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Superconvergence and a posteriori error estimation for triangular mixed finite elements

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Summary. In this paper,we prove superconvergence results for the vector variable when lowest order triangular mixed finite elements of Raviart-Thomas type [17] on uniform triangulations are used, i.e., that the $H(\text{div};\Omega)$-distance between the approximate solution and a suitable projection of the real solution is of higher order than the $H(\text{div};\Omega)$-error. We prove results for both Dirichlet and Neumann boundary conditions. Recently, Duran [9] proved similar results for rectangular mixed finite elements, and superconvergence along the Gauss-lines for rectangular mixed finite elements was considered by Douglas, Ewing, Lazarov and Wang in [11], [8] and [18]. The triangular case however needs some extra effort. Using the superconvergence results, a simple postprocessing of the approximate solution will give an asymptotically exact a posteriori error estimator for the $L^2(\Omega)$-error in the approximation of the vector variable.

Mathematics Subject Classification (1991): 65N30

1. Introduction

First, we consider the following elliptic problem with Dirichlet boundary conditions, defined on an open domain $\Omega \subset \mathbb{R}^2$ which can be triangulated uniformly (see Definition 2.4):

find a function $u \in H^2(\Omega)$ such that:

$$-\text{div}(A\nabla u) = f \text{ in } \Omega,$$

$$u = g \text{ on } \partial\Omega. \tag{1.1}$$

Here, A is a matrix-valued function on Ω such that there exists a $\beta > 0$ such that for all $\mathbf{q} \in [L^2(\Omega)]^2$

$$(A\mathbf{q}, \mathbf{q}) \geq \beta\|\mathbf{q}\|^2 \ . \tag{1.2}$$

$A(x)$ is supposed to be symmetric for all $x \in \Omega$ and the coefficients a_{ij} of A to be Lipschitz continuous. Let $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$.

An equivalent formulation of (1.1) as a system of first-order equations is given by:

find functions $u \in H^2(\Omega)$ and $\mathbf{p} \in H(\text{div};\Omega)$ such that:

$$\mathbf{p} = -A\nabla u \text{ in } \Omega,$$

$$\operatorname{div} \mathbf{p} = f \text{ in } \Omega,$$

$$u = g \text{ on } \partial\Omega. \tag{1.3}$$

In this paper, triangular mixed finite elements of lowest order Raviart-Thomas type [17] are used to approximate u and $\mathbf{p}$ simultaniously. For the approximation $\mathbf{p}_h$ of the vector field $\mathbf{p}$ we prove superconvergence results when a family of uniform triangulations of Ω is used, i.e. we prove that the $L^2(\Omega)$-distance between $\mathbf{p}_h$ and a suitable projection $\boldsymbol{\Pi}_h\mathbf{p}$ of $\mathbf{p}$ is of higher order than the $L^2(\Omega)$-norm of the error $\mathbf{p} - \mathbf{p}_h$. A similar result was recently proved by Duran [9] in the case of rectangular mixed finite elements, making use of the well-known lemma of Bramble and Hilbert [3]. Here, in the triangular case, the Bramble-Hilbert lemma will also be used, however, it takes extra effort before it can be applied.

Superconvergence has recently been studied intensively, for both conforming and mixed finite element methods. For conforming finite elements, see Goodsell and Whiteman for a treatment of linear [12] triangular and quadratic [13], [14] triangular elements. For mixed elements, see Douglas and Roberts [7], who prove superconvergence for the displacement variable on general triangulations. For superconvergence along the Gauss-lines in rectangular mixed finite element methods, see the papers of Douglas, Ewing, Lazarov and Wang [11], [8], [18].

The conditions on the triangulations that are used here are quite restrictive; however, also in conforming finite elements, similar conditions on the triangulations for obtaining global superconvergence results for the gradient are not unusual, and even proven necessary by Duran et al. in [10]. And, for example, the triangulations are exactly of the type that Brezzi et al. [5] use when approximating the semi-conductor device equations and Kaasschieter [15] in the computation of streamlines for potential flow problems. It should be stressed that triangular grids, in general, give more flexibility in the approximation process than rectangular grids, especially when it comes to refining, but in this area still much research needs to be done.

The superconvergence results will be used to obtain a higher order approximation for $\mathbf{p}$ by means of a simple post-processing of $\mathbf{p}_h$. As a consequence, a cheap asymptotically exact a posteriori error estimator for the $L^2(\Omega)$-error in $\mathbf{p}_h$ can be constructed, using arguments of Ainsworth and Craig [2].

The outline of this paper is as follows. In Sect. 2, we establish some notations, recall the mixed finite element method with lowest order triangular Raviart-Thomas elements and collect some well-known results. We mainly follow the notations and conventions of [17], from whom we also adopt the above mentioned projection operator $\boldsymbol{\Pi}_h$. Also, the types of triangulations to be used are described. In Sect. 3 we prove the main theorem on superconvergence of $\mathbf{p}_h$ to $\boldsymbol{\Pi}_h\mathbf{p}$ in the $L^2(\Omega)$-sense for the Dirichlet problem (1.1). In Sect. 4 we will formulate the homogeneous Neumann problem and show that, in this case, an even better rate of superconvergence holds. An a posteriori error estimator for the $L^2(\Omega)$-error in $\mathbf{p}_h$ will be considered in Sect. 5. In Sect. 6 some concluding remarks are made.

2. Spaces, mixed formulation and projections

2.1. Sobolev spaces

First of all, we consider some of the spaces involved. Denote by $H^k(\Omega)$ the usual Sobolev space of order k, i.e. the space of $L^2(\Omega)$-functions v with square-integrable generalized partial derivatives up to order k; this is a Hilbert space with respect to the norm

$$\|v\|_{k,\Omega} = \left(\sum_{j=0}^{k} |v|^2_{j,\Omega} \right)^{\frac{1}{2}}, \tag{2.1}$$

where

$$|v|_{j,\Omega} = \left(\sum_{|\alpha|=j} \int_\Omega (D^\alpha v)^{\mathrm{T}} (D^\alpha v) dx \right)^{\frac{1}{2}}. \tag{2.2}$$

Further, for non-integer s we can define the intermediate spaces $H^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ by means of interpolation of Sobolev spaces of integer order. For a thorough description of this procedure we refer to [16]. As usual, the spaces $H^s_0(\Omega)$ are defined as the $\|\cdot\|_{s,\Omega}$-completion of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. By $H(\text{div};\Omega)$ we denote the $L^2(\Omega)$-functions v with square-integrable weak divergence, which form a Hilbert space when normed by

$$\|v\|_{\text{div};\Omega} = (\|v\|^2_{0,\Omega} + \|\text{div } v\|^2_{0,\Omega})^{\frac{1}{2}}. \tag{2.3}$$

Finally, we define $L^\infty(\Omega)$ as the space of essentially bounded functions on Ω. We norm it with

$$\|v\|_{0,\infty,\Omega} = \text{ essup } \{x \in \Omega | v(x)\} \tag{2.4}$$

For more details on Sobolev space theory, we recommend Adams [1] and Lions and Magenes [16].

In the sequel, we will need some results on Sobolev spaces. They are formulated in the following lemmata. First of all, define Ω_h as the subset of points in Ω having (Euclidian) distance less than h from the boundary:

$$\Omega_h = \{x \in \Omega | \ \exists y \in \partial\Omega : \text{dist}(x, y) \le h\}. \tag{2.5}$$

Then we have the following results.

Lemma 2.1. *For v in $H^s_0(\Omega)$,where $0 \le s \le 1$, we have:*

$$\|v\|_{0,\Omega_h} \le Ch^s \|v\|_{s,\Omega}. \tag{2.6}$$

Proof. For $s = 0$, the inequality is obvious. For $s = 1$, the proof is first given for $v \in C^\infty_0(\Omega)$ by Taylor-expansion and then by completion for $H^1_0(\Omega)$. For $0 < s < 1$, the assertion follows from the interpolation inequality, for which we refer to [16], Proposition 2.3 pp. 29. □

Lemma 2.2. *For $v \in H^s(\Omega)$, where $0 \le s \le \frac{1}{2}$, we have:*

$$\|v\|_{0,\Omega_h} \le Ch^s \|v\|_{s,\Omega}. \tag{2.7}$$

Proof. For $0 \le s \le \frac{1}{2}$ it is valid that $H^s(\Omega) = H^s_0(\Omega)$ (see [16], Theorem 11.1, pp. 55). So the statement is a corollary of Lemma 2.1. □

Remark 2.3. Obviously, for $v \in H^s(\Omega)$ with $s > \frac{1}{2}$, we cannot improve the order of h in (2.7) as the counter example $v = 1$ shows.

2.2. Discretisation, triangulations and a priori estimates

Let $(\cdot,\cdot)$ be the standard $L^2(\Omega)$ inner product, and $< \cdot,\cdot >$ the corresponding one on $H^{\frac{1}{2}}(\partial\Omega)$. Further, let ν be the outer unit normal to the boundary. Then the weak formulation of (1.3) to be discretised is:

find $u \in L^2(\Omega)$ and $\mathbf{p} \in H(\text{div};\Omega)$ such that:

$$(A^{-1}\mathbf{p},\mathbf{q}) - (u, \text{div } \mathbf{q}) = - < g, \mathbf{q}^{\mathrm{T}}\nu > \text{ for all } \mathbf{q} \in H(\text{div};\Omega), \tag{2.8}$$

$$(v, \text{div } \mathbf{p}) = (f, v) \text{ for all } v \in L^2(\Omega). \tag{2.9}$$

Here, by $A^{-1}(x)$ we mean $(A(x))^{-1}$. The discretisation of (2.8) and (2.9) consists of choosing suitable finite dimensional subspaces V_h and $\mathbf{Q}_h$ of $L^2(\Omega)$ and $H(\text{div};\Omega)$ respectively. The discretised problem then runs as follows:

find $u_h \in V_h$ and $\mathbf{p}_h \in \mathbf{Q}_h$ such that:

$$(A^{-1}\mathbf{p}_h,\mathbf{q}_h) - (u_h, \text{div } \mathbf{q}_h) = - < g, \mathbf{q}_h^{\mathrm{T}}\nu > \text{ for all } \mathbf{q}_h \in \mathbf{Q}_h, \tag{2.10}$$

$$(v_h, \text{div } \mathbf{p}_h) = (f, v_h) \text{ for all } v_h \in V_h\ . \tag{2.11}$$

For V_h and $\mathbf{Q}_h$ we will consider here the lowest order Raviart-Thomas spaces, from which V_h is the space of piecewise constant functions relative to the triangulation $\mathcal{T}_h$ of Ω. As usual, h denotes the meshsize of the triangulation $\mathcal{T}_h$. The family of triangulations to be used in the sequel of this paper will be assumed regular and uniform:

Definition 2.4. *A triangulation $\mathcal{T}_h$ of Ω is said to be uniform if any two adjacent triangles of $\mathcal{T}_h$ form a parallelogram.*

Definition 2.5. *A family of triangulations $(\mathcal{T}_h)_h$ is said to be regular if the angles of the triangles are bounded away from* 0 *and* $\pi(-\pi)$ *when h tends to zero.*

For later use, we will introduce some notations. First, choose any triangle K from the triangulation $\mathcal{T}_h$. From the three outer unit vectors normal to the boundary ∂K of K, select two which are closest to orthogonal and denote them by $\mathbf{f}_1$ and $\mathbf{f}_2$. This procedure is in general not unique, we might for example also have come up with the pair $-\mathbf{f}_2$, $-\mathbf{f}_1$. Since we will only be interested in the directions of the vectors, this will appear to be no restriction.

Further, denote a parallelogram consisting of two triangles sharing a side with normal $\mathbf{f}_i$ by $N_{\mathbf{f}_i}$, $(i = 1, 2)$. For each $i = 1, 2$, the domain Ω can be partitioned into those parallelograms $N_{\mathbf{f}_i}$ and some resulting boundary triangles which we denote by $T_{\mathbf{f}_i}$. For an example of the definitions and notations concerning the triangulation, see Fig. 1.

The space $\mathbf{Q}_h$ can as usual be described as functions which are locally affine transforms of functions in the reference space $\hat{\mathbf{Q}}$, defined on the reference triangle $\hat{K}$ with corner points $(0,0), (1,0), (0,1)$ as follows:

$$\hat{\mathbf{Q}} = \{\hat{\mathbf{q}} | \exists a_0, a_1, a_2 \in \mathbb{R} : \hat{\mathbf{q}} : (x,y)^{\mathrm{T}} \longrightarrow (a_0 + a_1 x, a_2 + a_1 y)^{\mathrm{T}}.$$

Now, denote by $\mathcal{F}_K$ an affine transformation of the reference triangle onto K, then:

$$\mathbf{Q}_h = \{\mathbf{q}_h \in H(\text{div};\Omega) | \forall K \in \mathcal{T}_h : \exists \hat{\mathbf{q}} \in \hat{\mathbf{Q}} : \mathbf{q}_h|_K = \hat{\mathbf{q}} \circ \mathcal{F}_K^{-1}\} \tag{2.12}$$

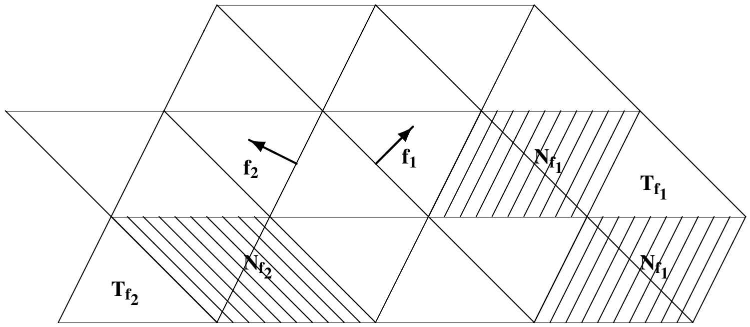

Fig. 1. A uniform triangulation of Ω

Remark 2.6. It is easy to see that for a function $\mathbf{q}_h \in \mathbf{Q}_h$, the divergence div $\mathbf{q}_h \in V_h$ and also that if $\mathbf{q}_h$ is divergence free, then it is a piecewise constant vector field. Further, the component of $\mathbf{q}_h$ normal to a side of a triangle is constant on and continuous across that side. As a matter of fact, $\mathbf{q}_h$ is determined uniquely by the values of those normal components on each side of each triangle in the triangulation $\mathscr{T}_h$ of Ω.

With this choice of V_h and $\mathbf{Q}_h$, one can prove [4] that the discrete system (2.10), (2.11) has a unique solution $(u_h, \mathbf{p}_h) \in V_h \times \mathbf{Q}_h$ and that the following optimal a priori error estimates hold [7], [17]:

$$\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega} \leq Ch|\mathbf{p}|_{1,\Omega}, \tag{2.13}$$

$$\|\mathrm{div}(\mathbf{p} - \mathbf{p}_h)\|_{0,\Omega} \leq Ch|\mathbf{p}|_{2,\Omega}. \tag{2.14}$$

Here and in the sequel, C is a constant independent of h and the functions to which the assertion applies, such as $\mathbf{p}$ in this case, which can have different values in different formulas.

2.3. Fortin interpolation

Error analysis often can be simplified by using projections of the solution $\mathbf{p}$ on the approximating space $\mathbf{Q}_h$. Here, we consider the so called Fortin interpolation operator $\boldsymbol{\Pi}_h$ which is also used by [17], [7] and [9]. Given a function $\mathbf{q} \in [H^1(\Omega)]^2$, it uses the average of the component normal to the side of each triangle over that side to define a function $\boldsymbol{\Pi}_h\mathbf{q} \in \mathbf{Q}_h$ in a unique (see Remark 2.6) way:

$$\boldsymbol{\Pi}_h : [H^1(\Omega)]^2 \longrightarrow \mathbf{Q}_h,$$

$$\int_{\partial K_i} (\boldsymbol{\Pi}_h\mathbf{q} - \mathbf{q})^{\mathrm{T}}\nu_i \, d\gamma = 0 \quad \text{for all sides } \partial K_i \text{ of all } K \in \mathscr{T}_h. \tag{2.15}$$

Here, ν_i is the outer unit normal to the side ∂K_i of K. For projection (2.15) one can prove [17]:

$$(\mathrm{div}(\mathbf{q} - \boldsymbol{\Pi}_\mathbf{h}\mathbf{q}), v_h) = 0 \quad \text{for all } v_h \in V_h, \tag{2.16}$$

$$\|\mathbf{q} - \boldsymbol{\Pi}_h\mathbf{q}\|_{0,\Omega} \leq Ch|\mathbf{q}|_{1,\Omega}. \tag{2.17}$$

We will also consider the $L^2(\Omega)$-orthogonal projection P_h on the approximating space V_h, which is characterized by the property that

$$(u - P_h u\,, v_h) = 0 \quad \text{for all } v_h \in V_h. \tag{2.18}$$

Standard approximation theory tells us that

$$\|u - P_h u\|_{0,\Omega} \le Ch|u|_{1,\Omega}. \tag{2.19}$$

Combining (2.16), (2.18) and (2.19) we easily notice that

$$\text{div } \boldsymbol{\Pi}_h \mathbf{q} = P_h \text{div}\, \mathbf{q}, \tag{2.20}$$

$$\|\text{div}(\mathbf{q} - \boldsymbol{\Pi}_\mathbf{h}\mathbf{q})\|_{0,\Omega} \le Ch|\text{div } \mathbf{q}|_{1,\Omega}, \tag{2.21}$$

from which (2.20) is expressed in the following commuting diagram:

$$\begin{tikzcd}
{[H^1(\Omega)]^2} \arrow[r, "\boldsymbol{\Pi}_h"] \arrow[d, "\text{div}"'] & \mathbf{Q}_h \arrow[d, "\text{div}"] \\
L^2(\Omega) \arrow[r, "P_h"'] & V_h
\end{tikzcd}$$

3. Superconvergence for the Dirichlet problem

In the previous section, it was stated in (2.13) that the $L^2(\Omega)$-error in the approximation $\mathbf{p}_h$ of the vector field $\mathbf{p}$ is of order h. In this section we will prove that the $L^2(\Omega)$-distance between $\mathbf{p}_h$ and the Fortin-projection $\boldsymbol{\Pi}_h\mathbf{p}$ of $\mathbf{p}$ is at least of order $h^{\frac{3}{2}}$, assuming that we use a regular family $(\mathcal{T}_h)_h$ of uniform triangulations of Ω.

First of all, subtracting (2.10) and (2.11) from (2.8) and (2.9) respectively, gives the error equations:

$$(A^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{q}_h) = (u - u_h, \text{div } \mathbf{q}_h) \quad \text{for all } \mathbf{q}_h \in \mathbf{Q}_h, \tag{3.1}$$

$$(\text{div}(\mathbf{p} - \mathbf{p}_h), v_h) = 0 \quad \text{for all } v_h \in V_h. \tag{3.2}$$

Combining (3.2) and (2.20), it follows that

$$\text{div } \mathbf{p}_h = P_h \text{div}\, \mathbf{p} = \text{div}\, \boldsymbol{\Pi}_h \mathbf{p}, \tag{3.3}$$

so that the vector field $\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}$ is divergence-free, and, as a consequence, a piecewise constant vectorfield (see Remark 2.6). Substituting this expression for $\mathbf{q}_h$ in (3.1) gives

$$(A^{-1}\mathbf{p}, \mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}) = (A^{-1}\mathbf{p}_h, \mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}). \tag{3.4}$$

By condition (1.2) on A and with equation (3.4) , we find that

$$\begin{aligned}
\|\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}\|_{0,\Omega}^2 &\le C(A^{-1}(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}), \mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}) \\
&= C(A^{-1}(\mathbf{p} - \boldsymbol{\Pi}_h\mathbf{p}), \mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}). \qquad (3.5)
\end{aligned}$$

This last expression will be estimated in the superconvergence-theorem 3.2. The following lemma will appear to be useful in the proof. Notice that uptill now, the special form of the triangulation has not been used.

Lemma 3.1. *Let N be a parallelogram and T_1, T_2 two triangles such that $N = T_1 \cup T_2$. Then, for all $\mathbf{r} \in [\mathscr{P}^1(N)]^2$, where $\mathscr{P}^1(N)$ is the space of polynomials of degree 1 in x and y on N, we have that*

$$\int_N (\mathbf{r} - \boldsymbol{\Pi}_{\mathbf{h}}\mathbf{r})\, dx\, dy = 0.$$

Proof. We may assume that N is centered around the origin and, since $\mathbf{r} = \boldsymbol{\Pi}_h\mathbf{r}$ whenever $\mathbf{r}$ is a constant, take $\mathbf{r} \in [\mathscr{P}^1(N)]^2$ zero in the origin and thus odd. But then is $\boldsymbol{\Pi}_h\mathbf{r}$ odd as well, with, as a consequence, $\int_{T_1}(\mathbf{r} - \boldsymbol{\Pi}_h\mathbf{r})dx\, dy = -\int_{T_2}(\mathbf{r} - \boldsymbol{\Pi}_h\mathbf{r})dx\, dy$. □

Now, (3.5) leads us to the main result of this section:

Theorem 3.2. *Assume that we have a regular family $(\mathscr{T}_h)_h$ of uniform triangulations. Let the solution u of the Dirichlet problem (1.1) be an element of $H^3(\Omega)$. Then we have*

$$\|\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}\|_{0,\Omega} \leq Ch^{\frac{3}{2}}(\|\mathbf{p}\|_{\frac{3}{2},\Omega} + h^{\frac{1}{2}}|\mathbf{p}|_{1,\Omega} + h^{\frac{1}{2}}|\mathbf{p}|_{2,\Omega}). \tag{3.6}$$

(Superconvergence of the vector field approximation to the vector field Fortin-projection)

Proof. First, we define a coordinate transformation F on $\mathbb{R}^2$ by taking the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ (see Sect. 2.2.) as the new basis, so, the matrix of F^{-1} has $\mathbf{f}_1$ and $\mathbf{f}_2$ as columns. Denote by $\mathbf{e}_1$ and $\mathbf{e}_2$ the standard basis vectors of $\mathbb{R}^2$ in respectively the x- and the y-direction. Then we have:

$$\begin{aligned}(A^{-1}(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}), \mathbf{p} - \boldsymbol{\Pi}_h\mathbf{p}) &= (F(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p})\ ,\ F^{-1^{\mathrm{T}}}A^{-1}(\mathbf{p} - \boldsymbol{\Pi}_h\mathbf{p}))\\ &= \sum_{K\in\mathscr{T}_h}\int_K (F(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}))^{\mathrm{T}}F^{-1^{\mathrm{T}}}A^{-1}(\mathbf{p} - \boldsymbol{\Pi}_h\mathbf{p})\, dx\, dy\\ &= \sum_{i,j=1}^{2}\mathbf{I}_{ij} \end{aligned} \tag{3.7}$$

where

$$\mathbf{I}_{ij} = \sum_{K\in\mathscr{T}_h}\int_K \mathbf{e}_i^{\mathrm{T}}F(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p})(\mathbf{e}_i^{\mathrm{T}}F^{-1^{\mathrm{T}}}A^{-1}\mathbf{e}_j)(\mathbf{p} - \boldsymbol{\Pi}_h\mathbf{p})^{\mathrm{T}}\mathbf{e}_j\, dx\, dy. \tag{3.8}$$

Now, for simplicity, consider only the sum $\mathbf{I}_{11}$. Let Ω be partitioned into parallelograms $N_{\mathbf{f}_1}$ and boundary triangles $T_{\mathbf{f}_1}$, as defined in Sect. 2.2. Denote the centre of a parallelogram $N_{\mathbf{f}_1}$ or a boundary triangle $T_{\mathbf{f}_1}$ by M, then, by the conditions on A in Sect. 1 and the normality of the columns of the matrix F^{-1}, we have:

$$\mathbf{e}_1^{\mathrm{T}}F^{-1^{\mathrm{T}}}A^{-1}\mathbf{e}_1(x) = \mathbf{e}_1^{\mathrm{T}}F^{-1^{\mathrm{T}}}A^{-1}\mathbf{e}_1(M) + O(h) \quad \text{on } N_{\mathbf{f}_1} \text{ resp. } T_{\mathbf{f}_1}. \tag{3.9}$$

Furthermore, since $\mathbf{e}_1^{\mathrm{T}}F(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p})$ is the component of $\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}$ normal to the shared side of the two triangles forming a parallelogram $N_{\mathbf{f}_1}$, it is continuous and thus (see Remark 2.6) constant on $N_{\mathbf{f}_1}$. So, rewriting the sum $\mathbf{I}_{11}$ as a sum over parallelograms $N_{\mathbf{f}_1}$, boundary triangles $T_{\mathbf{f}_1}$, and lower order terms, we find:

$$|\mathbf{I}_{11}| \leq \sum_{N_{\mathbf{f}_1}} |(\mathbf{e}_1^{\mathrm{T}} F^{-1^{\mathrm{T}}} A^{-1}\mathbf{e}_1)(M)\, \mathbf{e}_1^{\mathrm{T}} F(\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p}) \int_{N_{\mathbf{f}_1}} (\mathbf{p} - \boldsymbol{\Pi}_h \mathbf{p})^{\mathrm{T}} \mathbf{e}_1 \, dx\, dy|$$

$$+ \sum_{T_{\mathbf{f}_1}} |(\mathbf{e}_1^{\mathrm{T}} F^{-1^{\mathrm{T}}} A^{-1}\mathbf{e}_1)(M) \int_{T_{\mathbf{f}_1}} \mathbf{e}_1^{\mathrm{T}} F(\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p})(\mathbf{p} - \boldsymbol{\Pi}_h \mathbf{p})^{\mathrm{T}} \mathbf{e}_1 \, dx\, dy|$$

$$+ \sum_{K \in \mathscr{T}_h} Ch| \int_K \mathbf{e}_1^{\mathrm{T}} F(\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p})(\mathbf{p} - \boldsymbol{\Pi}_h \mathbf{p})^{\mathrm{T}} \mathbf{e}_1 \, dx\, dy|. \tag{3.10}$$

Now, denote by $\partial\Omega_{\mathbf{f}_1}$ the union of the boundary triangles $T_{\mathbf{f}_1}$. In bounding (3.10) we use the Cauchy-Schwarz inequality and the estimate

$$|\mathbf{e}_1^{\mathrm{T}} F(\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p})| \leq Ch^{-1} |\det(F)| \|\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,N_{\mathbf{f}_1}}, \tag{3.11}$$

which results in:

$$\begin{aligned} |\det(F)|^{-1} |\mathbf{I}_{11}| \quad &\leq \quad Ch^{-1} \|(\mathbf{e}_1^{\mathrm{T}} F^{-1^{\mathrm{T}}} A^{-1}\mathbf{e}_1)\|_\infty \|\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\Omega} \\ &\qquad \cdot \left(\sum_{N_{\mathbf{f}_1}} \left(\int_{N_{\mathbf{f}_1}} (\mathbf{p} - \boldsymbol{\Pi}_h \mathbf{p})^{\mathrm{T}} \mathbf{e}_1 \, dx\, dy \right)^2 \right)^{\frac{1}{2}} \\ &+ \|(\mathbf{e}_1^{\mathrm{T}} F^{-1^{\mathrm{T}}} A^{-1}\mathbf{e}_1)\|_\infty \|\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\partial\Omega_{\mathbf{f}_1}} \|\mathbf{p} - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\partial\Omega_{\mathbf{f}_1}} \\ &+ Ch \|\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\Omega} \|\mathbf{p} - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\Omega}. \end{aligned} \tag{3.12}$$

The next step in estimating $\mathbf{I}_{11}$ will be the application of the Bramble-Hilbert lemma [3] on the linear functional $\mathscr{F}$ on $[H^2(N_{\mathbf{f}_1})]^2$ defined by

$$\mathscr{F}(\mathbf{q}) = \int_{N_{\mathbf{f}_1}} (\mathbf{q} - \boldsymbol{\Pi}_h \mathbf{q})^{\mathrm{T}} \mathbf{e}_1 \, dx\, dy \quad , \mathbf{q} \in [H^2(N_{\mathbf{f}_1})]^2. \tag{3.13}$$

For this functional, the Cauchy-Schwarz inequality and (2.17) yield:

$$|\mathscr{F}(\mathbf{q})| \leq Ch \|\mathbf{q} - \boldsymbol{\Pi}_h \mathbf{q}\|_{0,N_{\mathbf{f}_1}} \leq Ch^2 |\mathbf{q}|_{1,N_{\mathbf{f}_1}}. \tag{3.14}$$

Now, since each parallelogram $N_{\mathbf{f}_1}$ is a translate of the parallelogram N of Lemma 3.1, one finds that $[\mathscr{P}^1(N_{\mathbf{f}_1})]^2 \subset \mathrm{Ker}(\mathscr{F})$, and a standard application of the Bramble-Hilbert lemma gives

$$|\mathscr{F}(\mathbf{q})| \leq Ch^3 |\mathbf{q}|_{2,N_{\mathbf{f}_1}} \quad , \text{for all } \mathbf{q} \in [H^2(N_{\mathbf{f}_1})]^2. \tag{3.15}$$

Combining (3.12), (2.17) and (3.15), we conclude that

$$\begin{aligned} |\mathbf{I}_{11}| \quad &\leq \quad C |\det(F)| \|\mathbf{p}_h - \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\Omega} (\|(\mathbf{e}_1^{\mathrm{T}} F^{-1^{\mathrm{T}}} A^{-1}\mathbf{e}_1)\|_\infty + 1) \\ &\qquad (h^2 |\mathbf{p}|_{2,\Omega} + h^2 |\mathbf{p}|_{1,\Omega} + h |\mathbf{p}|_{1,\partial\Omega_{\mathbf{f}_1}}). \end{aligned} \tag{3.16}$$

Since the family of triangulations was assumed to be regular (see Definition 2.5), we have that the determinant of F, which is the inverse of the surface of the parallelogram spanned by $\mathbf{f}_1$ and $\mathbf{f}_2$, is bounded by a constant independent of h. Now, Lemma 2.2 yields

$$|\mathbf{p}|_{1,\partial\Omega_{\mathbf{f}_1}} \leq |\mathbf{p}|_{1,\Omega_h} \leq Ch^{\frac{1}{2}} \|\mathbf{p}\|_{\frac{3}{2},\Omega} \tag{3.17}$$

This completes the estimation of $|\mathbf{I}_{11}|$. From the obvious fact that the sums $\mathbf{I}_{12}$, $\mathbf{I}_{21}$ and $\mathbf{I}_{22}$ can be estimated similarly, the proof of the theorem follows from (3.5). $\square$

Remark 3.3. Following the same lines as above, the result of this theorem can also be achieved for arbitrary polygonal domains Ω, when a regular family of triangulations is used which is uniform only on the part of Ω more than a constant times h away from the boundary.

4. Superconvergence for the Neumann problem

In this section, we will consider the homogeneous Neumann boundary value problem, i.e., we change the boundary condition of (1.1) into

$$\nabla u^{\mathrm{T}}\nu = \mathbf{p}^{\mathrm{T}}\nu = 0 \text{ on } \partial\Omega, \tag{4.1}$$

ν being the outer unit normal to $\partial\Omega$. Notice that this problem is not uniquely solvable, u can only be determined up to a constant value. The formulations of the weak and the discrete problem of course differ from those of the Dirichlet problem considered in the previous sections. In this section, we will give those formulations and prove better rates of superconvergence than in Sect. 3. To be specific, we will prove that the $L^2(\Omega)$-norm of $\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}$ is of order h^2.

First, define $H_0(\mathrm{div};\Omega)$ to be the subspace of $H(\mathrm{div};\Omega)$ consisting of functions $\mathbf{q}$ for which $\mathbf{q}^{\mathrm{T}}\nu$ equals zero on $\partial\Omega$, and $\mathbf{Q}_{0h}$ as the space $\mathbf{Q}_h \cap H_0(\mathrm{div};\Omega)$. Then the weak problem is the following:

find $u \in L^2(\Omega)$ and $\mathbf{p} \in H_0(\mathrm{div};\Omega)$ such that:

$$(A^{-1}\mathbf{p}, \mathbf{q}) - (u, \mathrm{div}\ \mathbf{q}) = 0 \text{ for all } \mathbf{q} \in H_0(\mathrm{div};\Omega), \tag{4.2}$$

$$(v, \mathrm{div}\ \mathbf{p}) = (f, v) \text{ for all } v \in L^2(\Omega). \tag{4.3}$$

As a consequence, we will consider the following discrete problem:

find $u_h \in V_h$ and $\mathbf{p}_h \in \mathbf{Q}_{0h}$ such that:

$$(A^{-1}\mathbf{p}_h, \mathbf{q}_h) - (u_h, \mathrm{div}\ \mathbf{q}_h) = 0 \text{ for all } \mathbf{q}_h \in \mathbf{Q}_{0h}, \tag{4.4}$$

$$(v_h, \mathrm{div}\ \mathbf{p}_h) = (f, v_h) \text{ for all } v_h \in V_h. \tag{4.5}$$

In solving this system in practice, one adds an extra equation in which the coefficient of one of the components of u_h is prescribed to obtain unique solvability of the resulting matrix-vector system. The a priori error estimates (2.13) and (2.14) are still valid.

The error equation (3.1) is now only valid for all $\mathbf{q}_h \in \mathbf{Q}_{0h}$, but this is not a restriction; since both $\mathbf{p}_h$ and $\boldsymbol{\Pi}_h\mathbf{p}$ are elements of $\mathbf{Q}_{0h}$, we can still substitute $\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}$ for $\mathbf{q}_h$ in (3.1) and thus (3.5) still holds. This leads to the following superconvergence result for the Neumann problem:

Theorem 4.1. *Let the solution u of the Neumann problem be an element of $H^3(\Omega)$. Assume that we have a regular family of triangulations $(\mathcal{T}_h)_h$ of Ω. Then*

$$\|\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}\|_{0,\Omega} \le Ch^2(|\mathbf{p}|_{1,\Omega} + |\mathbf{p}|_{2,\Omega}) \tag{4.6}$$

Proof. We claim that the proof of Theorem 3.2 can still be used, with one essential improvement: the summation over the boundary triangles (the second term in formula (3.10)) equals zero, since the term $\mathbf{e}_1^{\mathrm{T}}F(\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p})$ is the component of $\mathbf{p}_h - \boldsymbol{\Pi}_h\mathbf{p}$ normal to $\partial\Omega$ and also a constant on each triangle. The homogeneous Neumann boundary conditions on $\mathbf{p}_h$ and $\mathbf{p}$ and the definition of the Fortin projection yield that this normal component equals zero. $\square$

Remark 4.2. When we consider mixed boundary conditions, the weak and the discrete formulation use spaces $H_{BC}(\mathrm{div};\Omega)$ and $\mathbf{Q}_{hBC}$ instead of $H_0(\mathrm{div};\Omega)$ and $\mathbf{Q}_{0h}$. The functions in these spaces have normal components equal to zero only on the part of $\partial\Omega$ on which the Neumann boundary conditions are prescribed. One can easily conclude that superconvergence of order $h^{\frac{3}{2}}$ holds.

5. Post-processing and a posteriori error estimation

In Sect. 4.1 we will construct a post-processing mechanism for functions in $\mathbf{Q}_h$, which, when applied to the Fortin projection $\boldsymbol{\Pi}_h\mathbf{q}$ of a function $\mathbf{q} \in [H^2(\Omega)]^2$, will improve its approximation property. Further, we will use the superconvergence results of sections 3 and 4 to show that this post-processor also improves the order of approximation in the mixed finite element approximation. As a consequence, in Sect. 5.2, an asymptotically exact a posteriori error estimator for this approximation can be constructed.

5.1. Post-processing

First, we consider an arbitrary $\mathbf{q}_h \in \mathbf{Q}_h$. Once again, notice the continuity of the components normal to edges of triangles across those edges, but this time also the, in general, discontinuity of the components tangential to edges of triangles across those edges. This discontinuity of tangential components is the main feature of the post-processing, which defines a function $\mathbf{K}_h\mathbf{q}_h$ as follows (see also Fig. 2):

– In the midpoint P of each edge which has a triangle on both sides, take the average of the values of the approximation on both triangles:

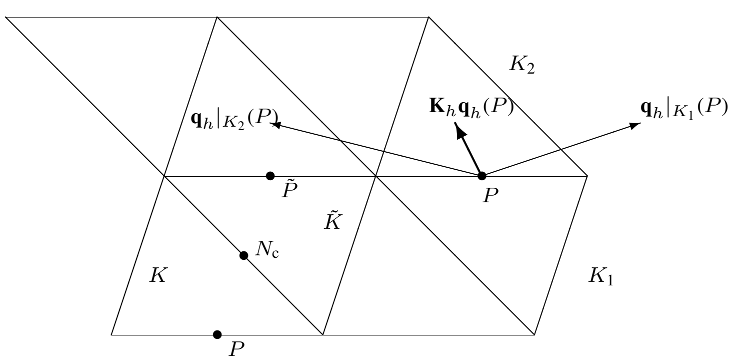

Fig. 2. Post-processing a function $\mathbf{q}_h \in \mathbf{Q}_h$

$$\mathbf{K}_h\mathbf{q}_h(P) = \frac{1}{2}(\mathbf{q}_h|_{K_1}(P) + \mathbf{q}_h|_{K_2}(P)). \tag{5.1}$$

(Notice that actually we are only postprocessing the tangential components here.)

– If we are dealing with a boundary edge, i.e. one which has only a triangle K on one side of that edge, then there exists at least one $\tilde{K} \in \mathscr{T}_h$ such that $N = K \cup \tilde{K}$ is a parallelogram. The straight line through the midpoint P of the boundary edge and the centre N_c of the parallelogram intersects the boundary of N in another point $\tilde{P}$. We will assume that $\mathbf{K}_h\mathbf{q}_h$ is already defined at N_c and $\tilde{P}$ of at least one of the parallelograms N associable to K. Then we choose such a parallelogram and define the value of $\mathbf{K}_h\mathbf{q}_h$ in P by linear extrapolation:

$$\mathbf{K}_h\mathbf{q}_h(P) = 2\mathbf{K}_h\mathbf{q}_h(N_c) - \mathbf{K}_h\mathbf{q}_h(\tilde{P}). \tag{5.2}$$

– On each triangle, the values of $\mathbf{K}_h\mathbf{q}_h$ are now defined by linear inter-(extra-)polation of the three values in the midpoints of the edges.

Now, apply this post-processing to $\boldsymbol{\Pi}_h\mathbf{q}$, where $\mathbf{q} \in [H^2(\Omega)]^2$. This gives rise to an approximation $\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}$ of $\mathbf{q}$, which is in general not uniquely defined. This will prove to be no problem, since the approximation properties of all the possible choices are sufficient for our goals. In the following theorem we will prove that the vector field $\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}$ is a higher order approximation of $\mathbf{q}$ than $\boldsymbol{\Pi}_h\mathbf{q}$ itself.

Theorem 5.1. *Let $\mathbf{q} \in [H^2(\Omega)]^2$, then for $\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}$ we have:*

$$\|\mathbf{q} - \mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}\|_{0,\Omega} \leq Ch^2|\mathbf{q}|_{2,\Omega}. \tag{5.3}$$

Proof. First, let $\mathbf{r}$ be an element of $[\mathscr{P}^1(\tilde{K})]^2$, where $\tilde{K}$ is the union of K and the triangles sharing a side with K . Then, using the same arguments as in Lemma 3.1 we find that

$$\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{r} = \mathbf{r} \quad \text{on } K \text{ , for all } \mathbf{r} \in [\mathscr{P}^1(\tilde{K})]^2. \tag{5.4}$$

Now, denote the midpoints of the sides of K by M_1, M_2, M_3 and the corner points by P_1, P_2, P_3. Then one can easily check that for all $\mathbf{q} \in [H^2(\Omega)]^2$, since $\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}$ is a linear function on K:

$$\begin{aligned} \|\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}\|_{0,\infty,K} &\leq 3\max\{\|\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}(M_1)\|_\infty, \cdots, \|\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}(M_3)\|_\infty\} \\ &\leq 3 \quad \|\boldsymbol{\Pi}_h\mathbf{q}\|_{0,\infty,\tilde{K}} \ . \end{aligned} \tag{5.5}$$

Since the Fortin projection $\boldsymbol{\Pi}_h\mathbf{q}$ is also a linear function on K, it is also valid that:

$$\|\boldsymbol{\Pi}_h\mathbf{q}\|_{0,\infty,K} \leq \max\{\|\boldsymbol{\Pi}_h\mathbf{q}(P_1)\|_\infty, \cdots, \|\boldsymbol{\Pi}_h\mathbf{q}(P_3)\|_\infty\}, \tag{5.6}$$

and since the angles between the normals of the sides of K are bounded away from 0 and $\pi(-\pi)$, we have:

$$\|\boldsymbol{\Pi}_h\mathbf{q}(P_i)\|_\infty \leq C\sum_{j=1}^{3}|\boldsymbol{\Pi}_h\mathbf{q}^{\mathrm{T}}\nu_j(P_i)| \leq C\sum_{j=1}^{3}\|\mathbf{q}^{\mathrm{T}}\nu_j\|_{0,\infty,\partial K_j} \leq C\|\mathbf{q}\|_{0,\infty,K}. \tag{5.7}$$

From (5.5), (5.6) and (5.7) we conclude that

$$\|\mathbf{K}_h\boldsymbol{\Pi}_h\mathbf{q}\|_{0,\infty,K} \leq C\|\mathbf{q}\|_{0,\infty,\tilde{K}},$$

so that, using (5.4), for all $\mathbf{r} \in [\mathscr{P}^1(\tilde{K})]^2$:

$$\begin{aligned}\|\mathbf{q} - \mathbf{K}_h \boldsymbol{\Pi}_h \mathbf{q}\|_{0,K} &\leq Ch\|\mathbf{q} - \mathbf{K}_h \boldsymbol{\Pi}_h \mathbf{q}\|_{0,\infty,K} = Ch\|(\mathbf{I} - \mathbf{K}_h \boldsymbol{\Pi}_h)(\mathbf{q} - \mathbf{r})\|_{0,\infty,K} \\ &\leq Ch\|\mathbf{q} - \mathbf{r}\|_{0,\infty,\tilde{K}}.\end{aligned}$$

Interpolation theory in Sobolev spaces (see Ciarlet [6] chapter 3), tells us that

$$\inf \{\mathbf{r} \in [\mathscr{P}^1(\tilde{K})]^2 : \|\mathbf{q} - \mathbf{r}\|_{0,\infty,\tilde{K}}\} \leq Ch|\mathbf{q}|_{2,\tilde{K}},$$

so the conclusion is that

$$\|\mathbf{q} - \mathbf{K}_h \boldsymbol{\Pi}_h \mathbf{q}\|_{0,K} \leq Ch^2|\mathbf{q}|_{2,\tilde{K}}. \tag{5.8}$$

We now obtain the statement by squaring (5.8), summing over all triangles $K \in \mathscr{T}_h$ and taking the square root. □

Combining the superconvergence results and the statements of the theorem in this section, it is easy to conclude that the post-processor $\mathbf{K}_h$ also improves the order of approximation of $\mathbf{p}_h$:

Corollary 5.2. *Assume that we have a regular family $(\mathscr{T}_h)_h$ of uniform triangulations. Let the solution u of (1.1) be an element of $H^3(\Omega)$. Then we have for the Dirichlet problem:*

$$\|\mathbf{p} - \mathbf{K}_h \mathbf{p}_h\|_{0,\Omega} \leq Ch^{\frac{3}{2}}(\|\mathbf{p}\|_{\frac{3}{2},\Omega} + h^{\frac{1}{2}}|\mathbf{p}|_{1,\Omega} + h^{\frac{1}{2}}|\mathbf{p}|_{2,\Omega}). \tag{5.9}$$

For the homogeneous Neumann problem we obtain

$$\|\mathbf{p} - \mathbf{K}_h \mathbf{p}_h\|_{0,\Omega} \leq Ch^2(|\mathbf{p}|_{1,\Omega} + |\mathbf{p}|_{2,\Omega}). \tag{5.10}$$

Proof. The triangle inequality

$$\|\mathbf{p} - \mathbf{K}_h \mathbf{p}_h\|_{0,\Omega} \leq \|\mathbf{p} - \mathbf{K}_h \boldsymbol{\Pi}_h \mathbf{p}\|_{0,\Omega} + \|\mathbf{K}_h(\boldsymbol{\Pi}_h \mathbf{p} - \mathbf{p}_h)\|_{0,\Omega}, \tag{5.11}$$

combined with the boundedness (independent of h) of the operator $\mathbf{K}_h$ on $\mathbf{Q}_h$ and the application of Theorems 3.2, 4.1 and 5.1 give the assertions. □

5.2. A posteriori error estimation

Assuming that $\mathbf{p}$ is not approximated exactly for some triangulation in the family $(\mathscr{T}_h)_h$ of triangulations of Ω, we have that the error $\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega}$ is also bounded from below by a positive constant times h. Using the results of Sect. 4.1, two triangle inequalities lead to the observation that $\varepsilon_{\mathbf{p}}$, defined by

$$\varepsilon_{\mathbf{p}}(h) = \|\mathbf{p}_h - \mathbf{K}_h \mathbf{p}_h\|_{0,\Omega}, \tag{5.12}$$

is an asymptotically exact [2] a posteriori error estimator for the corresponding error, i.e., the quotient of the error estimator and the error itself converges to 1 when h tends to zero. In fact, for the so-called effectivity-index $\eta_{\mathbf{p}}$, it is valid that

$$\eta_{\mathbf{p}}(h) = \frac{\varepsilon_{\mathbf{p}}(h)}{\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega}} = 1 + O(h^{\frac{1}{2}}) \quad (h \to 0), \tag{5.13}$$

whenever the Dirichlet problem is under consideration, and

$$\eta_{\mathbf{p}}(h) = \frac{\varepsilon_{\mathbf{p}}(h)}{\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega}} = 1 + O(h) \quad (h \to 0), \tag{5.14}$$

when we deal with the homogeneous Neumann problem.

6. Concluding remarks

We conclude this article with some remarks with respect to the results. First of all, from the proof of Theorem 3.2, it is not quite clear whether the order of the bound proved here is in fact optimal. Numerical experiments could give an indication if it would pay off to do some further research.

Second, we suspect that, as is the case in similar conforming finite element problems, perturbations on the uniform grid of the order $O(h^2)$ will not damage the superconvergence seriously. The asymptotic behaviour of the error estimators will still be the same. However, as in standard finite elements (see [10]), we might loose the superconvergence (in general) when we use for example criss-cross grids instead of uniform grids. Here too, numerical experiments could supply evidence for this, and a proof might follow from analysis of a counter example.

An interesting question is whether the superconvergence holds in some sense locally. Wheeler and Whiteman [19] prove for standard linear finite elements superconvergence on certain fixed subdomains which are triangulated uniformly. Local superconvergence might be of use in local error estimation and refinement strategies.

Finally, we notice that the post-processor suggested here can de easily implemented in existing mixed finite element codes; moreover, the computational costs of the actual error estimations is neglectible compared to the costs of solving the linear system associated to the discretization.

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