SUPERCONVERGENCE OF MIXED FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS

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ABSTRACT. In this paper, we investigate the superconvergence property of the numerical solution of a quadratic convex optimal control problem by using rectangular mixed finite element methods. The state and co-state variables are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. Some realistic regularity assumptions are presented and applied to error estimation by using an operator interpolation technique. We derive L^2 superconvergence properties for the flux functions along the Gauss lines and for the scalar functions at the Gauss points via mixed projections. Moreover, global L^2 superconvergence results are obtained by virtue of an interpolation postprocessing technique. Thus, based on these superconvergence estimates, some asymptotic exactness a posteriori error estimators are presented for the mixed finite element methods. Finally, some numerical examples are given to demonstrate the practical side of the theoretical results about superconvergence.

1. INTRODUCTION

Optimal control problems [27] are playing an increasingly important role in science and engineering. They have various application backgrounds in the operation of physical, social, and economic processes. Efficient numerical methods [32] are critical for successful applications of optimal control in practical areas. With its wide range of applications in science and engineering, the finite element method is an important class of methods for optimal control problems and plays a very important role in numerical method for these problems. Many researchers have made various contributions to the finite element methods for optimal control problems. For example, they have obtained a priori error estimates (see, e.g., [17], [18], [2]), a posteriori error estimates, based on which adaptive finite element methods can be constructed (see, e.g., [25], [28], [29]), and have also obtained some recent superconvergence and postprocessing results (see, e.g., [30], [33]). Most of this research has been, however, for the standard finite element methods for optimal controls.

In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical approximation of the state equations. Mixed finite element methods are appropriate for the state equations in such cases, since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. In computational optimal control, mixed finite element methods are not as widely used as in engineering simulations. Also, not much work exists on theoretical analysis of these methods in the literature, let alone on their superconvergence analysis.

Recently, in [9] and [10], we obtained sharp a posteriori error estimates and a priori error estimates of mixed finite element methods for quadratic optimal control problems. A posteriori error estimates for general convex optimal control was addressed in [11]. In this paper, we will prove the superconvergence of $\mathcal{O}(h^{1+s})$ (for some $0 < s \le 1$) for the control with the lowest order Raviart-Thomas mixed finite elements based on rectangulation, via adopting a different interpolation operator (see [30]). We shall derive the local and global L^2 superconvergence analysis of the rectangular lowest order Raviart-Thomas mixed finite element methods by applying an operator interpolation technique. Some asymptotically exact a posteriori error estimators will be presented for the mixed finite element methods. Some numerical examples will be given to confirm the theoretical results.

Compared with related work (e.g., [30]), the present paper provides the first theoretical analysis of superconvergence properties for optimal control problems when they are discretized by mixed finite element methods. We can obtain the superconvergence of $\mathcal{O}(h^{1+s})$ ($0 < s \le 1$) for the control variable which is approximated by piecewise constant functions. We develop a new technique to prove the superconvergence results under some realistic regularity assumptions by using operator interpolation. So far, such an application of the operator interpolation technique to error estimation has not been found in existing superconvergence literature.

We consider the following two dimensional elliptic optimal control problem:

$$(1.1) \qquad \min_{u \in U_{ad}} \left\{ \frac{1}{2}||\boldsymbol{p} - \boldsymbol{p}_d||^2 + \frac{1}{2}||y - y_d||^2 + \frac{\nu}{2}||u||^2 \right\}$$

subject to the state equation

$$(1.2) \qquad \mathrm{div}\boldsymbol{p} + a_0 y = u, \qquad \boldsymbol{p} = -A(x)\mathrm{grad}y, \qquad x \in \Omega,$$

with the boundary condition

$$(1.3) \qquad y = 0, \qquad x \in \partial\Omega,$$

where Ω is a rectangular domain, $\boldsymbol{p}_d$ and y_d are two known functions, $\boldsymbol{p}$, y are the state variables, u is the control variable, and $\nu > 0$ is fixed. We use $|| \cdot ||$ to denote the $L^2(\Omega)$ norm. Moreover, U_{ad} denotes the admissible set of the control variable, defined for real numbers a and b by

$$(1.4) \qquad U_{ad} = \{u \in L^2(\Omega) : \quad a \le u \le b \ \text{ a.e. in } \Omega\}.$$

The paper is organized as follows. In section 2, some realistic regularity assumptions are presented by using an operator interpolation technique. We construct a discretized scheme for the optimal control problem (1.1)–(1.3) by using mixed finite element methods and give its equivalent optimality conditions. Some projection operators and their approximated properties are also introduced. In section 3, we derive the local L^2 superconvergence analysis of the rectangular lowest order

Raviart-Thomas mixed finite element methods by applying an operator interpolation technique. In section 4, we carry out the global L^2 superconvergence of the rectangular lowest order Raviart-Thomas mixed finite element methods. Some asymptotic exact a posteriori error estimators are presented for the mixed finite element methods. In section 5, some numerical examples are given to demonstrate our theoretical results. Finally, we give conclusions and mention some possible future work in section 6.

Throughout the paper C will denote a generic positive constant that is independent of h but which will depend on the regularity of the optimal control problem (1.1)–(1.3).

2. Mixed methods for optimal control problems

In this section, we shall describe a mixed finite element discretization for the optimal control problem (1.1)–(1.3). We assume that the coefficient matrix function $A(x) = (a_{ij}(x))$ is symmetric with $a_{ij}(x) \in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_* |\xi|^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j, \qquad \forall\, (\xi, x) \in \mathbb{R}^2 \times \overline{\Omega}, \quad c_* > 0.$$

We require the data in the control problem to satisfy sufficient regularity.

Next, we introduce the co-state elliptic equation

$$(2.1) \qquad \operatorname{div}\boldsymbol{q} + a_0 z = y - y_d, \qquad \boldsymbol{q} = -A(x)(\operatorname{grad}z + \boldsymbol{p} - \boldsymbol{p}_d), \qquad x \in \Omega,$$

with the boundary condition

$$(2.2) \qquad\qquad z = 0, \qquad x \in \partial\Omega.$$

In general, one can expect the following regularity results for the convex polygonal domain.

Lemma 2.1. *If Ω is a convex polygonal domain, let the state variable y and the co-state variable z be the variational solutions of the elliptic problem (1.2)–(1.3) and the elliptic problem (2.1)–(2.2), respectively. Then, there exists $0 < s_0 \leq 1$ such that for $0 < s < s_0$,*

$$(2.3) \qquad\qquad ||y||_{H^{2+s}(\Omega)} \leq C(s)||u||_{H^s(\Omega)},$$

$$(2.4) \qquad\qquad ||z||_{H^{2+s}(\Omega)} \leq C(s)||y||_{H^s(\Omega)}.$$

It is known (see, e.g., [3] and [13]) that $s_0 = \min\{1, \pi/\omega - 1\}$ when both the state equation and the co-state equation are a Laplace equation and ω is the radian measure of the largest corner of the domain Ω ($\omega < \pi$). If the state equation and the co-state equation are variable coefficient equations, then s_0 is dependent on ω and the eigenvalue of the coefficient matrix $A(x)$ at the corner points. Some regularity analysis of elliptic problems with variable coefficients on polygonal domains are provided in [19, 21]. A special finite element method was proposed [19, 22] which has local refinement property and optimal error estimates by introducing a proper transformation of variables. Superconvergence and extrapolation are considered in [19, 20].

Thus, we make the following realistic assumption on the regularity of the solution of the optimal control problem (1.1)–(1.3) and the co-state problem (2.1)–(2.2):

$$(2.5) \qquad\qquad u \in W^{1,\infty}(\Omega), \qquad y,\ z \in H^{2+s}(\Omega),$$

for $0 < s \leq 1$. The operator interpolation technique will be used to obtain super-convergence results under the realistic regularity assumptions (2.5).

Let

$$\boldsymbol{V} = H(\mathrm{div}; \Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \ \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\}, \qquad W = L^2(\Omega).$$

The Hilbert space $\boldsymbol{V}$ is equipped with the following norm:

$$||\boldsymbol{v}||_{\mathrm{div}} = \left(||\boldsymbol{v}||^2 + ||\mathrm{div}\boldsymbol{v}||^2\right)^{1/2}.$$

Then, the weak formulation of the optimal control problem (1.1)–(1.3) is to find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$(2.6) \qquad \min_{u \in U_{ad}} \left\{ \frac{1}{2}||\boldsymbol{p} - \boldsymbol{p}_d||^2 + \frac{1}{2}||y - y_d||^2 + \frac{\nu}{2}||u||^2 \right\},$$

$$(2.7) \qquad (A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \ \boldsymbol{v} \in \boldsymbol{V},$$

$$(2.8) \qquad (\mathrm{div}\boldsymbol{p}, w) + (a_0 y, w) = (u, w), \qquad \forall \ w \in W,$$

where the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ is denoted by $(\cdot, \cdot)$. It is well-known (see, e.g., [27]) that the convex control problem (2.6)–(2.8) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.6)–(2.8) if and only if there exists a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(2.9) \qquad (A^{-1}\boldsymbol{p}, \boldsymbol{v}) - (y, \mathrm{div}\boldsymbol{v}) = 0, \qquad \forall \ \boldsymbol{v} \in \boldsymbol{V},$$

$$(2.10) \qquad (\mathrm{div}\boldsymbol{p}, w) + (a_0 y, w) = (u, w), \qquad \forall \ w \in W,$$

$$(2.11) \qquad (A^{-1}\boldsymbol{q}, \boldsymbol{v}) - (z, \mathrm{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \quad \forall \ \boldsymbol{v} \in \boldsymbol{V},$$

$$(2.12) \qquad (\mathrm{div}\boldsymbol{q}, w) + (a_0 z, w) = (y - y_d, w), \qquad \forall \ w \in W,$$

$$(2.13) \qquad (z + \nu u, \tilde{u} - u) \geq 0, \qquad \forall \ \tilde{u} \in U_{ad}.$$

Introducing the projection ([30])

$$(2.14) \qquad \Pi_{[a,b]}(f(x)) = \max(a, \min(b, f(x)))$$

from above optimality condition (2.13) and Remark 3 in [8], we have that

$$(2.15) \qquad u(x) = \Pi_{[a,b]}\left(-\frac{1}{\nu}z\right).$$

Let $\mathcal{T}_h$ denote a regular (in the sense of [12]) rectangular partition of the domain Ω, where $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denotes the order k Raviart-Thomas mixed finite element space [31]. To approximate the control, we use the following space of piecewise constant functions:

$$U_h = \{\tilde{u}_h \in U_{ad} : \ \tilde{u}_h|_T = \mathrm{constant}, \ T \in \mathcal{T}_h\}.$$

Before the mixed finite element scheme is given, we introduce the following Raviart-Thomas projection (see [14] and [31]):

$$\Pi_h \times P_h : \ \boldsymbol{V} \times W \to \boldsymbol{V}_h \times W_h,$$

which has the following properties:

(i) P_h is the local $L^2(\Omega)$ projection.

(ii) Π_h and P_h satisfy

$$\text{(2.16)} \qquad \text{div} \circ \Pi_h = P_h \circ \text{div},$$

$$\text{(2.17)} \qquad (\text{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v}), w_h) = 0, \qquad w_h \in W_h,$$

$$\text{(2.18)} \qquad (\text{div}\boldsymbol{v}_h, w - P_h w) = 0, \qquad \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

(iii) The following approximation properties hold (see [14] and [23]):

$$\text{(2.19)} \qquad ||\boldsymbol{v} - \Pi_h \boldsymbol{v}||_{0,\rho} \le Ch^r ||\boldsymbol{v}||_{r,\rho}, \qquad \frac{1}{\rho} < r \le k+1,$$

$$\text{(2.20)} \qquad ||\text{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v})||_{-t} \le Ch^{r+t}||\text{div}\boldsymbol{v}||_r, \qquad 0 \le r, \ t \le k+1,$$

$$\text{(2.21)} \qquad ||w - P_h w||_{-t,\rho} \le Ch^{r+t}||w||_{r,\rho}, \qquad 0 \le r, \ t \le k+1,$$

where $||\cdot||_{r,\rho}$ denotes the norm of the usual Sobolev space $W^{r,\rho}(\Omega)$ for $1 \le \rho \le +\infty$ and $r \ge 0$.

The mixed finite element approximation of (2.6)–(2.8) is to find $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ such that

$$\text{(2.22)} \qquad \min_{u_h \in U_h} \left\{ \frac{1}{2}||\boldsymbol{p}_h - \boldsymbol{p}_d||^2 + \frac{1}{2}||y_h - y_d||^2 + \frac{\nu}{2}||u_h||^2 \right\},$$

$$\text{(2.23)} \qquad (A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \text{div}\boldsymbol{v}_h) = 0, \qquad \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\text{(2.24)} \qquad (\text{div}\boldsymbol{p}_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h), \quad \forall \ w_h \in W_h.$$

The control problem (2.22)–(2.24) again has a unique solution $(\boldsymbol{p}_h, y_h, u_h)$, and a triplet $(\boldsymbol{p}_h, y_h, u_h) \in \boldsymbol{V}_h \times W_h \times U_h$ is the solution of (2.22)–(2.24) if and only if there is a co-state $(\boldsymbol{q}_h, z_h) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ satisfies the following discretized optimality conditions:

$$\text{(2.25)} \qquad (A^{-1}\boldsymbol{p}_h, \boldsymbol{v}_h) - (y_h, \text{div}\boldsymbol{v}_h) = 0, \qquad \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\text{(2.26)} \qquad (\text{div}\boldsymbol{p}_h, w_h) + (a_0 y_h, w_h) = (u_h, w_h), \qquad \forall \ w_h \in W_h,$$

$$\text{(2.27)} \qquad (A^{-1}\boldsymbol{q}_h, \boldsymbol{v}_h) - (z_h, \text{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h - \boldsymbol{p}_d, \boldsymbol{v}_h), \ \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\text{(2.28)} \qquad (\text{div}\boldsymbol{q}_h, w_h) + (a_0 z_h, w_h) = (y_h - y_d, w_h), \qquad \forall \ w_h \in W_h,$$

$$\text{(2.29)} \qquad (z_h + \nu u_h, \tilde{u}_h - u_h) \ge 0, \qquad \forall \ \tilde{u}_h \in U_h.$$

In the following analysis, we shall use some intermediate variables. For any control function $\tilde{u} \in U_{ad}$, we first define the state solution $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ which satisfies

$$\text{(2.30)} \qquad (A^{-1}\boldsymbol{p}(\tilde{u}), \boldsymbol{v}) - (y(\tilde{u}), \text{div}\boldsymbol{v}) = 0, \qquad \forall \ \boldsymbol{v} \in \boldsymbol{V},$$

$$\text{(2.31)} \qquad (\text{div}\boldsymbol{p}(\tilde{u}), w) + (a_0 y(\tilde{u}), w) = (\tilde{u}, w), \qquad \forall \ w \in W,$$

$$\text{(2.32)} \qquad (A^{-1}\boldsymbol{q}(\tilde{u}), \boldsymbol{v}) - (z(\tilde{u}), \text{div}\boldsymbol{v}) = -(\boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}), \ \forall \ \boldsymbol{v} \in \boldsymbol{V},$$

$$\text{(2.33)} \qquad (\text{div}\boldsymbol{q}(\tilde{u}), w) + (a_0 z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w), \qquad \forall \ w \in W.$$

Then, we define the discrete state solution $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))$ corresponding to $\tilde{u}$ which satisfies

$$\text{(2.34)} \qquad (A^{-1}\boldsymbol{p}_h(\tilde{u}), \boldsymbol{v}_h) - (y_h(\tilde{u}), \text{div}\boldsymbol{v}_h) = 0, \qquad \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\text{(2.35)} \qquad (\text{div}\boldsymbol{p}_h(\tilde{u}), w_h) + (a_0 y_h(\tilde{u}), w_h) = (\tilde{u}, w_h), \qquad \forall \ w_h \in W_h,$$

$$\text{(2.36)} \qquad (A^{-1}\boldsymbol{q}_h(\tilde{u}), \boldsymbol{v}_h) - (z_h(\tilde{u}), \text{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(\tilde{u}) - \boldsymbol{p}_d, \boldsymbol{v}_h), \ \forall \ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\text{(2.37)} \qquad (\text{div}\boldsymbol{q}_h(\tilde{u}), w_h) + (a_0 z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h), \qquad \forall \ w_h \in W_h.$$

With these definitions, the exact state and co-state solutions and their approximation can be written as:

$$(\boldsymbol{p}, y, \boldsymbol{q}, z) = (\boldsymbol{p}(u), y(u), \boldsymbol{q}(u), z(u)),$$

$$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h) = (\boldsymbol{p}_h(u_h), y_h(u_h), \boldsymbol{q}_h(u_h), z_h(u_h)).$$

3. L^2 SUPERCONVERGENCE ON RECTANGULAR MIXED FINITE ELEMENTS

From now on, we assume that the coefficient $A(x)$ in the optimal control problem (1.1)–(1.3) is a diagonal matrix function. Let $\mathcal{T}_h = \{T_i\}$ be a rectangular partition of Ω. In general, the partition $\mathcal{T}_h$ is nonuniform but regular [12]. Based on this partition, we consider the lowest order $k = 0$ Raviart-Thomas mixed finite element space $\boldsymbol{V}_h \times W_h$, namely,

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V} : \quad \forall T_i \in \mathcal{T}_h, \ \boldsymbol{v}|_{T_i} \in Q_{1,0}(T_i) \times Q_{0,1}(T_i)\},$$
$$W_h = \{w \in W : \quad \forall T_i \in \mathcal{T}_h, \ w|_{T_i} \in Q_{0,0}(T_i)\},$$

where $Q_{m,n}(T_i)$ indicates the space of polynomials of degree no more than m and n in x and y on T_i, respectively.

Thus, on each rectangle element $T_i \in \mathcal{T}_h$, the Gauss point is its center point S_i. For example,

$$\text{if} \quad T_i = [a_i, b_i] \times [c_i, d_i], \qquad \text{then} \quad S_i = ((a_i + b_i)/2, (c_i + d_i)/2).$$

As in [15], we define some semi-norms for $\boldsymbol{v} = (v_1, v_2) \in \boldsymbol{V}$ and $w \in W$ as follows:

(3.1)
$$|||w|||_0 = \left(\sum_{T_i \in \mathcal{T}_h} |T_i| \cdot w^2(S_i) \right)^{1/2}, \qquad |T_i| = \text{meas}(T_i),$$

$$|||v_1|||_1 = \left(\sum_{T_i \in \mathcal{T}_h} |d_i - c_i| \cdot \int_{a_i}^{b_i} v_1^2(x_1, (c_i + d_i)/2) dx_1 \right)^{1/2},$$

$$|||v_2|||_2 = \left(\sum_{T_i \in \mathcal{T}_h} |b_i - a_i| \cdot \int_{c_i}^{d_i} v_2^2((a_i + b_i)/2, x_2) dx_2 \right)^{1/2},$$

and

(3.2)
$$|||\boldsymbol{v}||| = |||v_1|||_1 + |||v_2|||_2.$$

It is clear from [15] that these semi-norms are equivalent to the L^2-norms for the functions from $\boldsymbol{V}_h$ and W_h, respectively.

For any smooth function $f(x) \in U_{ad}$, we define an interpolation function $f_I \in U_h$ (or $f_I \in W_h$):

(3.3)
$$f_I(x) = f(S_i) \qquad \text{if } x \in T_i,$$

where S_i is the center point of the rectangle T_i. Let f be a function belonging to $H^2(T_i)$ for all i. Then, by the Bramble-Hilbert lemma, we have [30]

(3.4)
$$\left| \int_{T_i} (f(x) - f(S_i)) dx \right| \leq C h^2 \sqrt{|T_i|} |f|_{H^2(T_i)}$$

and

(3.5)
$$\sum_i \left| \int_{T_i} (f(x) - f(S_i)) dx \right| \leq C h^2 \left(\sum_i |f|_{H^2(T_i)}^2 \right)^{1/2}.$$

From the regularity assumption (2.5) and the projection formula (2.15) of u, we can classify the rectangles $T_i \in \mathcal{T}_h$ into two sets K_1 and K_2:

$$K_1 = \{T_i : u \text{ only belongs to } W^{1,\infty}(T_i)\}, \qquad K_2 = \{T_i : u \in H^2(T_i)\}.$$

Clearly, the number of elements in K_1 grows for decreasing h. We make the following additional assumption for K_1 which is fulfilled in many practical cases (see the assumption (A3) in [30]):

$$(3.6) \qquad \qquad \text{meas}(K_1) \leq c \cdot h.$$

We shall prove that there exists the following superconvergence property between the interpolation function u_I and the discrete solution u_h of the exact control u. Before presenting the main theorem, we first state some known results and derive some lemmas.

Lemma 3.1. *Let $z_h(u)$ and $z_h(u_h)$ be the discrete solutions of* (2.34)–(2.37) *with $\tilde{u} = u$ and $\tilde{u} = u_h$, respectively. Then we have*

$$(3.7) \qquad \qquad (z_h(u_h) - z_h(u_I), u_I - u_h) \leq 0.$$

Proof. In the equations (2.34)–(2.37), we choose test functions $\tilde{u} = u_I$ and $\tilde{u} = u_h$, respectively, and get the difference equations:

$$(A^{-1}(\boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)), \boldsymbol{v}_h) - (y_h(u_h) - y_h(u_I), \text{div}\boldsymbol{v}_h) = 0,$$
$$(\text{div}(\boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)), w_h) + (a_0(y_h(u_h) - y_h(u_I)), w_h) = (u_h - u_I, w_h),$$
$$(A^{-1}(\boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)), \boldsymbol{v}_h) - (z_h(u_h) - z_h(u_I), \text{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I), \boldsymbol{v}_h),$$
$$(\text{div}(\boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)), w_h) + (a_0(z_h(u_h) - z_h(u_I)), w_h) = (y_h(u_h) - y_h(u_I), w_h),$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$. For above equations, we choose $w_h = z_h(u_h) - z_h(u_I)$ in the second equation, $\boldsymbol{v}_h = \boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)$ in the third equation, $w_h = y_h(u_h) - y_h(u_I)$ in the fourth equation, and $\boldsymbol{v}_h = \boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)$ in the first equation. Then we can deduce that

$$
\begin{aligned}
&(z_h(u_h) - z_h(u_I), u - u_h) \\
&= -(\text{div}(\boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)), z_h(u_h) - z_h(u_I)) \\
&\quad - (a_0(y_h(u_h) - y_h(u_I)), z_h(u_h) - z_h(u_I)) \\
&= -(A^{-1}(\boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)), \boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)) \\
&\quad - (\boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I), \boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)) \\
&\quad + (\text{div}(\boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)), y_h(u_h) - y_h(u_I)) \\
&\quad - (y_h(u_h) - y_h(u_I), y_h(u_h) - y_h(u_I)) \\
&\leq -(A^{-1}(\boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)), \boldsymbol{p}_h(u_h) - \boldsymbol{p}_h(u_I)) \\
&\quad + (\text{div}(\boldsymbol{q}_h(u_h) - \boldsymbol{q}_h(u_I)), y_h(u_h) - y_h(u_I)) = 0,
\end{aligned}
$$

which implies (3.7). $\qquad \qquad \square$

Lemma 3.2. *Assume that the regularity conditions* (2.5) *and* (3.6) *hold. Let u_I be the interpolation of the exact control u defined in* (3.3) *and $z(u_I)$ and $z(u)$ be the solutions of* (2.30)–(2.33) *with $\tilde{u} = u_I$ and $\tilde{u} = u$, respectively. Then we have*

$$(3.8) \qquad \|z(u_I) - z(u)\| \leq C \left(\|u\|_{W^{1,\infty}(\Omega)}, \|z\|_{H^2(\Omega)} \right) h^2 |\ln h|^{1/2}.$$

Proof. From the regularity of state and co-state functions, we use the strong forms of equations (1.2) and (2.1) to obtain the error equations

$$(3.9) \qquad -\operatorname{div}(A(x)\operatorname{grad}(y(u_I) - y(u))) + a_0(y(u_I) - y(u)) = u_I - u,$$

$$(3.10) \qquad -\operatorname{div}(A(x)(\operatorname{grad}(z(u_I) - z(u)) + \boldsymbol{p}(u_I) - \boldsymbol{p}(u)))$$
$$+ a_0(z(u_I) - z(u)) = y(u_I) - y(u),$$

which imply that

$$(3.11) \qquad ||z(u_I) - z(u)||_{H^1(\Omega)} \le C\Big(||y(u_I) - y(u)|| + ||\boldsymbol{p}(u_I) - \boldsymbol{p}(u)||\Big).$$

Now, we multiply (3.10) by $y(u_I) - y(u)$ to obtain that

$$||y(u_I) - y(u)||^2$$
$$= (y(u_I) - y(u), y(u_I) - y(u))$$
$$= -(\operatorname{div}(A(x)(\operatorname{grad}(z(u_I) - z(u)) + \boldsymbol{p}(u_I) - \boldsymbol{p}(u))), y(u_I) - y(u))$$
$$+ (a_0(z(u_I) - z(u)), y(u_I) - y(u))$$
$$= (A(x)\operatorname{grad}(z(u_I) - z(u)), \operatorname{grad}(y(u_I) - y(u)))$$
$$+ (A(x)(\boldsymbol{p}(u_I) - \boldsymbol{p}(u)), \operatorname{grad}(y(u_I) - y(u)))$$
$$+ (a_0(z(u_I) - z(u)), y(u_I) - y(u))$$
$$= (A(x)\operatorname{grad}(y(u_I) - y(u)), \operatorname{grad}(z(u_I) - z(u)))$$
$$- (\boldsymbol{p}(u_I) - \boldsymbol{p}(u), \boldsymbol{p}(u_I) - \boldsymbol{p}(u)) + (a_0(y(u_I) - y(u)), z(u_I) - z(u))$$
$$= (u_I - u, z(u_I) - z(u)) - (\boldsymbol{p}(u_I) - \boldsymbol{p}(u), \boldsymbol{p}(u_I) - \boldsymbol{p}(u)),$$

where we used $\boldsymbol{p}(u_I) - \boldsymbol{p}(u) = -A(x)\operatorname{grad}(y(u_I) - y(u))$ and (3.9). Thus, we have the following identity:

$$(3.12) \qquad ||y(u_I) - y(u)||^2 + ||\boldsymbol{p}(u_I) - \boldsymbol{p}(u)||^2 = (u_I - u, z(u_I) - z(u)).$$

Now, we define a standard piecewise linear function space

$$(3.13) \qquad S_h = \{r_h(x) \in C(\overline{\Omega}): \quad r_h \in Q_{1,1}(T_i), \ \forall \ T_i \in \mathcal{T}_h\},$$

and a standard $H^1(\Omega)$-orthogonal projection $Q_h \colon C(\overline{\Omega}) \to S_h$, which satisfies: for any $\psi \in C(\overline{\Omega})$

$$(3.14) \qquad (\operatorname{grad}(\psi - Q_h\psi), \operatorname{grad} r_h) = 0, \qquad \forall \ r_h \in S_h.$$

By standard finite element analysis, the projection Q_h has the following approximate property and stable property:

$$(3.15) \qquad ||\psi - Q_h\psi|| \le Ch||\psi||_{H^1(\Omega)}$$

and

$$(3.16) \qquad ||Q_h\psi||_{H^1(\Omega)} \le C||\psi||_{H^1(\Omega)}.$$

Set $r = z(u_I) - z(u)$ and $r_h = Q_h(z(u_I) - z(u))$. We can write the right side of (3.12) as follows:

$$(3.17) \qquad (u_I - u, z(u_I) - z(u)) = (u_I - u, r - r_h) + (u_I - u, r_h).$$

It follows from (3.15) that

$$(3.18) \ (u_I - u, r - r_h) \le ||u_I - u|| \cdot ||r - r_h|| \le Ch^2 ||u||_{W^{1,\infty}(\Omega)} \cdot ||z(u_I) - z(u)||_{H^1(\Omega)}.$$

We decompose the integral $(u_I - u, r_h)$ into two parts:

$$(3.19) \qquad (u_I - u, r_h) = \int_{K_1} (u_I - u) \cdot r_h dx + \int_{K_2} (u_I - u) \cdot r_h dx.$$

Since $r_h \in Q_{1,1}(T_i)$ for any rectangle T_i, we then have

$$(3.20) \qquad \int_{T_i} u_I r_h dx = \int_{T_i} u(S_i) r_h dx = \int_{T_i} u(S_i) r_h(S_i) dx.$$

By means of assumption (3.6) and (3.16), we obtain that

$$
\begin{aligned}
\left| \int_{K_1} (u_I - u) r_h dx \right| &\leq \sum_{T_i \in K_1} \left| \int_{T_i} (u(S_i) - u(x)) \cdot r_h(x) dx \right| \\
&\leq C \sum_{T_i \in K_1} h |u|_{1,\infty,T_i} \cdot |r_h|_{0,\infty,T_i} \cdot |T_i| \\
&\leq Ch^2 |u|_{1,\infty} \cdot |r_h|_{0,\infty} \leq Ch^2 |u|_{1,\infty} \cdot |\ln h|^{1/2} \|r_h\|_{H^1(\Omega)} \\
(3.21) \qquad &\leq Ch^2 |\ln h|^{1/2} \|u\|_{W^{1,\infty}(\Omega)} \cdot \|z(u_I) - z(u)\|_{H^1(\Omega)}.
\end{aligned}
$$

Here, we used the well known inequality that

$$(3.22) \qquad \|r_h\|_{L^\infty(\Omega)} \leq C |\ln h|^{1/2} \|r_h\|_{H^1(\Omega)}.$$

By using (3.20) and (3.5), we obtain that

$$
\begin{aligned}
\left| \int_{K_2} (u_I - u) r_h dx \right| &\leq \sum_{T_i \in K_2} \left| \int_{T_i} (u(S_i) - u(x)) \cdot r_h(x) dx \right| \\
&= \sum_{T_i \in K_2} \left| \int_{T_i} \left[(ur_h)(S_i) - (ur_h)(x) \right] dx \right| \\
&\leq Ch^2 \left(\sum_{T_i \in K_2} |ur_h|^2_{H^2(T_i)} \right)^{1/2} \\
&\leq Ch^2 \left(\sum_{T_i \in K_2} |u|^2_{H^2(T_i)} \cdot |r_h|^2_{H^1(T_i)} \right)^{1/2}.
\end{aligned}
$$

From (2.15), we obtain that

$$|u|_{H^2(T_i)} \leq C |z|_{H^2(T_i)},$$

which was discussed in [30] in detail. Applying the estimate in H^1 norm for L^2 projections (see [5]), we have

$$|r_h|_{H^1(T_i)} = |Q_h(z(u_I) - z(u))|_{H^1(T_i)} \leq C |z(u_I) - z(u)|_{H^1(T_i)}.$$

Therefore,

$$
\begin{aligned}
\left| \int_{K_2} (u_I - u) r_h dx \right| &\leq Ch^2 \left(\sum_{T_i \in K_2} |z|^2_{H^2(T_i)} \cdot |z(u_I) - z(u)|^2_{H^1(T_i)} \right)^{1/2} \\
(3.23) \qquad &\leq Ch^2 \cdot \|z\|_{H^2(\Omega)} \cdot |z(u_I) - z(u)|_{H^1(\Omega)}.
\end{aligned}
$$

Finally, we can combine the estimates (3.12), (3.17)–(3.19), and (3.21)–(3.23) to derive

$$||y(u_I) - y(u)||^2 + ||\boldsymbol{p}(u_I) - \boldsymbol{p}(u)||^2$$
$$\leq Ch^2|\ln h|^{1/2} \cdot ||z(u_I) - z(u)||_{H^1(\Omega)}$$
(3.24)
$$\leq Ch^2|\ln h|^{1/2} \cdot (||y(u_I) - y(u)|| + ||\boldsymbol{p}(u_I) - \boldsymbol{p}(u)||),$$

where $C = C\left(||u||_{W^{1,\infty}(\Omega)}, ||z||_{H^2(\Omega)}\right)$, and we use the stable property (3.11) in the last step. The above inequality implies the result (3.8). $\qquad\square$

By using the standard superconvergence results of the mixed finite element methods ([15, 16]), we have the following results.

Lemma 3.3. *For any function* $\tilde{u} \in U_{ad}$, *let* $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u}))$ *and* $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))$ *be the solutions of* (2.30)–(2.33) *and* (2.34)–(2.37), *respectively, for the lowest order Raviart-Thomas mixed finite elements. If the regularity condition*

$$y(\tilde{u}), \ z(\tilde{u}) \in H^1(\Omega) \qquad \boldsymbol{p}(\tilde{u}), \ \boldsymbol{q}(\tilde{u}) \in (H^2(\Omega))^2$$

holds, then we have

(3.25)
$$||P_h y(\tilde{u}) - y_h(\tilde{u})|| + ||\Pi_h \boldsymbol{p}(\tilde{u}) - \boldsymbol{p}_h(\tilde{u})||$$
$$\leq Ch^2 \left(||\boldsymbol{p}(\tilde{u})||_{H^2(\Omega)} + ||y(\tilde{u})||_{H^1(\Omega)}\right),$$
(3.26)
$$||P_h z(\tilde{u}) - z_h(\tilde{u})|| + ||\Pi_h \boldsymbol{q}(\tilde{u}) - \boldsymbol{q}_h(\tilde{u})||$$
$$\leq Ch^2 \left(||\boldsymbol{q}(\tilde{u})||_{H^2(\Omega)} + ||z(\tilde{u})||_{H^1(\Omega)}\right).$$

Lemma 3.4. *Assume that the regularity condition* (2.5) *holds. Let* $z(u_I)$ *and* $z_h(u_I)$ *be the solutions of* (2.30)–(2.33) *and* (2.34)–(2.37) *with* $\tilde{u} = u_I$, *respectively. Then we have*

(3.27) $\ (z_h(u_I) - z(u_I), u_I - u_h) \leq Ch^{1+s}||z||_{H^{2+s}(\Omega)} \cdot ||u_I - u_h||, \qquad 0 < s \leq 1.$

Proof. It is easy to see that

(3.28)
$$(z_h(u_I) - z(u_I), u_I - u_h) = (z_h(u_I) - P_h z(u_I), u_I - u_h).$$

On the one hand, under the condition $z(u_I) \in H^3(\Omega)$, namely

$$z(u_I) \in H^1(\Omega) \qquad \boldsymbol{q}(u_I) \in (H^2(\Omega))^2,$$

by applying the superconvergence result (3.26) in Lemma 3.3 with $\tilde{u} = u_I$, we obtain that

$$||z_h(u_I) - P_h z(u_I)|| \leq Ch^2 \left(||\boldsymbol{q}(u_I)||_{H^2(\Omega)} + ||z(u_I)||_{H^1(\Omega)}\right)$$
(3.29)
$$\leq Ch^2 ||z(u_I)||_{H^3(\Omega)}.$$

On the other hand, by the standard priori error estimate, we have

(3.30)
$$||z_h(u_I) - P_h z(u_I)|| \leq Ch ||z(u_I)||_{H^2(\Omega)}.$$

It easy to see that the equations (2.34)–(2.37) define a linear operator $z(u_I) \longrightarrow z_h(u_I)$ and P_h defines a linear operator $z(u_I) \longrightarrow P_h z(u_I)$. Let L be the linear operator $z(u_I) \longrightarrow z_h(u_I) - P_h z(u_I)$, namely

$$L z(u_I) = z_h(u_I) - P_h z(u_I).$$

Then, by the interpolation theory in Sobolev space (see [1]), it follows from (3.29)–(3.30) that

$$
\begin{aligned}
||L||_{\mathcal{L}(H^{2+s}(\Omega)\to L^2(\Omega))} &\leq ||L||^s_{\mathcal{L}(H^3(\Omega)\to L^2(\Omega))}||L||^{1-s}_{\mathcal{L}(H^2(\Omega)\to L^2(\Omega))} \\
&\leq C(h^2)^s \cdot h^{1-s} = Ch^{1+s},
\end{aligned}
$$

which implies

$$
\begin{aligned}
||z_h(u_I) - P_h z(u_I)|| &= ||Lz(u_I)|| \\
&\leq ||L||_{\mathcal{L}(H^{2+s}(\Omega)\to L^2(\Omega))} \cdot ||z(u_I)||_{H^{2+s}(\Omega)} \\
&\leq Ch^{1+s}||z(u_I)||_{H^{2+s}(\Omega)} \\
(3.31) \qquad &\leq Ch^{1+s}||z||_{H^{2+s}(\Omega)}.
\end{aligned}
$$

Then, the requested result (3.27) follows from (3.28) and (3.31). $\qquad\square$

Lemma 3.5. *Let u_h be the solution of* (2.25)–(2.29) *and u_I and z_I the interpolation functions of u and z defined in* (3.3)*, respectively. Then we have*

$$
(3.32) \qquad (z - z_I, u_I - u_h) \leq Ch^2||z||_{H^2(\Omega)} \cdot ||u_I - u_h||.
$$

Proof. It follows from the regularity of the elliptic equations (1.2), (2.1), and (3.4) that

$$
\begin{aligned}
(z &- z_I, u_I - u_h) \\
&= \sum_i \int_{T_i} (z(x) - z_I(x)) \cdot (u_I(x) - u_h(x))dx \\
&= \sum_i (u_I(S_i) - u_h(S_i)) \int_{T_i} (z(x) - z(S_i))dx \\
&\leq \sum_i Ch^2 |u_I(S_i) - u_h(S_i)| \sqrt{|T_i|} \cdot |z|_{H^2(T_i)} \\
&\leq Ch^2 \cdot ||u_I - u_h|| \cdot |z|_{H^2(\Omega)},
\end{aligned}
$$

which implies the estimate (3.32). $\qquad\square$

Now, we are able to derive our first main result.

Theorem 3.1. *Assume that the regularity conditions* (2.5) *and* (3.6) *hold. Let u_I be the interpolation of the exact control u defined in* (3.3) *and u_h be the solution of* (2.25)–(2.29)*. Then we have the estimate*

$$
(3.33) \qquad ||u_I - u_h|| \leq Ch^{1+s}, \qquad 0 < s \leq 1.
$$

Proof. First, we derive a variational inequality for the function u_I. Since the relation (2.13) is true for all $\tilde{u} \in U_{ad}$, we have

$$
(z(x) + \nu u(x)) \cdot (\tilde{u} - u(x)) \geq 0, \qquad \forall\ \tilde{u} \in [a,b], \quad \forall\ x \in \Omega.
$$

We apply this formula for $x = S_i$ and $\tilde{u} = u_h(S_i)$. This is correct because of the continuity of u, z, and u_h in these points $\{S_i\}$. That is,

$$
(z(S_i) + \nu u(S_i)) \cdot (u_h(S_i) - u(S_i)) \geq 0, \qquad \forall\ S_i.
$$

It follows from the definition (3.3) of u_I that

$$
(z(S_i) + \nu u_I(S_i)) \cdot (u_h(S_i) - u_I(S_i)) \geq 0, \qquad \forall\ S_i.
$$

We integrate this inequality over T_i and add up over all i to obtain that

(3.34) $(z_I + \nu u_I, u_h - u_I) \geq 0.$

Now, in (2.29), we choose the test function $\tilde{u}_h = u_I$ to obtain that

(3.35) $(z_h(u_h) + \nu u_h, u_I - u_h) \geq 0.$

By adding these two inequalities (3.34) and (3.35), we have

$$(z_h(u_h) - z_I + \nu(u_h - u_I), u_I - u_h) \geq 0.$$

Hence,

$$
\begin{aligned}
\nu\|u_I - u_h\|^2 \quad \leq \quad & (z_h(u_h) - z_I, u_I - u_h) \\
= \quad & (z_h(u_h) - z_h(u_I), u_I - u_h) + (z_h(u_I) - z(u_I), u_I - u) \\
(3.36) \qquad & + (z(u_I) - z(u), u_I - u_h) + (z - z_I, u_I - u_h).
\end{aligned}
$$

Then we combine Lemma 3.1, Lemma 3.4, Lemma 3.2, and Lemma 3.5 to deduce the superconvergence result (3.33). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Furthermore, we can establish the following superconvergence result for state and co-state. But, due to the low regularity of the control, the accuracy is only of the order $h^{1+\min(s,1/2)}$.

Theorem 3.2. *Assume that the regularity conditions* (2.5) *and* (3.6) *hold. Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ *be the exact solutions defined in* (2.9)–(2.13) *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the exact solutions of* (2.25)–(2.29). *Then we have*

(3.37) $\|\Pi_h \boldsymbol{p} - \boldsymbol{p}_h\|_{\mathrm{div}} + \|P_h y - y_h\| \leq C h^{1+\min(s,1/2)},$

(3.38) $\|\Pi_h \boldsymbol{q} - \boldsymbol{q}_h\|_{\mathrm{div}} + \|P_h z - z_h\| \leq C h^{1+\min(s,1/2)},$

for $0 < s \leq 1$.

Proof. It follows from (2.9)–(2.12) and (2.25)–(2.28) that we have the error equations:

$$
\begin{aligned}
(A^{-1}(\boldsymbol{p} - \boldsymbol{p}_h), \boldsymbol{v}_h) - (y - y_h, \mathrm{div}\boldsymbol{v}_h) &= 0, \\
(\mathrm{div}(\boldsymbol{p} - \boldsymbol{p}_h), w_h) + (a_0(y - y_h), w_h) &= (u - u_h, w_h), \\
(A^{-1}(\boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{v}_h) - (z - z_h, \mathrm{div}\boldsymbol{v}_h) &= -(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{v}_h), \\
(\mathrm{div}(\boldsymbol{q} - \boldsymbol{q}_h), w_h) + (a_0(z - z_h), w_h) &= (y - y_h, w_h),
\end{aligned}
$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$. By using the definitions of projections Π_h and P_h, the above equations can be rewritten as follows:

$$
\begin{aligned}
(A^{-1}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h), \boldsymbol{v}_h) - (P_h y - y_h, \mathrm{div}\boldsymbol{v}_h) &= \phi_1(\boldsymbol{v}_h), \\
(\mathrm{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h), w_h) + (a_0(P_h y - y_h), w_h) &= \psi_1(w_h), \\
(A^{-1}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{v}_h) - (P_h z - z_h, \mathrm{div}\boldsymbol{v}_h) &= \phi_2(\boldsymbol{v}_h), \\
(\mathrm{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h), w_h) + (a_0(P_h z - z_h), w_h) &= \psi_2(w_h),
\end{aligned}
$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, where

$$
\begin{aligned}
\phi_1(\boldsymbol{v}_h) &= -(A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h), \\
\psi_1(w_h) &= (u - u_h, w_h) - (a_0(y - P_h y), w_h), \\
\phi_2(\boldsymbol{v}_h) &= -(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{v}_h) - (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h), \\
\psi_2(w_h) &= (y - y_h, w_h) - (a_0(z - P_h z), w_h).
\end{aligned}
$$

Since the terms $\phi_1(\boldsymbol{v}_h)$, $\psi_1(w_h)$, $\phi_2(\boldsymbol{v}_h)$, and $\psi_2(w_h)$ can be regarded as linear functionals of $\boldsymbol{v}_h$ and w_h defined on $\boldsymbol{V}_h$ and W_h, respectively, then we know from the stability result of [7] that

$$(3.39) \quad ||\Pi_h \boldsymbol{p} - \boldsymbol{p}_h||_{\text{div}} + ||P_h y - y_h|| \leq C \left\{ \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{|\phi_1(\boldsymbol{v}_h)|}{||\boldsymbol{v}_h||_{\text{div}}} + \sup_{w_h \in W_h} \frac{|\psi_1(w_h)|}{||w_h||} \right\},$$

$$(3.40) \quad ||\Pi_h \boldsymbol{q} - \boldsymbol{q}_h||_{\text{div}} + ||P_h z - z_h|| \leq C \left\{ \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{|\phi_2(\boldsymbol{v}_h)|}{||\boldsymbol{v}_h||_{\text{div}}} + \sup_{w_h \in W_h} \frac{|\psi_2(w_h)|}{||w_h||} \right\}.$$

It is easy to see that

$$(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{v}_h) = (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h) + (\Pi_h \boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{v}_h),$$
$$(y - y_h, w_h) = (y - P_h y, w_h) + (P_h y - y_h, w_h) = (P_h y - y_h, w_h).$$

By the standard superconvergence of the mixed finite element methods, we have

$$(3.41) \qquad (a_0(y - P_h y), w_h) \leq Ch^2 ||y||_{H^1(\Omega)} ||w_h||,$$
$$(3.42) \qquad (a_0(z - P_h z), w_h) \leq Ch^2 ||z||_{H^1(\Omega)} ||w_h||.$$

On the one hand, under the condition $y,\ z \in H^3(\Omega)$, applying the integral identity technique ([26] and [15]), we see that

$$(3.43) \qquad (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h) \leq Ch^2 ||y||_{H^3(\Omega)} ||\boldsymbol{v}_h||,$$
$$(3.44) \qquad (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h) \leq Ch^2 ||z||_{H^3(\Omega)} ||\boldsymbol{v}_h||,$$
$$(3.45) \qquad (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h) \leq Ch^2 ||y||_{H^3(\Omega)} ||\boldsymbol{v}_h||.$$

On the other hand, applying the standard error estimates of the mixed finite element methods and the approximation properties of the projection operators P_h and Π_h, we have that

$$(3.46) \qquad (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h) \leq Ch ||y||_{H^2(\Omega)} ||\boldsymbol{v}_h||,$$
$$(3.47) \qquad (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h) \leq Ch ||z||_{H^2(\Omega)} ||\boldsymbol{v}_h||,$$
$$(3.48) \qquad (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h) \leq Ch ||y||_{H^2(\Omega)} ||\boldsymbol{v}_h||.$$

Then, by interpolation theory, under the regularity conditions (2.5), we obtain that

$$(3.49) \qquad (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h) \quad \leq \quad Ch^{1+s} ||y||_{H^{2+s}(\Omega)} ||\boldsymbol{v}_h||,$$
$$(3.50) \qquad (A^{-1}(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h) \quad \leq \quad Ch^{1+s} ||z||_{H^{2+s}(\Omega)} ||\boldsymbol{v}_h||,$$
$$(3.51) \qquad (\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{v}_h) \quad \leq \quad Ch^{1+s} ||y||_{H^{2+s}(\Omega)} ||\boldsymbol{v}_h||.$$

Here, we only give the proof of (3.49). We define a linear functional

$$Ty = (A^{-1}(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h) = (A^{-1}((-A\text{grad}y) - \Pi_h(-A\text{grad}y)), \boldsymbol{v}_h).$$

Then, it follows from (3.43) and (3.46) that

$$\begin{aligned} ||T||_{\mathcal{L}(H^{2+s}(\Omega) \to \mathbb{R})} \quad &\leq \quad ||T||^s_{\mathcal{L}(H^3(\Omega) \to \mathbb{R})} ||T||^{1-s}_{\mathcal{L}(H^2(\Omega) \to \mathbb{R})} \\ &\leq \quad C(h^2 ||\boldsymbol{v}_h||)^s \cdot (h||\boldsymbol{v}_h||)^{1-s} = Ch^{1+s} ||\boldsymbol{v}_h||, \end{aligned}$$

which implies (3.49). We can similarly prove (3.50) and (3.51).

Note that

$$\begin{aligned} (u - u_h, w_h) \quad &= \quad (u - u_I, w_h) + (u_I - u_h, w_h) \\ (3.52) \qquad &= \quad (u - u_I, w_h)_{K_1} + (u - u_I, w_h)_{K_2} + (u_I - u_h, w_h). \end{aligned}$$

It follows from the assumption (3.6) and (3.5) that

$$
\begin{aligned}
(u - u_I, w_h)_{K_1} &\leq \sum_{T_i \in K_1} \int_{T_i} |u(x) - u(S_i)| \cdot |w_h| dx \\
&\leq Ch \sum_{T_i \in K_1} ||u||_{1,\infty,T_i} \int_{T_i} |w_h| dx \\
&\leq Ch ||u||_{1,\infty} \sum_{T_i \in K_1} \sqrt{|T_i|} \cdot ||w_h||_{T_i} \\
(3.53) \qquad &\leq Ch^{3/2} ||u||_{1,\infty} \cdot ||w_h||, \\
(u - u_I, w_h)_{K_2} &\leq \sum_{T_i \in K_2} |w_h| \cdot \left| \int_{T_i} (u(x) - u(S_i)) dx \right| \\
&\leq Ch^2 \sum_{T_i \in K_2} ||u||_{H^2(T_i)} \cdot \sqrt{|T_i|} \cdot |w_h| \\
(3.54) \qquad &\leq Ch^2 ||u||_{H^2(\Omega)} \cdot ||w_h||.
\end{aligned}
$$

By using Theorem 3.1, we clearly see that

$$
(3.55) \qquad (u_I - u_h, w_h) \leq ||u_I - u_h|| \cdot ||w_h|| \leq Ch^{1+s} ||w_h||.
$$

We apply the estimates (3.52)–(3.55), (3.41), and (3.49) in (3.39) to obtain the result (3.37). Similarly, we can apply the estimates (3.42), (3.50)–(3.51), and (3.37) in (3.40) to obtain the result (3.38). □

Lemma 3.6. *For $0 < s \leq 1$, assume that $\boldsymbol{v} \in (H^{1+s}(\Omega))^2 \cap \boldsymbol{V}$, $w \in H^2(\Omega)$, and w_I is the interpolation function of w defined in (3.3). Then we have for some constant $C > 0$ that*

$$
(3.56) \qquad |||\boldsymbol{v} - \Pi_h \boldsymbol{v}||| \leq Ch^{1+s} ||\boldsymbol{v}||_{H^{1+s}(\Omega)}, \qquad 0 < s \leq 1,
$$

$$
(3.57) \qquad ||P_h w - w_I|| \leq Ch^2 ||w||_{H^2(\Omega)}.
$$

Proof. From [15], if $\boldsymbol{v} \in (H^2(\Omega))^2 \cap \boldsymbol{V}$ and $w \in H^2(\Omega)$, then we have

$$
(3.58) \qquad |||\boldsymbol{v} - \Pi_h \boldsymbol{v}||| \leq Ch^2 ||\boldsymbol{v}||_{H^2(\Omega)},
$$

$$
(3.59) \qquad ||P_h w - w_I|| \leq Ch^2 ||w||_{H^2(\Omega)}.
$$

The estimate (3.56) can be proved by interpolation theory. □

Corollary 3.1. *Assume that the regularity conditions (2.5) and (3.6) hold. Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ be the exact solutions defined in (2.9)–(2.13) and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ be the exact solutions of (2.25)–(2.29). Then we have*

$$
(3.60) \qquad |||\boldsymbol{p} - \boldsymbol{p}_h||| + |||y - y_h|||_0 \leq Ch^{1+\min(s,1/2)},
$$

$$
(3.61) \qquad |||\boldsymbol{q} - \boldsymbol{q}_h||| + |||z - z_h|||_0 \leq Ch^{1+\min(s,1/2)},
$$

$$
(3.62) \qquad |||u - u_h|||_0 \leq Ch^{1+s},
$$

for $0 < s \leq 1$.

Proof. First, the errors are decomposed as

$$\boldsymbol{p} - \boldsymbol{p}_h = (\boldsymbol{p} - \Pi_h \boldsymbol{p}) + (\Pi_h \boldsymbol{p} - \boldsymbol{p}_h),$$
$$y - y_h = (y - P_h y) + (P_h y - y_h),$$
$$\boldsymbol{q} - \boldsymbol{q}_h = (\boldsymbol{q} - \Pi_h \boldsymbol{q}) + (\Pi_h \boldsymbol{q} - \boldsymbol{q}_h),$$
$$z - z_h = (z - P_h z) + (P_h z - z_h),$$

and note that

$$|||y - P_h y|||_0 = |||y_I - P_h y|||_0 = ||y_I - P_h y||,$$
$$|||z - P_h z|||_0 = |||z_I - P_h z|||_0 = ||z_I - P_h z||,$$
$$|||u - u_h|||_0 = |||u_I - u_h|||_0 = ||u_I - u_h||.$$

Then by Lemma 3.6 and Theorems 3.1 and 3.2 we can prove the results of (3.60)–(3.62). $\qquad\square$

4. Global L^2 superconvergence by postprocessing

In this section, we will apply a higher order interpolation postprocessing method presented by Lin and Yan [26] to obtain global superconvergence for the mixed finite element approximations. As by-products, these superconvergence results can be utilized to form a class of useful a posteriori error estimators to assess the accuracy of the mixed finite element solutions in applications for the optimal control problems considered in this paper.

We construct a larger rectangular elements partition $\mathcal{T}_{2h}$, which is the coarse meshes of $\mathcal{T}_h$. That is, each element τ of $\mathcal{T}_{2h}$ is composed of four neighboring rectangular elements of $\mathcal{T}_h$. Based on this coarse meshes, we denote $\boldsymbol{V}_{2h} \times W_{2h}$ to express the order $k = 1$ Raviart-Thomas mixed finite element spaces:

$$\boldsymbol{V}_{2h} = \{\boldsymbol{v} \in \boldsymbol{V} : \quad \forall\, \tau \in \mathcal{T}_{2h},\ \boldsymbol{v}|_\tau \in Q_{2,1}(\tau) \times Q_{1,2}(\tau)\},$$
$$W_{2h} = \{w \in W : \quad \forall\, \tau \in \mathcal{T}_{2h},\ w|_\tau \in Q_{1,1}(\tau)\},$$

and the related Raviart-Thomas projection (see [14] and [31]):

$$\Pi_{2h} \times P_{2h} : \quad \boldsymbol{V} \times W \to \boldsymbol{V}_{2h} \times W_{2h},$$

which satisfies the following properties [26]:

(i) $P_{2h} P_h = P_{2h}$ and $||P_{2h} w_h|| \leq C||w_h||$, for all $w_h \in W_h$.

(ii) $\Pi_{2h} \Pi_h = \Pi_{2h}$ and $||\Pi_{2h} \boldsymbol{v}_h||_{\mathrm{div}} \leq C||\boldsymbol{v}_h||_{\mathrm{div}}$, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

By using the interpolation operators Π_{2h} and P_{2h} and their properties, we immediately obtain the following global superconvergence theorem.

Theorem 4.1. *Assume that the regularity conditions* (2.5) *and* (3.6) *hold. Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ *be the exact solutions defined in* (2.9)–(2.13) *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the exact solutions of* (2.25)–(2.29). *Then we have*

(4.1) $$||\boldsymbol{p} - \Pi_{2h} \boldsymbol{p}_h||_{\mathrm{div}} + ||y - P_{2h} y_h|| \leq Ch^{1+\min(s,1/2)},$$

(4.2) $$||\boldsymbol{q} - \Pi_{2h} \boldsymbol{q}_h||_{\mathrm{div}} + ||z - P_{2h} z_h|| \leq Ch^{1+\min(s,1/2)},$$

for $0 < s \leq 1$.

Proof. From property (i) of the operator P_{2h}, we find that

$$y - P_{2h}y_h = y - P_{2h}y + P_{2h}(P_hy - y_h).$$

Therefore, it follows from the approximation property, property (i) of the operator P_{2h}, and Theorem 3.2 that

$$||y - P_{2h}y_h|| \leq ||y - P_{2h}y|| + C||P_hy - y_h|| \leq Ch^{1+\min(s,1/2)},$$

which implies (4.1). Similarly, we can obtain (4.2). $\square$

Furthermore, by using Lemma 3.3, it is obvious to see that

Lemma 4.1. *For any function* $\tilde{u} \in U_{ad}$, *let* $(\boldsymbol{p}(\tilde{u}), y(\tilde{u}), \boldsymbol{q}(\tilde{u}), z(\tilde{u}))$ *and* $(\boldsymbol{p}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{q}_h(\tilde{u}), z_h(\tilde{u}))$ *be the solutions of* (2.30)–(2.33) *and* (2.34)–(2.37), *respectively, for the lowest order Raviart-Thomas mixed finite elements. If the regularity condition*

$$y(\tilde{u}),\ z(\tilde{u}) \in H^1(\Omega), \qquad \boldsymbol{p}(\tilde{u}),\ \boldsymbol{q}(\tilde{u}) \in (H^2(\Omega))^2$$

holds, then we have

$$(4.3) \qquad ||p(\tilde{u}) - P_{2h}p_h(\tilde{u})|| + ||\boldsymbol{p}(\tilde{u}) - \Pi_{2h}\boldsymbol{p}_h(\tilde{u})|| \leq Ch^2,$$

$$(4.4) \qquad ||z(\tilde{u}) - P_{2h}z_h(\tilde{u})|| + ||\boldsymbol{q}(\tilde{u}) - \Pi_{2h}\boldsymbol{q}_h(\tilde{u})|| \leq Ch^2.$$

Proof. For brevity, we only give the proof of (4.4). From property (i) of the operator P_{2h}, we find that

$$z(\tilde{u}) - P_{2h}z_h(\tilde{u}) = z(\tilde{u}) - P_{2h}z(\tilde{u}) + P_{2h}(P_hz(\tilde{u}) - z_h(\tilde{u})).$$

Therefore, it follows from the approximation property, property (i) of the operator P_{2h}, and Lemma 3.3 that

$$||z(\tilde{u}) - P_{2h}z_h(\tilde{u})|| \leq ||z(\tilde{u}) - P_{2h}z(\tilde{u})|| + C||P_hz(\tilde{u}) - z_h(\tilde{u})|| \leq Ch^2,$$

which implies (4.4). $\square$

Then, in order to improve the accuracy of the control approximation on a global scale, as in [30] we construct a postprocessing projection operator of the discrete co-state to the admissible set

$$(4.5) \qquad \hat{u}(x) = \Pi_{[a,b]}\left(-\frac{1}{\nu}(P_{2h}z_h(u_h))(x)\right),$$

where $\Pi_{[a,b]}$ was defined in (2.14). Now, we can prove the following global superconvergence result.

Theorem 4.2. *Assume that the regularity conditions* (2.5) *and* (3.6) *hold. Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ *be the exact solutions defined in* (2.9)–(2.13) *and* $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the exact solutions of* (2.25)–(2.29). *Then we have*

$$(4.6) \qquad ||u - \hat{u}|| \leq Ch^{1+s}, \qquad 0 < s \leq 1.$$

Proof. We note that the projection $\Pi_{[a,b]}$ defined in (2.14) is Lipschitz continuous with constant 1 from $L^2(\Omega)$ to $L^2(\Omega)$. From (2.15) and (4.5), we obtain by the triangle inequality

$$
\begin{aligned}
||u - \hat{u}|| &\leq C||z - P_{2h}z_h|| \\
&\leq C\Big(||z(u) - z(u_I)|| + ||z(u_I) - P_{2h}z_h(u_I)|| \\
&\quad + ||P_{2h}z_h(u_I) - P_{2h}z_h(u_h)||\Big).
\end{aligned}
$$

(4.7)

We firstly apply Lemma 3.2 to obtain that

(4.8) $$||z(u) - z(u_I)|| \leq Ch^2|\ln h|^{1/2}.$$

Then, from the approximation property of the operator P_{2h}, property (i) of the operator P_{2h}, and (3.31), we have

$$
\begin{aligned}
||z(u_I) - P_{2h}z_h(u_I)|| &\leq ||z(u_I) - P_{2h}z(u_I)|| + ||P_{2h}z(u_I) - P_{2h}z_h(u_I)|| \\
&\leq Ch^2||z(u_I)||_{H^2(\Omega)} + ||P_{2h}P_h z(u_I) - P_{2h}z_h(u_I)|| \\
&\leq Ch^2||z(u_I)||_{H^2(\Omega)} + ||P_h z(u_I) - z_h(u_I)|| \\
&\leq Ch^{1+s}||z||_{H^{2+s}(\Omega)}.
\end{aligned}
$$

(4.9)

Next, it remains to bound the third term of above inequality (4.7). By property (i), we have

(4.10) $$||P_{2h}z_h(u_I) - P_{2h}z_h(u_h)|| \leq C||z_h(u_I) - z_h(u_h)||.$$

Similar to the proof of Lemma 3.1, we use (2.34)–(2.37) to obtain the error equations

$$
\begin{aligned}
&(A^{-1}(\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h)), \boldsymbol{v}_h) - (y_h(u_I) - y_h(u_h), \operatorname{div}\boldsymbol{v}_h) = 0, \\
&(\operatorname{div}(\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h)), w_h) + (a_0(y_h(u_I) - y_h(u_h)), w_h) = (u_I - u_h, w_h), \\
&(A^{-1}(\boldsymbol{q}_h(u_I) - \boldsymbol{q}_h(u_h)), \boldsymbol{v}_h) - (z_h(u_I) - z_h(u_h), \operatorname{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h), \boldsymbol{v}_h), \\
&(\operatorname{div}(\boldsymbol{q}_h(u_I) - \boldsymbol{q}_h(u_h)), w_h) + (a_0(z_h(u_I) - z_h(u_h)), w_h) = (y_h(u_I) - y_h(u_h), w_h),
\end{aligned}
$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$. We use the stability property of the saddle-point problem to obtain that

$$
\begin{aligned}
||z_h(u_I) - z_h(u_h)|| &+ ||\boldsymbol{q}_h(u_I) - \boldsymbol{q}_h(u_h)|| \\
&\leq C\left(||y_h(u_I) - y_h(u_h)|| + ||\boldsymbol{p}_h(u_I) - \boldsymbol{p}_h(u_h)||\right) \\
&\leq C||u_I - u_h|| \leq Ch^{1+s},
\end{aligned}
$$

(4.11)

where the last step was derived by using Theorem 3.1.

Finally, we have established (4.6) from (4.7) combined with (4.8)–(4.11).   □

It is of great importance for a mixed finite element method to have a computable a posteriori error estimator by which we can evaluate the accuracy of the mixed finite element solutions in applications. One way to construct error estimators is to employ certain superconvergence properties of the finite element solutions. Thus, based on the above global superconvergence, we can obtain the following recovery type a posteriori error estimates for the control problems.

Theorem 4.3. *Assume that the regularity conditions (2.5) and (3.6) hold. Let* $(\boldsymbol{p}, y, \boldsymbol{q}, z, u) \in (\boldsymbol{V} \times W)^2 \times U_{ad}$ *be the exact solutions defined in (2.9)–(2.13) and*

$(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h$ *be the exact solutions of* (2.25)–(2.29). *Then we have*

$$||\boldsymbol{p} - \boldsymbol{p}_h||_{\mathrm{div}} = ||\Pi_{2h}\boldsymbol{p}_h - \boldsymbol{p}_h||_{\mathrm{div}} + O(h^{1+\min(s,1/2)}),$$
$$||y - y_h|| = ||P_{2h}y_h - y_h|| + O(h^{1+\min(s,1/2)}),$$
$$||\boldsymbol{q} - \boldsymbol{q}_h||_{\mathrm{div}} = ||\Pi_{2h}\boldsymbol{q}_h - \boldsymbol{q}_h||_{\mathrm{div}} + O(h^{1+\min(s,1/2)}),$$
$$||z - z_h|| = ||P_{2h}z_h - z_h|| + O(h^{1+\min(s,1/2)}),$$
$$||u - u_h|| = ||\hat{u} - u_h|| + O(h^{1+s}),$$

for $0 < s \leq 1$. *Furthermore, there hold*

$$\lim_{h \to 0} \frac{||\Pi_{2h}\boldsymbol{p}_h - \boldsymbol{p}_h||_{\mathrm{div}}}{||\boldsymbol{p} - \boldsymbol{p}_h||_{\mathrm{div}}} = 1,$$
$$\lim_{h \to 0} \frac{||P_{2h}y_h - y_h||}{||y - y_h||} = 1,$$
$$\lim_{h \to 0} \frac{||\Pi_{2h}\boldsymbol{q}_h - \boldsymbol{q}_h||_{\mathrm{div}}}{||\boldsymbol{q} - \boldsymbol{q}_h||_{\mathrm{div}}} = 1,$$
$$\lim_{h \to 0} \frac{||P_{2h}z_h - z_h||}{||z - z_h||} = 1,$$
$$\lim_{h \to 0} \frac{||\hat{u} - u_h||}{||u - u_h||} = 1.$$

Proof. The proof follows easily from Theorems 4.1-4.2 and the composed relations as follows:

$$\boldsymbol{p} - \boldsymbol{p}_h = (\Pi_{2h}\boldsymbol{p}_h - \boldsymbol{p}_h) + (\boldsymbol{p} - \Pi_{2h}\boldsymbol{p}_h),$$
$$y - y_h = (P_{2h}y_h - y_h) + (y - P_{2h}y_h),$$
$$\boldsymbol{q} - \boldsymbol{q}_h = (\Pi_{2h}\boldsymbol{q}_h - \boldsymbol{q}_h) + (\boldsymbol{q} - \Pi_{2h}\boldsymbol{q}_h),$$
$$z - z_h = (P_{2h}z_h - z_h) + (z - P_{2h}z_h),$$
$$u - u_h = (\hat{u} - u_h) + (u - \hat{u}).$$

$\square$

Therefore, the recovery type a posteriori error estimators defined above are asymptotically exact if the conditions for superconvergence are valid.

5. Numerical tests

We present below two examples to illustrate the superconvergence theoretical results of the control. In both cases, the Laplace operator $-\Delta$ was chosen for the elliptic operator $A(x)$. The first example is based on Example 1 of [30] with some modification. In the second example, we take the obstacle function not as a constant, but with some weak discontinuities, and find the same superconvergence phenomenon in the first example.

Actually, our numerical example satisfies the stronger regularity y, $z \in H^3(\Omega)$. The numerical results will show that the approximated solution of the control possesses almost two order superconvergences for the approximation space of piecewise constant functions.

Both optimization problems were solved numerically by projected gradient methods, with codes developed based on AFEPack [24]. The discretization was already

described in previous sections: The control function u is discretized by piecewise constant functions, whereas the state $(y, \boldsymbol{p})$ and the co-state $(z, \boldsymbol{q})$ were approximated by the lowest order Raviart-Thomas mixed finite element functions. In our examples, we choose the domain $\Omega = [0, 1] \times [0, 1]$. We use the uniform rectangle partitions.

Example 1. We consider the following two dimensional elliptic optimal control problem:

$$(5.1) \qquad \min_{u \in U_{ad}} \left\{ \frac{1}{2} ||\boldsymbol{p} - \boldsymbol{p}_d||^2 + \frac{1}{2} ||y - y_d||^2 + \frac{\nu}{2} ||u||^2 \right\}$$

subject to the state equation

$$(5.2) \qquad \operatorname{div} \boldsymbol{p} + a_0 y = u + f, \qquad \boldsymbol{p} = -A(x) \operatorname{grad} y, \qquad x \in \Omega,$$

which is written in the form of the first order system with the boundary condition

$$(5.3) \qquad y = 0, \qquad x \in \partial\Omega,$$

where $\Omega = [0, 1] \times [0, 1]$. Here we choose $A(x) \equiv 1$ and $a_0 \equiv 0$. Then the state equation may be restated as

$$(5.4) \qquad \operatorname{div} \boldsymbol{p} = u + f, \qquad \boldsymbol{p} = -\operatorname{grad} y, \qquad x \in \Omega.$$

Next, we introduce the co-state elliptic equation

$$(5.5) \qquad \operatorname{div} \boldsymbol{q} = y - y_d, \qquad \boldsymbol{q} = -(\operatorname{grad} z + \boldsymbol{p} - \boldsymbol{p}_d), \qquad x \in \Omega,$$

with the boundary condition

$$(5.6) \qquad z = 0, \qquad x \in \partial\Omega.$$

Now, we define the optimal state function by

$$y(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$$

and the source function f is given by

$$f(x_1, x_2) = \begin{cases} u_f(x_1, x_2) - a, & \text{if } u_f(x_1, x_2) < a, \\ 0, & \text{if } u_f(x_1, x_2) \in [a, b], \\ u_f(x_1, x_2) - b, & \text{if } u_f(x_1, x_2) > b, \end{cases}$$

with $u_f(x_1, x_2) = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2)$. Due to the state equation (5.4), we obtain for the exact optimal control function u

$$u(x_1, x_2) = \begin{cases} a, & \text{if } u_f(x_1, x_2) < a, \\ u_f(x_1, x_2), & \text{if } u_f(x_1, x_2) \in [a, b], \\ b, & \text{if } u_f(x_1, x_2) > b. \end{cases}$$

For the optimal co-state function z, we find

$$z(x_1, x_2) = -2\pi^2 \nu \sin(\pi x_1) \sin(\pi x_2).$$

It follows from the co-state equation (5.5) that

$$y - y_d = -\operatorname{div}(\operatorname{grad} z + \boldsymbol{p} - \boldsymbol{p}_d) = -4\pi^4 \nu \sin(\pi x_1) \sin(\pi x_2) - \operatorname{div}(\boldsymbol{p} - \boldsymbol{p}_d)$$

or

$$\operatorname{div} \boldsymbol{p}_d + y_d = y + u + f + 4\pi^4 \nu \sin(\pi x_1) \sin(\pi x_2).$$

Thus, the desired state variables are given by

$$y_d(x_1, x_2) = y + 2\pi^4 \nu \sin(\pi x_1) \sin(\pi x_2) = (1 + 2\pi^4 \nu) \sin(\pi x_1) \sin(\pi x_2)$$
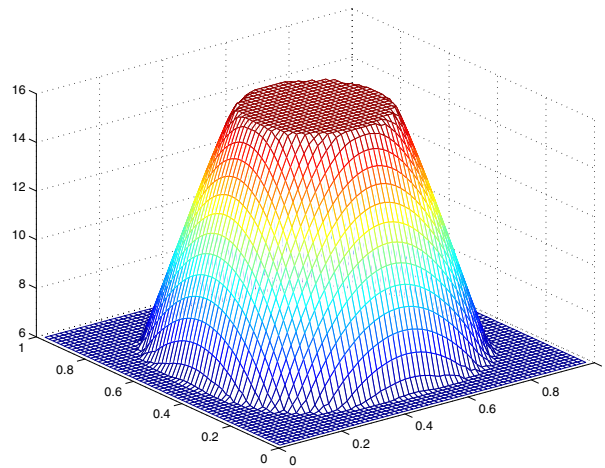
FIGURE 1. The profile of the numerical solution of Example 1 on the 64×64 mesh grid.

and

$$\text{div}\boldsymbol{p}_d = u + f + 2\pi^4\nu\sin(\pi x_1)\sin(\pi x_2) = -\text{div}(\text{grad}(y - \pi^2\nu\sin(\pi x_1)\sin(\pi x_2))).$$

We may choose

$$\begin{aligned}
\boldsymbol{p}_d(x_1, x_2) &= -\text{grad}(y - \pi^2\nu\sin(\pi x_1)\sin(\pi x_2)) \\
&= (-(\pi - \pi^3\nu)\cos(\pi x_1)\sin(\pi x_2), -(\pi - \pi^3\nu)\sin(\pi x_1)\cos(\pi x_2)).
\end{aligned}$$

In numerical implementation, we set $a = 6$ and $b = 16$ to make both the lower and the upper constraints active. The profile of the numerical solution is plotted in Figure 1 and the error $\|u_I - u_h\|$ and $\|u - \hat{u}\|$ obtained on a sequence of uniformly refined meshes are presented in Table 1. The superconvergence phenomenon can clearly be observed from the data.

resolution	$\|u_I - u_h\|$	order	$\|u - \hat{u}\|$	order
16×16	3.239e-02	-	1.494e-01	-
32×32	8.457e-03	1.94	3.763e-02	1.99
64×64	2.058e-03	2.04	9.387e-03	2.00
128×128	5.218e-04	1.98	2.354e-03	2.00

TABLE 1. The error of the numerical solution of Example 1 on a sequence of uniformly refined mesh grid.

Example 2. In this example, we investigate the case of the nonconstant control constraints. We replace a and b in Example 1 by two specific functions as follows:

$$a(x_1, x_2) = 5 + \frac{5}{\sqrt{2}}|x_1 - x_2|,$$

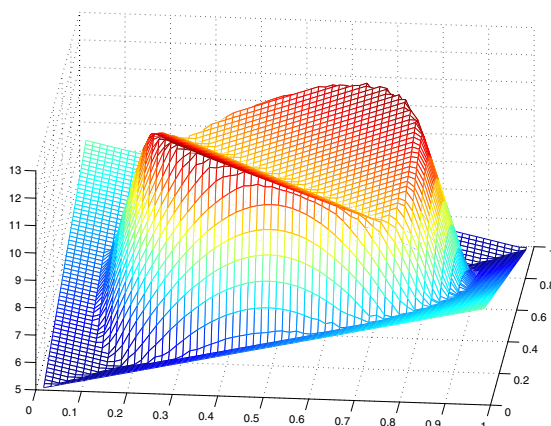$$b(x_1, x_2) = 10 + \frac{8}{\sqrt{2}}|x_1 + x_2 - 1|.$$

FIGURE 2. The profile of the numerical solution of Example 2 on the 64×64 mesh grids.

Then not only the constraints depend on the coordinates (x_1, x_2), but also there are some weak discontinuities in both constraints. The profile of the numerical solution is presented in Figure 2. From the error data on the uniformly refined meshes, as listed in Table 2, it can be seen that the superconvergence remains, even though there is the introduction of the weak discontinuities in the input data.

resolution	$\|u_I - u_h\|$	order	$\|u - \hat{u}\|$	order
16×16	1.995e-02	-	9.109e-02	-
32×32	5.056e-03	1.98	2.289e-02	1.99
64×64	1.261e-03	2.00	5.712e-03	2.00
128×128	3.205e-04	1.98	1.435e-03	1.99

TABLE 2. The error of the numerical solution of Example 2 on a sequential of uniformly refined mesh grids.

6. Conclusion and future work

The present paper discussed the rectangular lowest order Raviart-Thomas mixed finite element methods for the optimal control problem (1.1)–(1.3). By applying the superconvergence results (see [15] and [16]) of standard mixed finite element methods, we have established some local and global superconvergence results in L^2 norm both for the state and the co-state discrete solutions with convergence order $h^{1+\min(s,1/2)}$), and for the control approximation with convergence order h^{1+s}, where $0 < s \leq 1$.

These results can be extended to mixed finite element methods on quadrilaterals. Moreover, one may consider the piecewise point behavior of the approximations and L^∞-norm superconvergence properties for the flux functions along the Gauss lines and for the scalar functions at the Gauss points. We have used piecewise constant

functions to approximate the control variable. In our future work, we shall use the standard linear element space to approximate the control function.

Furthermore, we shall investigate the case that a and b in the admissible control set U_{ad} are some smooth functions (not constants, as in the present paper; see (1.4)) or the set

$$(6.1) \qquad U_{ad} = \left\{ u \in L^2(\Omega) : \quad \int_\Omega u(x)dx \geq 0 \right\}.$$

Acknowledgments

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