# SUPERCONGRUENCES FOR TRUNCATED ${ }_{n+1} F_{n}$ HYPERGEOMETRIC SERIES WITH APPLICATIONS TO CERTAIN WEIGHT THREE NEWFORMS 

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#### Abstract

We prove general results on supercongruences between values of truncated ${ }_{n+1} F_{n}$ hypergeometric functions and their character analogs. As a consequence of the main results of this paper, we prove Beukers-type supercongruences for certain weight three newforms.


## 1. Introduction

In RV1], Fernando Rodriguez-Villegas discovered numerically a number of Beukers-type supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \leq 3$. Specifically, he observed supercongruences between the truncated fundamental period of the Picard-Fuchs differential equation of the manifold and an expression derived from the number of its $\mathbb{F}_{p}$-points. This had been motivated by his joint work with Candelas and de la Ossa [OV]. Here we prove general results on supercongruences between values of truncated ${ }_{n+1} F_{n}$ hypergeometric functions and their character analogs. As a consequence of these results, we prove some of the observed supercongruences for manifolds of dimension $d=2$. Supercongruences of this type were first observed by Beukers $[B]$ in connection with the Apéry numbers used in the proof of the irrationality of $\zeta(3)$. Ahlgren and Ono AO proved Beukers' supercongruence conjecture relating Apéry numbers to the coefficients of a certain weight four newform.

In RV1] and RV2], Rodriguez-Villegas identified four modular K3 surfaces with potential supercongruences. We define Dedekind's eta function by the infinite product:

$$
\begin{equation*}
\eta(z):=q^{\frac{1}{24}} \prod_{n=1}^{\infty}\left(1-q^{n}\right), q:=e^{2 \pi i z} \tag{1.1}
\end{equation*}
$$

We then define the integers $a(n), b(n)$, and $c(n)$ by

$$
\begin{equation*}
\sum_{n=1}^{\infty} a(n) q^{n}:=\eta^{6}(4 z) \in S_{3}\left(\Gamma_{0}(16),\left(\frac{-4}{d}\right)\right), \tag{1.2}
\end{equation*}
$$

[^0]\[

$$
\begin{align*}
& \sum_{n=1}^{\infty} b(n) q^{n}:=\eta^{3}(6 z) \eta^{3}(2 z) \in S_{3}\left(\Gamma_{0}(12),\left(\frac{-3}{d}\right)\right),  \tag{1.3}\\
& \sum_{n=1}^{\infty} c(n) q^{n}:=\eta^{2}(8 z) \eta(4 z) \eta(2 z) \eta^{2}(z) \in S_{3}\left(\Gamma_{0}(8),\left(\frac{-2}{d}\right)\right) \tag{1.4}
\end{align*}
$$
\]

These weight three newforms are related to modular K3 surfaces. They are extensively studied in SB , where, among other results, the authors prove several modulo $p$ congruences. From [RV1] and RV2] we are able to formulate the following:
Conjecture. If $p \geq 5$ is a prime, then

$$
\begin{align*}
& \sum_{n=0}^{p-1} \frac{(2 n)!^{3}}{n!^{6}} 64^{-n} \equiv a(p) \quad\left(\bmod p^{2}\right),  \tag{1.5}\\
& \sum_{n=0}^{p-1} \frac{(3 n)!(2 n)!}{n!^{5}} 108^{-n} \equiv b(p) \quad\left(\bmod p^{2}\right),  \tag{1.6}\\
& \sum_{n=0}^{p-1} \frac{(4 n)!}{n!^{4}} 256^{-n} \equiv c(p) \quad\left(\bmod p^{2}\right),  \tag{1.7}\\
& \sum_{n=0}^{p-1} \frac{(6 n)!}{(3 n)!n!^{3}} 1728^{-n} \equiv \gamma(p) a(p) \quad\left(\bmod p^{2}\right), \tag{1.8}
\end{align*}
$$

where $\gamma(p):=-1$ if $p \equiv 5(\bmod 12)$ and $\gamma(p):=1$ otherwise.
It should be noted that (1.5) has already been proved by several individuals including Ishikawa [I], Van Hamme [vH], and Ahlgren [A]. The numbers 64, 108, 256, 1728 are called the conifold points (see RV1). Here we prove several cases of these conjectures.

To state our results, we recall basic facts about characters and Jacobi sums and introduce some notation. We denote by $\mathbb{F}_{q}$ the finite field with $q=p^{r}$ elements, where $p$ is a prime. We extend all multiplicative characters $\chi: \mathbb{F}_{q}^{\times} \rightarrow \mathbb{C}_{p}$, including the trivial character $\epsilon_{q}$, to $\mathbb{F}_{q}$ by setting $\chi(0):=0$. If $A$ and $B$ are two characters on $\mathbb{F}_{q}$, then we define $\binom{A}{B}$ in terms of the Jacobi sum by

$$
\begin{equation*}
\binom{A}{B}:=\frac{B(-1)}{q} J_{r}(A, \bar{B})=\frac{B(-1)}{q} \sum_{x \in \mathbb{F}_{q}} A(x) \bar{B}(1-x), \tag{1.9}
\end{equation*}
$$

where $J_{r}(\cdot, \cdot)$ is a Jacobi sum over $\mathbb{F}_{p^{r}}$. We recall some useful properties of binomial coefficients ([G], (2.6)-(2.7)):

$$
\begin{equation*}
\binom{A}{B}=\binom{A}{A \bar{B}} \text { and }\binom{A}{B}=\binom{B \bar{A}}{B} B(-1) . \tag{1.10}
\end{equation*}
$$

If $A_{0}, A_{1}, \ldots A_{n}$, and $B_{1}, B_{2}, \ldots B_{n}$ are characters on $\mathbb{F}_{q}$ and if $x \in \mathbb{F}_{q}$, then Greene [G] defines ${ }_{n+1} F_{n}$ Gaussian hypergeometric series by

$$
{ }_{n+1} F_{n}\left(\left.\begin{array}{cccc}
A_{0}, & A_{1}, & \ldots, & A_{n}  \tag{1.11}\\
& B_{1}, & \ldots, & B_{n}
\end{array} \right\rvert\, x\right)_{q}:=\frac{q}{q-1} \sum_{\chi}\binom{A_{0} \chi}{\chi}\binom{A_{1} \chi}{B_{1} \chi} \ldots\binom{A_{n} \chi}{B_{n} \chi} \chi(x),
$$

where the sum runs over all characters $\chi$ on $\mathbb{F}_{q}$. We note that this definition lives in some extension of $\mathbb{Q}_{p}$. For certain choices of characters, the right-hand side actually is in $\mathbb{Z}_{p}$.

If $m$ is a positive integer, then we define the truncated hypergeometric series by

$$
{ }_{n+1} F_{n}\left(\left.\begin{array}{cccc}
a_{0}, & a_{1}, & \ldots, & a_{n}  \tag{1.12}\\
& b_{1}, & \ldots, & b_{n}
\end{array} \right\rvert\, x\right)_{\operatorname{tr}(m)}:=\sum_{k=0}^{m-1} \frac{\left(a_{0}\right)_{k}\left(a_{1}\right)_{k} \cdots\left(a_{n}\right)_{k}}{k!\left(b_{1}\right)_{k} \cdots\left(b_{n}\right)_{k}} x^{k}
$$

where $(a)_{k}:=a(a+1) \cdots(a+k-1)$.
If $n \in \mathbb{N}$, we define the $p$-adic $\Gamma$-function on the ring $\mathbb{Z}_{p}$ of $p$-adic integers by

$$
\begin{equation*}
\Gamma_{p}(n):=(-1)^{n} \prod_{j<n, p \nmid j} j \text { and } \Gamma_{p}(x):=\lim _{n \rightarrow x} \Gamma_{p}(n), x \in \mathbb{Z}_{p} \tag{1.13}
\end{equation*}
$$

where in the limit we take any sequence of positive integers that approaches $x$ in the $p$-adic sense. We recall three basic properties of the $p$-adic $\Gamma$-function. If $p \geq 5$ is a prime and $x, y \in \mathbb{Z}_{p}$, then the following are true. We have

$$
\Gamma_{p}(x+1)= \begin{cases}-x \Gamma_{p}(x) & \text { if } x \in \mathbb{Z}_{p}^{*}  \tag{1.14}\\ -\Gamma_{p}(x) & \text { if } x \in p \mathbb{Z}_{p}\end{cases}
$$

If $n \geq 1$, then

$$
\begin{equation*}
x \equiv y \quad\left(\bmod p^{n}\right) \Rightarrow \Gamma_{p}(x) \equiv \Gamma_{p}(y) \quad\left(\bmod p^{n}\right) \tag{1.15}
\end{equation*}
$$

If $R(x)$ denotes the reduction of $x$ modulo $p$ to the range $\{1, \ldots, p\}$, then

$$
\begin{equation*}
\Gamma_{p}(x) \Gamma_{p}(1-x)=(-1)^{R(x)} \tag{1.16}
\end{equation*}
$$

We are now able to state the results of this paper. Let $\phi_{q}$ denote the character of order 2 on $\mathbb{F}_{q}$, and let $\epsilon_{q}$ denote the trivial character on $\mathbb{F}_{q}$. In the sequel we shall drop the subscript $q$ since it will be obvious from the context.

Theorem 1. If $p$ is a prime, $p \equiv 1\left(\bmod d_{i}\right)$ with $1 \leq m_{i}<d_{i}, \rho_{i}$ is a character of order $d_{i}$ on $\mathbb{F}_{p}$, and $\sum_{i=1}^{n+1} \frac{m_{i}}{d_{i}} \geq n-1$, then

$$
\begin{aligned}
{ }_{n+1} F_{n} & \left(\begin{array}{cccc}
\frac{m_{1}}{d_{1}}, & \frac{m_{2}}{d_{2}}, & \ldots, & \left.\frac{m_{n+1}}{d_{n+1}} \right\rvert\, \\
1, & \ldots, & 1
\end{array}\right)_{t r(p)} \\
& \equiv(-1)^{n} p^{n} \cdot{ }_{n+1} F_{n}\left(\left.\begin{array}{cccc}
\rho_{1}^{m_{1}}, & \rho_{2}^{m_{2}}, & \ldots, & \rho_{n+1}^{m_{n+1}} \\
& \epsilon_{p}, & \ldots, & \epsilon_{p}
\end{array} \right\rvert\,\right)_{p}-\delta \cdot p \quad\left(\bmod p^{2}\right)
\end{aligned}
$$

$$
\text { where } \delta:= \begin{cases}0 & \text { if } \sum_{i=1}^{n+1} \frac{m_{i}}{d_{i}}>n-1 \\ \prod_{i=1}^{n+1} \Gamma_{p}\left(1-\frac{m_{i}}{d_{i}}\right) & \text { if } \sum_{i=1}^{n+1} \frac{m_{i}}{d_{i}}=n-1 .\end{cases}
$$

Corollary 1. If $p$ is a prime, $p \equiv 1\left(\bmod d_{i}\right), 1 \leq m_{i}<d_{i}$, and $\rho_{i}$ is a character of order $d_{i}$ on $\mathbb{F}_{p}$, then

Corollary 2. If $p$ is a prime, $p \equiv 1(\bmod d), 1 \leq m<d$, and $\rho$ is a character of order d on $\mathbb{F}_{p}$, then

$$
{ }_{3} F_{2}\left(\begin{array}{ccc}
\frac{1}{2}, & \frac{m}{d}, & \left.1-\frac{m}{d} \right\rvert\, 1 \\
1, & 1
\end{array}\right)_{\operatorname{tr}(p)} \equiv p^{2} \cdot{ }_{3} F_{2}\left(\begin{array}{ccc}
\phi_{q}, & \rho^{m}, & \bar{\rho}^{m} \mid \\
& \epsilon_{p}, & \epsilon_{p}
\end{array}\right)_{p} \quad\left(\bmod p^{2}\right)
$$

$$
\begin{aligned}
& { }_{4} F_{3}\left(\begin{array}{ccccc}
\frac{m_{1}}{d_{1}}, 1-\frac{m_{1}}{d_{1}}, & \frac{m_{2}}{d_{2}}, & 1-\frac{m_{2}}{d_{2}} & & 1 \\
1, & 1, & 1
\end{array}\right)_{\operatorname{tr}(p)} \\
& \equiv-p^{3} \cdot{ }_{4} F_{3}\left(\begin{array}{cccc}
\rho_{1}{ }^{m_{1}}, & {\overline{\rho_{1}}}^{m_{1}}, & \rho_{2}{ }^{m_{2}}, & {\overline{\rho_{2}}}^{m_{2}} \\
& \epsilon_{p}, & \epsilon_{p}, & \epsilon_{p}
\end{array}\right)_{p} \\
& -(-1)^{\frac{m_{1}}{d_{1}}(p-1)+\frac{m_{2}}{d_{2}}(p-1)} p \quad\left(\bmod p^{2}\right) .
\end{aligned}
$$

Theorem 2. If $p$ is a prime, $p \equiv-1\left(\bmod d_{i}\right), 1 \leq m_{i}<d_{i}$, and $\rho_{i}$ is a character of order $d_{i}$ on $\mathbb{F}_{p^{2}}$, then

$$
\begin{aligned}
{ }_{4} F_{3} & \left(\begin{array}{cccc}
\frac{m_{1}}{d_{1}}, & 1-\frac{m_{1}}{d_{1}}, & \frac{m_{2}}{d_{2}}, & \left.1-\frac{m_{2}}{d_{2}} \right\rvert\, 1 \\
1, & 1, & 1
\end{array}\right)_{\operatorname{tr}(p)}^{2} \\
& \equiv-p^{6} \cdot{ }_{4} F_{3}\left(\begin{array}{cccc}
\rho_{1}{ }^{m_{1}}, & \bar{\rho}_{1} m_{1} & \rho_{2}{ }^{m_{2}}, & {\overline{\rho_{2}}}^{m_{2}} \mid 1 \\
\epsilon_{p^{2}}, & \epsilon_{p^{2}}, & \epsilon_{p^{2}} & 1
\end{array}{ }_{p^{2}}\right.
\end{aligned} \quad\left(\bmod p^{2}\right) . . ~ \$
$$

Theorem 3. If $p$ is a prime, $p \equiv-1(\bmod d), 1 \leq m<d$, and $\rho$ is a character of order $d$ on $\mathbb{F}_{p^{2}}$, then

$$
{ }_{3} F_{2}\left(\begin{array}{ccc}
\frac{1}{2}, & \frac{m}{d}, & \left.1-\frac{m}{d} \right\rvert\, \\
1, & 1
\end{array}\right)_{\operatorname{tr}(p)}^{2} \equiv p^{4} \cdot{ }_{3} F_{2}\left(\begin{array}{ccc}
\phi_{q}, & \rho^{m}, & \bar{\rho}^{m} \\
& \epsilon_{p^{2}}, & \epsilon_{p^{2}}
\end{array}\right)_{p^{2}} \quad\left(\bmod p^{2}\right) .
$$

For (1.5) - (1.8), we are able to prove the following.
Theorem 4. Let $p \geq 5$ be a prime.
(1) We have $\sum_{n=0}^{p-1} \frac{(2 n)!^{3}}{n!^{6}} 64^{-n} \equiv a(p)\left(\bmod p^{2}\right)$.
(2) If $p \equiv 1(\bmod 3)$, then $\sum_{n=0}^{p-1} \frac{(3 n)!(2 n)!}{n!^{5}} 108^{-n} \equiv b(p)\left(\bmod p^{2}\right)$.

$$
\text { If } p \equiv 2(\bmod 3), \text { then }\left(\sum_{n=0}^{p-1} \frac{(3 n)!(2 n)!}{n!^{5}} 108^{-n}\right)^{2} \equiv b(p)^{2}\left(\bmod p^{2}\right)
$$

(3) If $p \equiv 1(\bmod 4)$, then $\sum_{n=0}^{p-1} \frac{(4 n)!}{n!!^{4}} 256^{-n} \equiv c(p)\left(\bmod p^{2}\right)$.

$$
\text { If } p \equiv 3(\bmod 4), \text { then }\left(\sum_{n=0}^{p-1} \frac{(4 n)!}{n!^{4}} 256^{-n}\right)^{2} \equiv c(p)^{2}\left(\bmod p^{2}\right)
$$

(4) If $p \equiv 1(\bmod 6)$, then $\sum_{n=0}^{p-1} \frac{(6 n)!}{(3 n)!n!3} 1728^{-n} \equiv a(p)\left(\bmod p^{2}\right)$.

$$
\text { If } p \equiv 5(\bmod 6), \text { then }\left(\sum_{n=0}^{p-1} \frac{(6 n)!}{(3 n)!n!^{3}} 1728^{-n}\right)^{2} \equiv a(p)^{2}\left(\bmod p^{2}\right)
$$

In sections 2 and 3, we prove Theorems 1-3 using the method of proof in M2] (i.e. we use basic character theory, the Gross-Koblitz formula [GK], and properties of the $p$-adic $\Gamma$-function). For these proofs, the arguments are similiar enough to those in [M2] that we only point out the changes made in the strategy. The key change is in dealing with the strange combinatorial expressions involving harmonic numbers that we encounter. In [M2], using Wilf-Zeilberger theory, the author evaluated two families of expressions explicitly in terms of $p$ (see (5.28), (6.21)). Here, by writing the expressions in a different way and by using new techniques, we avoid WZ-theory, Corollary 2 is immediate and Corollary 1 uses (1.16).

In section 4, we prove Theorem 4 using Corollary 2 and Theorem 3. In addition, we need to evaluate the Gaussian hypergeometric series in terms of the trace of Frobenius. To accomplish this we borrow an idea from Ono [O] and use a character analog of Whipple's theorem for classical ${ }_{3} F_{2}$ hypergeometric series. This analog was found by Greene [G], and it yields an expression in terms of Jacobi sums. Using several theorems of Berndt, Evans, and Williams ([BE, [BEW]), and a theorem of Beukers and Stienstra [SB], we evaluate these Jacobi sums in terms of the coefficients of the respective weight three modular forms.

## 2. Proof of Theorem 1

We begin this section with a lemma and a proposition. The proof of the lemma is trivial.

Lemma 2.1. If $p \geq 5$ is a prime and $n \geq 1$, then

$$
\sum_{k=1}^{p-1} k^{n} \equiv \begin{cases}0 & (\bmod p) \text { if } p-1 \nmid n \\ -1 & (\bmod p) \text { if } p-1 \mid n\end{cases}
$$

Proposition 2.2. Let $m$ and $d$ be integers such that $1 \leq m<d$. If $p \equiv 1(\bmod d)$ is a prime, then define $r$ such that $p=d r+1$.
(1) If $0 \leq j \leq m r$, then $\left(\frac{m}{d}\right)_{j} \equiv \Gamma_{p}\left(1-\frac{m}{d}\right)((d-m) r+j)$ ! $(\bmod p)$.
(2) If $m r+1 \leq j \leq p-2$, then $\left(\frac{m}{d}\right)_{j}\left(\frac{d}{m p}\right) \equiv \Gamma_{p}\left(1-\frac{m}{d}\right) \frac{(d-m) r+j)!}{p}(\bmod p)$.

Proof of Proposition 2.2. We first prove (1). From Proposition (1.14), we have

$$
\begin{equation*}
\Gamma_{p}\left(\frac{m}{d}+j\right)=(-1)^{j}\left(\frac{m}{d}\right)_{j} \Gamma_{p}\left(\frac{m}{d}\right) \tag{2.1}
\end{equation*}
$$

Using (1.15) and (1.13), we obtain

$$
\begin{equation*}
\Gamma_{p}\left(\frac{m}{d}+j\right) \equiv \Gamma_{p}((d-m) r+1+j) \equiv(-1)^{(d-m) r+1+j}((d-m) r+j)!\quad(\bmod p) \tag{2.2}
\end{equation*}
$$

We then equate the two expressions and use Proposition (1.16).
For (2), the argument is similar. We use Proposition (1.14) to obtain

$$
\begin{equation*}
\Gamma_{p}\left(\frac{m}{d}+j\right)=(-1)^{j} \cdot \frac{d}{m p} \cdot\left(\frac{m}{d}\right)_{j} \Gamma_{p}\left(\frac{m}{d}\right) \tag{2.3}
\end{equation*}
$$

and we use Proposition (1.15) and (1.13) to obtain
$\Gamma_{p}\left(\frac{m}{d}+j\right) \equiv \Gamma_{p}((d-m) r+1+j) \equiv(-1)^{(d-m) r+1+j} \cdot \frac{1}{p} \cdot((d-m) r+j)!\quad(\bmod p)$.
We note that the expressions in (2.3) and (2.4) are $p$-integral. The terms with the $p$ 's in their denominator are only present to cancel out their reciprocals.

Proof of Theorem 1. Recalling the notation of Theorem 1, we define $r_{i}:=\frac{p-1}{d_{i}}$. We also define the harmonic number $H_{n}:=1+\frac{1}{2}+\cdots+\frac{1}{n}$. Without loss of generality, we can assume $m_{1} r_{1} \leq m_{2} r_{2} \leq \cdots \leq m_{n+1} r_{n+1}$. Using basic character theory, the Gross-Koblitz formula, and $p$-adic $\Gamma$-function properties, we follow the method of proof in [M2, section 5] to obtain

$$
\begin{align*}
& (-1)^{n} p^{n} \cdot{ }_{n+1} F_{n}\left(\begin{array}{rrr}
\rho_{1}^{m_{1}}, & \rho_{2}^{m_{2}}, \ldots, & \rho_{n+1}^{m_{n+1}} \\
\epsilon_{p}, & \ldots, & \epsilon_{p}
\end{array}\right)_{p} \equiv p\left\{\sum_{m_{1} r_{1}+1}^{m_{2} r_{2}}\left(\prod_{i=1}^{n+1} \frac{\left(\frac{m_{i}}{d_{i}}\right)_{j}}{j!}\right)\left(\frac{j d_{1}}{m_{1} p}\right)\right.  \tag{2.5}\\
& \left.+\sum_{j=0}^{m_{1} r_{1}}\left[\prod_{i=1}^{n+1} \frac{\left(\frac{m_{i}}{d_{i}}\right)_{j}}{j!}\right]\left[1+j \cdot\left[\sum_{i=1}^{n+1}\left(H_{\left(d_{i}-m_{i}\right) r_{i}+j}-H_{j}\right)\right]\right]\right\} \\
& +\sum_{j=0}^{m_{2} r_{2}} \prod_{i=1}^{n+1} \frac{\left(\frac{m_{i}}{d_{i}}\right)_{j}}{j!}\left(\bmod p^{2}\right) .
\end{align*}
$$

This is analogous to (5.25) in [M2. If $m_{1} r_{1}=m_{2} r_{2}$, then only the second sum in the braces is present. Using Proposition 2.2 and arguing as we did for (5.27) in
(M2] yields

$$
\begin{align*}
& (-1)^{n} p^{n} \cdot{ }_{n+1} F_{n}\left(\begin{array}{cccc}
\rho_{1}^{m_{1}}, & \rho_{2}^{m_{2}}, & \ldots, & \rho_{n+1}^{m_{n+1}} \mid \\
\epsilon_{p}, & \ldots, & \epsilon_{p}
\end{array}\right)_{p} \\
& \equiv \equiv \sum_{j=0}^{m_{2} r_{2}} \prod_{i=1}^{n+1} \frac{\left(\frac{m_{i}}{d_{i}}\right)_{j}}{j!}+p \cdot\left(\prod_{i=1}^{n+1} \Gamma_{p}\left(1-\frac{m_{i}}{d_{i}}\right)\right) \cdot A \quad\left(\bmod p^{2}\right) \tag{2.6}
\end{align*}
$$

where

$$
\begin{equation*}
A:=\sum_{j=0}^{m_{2} r_{2}}\left[\prod_{i=1}^{n+1} \frac{\left(\left(d_{i}-m_{i}\right) r_{i}+j\right)!}{j!}\right] \cdot\left[1+j \cdot \sum_{i=1}^{n+1}\left(H_{\left(d_{i}-m_{i}\right) r_{i}+j}-H_{j}\right)\right] \tag{2.7}
\end{equation*}
$$

We determine when $A \equiv 0(\bmod p)$ and when $A \equiv 1(\bmod p)$. We can extend the sum in $A$ to $p-1$ to obtain

$$
\begin{equation*}
A \equiv \sum_{j=0}^{p-1}\left[\prod_{i=1}^{n+1} \frac{\left(\left(d_{i}-m_{i}\right) r_{i}+j\right)!}{j!}\right] \cdot\left[1+j \cdot \sum_{i=1}^{n+1}\left(H_{\left(d_{i}-m_{i}\right) r_{i}+j}-H_{j}\right)\right] \quad(\bmod p) \tag{2.8}
\end{equation*}
$$

In other words, for $j \geq m_{2} r_{2}+1$ the factorials for $i=1$ and $i=2$ each contain a factor of $p$; moreover, at most one $p$ is cancelled by a term of the harmonic number. Noting that

$$
\begin{equation*}
\left(\left(d_{i}-m_{i}\right) r_{i}+j\right)!/ j!=(j+1)_{\left(d_{i}-m_{i}\right) r_{i}} \tag{2.9}
\end{equation*}
$$

we can rewrite the right-hand side:

$$
\begin{equation*}
A \equiv \sum_{j=0}^{p-1} \frac{d}{d j}\left[j \prod_{i=1}^{n+1}(j+1)_{\left(d_{i}-m_{i}\right) r_{i}}\right] \quad(\bmod p) \tag{2.10}
\end{equation*}
$$

Define the polynomial $p(j) \in \mathbb{Z}[j]$ by

$$
\begin{equation*}
p(j):=\frac{d}{d j}\left[j \prod_{i=1}^{n+1}(j+1)_{\left(d_{i}-m_{i}\right) r_{i}}\right]=\sum_{k=0}^{D} a_{k} j^{k}, \text { where } D:=\sum_{i=1}^{n+1}\left(d_{i}-m_{i}\right) r_{i} \tag{2.11}
\end{equation*}
$$

to obtain

$$
\begin{equation*}
A \equiv \sum_{j=0}^{p-1}\left(a_{0}+\sum_{k=1}^{D} a_{k} j^{k}\right) \equiv \sum_{k=1}^{D} a_{k} \sum_{j=1}^{p-1} j^{k} \quad(\bmod p) \tag{2.12}
\end{equation*}
$$

We consider the case $D<2(p-1)$. By Lemma 2.1 , the only $k$ we need to be concerned with in the $j$ summation is $k=p-1$. Since $p(j)$ is a derivative, the coefficient $a_{p-1}$ will contain a factor of $p$. Hence in this case, $A \equiv 0(\bmod p)$. We consider the case $D=2(p-1)$. Using the above information, the only $k$ we need to concern ourselves with is $k=2(p-1)$. Since $p(j)$ is a derivative of a monic polynomial, it follows that $a_{2(p-1)}=(2 p-1)$. Using Lemma 2.1, we find that $A \equiv 1(\bmod p)$. Since we can extend the first sum in (2.6) from $m_{2} r_{2}$ to $p-1$, the theorem follows.

## 3. Proofs of Theorems 2 and 3

Proof of Theorem 3. We recall the notation of Theorem 3. We define $n$ such that $p=d(n+1)-1$ and define $N_{1}:=m(n+1)-1, N_{2}:=(d-m)(n+1)-1$. Using the method of proof in [M2, section 6], and using the appropriate analog of Proposition 2.2 , we obtain

$$
\begin{align*}
& p^{4} \cdot{ }_{3} F_{2}\left(\begin{array}{ccc}
\phi, & \rho^{m}, & \bar{\rho}^{m} \mid 1 \\
\epsilon_{p^{2}}, & \epsilon_{p^{2}}
\end{array}\right)_{p^{2}} \equiv\left(\sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_{k}\left(\frac{m}{d}\right)_{k}\left(1-\frac{m}{d}\right)_{k}}{k!^{3}}\right)^{2}  \tag{3.1}\\
& +2 \cdot p \cdot\left(\sum_{k=0}^{N_{1}} k \cdot \frac{\left(\frac{1}{2}\right)_{k}\left(\frac{m}{d}\right)_{k}\left(1-\frac{m}{d}\right)_{k}}{k!^{3}}\right) \cdot\left(\Gamma_{p}\left(\frac{1}{2}\right) \Gamma_{p}\left(\frac{m}{d}\right) \Gamma_{p}\left(1-\frac{m}{d}\right)\right) \cdot B \quad\left(\bmod p^{2}\right),
\end{align*}
$$

where

$$
\begin{equation*}
B:=\left(\sum_{j=0}^{\frac{p-1}{2}} \frac{\left(\frac{p-1}{2}+j\right)!}{j!} \frac{\left(N_{1}+j\right)!}{j!} \frac{\left(N_{2}+j\right)!}{j!}\left[H_{\frac{p-1}{2}+j}+H_{N_{1}+j}+H_{N_{2}+j}-3 H_{j}\right]\right) \tag{3.2}
\end{equation*}
$$

We point out that line (3.1) is similar to (6.21) of M2]. We note that the case where $m=1, d=p+1$ is handled like it is in (6.15) of [M2]. Arguing as in section 2 , we obtain

$$
\begin{equation*}
B \equiv \sum_{j=0}^{p-1} \frac{d}{d j}\left[(j+1)_{\frac{p-1}{2}}(j+1)_{N_{1}}(j+1)_{N_{2}}\right] \quad(\bmod p) \tag{3.3}
\end{equation*}
$$

If we let $d$ be the degree of the polynomial in terms of $j$, we see that $d<2(p-1)$. Arguing as in section 2, we have that $B \equiv 0(\bmod p)$.

Proof of Theorem 2. We recall the notation of Theorem 2. We define $n_{i}$ such that $p=d_{i}\left(n_{i}+1\right)-1$. Without lost of generality we assume $m_{1} / d_{1} \leq m_{2} / d_{2} \leq 1 / 2$. We define $R_{i}:=m_{i}\left(n_{i}+1\right)-1$ and $S_{i}:=\left(d_{i}-m_{i}\right)\left(n_{i}+1\right)-1$. We note $R_{1} \leq R_{2}$. Using the method of proof in [M2, section 6], and using the appropriate analog of Proposition 2.2, we obtain

$$
\begin{align*}
&-p^{6} \cdot{ }_{4} F_{3}\left(\begin{array}{ccc}
\rho_{1} m_{1}, & \bar{\rho}_{1} m_{1} & \rho_{2} m_{2}, \\
\epsilon_{p^{2}}, & \epsilon_{p^{2}}, & \epsilon_{p^{2}} m_{2}
\end{array}\right)_{p^{2}} \\
& \equiv\left(\sum_{k=0}^{R_{2}} \frac{\left(\frac{m_{1}}{d_{1}}\right)_{k}\left(\frac{d_{1}-m_{1}}{d_{1}}\right)_{k}\left(\frac{m_{2}}{d_{2}}\right)_{k}\left(\frac{d_{2}-m_{2}}{d_{2}}\right)_{k}}{k!^{3}}\right)^{2}  \tag{3.4}\\
&+2 \cdot p \cdot\left(\sum_{k=0}^{R_{1}} k \cdot \frac{\left(\frac{m_{1}}{d_{1}}\right)_{k}\left(\frac{d_{1}-m_{1}}{d_{1}}\right)_{k}\left(\frac{m_{2}}{d_{2}}\right)_{k}\left(\frac{d_{2}-m_{2}}{d_{2}}\right)_{k}}{k!^{3}}\right) \\
& \cdot\left(\Gamma_{p}\left(\frac{m_{1}}{d_{1}}\right) \Gamma_{p}\left(\frac{d_{1}-m_{1}}{d_{1}}\right) \Gamma_{p}\left(\frac{m_{2}}{d_{2}}\right) \Gamma_{p}\left(\frac{d_{2}-m_{2}}{d_{2}}\right)\right) \cdot C \quad\left(\bmod p^{2}\right)
\end{align*}
$$

where
(3.5)

$$
C:=\sum_{j=0}^{R_{2}} \frac{\left(R_{1}+j\right)!}{j!} \frac{\left(S_{1}+j\right)!}{j!} \frac{\left(R_{2}+j\right)!}{j!} \frac{\left(S_{2}+j\right)!}{j!}\left[H_{R_{1}+j}+H_{S_{1}+j}+H_{R_{2}+j}+H_{S_{2}+j}-4 H_{j}\right]
$$

The $m_{1}=1, d_{1}=p+1$ case and the $m_{1}=m_{2}=1, d_{1}=d_{2}=p+1$ case are handled like they are in [M2]. Arguing as we did in the proof of Theorem 3, we find that $C \equiv 0(\bmod p)$.

## 4. Proof of Theorem 4

We begin with a theorem of Beukers and Stienstra that describes the coefficients of the three modular forms in question. We recall the modular forms (1.2)-(1.4).

Theorem (SB, 14.2]). If we define $\Phi_{4}(p):=a(p), \Phi_{3}(p):=b(p), \Phi_{2}(p):=c(p)$, then the p-th coefficients of the modular forms are given by

$$
\Phi_{M}(p)= \begin{cases}0 & \text { if }\left(\frac{-M}{p}\right)=-1 \\ 4 a^{2}-2 p & \text { if }\left(\frac{-M}{p}\right)=1, p=a^{2}+M b^{2}\end{cases}
$$

We rewrite the conjecture to motivate the use of Corollary 2 and Theorem 3.
Conjecture. If $p \geq 5$ is a prime and $\gamma(p)$ is as before, then

$$
\begin{align*}
& \left.{ }_{3} F_{2}\left(\begin{array}{ccc}
\frac{1}{2}, & \frac{1}{2}, & \left.\frac{1}{2} \right\rvert\, \\
& 1, & 1
\end{array}\right) \quad 1\right)_{\operatorname{tr}(p)} \equiv a(p) \quad\left(\bmod p^{2}\right)  \tag{4.1}\\
& { }_{3} F_{2}\left(\begin{array}{cccc}
\frac{1}{2}, & \frac{1}{3}, & \left.\frac{2}{3} \right\rvert\, & 1 \\
& 1, & 1
\end{array}\right)_{\operatorname{tr}(p)} \equiv b(p) \quad\left(\bmod p^{2}\right)  \tag{4.2}\\
& { }_{3} F_{2}\left(\begin{array}{cccc}
\frac{1}{2}, & \frac{1}{4}, & \left.\frac{3}{4} \right\rvert\, & 1 \\
& 1, & 1
\end{array}\right)_{\operatorname{tr}(p)} \equiv c(p) \quad\left(\bmod p^{2}\right)  \tag{4.3}\\
& { }_{3} F_{2}\left(\begin{array}{cccc}
\frac{1}{2}, & \frac{1}{6}, & \left.\frac{5}{6} \right\rvert\, & 1 \\
& 1, & 1
\end{array}\right){ }_{\operatorname{tr}(p)} \equiv \gamma(p) a(p) \quad\left(\bmod p^{2}\right) \tag{4.4}
\end{align*}
$$

From the new formulation, we see that we need to use Corollary 2 and Theorem 3 where $m=1$ and $d=2,3,4$ or 6 . For primes $p$ where $p \equiv 1(\bmod d)$ we use Corollary 2 , for primes $p$ where $p \equiv-1(\bmod d)$ we use Theorem 3. The proof of Theorem 4 is thus reduced to evaluating the ${ }_{3} F_{2}$ Gaussian hypergeometric series. First, we state a theorem which is a special case of Greene ([G], 4.38(ii)). The corollary follows from the two binomial coefficient properties (1.10).

Theorem ([G]). If $B$ is a nontrivial character on $\mathbb{F}_{q}$, then

$$
{ }_{3} F_{2}\left(\begin{array}{ccc}
\phi, & B, & \bar{B} \\
& \epsilon_{q}, & \epsilon_{q}
\end{array}\right)_{q}=B(-1) \begin{cases}0 & \text { if } B \neq \square \\
\binom{\chi}{\phi}\binom{\chi}{\phi \chi}+\binom{\phi \chi}{\phi}\binom{\phi \chi}{\chi} & \text { if } B=\chi^{2}\end{cases}
$$

Corollary 4.1. If $B$ is a nontrivial character on $\mathbb{F}_{q}$, then

$$
q^{2} \cdot{ }_{3} F_{2}\left(\begin{array}{lll}
\phi, & B, & \bar{B} \\
& \epsilon_{q}, & \epsilon_{q}
\end{array}\right)_{q}=B(-1) \begin{cases}0 & \text { if } B \neq \square \\
J_{r}(\chi, \phi)^{2}+J_{r}(\bar{\chi}, \phi)^{2} & \text { if } B=\chi^{2}\end{cases}
$$

The following two propositions evaluate the Gaussian hypergeometric series in Corollary 2 and Theorem 3, respectively. Theorem 4 is then immediate. We recall (1.2)-(1.4) and define $\alpha_{2}(p):=a(p), \alpha_{3}(p):=b(p), \alpha_{4}(p):=c(p)$, and $\alpha_{6}(p):=$ $a(p)$.

Proposition 4.2. Fix a $d, d \in\{2,3,4,6\}$. Let $p$ be a prime, $p \equiv 1(\bmod d)$. If $\rho_{d}$ is a character of order $d$ on $\mathbb{F}_{p}$, then

$$
p^{2} \cdot{ }_{3} F_{2}\left(\begin{array}{ccc}
\phi, & \rho_{d}, & \overline{\rho_{d}} \\
& \epsilon_{p}, & \epsilon_{p}
\end{array}\right)_{p}=\alpha_{d}(p) .
$$

Proof of Proposition 4.2. This method comes from Ono [O], where he does the case $d=2$. We have two cases. For the first case we consider $p, p \equiv d+1(\bmod 2 d)$. Here $\rho_{d}$ is not a square, so the Gaussian hypergeometric series evaluates to zero. Using the theorem of $[\mathrm{SB}]$ and basic Legendre symbol properties, we have that $\alpha_{d}(p)=0$. For $d=3$ this case is vacuous.

For the second case we consider $p, p \equiv 1(\bmod 2 d)$. Here $\rho_{d}=\chi^{2}$ for some character $\chi$. We consider $d=4$. By [BEW. Theorem 3.3.1],

$$
J_{1}(\chi, \phi)^{2}+J_{1}(\bar{\chi}, \phi)^{2}=(a+i b \sqrt{2})^{2}+(a-i b \sqrt{2})^{2}=4 a^{2}-2 p
$$

where $p=a^{2}+2 b^{2}$. For $d=2,3$ and 6 we use BEW Theorems 3.2.1, 3.1.1, and 3.5.2], respectively. For each $d$, we use [SB] and see that this equals $\alpha_{d}(p)$.

Proposition 4.3. Fix a $d, d \in\{3,4,6\}$. Let $p$ be a prime, $p \equiv-1(\bmod d)$. If $\rho_{d}$ is a character of order $d$ on $\mathbb{F}_{p^{2}}$, then

$$
p^{4} \cdot{ }_{3} F_{2}\left(\begin{array}{ccc}
\phi, & \rho_{d}, & \overline{\rho_{d}} \\
& \epsilon_{p^{2}}, & \epsilon_{p^{2}}
\end{array}\right)_{p^{2}}=\alpha_{d}(p)^{2}-(-1)^{\frac{p-(d-1)}{d}} 2 p^{2} .
$$

Proof of Proposition 4.3. We note that $\rho_{d}$ is always a square. We have two cases to consider. For the first case, we consider $p$ with $p \equiv-1(\bmod 2 d)$. Using the theorem of $[\mathrm{SB}]$ we have that $\alpha_{d}(p)=0$. By [BE, Theorem 2.14],

$$
J_{2}(\chi, \phi)^{2}+J_{2}(\bar{\chi}, \phi)^{2}=2 p^{2}
$$

For the second case, we consider $p$ with $p \equiv d-1(\bmod 2 d)$. We consider $d=4$. By [BE, Theorem 4.6],
$J_{2}(\chi, \phi)^{2}+J_{2}(\bar{\chi}, \phi)^{2}=(a+i b \sqrt{2})^{4}+(a-i b \sqrt{2})^{4}=\left(4 a^{2}-2 p\right)^{2}-2 p^{2}=c(p)^{2}-2 p^{2}$, where $p=a^{2}+2 b^{2}$, and the last equality follows from SB . For $d=3$ this case is vacuous. For $d=6$ we use [BE, Theorem 4.10].

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