + \sum_{j=\ell'-\ell+1}^{N-n} |\phi_j^{**(2)}(z)|^2, \end{aligned} \tag{2.14}$$

where $a^{(1)}(z)$ is an $(n-1)$-vector valued linear function in z without constant term. Summarizing the above, we have

Lemma 2.2. *Let* $\ell' \geq \ell \geq 0$ *and* $N \geq n > 1$. *Assume that* $F = (f, \phi, g)$ *is a holomorphic map from* M, *a neighborhood of* 0 *in* $\mathbb{H}^n_\ell$, *into* $\mathbb{H}^N_{\ell,\ell',n}$ *with* $F(0) = 0$ *and* $g_w(0) = \lambda > 0$. *Then, there exists* $\tau \in \mathrm{Aut}_0(\mathbb{H}^N_{\ell,\ell',n})$ *such that* $F^{**} = \tau \circ F$ *has the normalization with the Gauss–Chern–Moser identity as in (2.14). Further, when* $\ell > 0$, *and either* $\ell' = \ell$ *or*$N - n = \ell' - \ell$, *we also have* $\phi^{**(2)}(z) \equiv 0$.

Proof of Lemma 2.2. From the above discussion preceding the statement of the lemma, it suffices to take $\tau := G_{r,\mathbf{a}} \circ \tau^*_{\lambda,\tilde{A}}$, where $G_{r,\mathbf{a}}$ and $\tau^*_{\lambda,\tilde{A}}$ are given by (2.11) and (2.12). It remains only to prove the last statement in the lemma. Indeed, when $\ell' = \ell$ or $N - n = \ell' - \ell$, we have from the last identity in (2.14) $\pm \sum_{j=1}^{N-n} |\phi_j^{**(2)}|^2 = |z|^2_\ell < a^{(1)}(z), \overline{z} >_\ell$. Since $\ell > 0$, the equation $|z|^2_\ell = 0$ defines a real analytic hypersurface in $\mathbb{C}^{n-1} \setminus \{0\}$. Hence, it is a uniqueness set for holomorphic functions. Since on the set defined by $|z|^2_\ell = 0$, we necessarily have $\phi_j^{**(2)}(z) = 0$, we conclude that $\phi_j^{**(2)}(z) \equiv 0$ for any j. This, together with (2.14), completes the proof of Lemma 2.2. q.e.d.

3. Application of the group structure to the Gauss–Chern–Moser equation

In this section, we prove the following result.

Theorem 3.1. *Let* $F = (f, \phi, g)$ *be a holomorphic map from a neighborhood* M *of* 0 *in* $\mathbb{H}^n_\ell$ *into* $\mathbb{H}^N_{\ell,\ell',n}$ *with* $\ell' \geq \ell > 0$, $N \geq n > 1$. *Suppose that* F *satisfies the normalization condition (2.14). (Namely, the second normalization* F^{**} *of* F *is the same as* F*). Assume that either* $\ell' = \ell$ *or* $\ell' - \ell = N - n$. *Then,* $F(z, w) = (z, 0, \ldots, 0, w)$.

Combining Theorem 3.1 with Lemma 2.2 and observing that $G_{r,\mathbf{a}}$ and $\tau_{\lambda,\tilde{A}}$ are linear fractional, we can easily get the following:

Corollary 3.1′. *Let* $F = (f, \phi, g)$ *be a holomorphic map from a neighborhood* M *of* 0 *in* $\mathbb{H}^n_\ell$ *into* $\mathbb{H}^N_{\ell,\ell',n}$ $(\ell' \geq \ell > 0, N \geq n > 1)$. *Suppose that* $F(0) = 0$ *and* $g_w(0) > 0$. *Assume that either* $\ell' = \ell$ *or* $\ell' - \ell = N - n$. *Then,* F *is a linear fractional map,* $F(z, w) = Q(z, w)/(1 + q(z, w))$ *with* $Q(z, w)$ *a vector valued linear polynomial and* $q(z, w)$ *a linear scalar polynomial vanishing at* 0.

Proof of Theorem 3.1. For any $p \in M$ close to the origin, we can associate a map F_p from a small neighborhood of 0 in $\mathbb{H}^n_\ell$ to $\mathbb{H}^N_{\ell,\ell',n}$ with

$F_p(0)=0$, defined by

$$F_p=\tau_p^F\circ F\circ\sigma_p^0=(f_p,\phi_p,g_p), \tag{3.1}$$

where for each $p=(z_0,w_0)\in M$, we write $\sigma^0_{(z_0,w_0)}\in \mathrm{Aut}(\mathbb{H}^n_\ell)$ for the map sending (z,w) to $(z+z_0,w+w_0+2i\langle z,\overline{z_0}\rangle_\ell)$ and we define $\tau^F_{(z_0,w_0)}\in\mathrm{Aut}(\mathbb{H}^N_{\ell,\ell',n})$ by

$$\begin{aligned}&\tau^F_{(z_0,w_0)}(z^*,w^*)\\ &\quad:=\left(z^*-\widetilde{f}(z_0,w_0),w^*-\overline{g(z_0,w_0)}-2i\langle z^*,\overline{\widetilde{f}(z_0,w_0)}\rangle_{\ell,\ell',n}\right).\end{aligned}$$

Here, we used again the notation $F=(\widetilde{f},g)$ as in the previous section. Notice that $\sigma_p^0(0)=p$ and $\tau_p^F(F(p))=0$. Consistent with the notation in Section 2, we write

$$\lambda(p)=(g_p)_w(0)=g_w(p)-2i\left\langle \widetilde{f}_w(p),\overline{\widetilde{f}(p)}\right\rangle_{\ell,\ell',n}. \tag{3.2}$$

Then, for p close to 0, one still has $\lambda(p)>0$. Now, a direct computation shows that, for $1\le l,\ r,\ s\le n-1$, we have

$$\begin{aligned}&\alpha_l(p):=\left(\frac{\partial\widetilde{f}_p}{\partial z_l}\right)\Big|_0\\ &\quad=\left(\frac{\partial f_{p,1}}{\partial z_l},\ldots\frac{\partial f_{p,n-1}}{\partial z_l},\frac{\partial\phi_{p,1}}{\partial z_l},\ldots\frac{\partial\phi_{p,N-n}}{\partial z_l}\right)\Big|_0=L_l(\widetilde{f})(p),\end{aligned} \tag{3.2$'$}$$

$$\begin{aligned}&\frac{\partial^2\widetilde{f}_p}{\partial z_r\partial z_s}(0)\\ &\quad=\left(\frac{\partial^2 f_{p,1}}{\partial z_r\partial z_s},\ldots,\frac{\partial^2 f_{p,n-1}}{\partial z_r\partial z_s},\frac{\partial^2\phi_{p,1}}{\partial z_l\partial z_s},\ldots,\frac{\partial^2\phi_{p,N-n}}{\partial z_l\partial z_s}\right)\Big|_0=L_rL_s(\widetilde{f})(p)\end{aligned} \tag{3.2$''$}$$

By Lemma 2.1$'$, the rank of $\{\alpha_1(p),\ldots,\alpha_{n-1}(p)\}$ is $(n-1)$. Consistent with the notation of the previous section, we write $A(p)$ for the $(n-1)\times(N-1)$ matrix, whose j^{th}-row is the vector $\alpha_j(p)$. As in Section 2, we can choose again $\alpha_j(p)$ for $n\le j\le N-1$ such that

$$\widetilde{A}(p)\,E_{(\ell,\ell',n,N-1)}\,\overline{\widetilde{A}(p)}^t=\lambda(p)E_{(\ell,\ell',n,N-1)}, \tag{3.3}$$

where the $(N-1)\times(N-1)$ matrix $\widetilde{A}(p)$ has $\alpha_j(p)$ as its j^{th}-row. Define, as in Section 2,

$$F_p^*=(\widetilde{f_p}^*,g_p^*)=((f_p)^*_1,\ldots,(f_p)^*_{n-1},(\phi_p)^*_1,\ldots,(\phi_p)^*_{N-n},g_p^*)$$

by $F_p^* := (\widetilde{f_p}^{\,*}, g_p^*) = \left(\widetilde{f_p}\,\widetilde{A}(p)^{-1}, \lambda(p)^{-1}g_p\right)$. Define $\mathbf{a}(p)$, $r(p)$ and $G(p)$ in a similar way as in Section 2. Then, we arrive at the second normalization $F_p^{**} = G(p)\circ F_p^*$ of F_p. Hence by Lemma 2.2, we have

$$(\phi_p^{**})^{(2)}(z) \equiv 0, \quad (f_p^{**})^{(2)}(z) \equiv 0. \tag{3.4}$$

Making use of (3.3), we have

$$\widetilde{A}(p)^{-1} = \lambda(p)^{-1} E_{(\ell,\ell',n,N-1)}\, \overline{\widetilde{A}(p)^t}\, E_{(\ell,\ell',n,N-1)}.$$

Hence,

$$\begin{aligned}\widetilde{f_p^*} &= \lambda(p)^{-1}\widetilde{f_p}\, E_{(\ell,\ell',n,N-1)}\overline{\widetilde{A}(p)^t}\, E_{(\ell,\ell',n,N-1)}\\ &= \lambda(p)^{-1}\widetilde{f_p}\, E_{(\ell,\ell',n,N-1)}\overline{D(p)},\end{aligned}$$

with

$$D(p) = \widetilde{A}(p)^t\, E_{(\ell,\ell',n,N-1)} = (D_1(p), \dots, D_{N-1}(p)),$$

where, for each j, $D_j(p) = \pm\alpha_j^t(p)$ are column vectors. Hence,

$$(\phi_j^*)_p = \frac{1}{\lambda(p)}\left\langle \widetilde{f_p}, \overline{D_{j+n-1}(p)}\right\rangle_{\ell,\ell',n}. \tag{3.5}$$

Write

$$\mathbf{a}(p) = \frac{\partial \widetilde{f_p^*}}{\partial w}(0) = (a_1(p), \dots, a_{N-1}(p)).$$

By (2.12), we have, for $1 \le j \le N-n$,

$$\begin{aligned}&(\phi_p^{**})_j \\ &\quad= \frac{(\phi_p^*)_j - a_{n-1+j}(p)g_p^*}{1 + 2i\langle \widetilde{f_p^*}, \overline{\mathbf{a}(p)}\rangle_{\ell,\ell',n} - (-r(p) + i\langle \mathbf{a}(p), \overline{\mathbf{a}(p)}\rangle_{\ell,\ell',n})g_p^*}.\end{aligned} \tag{3.6}$$

Collecting coefficients of $z_r z_s$,$1 \le r, s \le n-1$, in the Taylor expansion at 0 of the right-hand side of (3.6), we get by (3.4) and (3.5)

$$\left\langle \frac{\partial^2 \widetilde{f_p}}{\partial z_r \partial z_s}(0), \overline{D_{j+n-1}^t(p)}\right\rangle_{\ell,\ell',n} = 0, \quad 1 \le j \le N-n,$$

and hence, by the definition of $D_j(p)$, we have

$$\left\langle \frac{\partial^2 \widetilde{f_p}}{\partial z_r \partial z_s}(0), \overline{\alpha_{j+n-1}(p)}\right\rangle_{\ell,\ell',n} = 0, \quad 1 \le j \le N-n.$$

By the orthogonality relation (3.3), we conclude that

$$\frac{\partial^2 \widetilde{f_p}}{\partial z_r \partial z_s}(0) \in \mathrm{Span}\{\alpha_j(p)\}_{j=1}^{n-1}. \tag{3.7}$$

Hence, from (3.2′) and (3.2″), we conclude that there are unique $d_j^{rs}(p)$ such that

$$L_r L_s \widetilde{f}(p) = \sum_{j=1}^{n-1} d_j^{rs}(p) L_j(\widetilde{f})(p). \tag{3.8}$$

Since $A(p)$ has rank $(n-1)$ and the map F is holomorphic near 0, it is easy to see that the $d_j^{rs}(p)$ depend real analytically on p in a neighborhood of 0 in $\mathbb{H}_\ell^n$. Considering only the ϕ-components in (3.8), we have:

$$L_r L_s \phi(p) = \sum_{j=1}^{n-1} d_j^{rs}(p) L_j(\phi)(p). \tag{3.8′}$$

Applying $\overline{L_s}$ and $\overline{L_r L_s}$ to (3.8′), respectively, we conclude that there is a matrix valued function $\Psi(p)$, with elements depending real analytically on p, such that

$$D^2\phi(p) = D\phi(p)\,\Psi(p). \tag{3.9}$$

Here, $D\phi$ and $D^2\phi$ represent, respectively, all the first and the second partial derivatives of ϕ. Using the normalization condition $\phi(0) = 0$, $D\phi(0) = 0$ and appying a standard uniqueness argument for the complete ODE system (3.9), we conclude that $\phi \equiv 0$. Now, since the ϕ component of the map $F = (f, \phi, g)$ vanishes identically in a neighborhood of 0 in $\mathbb{H}_\ell^n$, and since by assumption of Theorem 3.1, we have the normalization $f(z, w) = z + o_{\mathrm{wt}}(3)$, $g(z, w) = w + o_{\mathrm{wt}}(4)$, we can apply the the equi-dimensional result of Chern–Moser [5] to the map (f, g) (which maps a neighborhood of 0 in $\mathbb{H}_\ell^n$ into $\mathbb{H}_\ell^n$) to conclude that $(f(z, w), g(z, w)) \equiv (z, w)$. This completes the proof of Theorem 3.1. q.e.d.

4. A Hopf lemma for holomorphic maps

We keep the notation of Section 3. In this section, we first prove the following lemma.

Lemma 4.1. *Let $F = (f, \phi, g)$ be a holomorphic map from a neighborhood M of 0 in $\mathbb{H}_\ell^n$ into $\mathbb{H}_{\ell,\ell',n}^N$, $\ell' \geq \ell > 0, N \geq n > 1$, with $F(0) = 0$. Assume that either $\ell' < n-1$ or $N - \ell' - 1 < n - 1$. For each $p \in M$, let $F_p = (\widetilde{f}_p, g_p)$ be defined as in (3.1). If $(g_p)_w(0) = 0$ for all p sufficiently close to the origin, then there exists a constant $(N - \ell' - 1) \times \ell'$ complex* *matrix* V *with* $V\,\overline{V}^t = \mathrm{Id}_{N-\ell'-1}$ *such that*

$$g \equiv 0, \quad (f_1, \ldots, f_\ell, f_n, \ldots, f_{n+\ell'-\ell-1}) \equiv (f_{\ell+1}, \ldots, f_{n-1}, f_{n+\ell'-\ell}, \ldots, f_{N-1})\, V. \tag{4.1}$$

Before proceeding with the proof of Lemma 4.1, we first give the following elementary lemma.

Lemma 4.2. *Suppose* A *is a complex* $(n-1)\times(N-1)$ *matrix,* $N \geq n > 1$, *satisfying*

$$A\,E_{(\ell',N-1)}\,\overline{A^t} = 0. \tag{$*$}$$

Assume that either $0 < \ell' < n-1$ *or* $N-\ell'-1 < n-1$. *Then, the rank of* A *is strictly less than* $(n-1)$.

Proof of Lemma 4.2. Write α_j, $1 \leq j \leq n-1$, for the row vector of A and write $\alpha_j = (x^{(j)}, y^{(j)})$ with $x^{(j)}$ an ℓ'-row vector whose components are the first ℓ' elements of α_j and $y^{(j)}$ the row vector with the remaining components of α_j. The assumption $(*)$ then implies that, for $1 \leq j, k \leq n-1$,

$$\langle x^{(j)}, \overline{x^{(k)}}\rangle_0 = \langle y^{(j)}, \overline{y^{(k)}}\rangle_0. \tag{$**$}$$

The rest of the assumption of the lemma indicates that either the $\{x^{(j)}\}_{j=1}^{n-1}$ or the $\{y^{(j)}\}_{j=1}^{n-1}$ are lineraly dependent. Without loss of generality, we may assume the former. Hence, there exists a non-zero sequence $\{a_1, \ldots, a_{n-1}\}$ of complex number such that $\sum_{j=1}^{n-1} a_j x^{(j)} = 0$. Then, $(**)$ easily implies that $\sum_{j=1}^{n-1} a_j y^{(j)} = 0$. Hence, $\sum_{j=1}^{n-1} a_j \alpha_j = 0$, which completes the proof of the lemma. q.e.d.

Proof of Lemma 4.1. It follows from (3.1) that we have

$$g_p = g \circ \sigma_p^0 - \overline{g(p)} - 2i\langle \widetilde{f} \circ \sigma_p^0, \overline{\widetilde{f}(p)}\rangle_{\ell,\ell',n}. \tag{4.2}$$

Hence, it follows that

$$(g_p)_w(0) = g_w(z_0, w_0) - 2i\langle \widetilde{f}_w(z_0, w_0), \overline{\widetilde{f}(z_0, w_0)}\rangle_{\ell,\ell',n},$$

where $p = (z_0, w_0) \in M$. Therefore, by the assumption in Lemma 4.1 and taking the complexification of (4.2), we get the following equation

$$g_w(z, w) = 2i\langle \widetilde{f}_w(z, w), \overline{\widetilde{f}}(\chi, w - 2i\langle z, \chi\rangle_\ell)\rangle_{\ell,\ell',n}$$

where $z \in \mathbb{C}^{n-1}, w \in \mathbb{C}, \chi \in \mathbb{C}^{n-1}$ are independent variables near the origin. Equivalently, we have, for $z \in \mathbb{C}^{n-1}, \tau \in \mathbb{C}, \chi \in \mathbb{C}^n$ near the origin,

$$g_w(z, \tau + 2i\langle z, \chi\rangle_\ell) = 2i\langle \widetilde{f}_w(z, \tau + 2i\langle z, \chi\rangle_\ell), \overline{\widetilde{f}}(\chi, \tau)\rangle_{\ell,\ell',n}. \tag{4.3}$$

Letting $\chi=0, \tau=0$, we get from (4.3)

$$g_w(z,0)\equiv 0. \tag{4.4}$$

Applying $\partial/\partial\chi_j$, $j=1,\ldots,n-1$, to (4.3) and letting $\chi=0,\tau=0$, we obtain

$$2i\delta_{j,\ell}z_j g_{w^2}(z,0)=2i\langle \widetilde{f}_w(z,0),\overline{\widetilde{f}_{z_j}}(0)\rangle_{\ell,\ell',n}. \tag{4.5}$$

Since we assumed that $g_w(0)=\lambda=0$,$\{f_{z_j}(0)\}_{j=1}^{n-1}$ are linearly dependent by (2.8) and a slight variant version of Lemma 4.2. Hence, there is a non-zero $(n-1)$-tuple $(a_1,\ldots,a_{n-1})$ such that $\sum_{j=1}^{n-1}a_j\widetilde{f}_{z_j}(0)=0$. It thus follows from (4.5) that

$$\left(\sum_{j=1}^{n-1}\overline{a_j}\delta_{j,\ell}z_j\right)g_{w^2}(z,0)\equiv 0.$$

Since $\sum_j\overline{a_j}\delta_{j,\ell}z_j\not\equiv 0$, we conclude that $g_{w^2}(z,0)\equiv 0$. In particular, we have $g_{w^2}(0)=0$. Applying the previous argument to F_p, we then also have $(g_p)_{w^2}(0)=0$ for p close to 0. Applying $\partial^2/\partial w^2$ to (4.2) and evaluating at 0, we obtain

$$g_{w^2}(z,w)=2i\langle\widetilde{f}_{w^2}(z,w),\overline{\widetilde{f}(z,w)}\rangle_{\ell,\ell',n}$$

for $(z,w)\in M$ close to the origin. Hence, after complexification, we get

$$g_{w^2}(z,\tau+2i\langle z,\chi\rangle_\ell)=2i\langle\widetilde{f}_{w^2}(z,\tau+2i\langle z,\chi\rangle_\ell),\overline{\widetilde{f}}(\chi,\tau)\rangle_{\ell,\ell',n}. \tag{4.6}$$

Applying $\partial/\partial\chi_j$ to (4.6) and letting $\chi=0,\tau=0$, we have, with the same argument as above, that

$$g_{w^3}(z,0)\equiv 0.$$

By an induction argument, we conclude that $g_{w^k}(z,0)\equiv 0$ for any $k\geq 1$. Together with (2.5), we have proved that $g\equiv 0$. Considering F_p instead of F, we also have $g_p\equiv 0$ for $p\in M$ close to 0. By (4.2), this gives that $\langle\widetilde{f}\circ\sigma_p^0(z,w),\overline{\widetilde{f}(p)}\rangle_{\ell,\ell',n}\equiv 0$ for any $(z,w)\in\mathbb{C}^n$ close to 0, and any $p\in M$ close to 0. Since σ_p^0 is an automorphism, we also have $\langle\widetilde{f}(z,w),\overline{\widetilde{f}(p)}\rangle_{\ell,\ell',n}\equiv 0$ for (z,w) and p as before. Since $p\mapsto\langle\overline{\widetilde{f}(z,w)},\widetilde{f}(p)\rangle_{\ell,\ell',n}$ is holomorphic in p and since M is a uniqueness set for holomorphic functions, in particular, we get $\langle\widetilde{f}(z,w),\overline{\widetilde{f}(z,w)}\rangle_{\ell,\ell',n}\equiv 0$ for $(z,w)\in\mathbb{C}^n$ close to the origin. We complete the proof the Lemma by applying a result of D'Angelo [7] (Proposition 3, p. 102). q.e.d.

Next, we give the following version of the Hopf lemma for holomorphic maps:

Lemma 4.3. *Let* $F = (f, \phi, g)$ *be a holomorphic map sending a neighborhood* M *of* 0 *in* $\mathbb{H}^n_\ell$ *into* $\mathbb{H}^N_{\ell,\ell',n}$*, with* $F(0) = 0$*,* $0 < \ell < n-1$*,* $\ell' \geq \ell$*, and* $N \geq n > 1$*. Assume that* $g_w(0) = 0$ *and either* $\ell' = \ell$ *or* $\ell' - \ell = N - n$*. If either* $(g_p)_w(0) \geq 0$ *for any* $p \in M$ *close to* 0*, or* $\ell' = \ell \leq (n-1)/2$*, then* (4.1) *holds.*

Proof of Lemma 4.3. We shall prove the lemma by contradiction. Assume that one of the two conditions in (4.1) does not hold. By Lemma 4.1, there would be a point $p \in M$ arbitrarily close to the origin such that $(g_p)_w(0) \neq 0$. By Lemma 2.1 (b), we necessarily have $(g_p)_w(0) > 0$ in the case $\ell' = \ell < (n-1)/2$. When $\ell' = \ell = (n-1)/2$, we have either $(g_p)_w(0) > 0$ or $(g_p \circ \sigma_{00})_w(0) > 0$. (See (2.7) for the definition of σ_{00}.) Hence, by Corollary 3.1′, F_p (or $F_p \circ \sigma_{00}$), and thus F, must be linear fractional. That is, $F(Z) = CZ/(1 + q(Z))$ with C an $(n \times N)$ complex constant matrix and $Q(Z)$ a vector valued linear polynomial vanishing at 0. Since $g(z, 0) \equiv 0$ and $g_w(0) = 0$, we conclude immediately that $g \equiv 0$. We, therefore, have on M the identity

$$\sum_{j=1}^{\ell} |f_j|^2 + \sum_{j=n}^{\ell'-\ell+n-1} |f_j|^2 = \sum_{j=\ell+1}^{n-1} |f_j|^2 + \sum_{j=\ell'-\ell+n}^{N-1} |f_j|^2. \tag{4.7}$$

Claim 4.4. Suppose that $\sum_{1\leq j\leq m_1} |h_j|^2 = \sum_{1\leq j\leq m_2} |k_j|^2$ on M, where h_j and k_j are homogeneous first order holomorphic polynomials, and m_1, m_2 are positive integers. Then, $\sum_{1\leq j\leq m_1} |h_j|^2 \equiv \sum_{1\leq j\leq m_2} |k_j|^2$ on $\mathbb{C}^n$.

Proof of Claim 4.4. Write $h_j = za^t_j + c_j w$ and $k_j = zb^t_j + d_j w$, where a_j, b_j are $(n-1)$-vectors and c_j, d_j are complex numbers. Then, we have

$$\begin{aligned}\sum_{j=1}^{m_1} &\left(za^t_j + c_j(u + i|z|^2_\ell)\right)\left(\overline{za^t_j} + \overline{c_j}(u - i|z|^2_\ell)\right) \\ &= \sum_{j=1}^{m_2} \left(zb^t_j + d_j(u + i|z|^2_\ell)\right)\left(\overline{zb^t_j} + \overline{d_j}(u - i|z|^2_\ell)\right).\end{aligned}$$

Identifying terms of weighted degree 2, 3, and 4, we can easily see that the above also holds if we replace $|z|^2_\ell$ by an independent variable t. This completes the proof of Claim 4.4. q.e.d.

We now return to (4.7). After multiplying by the common denominator $|1 + q(Z)|^2$ and applying Claim 4.4, we conclude that (4.7) holds in a neighborhood of the origin U in $\mathbb{C}^n$. Hence, since $g \equiv 0$, we have

$F(U) \subset \mathbb{H}^N_{\ell,\ell',n}$. In particular, with the choice of the point $p \in M$ as above, we have for (z,w) in a neighborhood of 0 in $\mathbb{C}^n$

$$(g_p)(z,w) \equiv \overline{(g_p)(z,w)} + 2i\langle (\widetilde{f_p})(z,w), \overline{(\widetilde{f_p})(z,w)}\rangle_{\ell,\ell',n}. \tag{4.8}$$

Differentiating (4.8) with respect to w and evaluating at $(z,w) = 0$, we obtain a contradiction to the fact that $(g_p)_w(0) \neq 0$. The proof of Lemma 4.2 is complete. q.e.d.

5. Proofs of the main theorems

We now complete the proofs of the theorems stated in the introduction.

Proofs of Theorem 1.6 *and* 1.8. We give here only the proof of Theorem 1.6, since the proof of Theorem 1.8 follows from the same arguments. Let F be as in Theorem 1.6. Assume that $\lambda = g_w(0) \neq 0$. Recall that λ is real-valued. Then, we know by Lemma 2.1 that $\lambda > 0$ when $\ell < (n-1)/2$. Also, a simple computation shows that when F preserves sides, $g_w(0) > 0$. (See the argument below, especially, (5.2) and (5.4).) Hence, by Lemma 2.2 and Theorem 3.1, there is a $\tau \in \mathrm{Aut}_0(\mathbb{H}^N_\ell)$ such that $\tau \circ F(z,w) \equiv (z,0,w)$. The same conclusion holds when $\ell = (n-1)/2$ and $\lambda > 0$. If $\ell = (n-1)/2$ and $\lambda < 0$, applying the above argument to $F \circ \sigma_{00}$, (see (2.7)), there exists a $\tau \in \mathbb{H}^N_{\ell,\ell',n}$ such that $\tau \circ F \circ \sigma_{00}(z,w) = (z,0,w)$. Hence, $\tau \circ F(z,w) = (z_{\ell+1}, \ldots, z_{n-1}, z_1, \ldots, z_\ell, 0, \ldots, 0, -w)$. This completes the proof of Part (i) of Theorem 1.6. The proof of Part (ii) of Theorem 1.6 follows from Lemma 4.3, and the observation that if F preserves sides, then so does F_p for any $p \in M$ close to 0, and hence $(g_p)_w(0) \geq 0$. q.e.d.

Proof of Theorem 1.1. Let F be the holomorphic map given in Theorem 1.1. Since the Levi form of the boundary of $\mathbb{B}^n_\ell$ has at least one negative eigenvalue at any point, by making use of a result of Siu and Ivashkovich (see [**22**], [**17**]), we can assume that F extends to a holomorphic map in a neighborhood of p in $\mathbb{CP}^n$ into $\mathbb{CP}^N$. Hence, by the assumption of the theorem, F sends a piece of the boundary of $\mathbb{B}^n_\ell$ into the boundary of $\mathbb{B}^N_\ell$. Next, since $U(N+1,\ell+1)$ acts transitively on the boundary of $\mathbb{B}^N_\ell$, after composing F by automorphisms, we can assume that $p = [1,0,\ldots,0,1]$ and $F([1,0,\ldots,0,1]) = [1,0,\ldots,0,1]$. Now for $(z,w) = (z_1,\ldots,z_{n-1},w) \in \mathbb{C}^n$, let

$$\Psi_n(z,w) \colon = [i+w, 2z, i-w] \in \mathbb{CP}^n \tag{5.1}$$

be the Cayley transformation which biholomorphically maps the generalized Siegel upper-half space $\mathbb{S}^n_\ell$ and its boundary $\mathbb{H}^n_\ell$ into $\mathbb{B}^n_\ell \setminus \{[z_0, \ldots, z_n] : \ z_0 + z_n = 0\}$ and $\partial\mathbb{B}^n_\ell \setminus \{[z_0, \ldots, z_n] : \ z_0 + z_n = 0\}$, respectively. Let $\hat{F} := \Psi_N^{-1} \circ F \circ \Psi_n$. Then, $\hat{F}$ maps an open neighborhood M of 0 in $\mathbb{H}^n_\ell$ into $\mathbb{H}^N_\ell$ and $\hat{F}(0) = 0$. For $Z^* = (z_1^*, \ldots, z_{N-1}^*, w^*) \in \mathbb{C}^N$,$w^* = u^* + iv^*$, let

$$\rho(Z^*, \overline{Z^*}) = -v^* + \sum_{j=1}^{\ell} |z_j^*|^2 + \sum_{j=\ell+1}^{N-1} |z_j^*|^2. \tag{5.2}$$

Then, by the assumption on F, we have $\rho(\hat{F}_p(z,w), \overline{\hat{F}_p(z,w)}) < 0$ for $(z,w) \in \mathbb{S}^n_\ell$ close to 0 and $p \in M$ close to 0. In particular,

$$\rho(\hat{F}_p(0,v), \overline{\hat{F}_p(0,v)}) < 0$$

for small positive v, and hence

$$\frac{\partial}{\partial v}\left[\rho(\hat{F}_p(0,v), \overline{\hat{F}_p(0,v)})\right]\Big|_{v=0} \leq 0. \tag{5.3}$$

As in Section 3, we write $\hat{F}_p = (\widetilde{\hat{f}}_p, \hat{g}_p)$. Since $\hat{g}_p(0)$ is real valued, combining (5.3) with the Cauchy–Riemann equation, we obtain:

$$(\hat{g}_p)_w(0) = \frac{\partial \operatorname{Im}(\hat{g}_p)}{\partial v}(0) \geq 0. \tag{5.4}$$

Since F does not map $U_p \cap \mathbb{B}^n_\ell$ into $\partial\,\mathbb{B}^N_\ell$ and hence $\hat{F}$ does not map a neighborhood of 0 in $\mathbb{C}^n$ into $\mathbb{H}^N_\ell$, it follows from (5.4) and Lemma 4.3 that we necessarily have $(\hat{g})_w(0) > 0$. By Corollary 3.1$'$, we thus conclude that $\hat{F}(z,w)$ is linear fractional. Since $F = \Psi_N \circ \hat{F} \circ \psi_n^{-1}$, we conclude that F is a linear map in the homogeneous coordinates $Z = [z_0, \ldots, z_n]$, namely, $F[Z] = Z \cdot C$ with C an $(n+1) \times (N+1)$ complex matrix. Since F sends a piece of $\partial\,\mathbb{B}^n_\ell$ near $[1, 0, \ldots, 0, 1]$ into $\partial\,\mathbb{B}^N_\ell$, we conclude that $C\,E_{(\ell+1,N+1)}\,\overline{C}^t = E_{(\ell+1,n+1)}$. We extend C to an $(N+1) \times (N+1)$- matrix $\widetilde{C}$ as in the proof of Lemma 2.1$'$, (see (2.8)–(2.10)$'$) such that $\widetilde{C}\,E_{(\ell+1,N+1)}\,\overline{\widetilde{C}}^t = E_{(\ell+1,N+1)}$ and define $\tau(Z^*) = Z^*\,\widetilde{C}^{-1}$ for $Z^* \in \mathbb{CP}^N$. Since we have $\widetilde{C}^{-1} = E_{(\ell+1,N+1)}\,\overline{\widetilde{C}}^t\,E_{(\ell+1,N+1)}$, we can easily see that $\tau \circ F[Z] = [Z, 0]$. This completes the proof of Theorem 1.1. q.e.d.

Proof of Theorem 1.4. Since Corollary 3.1$'$ and Lemma 4.3 apply also to the case $N - n = \ell' - \ell$, The proof of Theorem 1.4 follows the same lines as those for the proof of Theorem 1.1. We omit repeating the details here. q.e.d.

Remark 5.1. The argument used here to prove Theorem 1.1, in the case $N = n > 1$, also gives the fact that any proper holomorphic self-map of $\mathbb{B}^n_\ell$, $n > 1$, $\ell > 0$, is an element in $U(n+1, \ell+1)$. (For the case of $\ell = 0$, this is the well-known theorem of Alexander [**1**].)

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References

[1] H. Alexander, *Proper holomorphic maps in* $\mathbf{C}^n$, Indiana Univ. Math. J. **26** (1977) 137–146, MR 0422699, Zbl 0391.32015.

[2] M.S. Baouendi, P. Ebenfelt, & L.P. Rothschild, *Real Submanifolds in Complex Space and Their Mappings*, Princeton Math., **47**, Princeton Univ. Press, Princeton, NJ, 1999, MR 1668103, Zbl 0944.32040.

[3] M.S. Baouendi & L.P. Rothschild, *Geometric properties of mappings between hypersurfaces in complex spaces*, J. Differential Geom. **3** (1990) 473–499, MR 1037411, Zbl 0702.32014.

[4] H. Cao & N. Mok, *Holomorphic immersions between compact hyperbolic space forms*, Invent. Math. **100** (1990) 49–61, MR 1037142, Zbl 0698.53035.

[5] S.S. Chern & J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974) 219–271, MR 0425155, Zbl 0302.32015.

[6] J. Cima & T.J. Suffridge, *Boundary behavior of rational proper maps*, Duke Math. J. **60** (1990) 135–138, MR 1047119, Zbl 0694.32016.

[7] J. D'Angelo, *Several Complex Variables and the Geometry of Hypersurfaces*, Studies in Advanced Math., CRC Press, Boca Raton, 1993, MR 1224231, Zbl 0854.32001.

[8] J. D'Angelo, *Proper holomorphic mappings between balls of different dimensions*, Michigan Math. J. **35** (1988) 83–90, MR 0931941, Zbl 0651.32014.

[9] P. Ebenfelt, X. Huang, & D. Zaitsev, *The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics*, Amer. J. Math. **127** (2005) 169–191, MR 2115664.

[10] J. Faran, *Maps from the two ball to the three ball*, Invent. Math. **68** (1982) 441–475, MR 0669425, Zbl 0519.32016.

[11] J. Faran, *On the linearity of proper maps between balls in the lower dimensional case*, J. Differential Geom. **24** (1986) 15–17, MR 0857373, Zbl 0592.32018.

[12] F. Forstnerič, *Extending proper holomorphic mappings of positive codimension*, Invent. Math. **95** (1989) 31–62, MR 0969413, Zbl 0633.32017.

[13] F. Forstnerič, *Proper holomorphic mappings: a survey*, in 'Several complex variables' (Stockholm, 1987/1988), Math. Notes, **38**, Princeton University Press, Princeton, NJ, 1993, 297–363, MR 1207867, Zbl 0778.32008.

[14] X. Huang, *On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions*, J. Differential Geom. **51** (1999) 13–33, MR 1703603, Zbl 1042.32008.

[15] X. Huang, *On a semi-rigidity property for holomorphic maps*, Asian J. Math. **7** (2003) 463–492, MR 2074886.

[16] X. Huang, *On some problems in several complex variables and CR geometry*, First International Congress of Chinese Mathematicians (Beijing, 1998), AMS/IP Stud. Adv. Math., **20**, Amer. Math. Soc., Providence, RI, 2001, 383–396, MR 1830195, Zbl 1048.32022.

[17] S. Ivashkovich, *The Hartogs–type extension theorem for meromorphic maps into compact Kähler manifolds*, Invent. Math. **109** (1992) 47–54, MR 1168365, Zbl 0738.32008.

[18] N. Mok, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Mathematics, **6**, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989, MR 1081948, Zbl 0912.32026.

[19] N. Mok, Y.T. Siu, & S.K. Yeung, *Geometric superrigidity,* Invent. Math. **113** (1993) 57–83, MR 1223224, Zbl 0808.53043.

[20] J. Moser, *Analytic surfaces in* $\mathbb{C}^2$ *and their local hull of holomorphy*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985) 397–410, MR 0802502, Zbl 0585.32007.

[21] H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo, II **23** (1907) 185–220, JFM 38.0459.02.

[22] Y.-T. Siu, *Extension of meromorphic maps into Kähler manifolds*, Ann. of Math. (2) **102** (1975) 421–462, MR 0463498, Zbl 0318.32007.

[23] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of* n *complex variables*, J. Math. Soc. Japan **14** (1962) 397–429, MR 0145555, Zbl 0113.06303.

[24] Z.-H. Tu, *Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains*, Math. Z. **240** (2002) 13–35, MR 1906705, Zbl 1020.32010.

[25] S. Webster, *On mapping an* n*-ball into an* $(n+1)$*-ball in the complex space*, Pacific J. Math. **81** (1979) 267–272, MR 0543749.

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