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SUPERDIFFUSIONS AND PARABOLIC NONLINEAR DIFFERENTIAL EQUATIONS[1]

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We establish connections between superdiffusion processes and one class of nonlinear parabolic differential equations. Analytic results due to Brezis and Friedman, Baras and Pierre and others are used to investigate the graphs of superdiffusions. A survey of the literature and general comments are presented in Section 4.

1. Probabilistic solution of the first boundary value problem.

1.1. We consider positive solutions of a differential equation

$$\dot{v}(r,x) + Lv(r,x) - v(r,x)^{\alpha} = -\rho(r,x) \quad \text{for } (r,x) \in Q, \tag{1.1}$$

where $1 < \alpha \leq 2$, $\dot{v} = \partial v/\partial r$, L is an elliptic differential operator in $E = \mathbb{R}^d$ (with coefficients depending on r and x), Q is a domain in $S = \mathbb{R} \times E$ and ρ is a positive function in Q. (Part of the results hold for a more general equation with the term v^{α} replaced by a function ψ of a class described in [11] and [12].)

We assume that: (a) the coefficients of the operator

$$L = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} \tag{1.2}$$

are bounded and smooth (it is sufficient that b_i are continuously differentiable and a_{ij} are twice continuously differentiable); (b) there exists a constant $\gamma > 0$ such that

$$\sum_{i,j} a_{ij} u_i u_j \geq \gamma \sum_i u_i^2 \tag{1.3}$$

for all $(r,x) \in S$, $u_1, \dots, u_d \in \mathbb{R}$.

Under these conditions, there exists a Markov process $\xi = (\xi_t, \Pi_{r,x})$ in E with continuous paths such that, for every bounded continuous function f on E,

$$F(r,x) = \Pi_{r,x} f(\xi_t) \tag{1.4}$$

is the unique bounded solution of the equation

$$\dot{F} + LF = 0 \quad \text{in } (-\infty, t) \times E \tag{1.5}$$

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with the property

$$F(r,x) \to f(x) \quad \text{as } r \uparrow t. \tag{1.6}$$

It can be constructed by using either Itô's stochastic differential equations or the fundamental solution of (1.5) (see, for instance, [24] or [9]). We say that ξ is the *diffusion with generator L*.

Denote by $\mathscr{B}$ the Borel σ-algebra in E, by M the set of all finite measures on $\mathscr{B}$ and by $\mathscr{M}$ the σ-algebra in M generated by the functions $f_B(\mu) = \mu(B)$, $B \in \mathscr{B}$. Let $\mathscr{B}_S$ stand for the Borel σ-algebra in S and let $\mathfrak{M}_Q$ be the set of all finite measures on $\mathscr{B}_S$ such that $\mu(Q^c) = 0$. Put $\mu \in \mathfrak{M}_Q^0$ if the support of μ is compact and is contained in Q. We drop the subscript Q if $Q = S$. For every interval Δ, we put $S_\Delta = \Delta \times E$; here $S_{\le t}$ means S_Δ with $\Delta = (-\infty, t]$.

According to [10], [12], [15], there exists a Markov process $(X_t, P_{r,\nu})$ in $(M, \mathscr{M})$ such that:

1.1.A. If $f^t(x)$ is a bounded continuous function on S, then $\langle f^t, X_t\rangle$ is, a.s., right continuous in t (here $\langle v, \nu\rangle$ means the integral of v with respect to ν).

1.1.B. For every $r < t \in \mathbb{R}$, $\nu \in M$, $f \in \mathscr{B}$,

$$P_{r,\nu} \exp\langle -f, X_t\rangle = \exp\langle -v^r, \nu\rangle, \tag{1.7}$$

where $v^r(x) = v(r,x)$ is a solution of the integral equation

$$v(r,x) + \Pi_{r,x} \int_r^t v(s,\xi_s)^\alpha \, ds = \Pi_{r,x} f(\xi_t) \quad \text{for } r < t. \tag{1.8}$$

For every coanalytic set Q, the formula

$$\tau_r = \inf\{s\colon s \ge r, (s,\xi_s) \notin Q\} \tag{1.9}$$

determines a family of stopping times which we call the first exit times from Q. Two $\mathfrak{M}$-valued random measures X_τ and Y_τ are associated with this family. Moreover, the family of measures $\{P_{r,\nu}, r \in \mathbb{R}, \nu \in M\}$ is contained in a bigger family $\{P_\mu, \mu \in \mathfrak{M}\}$ such that, for every P_μ,

$$P_\mu \exp\{-\langle \rho, Y_\tau\rangle - \langle f, X_\tau\rangle\} = \exp\langle -v, \mu\rangle, \tag{1.10a}$$

where

$$\begin{aligned} &v(r,x) + \Pi_{r,x} \int_r^{\tau_r} v(s,\xi_s)^\alpha \, ds \\ &\quad = \Pi_{r,x}\left[\int_r^{\tau_r} \rho(s,\xi_s)\, ds + f(\tau_r, \xi_{\tau_r})\right] \quad \text{for } (r,x) \in Q, \\ &v(r,x) = f(r,x) \quad \text{for } (r,x) \notin Q. \end{aligned} \tag{1.11a}$$

Here ρ is an arbitrary positive Borel function on Q. If $\rho = 0$, μ is the image of ν under the mapping $x \to (r,x)$ and $Q = S_{<t}$, then (1.10a)–(1.11a) imply (1.7)–(1.8). Note that $X_\tau = \mu$ and $Y_\tau = 0$ P_μ-a.s. if $\mu(Q) = 0$.

We call the collection $X = (X_\tau, X_\tau; P_\mu)$ the *superdiffusion with parameters* (L, α).

The superdiffusion X can be obtained by a passage to the limit from a branching particle system as the mass β of individual particles tends to 0 (see [11] and [12]).

For every set $U \subset S$ we define the t-section of U by the formula $U_t = \{x: (t, x) \in U\}$. Condition 1.1.A implies that:

1.1.A′. For every open set U, $X_t(U_t)$ is, a.s., lower right semicontinuous in t, that is, for every $r \geq 0$,

$$\liminf_{t \to r, t > r} X_t(U_t) \geq X_r(U_r).$$

1.2. Let Q be an open set in S. A part of ξ in Q is obtained from ξ by reducing the life interval $[\alpha, \zeta)$ to $[\alpha, \tau_\alpha)$, where τ_r are the first exit times from Q (see [9], Chapter 3, Section 2). This is a Markov process $\tilde{\xi} = (\tilde{\xi}_t, \Pi_{r,x})$ on Q with the generator $\tilde{L}$ equal to the restriction of L to Q. The state space Q_t (= the t-section of Q) depends on t and it is nonempty only for $t \in \tilde{\Delta}$, where $\tilde{\Delta}$ is the projection of Q on $\mathbb{R}$.

Note that the first exit times $\tau_r(t)$ from $Q_{<t} = Q \cap S_{<t}$ are equal to $\tau_r \wedge t$ for $r < t$. The restriction $\tilde{X}_t$ of $X_{\tau(t)}$ to Q is a measure concentrated on Q_t. It follows from (1.10a)–(1.11a) that, for $r < t$,

$$P_{r,\nu} \exp\langle -f, X_{\tau(t)}\rangle = \exp\langle -v^r, \nu\rangle, \tag{1.10b}$$

where

$$v(r,x) + \Pi_{r,x} \int_r^{\tau_r \wedge t} v(s, \xi_s)^\alpha \, ds = \Pi_{r,x} f(\tau_r \wedge t, \xi_{\tau_r \wedge t}) \quad \text{for } r < t. \tag{1.11b}$$

If $f = 0$ on Q^c and $\mu \in \mathfrak{M}_Q$, then formulae (1.10b), (1.11b) are identical to (1.7), (1.8) with ξ_t, X_t replaced by $\tilde{\xi}_t$ and $\tilde{X}_t$. Each $\tilde{X}_t$ is defined only up to equivalence. They can be chosen in such a way that $\langle f^t, \tilde{X}_t\rangle$ is right continuous P_μ-a.s. for every bounded continuous function f and every $\mu \in \mathfrak{M}_Q$. The process $\tilde{X} = (\tilde{X}_t, P_\mu)$, where $\mu \in \mathfrak{M}_Q$ can be interpreted as a superdiffusion with parameters $(\tilde{L}, \alpha)$. We call $\tilde{X}$ the *part of X in Q*.

If Q is an open set, then the random measure Y_τ in formula (1.10) can be expressed through the part of X on Q by the formula $Y_\tau(dt, dx) = dt\,\tilde{X}_t(dx)$. (This follows from Theorem 1.6 in [12].) The measure $Y(dt, dx) = dt\,X_t(dx)$ corresponds to $Q = S$.

According to the Addendum to [12], if $Q^1 \subset Q^2$ and if $\tilde{X}^i$ is the part of X in Q^i, then $X_t^1(B) \leq X_t^2(B)$ for all $t \in \mathbb{R}$, $B \in \mathscr{B}$.

1.3. We refer to the Appendix for the definitions of the regular part $\partial_r Q$ of ∂Q, total subsets T of ∂Q, regular domains and the classes $\mathbb{C}^1(Q)$ and $\mathbb{C}^2(Q)$. When we say that v is a solution of the equation (1.1) we mean that

$v \in \mathbb{C}^2(Q)$. We say that v satisfies a boundary condition $v = f$ on $B \subset \partial Q$ if

$$v(r,x) \to f(t,a) \quad \text{as } (r,x) \to (t,a) \in B, (r,x) \in Q.$$

THEOREM 1.1. *Let X be a superdiffusion with parameters (L, α). Let Q be a bounded open set and let τ_r be the first exit times from Q. If $\rho \geq 0$ is bounded and belongs to $\mathbb{C}^1(Q)$ and if $f \geq 0$ is a bounded function on ∂Q, then the function*

$$v(r,x) = -\log P_{\delta_{r,x}} \exp\{-\langle \rho, Y_\tau \rangle - \langle f, X_\tau \rangle\} \tag{1.12}$$

[$\delta_{r,x}$ is the unit measure concentrated at (r,x)] is a solution of (1.1). *If $(t,a) \in \partial Q$ is regular and if f is continuous at (t,a), then*

$$v(r,x) \to f(t,a) \quad as\ (r,x) \to (t,a). \tag{1.13}$$

If Q is regular, then (1.12) *is the unique solution of* (1.1) *which satisfies* (1.13) *at all $(t,a) \in \partial_r Q$.*

For every $\mu \in \mathfrak{M}$,

$$P_\mu \exp\{-\langle \rho, Y_\tau \rangle - \langle f, X_\tau \rangle\} = \exp\langle -v, \mu \rangle. \tag{1.14}$$

PROOF. The equation (1.11) can be rewritten as

$$v + F_1 = h + F, \tag{1.15}$$

where h and F are given by (A.4) and (A.7) and

$$F_1(r,x) = \Pi_{r,x} \int_r^{\tau_r} v(s, \xi_s)^\alpha \, ds.$$

Clearly, h and F are bounded and therefore v is also bounded. By A.1.D, F and F_1 belong to $\mathbb{C}^1(Q)$. By A.1.C, $h \in \mathbb{C}^2(Q)$. We conclude from (1.15) that $v \in \mathbb{C}^1(Q)$. Hence $v^\alpha \in \mathbb{C}^1(Q)$ and, by A.1.D, $F_1 \in \mathbb{C}^2(Q)$. Now (A.5), (A.10) and (1.15) imply (1.1). Formula (1.13) follows from (A.6) and (A.8), and (1.14) is an implication of (1.10). If Q is regular, then the conditions (1.1) and (1.13) determine v uniquely by Lemma A.1. □

LEMMA 1.1 (The mean value property). *Suppose that τ_r are the first exit times from a bounded domain Q and $T \subset \partial_r Q$ is a total subset of ∂Q. If v satisfies the equation*

$$\dot{v}(r,x) + Lv(r,x) - v(r,x)^\alpha = 0 \quad in\ Q \tag{1.16}$$

and if it is bounded continuous in $Q \cup T$, then

$$P_{\delta_{r,x}} \exp\langle -v, X_\tau \rangle = e^{-v(r,x)} \quad in\ Q. \tag{1.17}$$

PROOF. By Theorem 1.1, the function

$$\tilde{v}(r,x) = -\log P_{\delta_{r,x}} \exp\langle -v, X_\tau \rangle$$

is a solution of (1.16) and $\tilde{v} = v$ on T. It follows from (1.11) that $\tilde{v}$ is bounded. By Lemma A.1, $\tilde{v} = v$ in Q. □

THEOREM 1.2. *Let v_n be positive solutions of* (1.16) *and let $v_n \to v$ pointwise in Q. Then v is a solution of* (1.16).

Suppose B is a relatively open subset of ∂Q, all points of B are regular and f is a bounded continuous function on B. If v_n satisfy the boundary condition

$$v_n = f \quad on\ B, \tag{1.18}$$

then the same condition holds for v.

PROOF. Let $(r^0, x^0) \in Q$. If ε is sufficiently small, then all functions v_n are uniformly bounded in $\overline{U}_\varepsilon(r^0, x^0)$ by Theorem A.2, and they have the mean value property (1.17) by Lemma 1.1. By the dominated convergence theorem, (1.17) holds for v, and, by Theorem 1.1, v satisfies (1.16) in $U_\varepsilon(r^0, x^0)$.

To prove the second part of the theorem, denote by c the supremum of f on B and consider ε defined in Theorem A.3. Set $Q_\varepsilon = U_\varepsilon(r^0, x^0) \cap Q$, $\Gamma = \partial_r Q_\varepsilon \cap \partial_r Q$. Note that v_n are uniformly bounded on $\overline{Q}_\varepsilon \cap Q$ and that $v_n \to \tilde{f}$ pointwise on $T = \Gamma \cup (Q \cap \partial_r U_\varepsilon)$, where $\tilde{f} = f$ on Γ, $\tilde{f} = v$ on $T \setminus \Gamma$. Clearly, $T \subset \partial_r Q_\varepsilon$ and it is a total subset of ∂Q_ε and v_n are continuous on $Q_\varepsilon \cap T$. By Lemma 1.1, the mean value property (1.17) holds in Q_ε and T for all functions v_n. By the dominated convergence theorem, it holds for v, and, by Theorem 1.1, $v(r, x) \to \tilde{f}(r^0, x^0) = f(r^0, x^0)$ as $(r, x) \to (r^0, x^0)$. □

THEOREM 1.3. *Suppose that Q is a bounded regular domain, τ_r are the first exit times from Q and f is a continuous mapping from $\partial_r Q$ to $[0, \infty]$. Then the formula*

$$v(r, x) = -\log P_{r, \delta_x} \exp\langle -f, X_\tau \rangle, \qquad (r, x) \in Q, \tag{1.19}$$

determines the minimal positive solution of the equation (1.16) *subject to the boundary condition*

$$v = f \quad on\ \partial_r Q. \tag{1.20}$$

For every $\mu \in \mathfrak{M}$,

$$P_\mu \exp\langle -f, X_\tau \rangle = e^{-\langle v, \mu \rangle}. \tag{1.21}$$

PROOF. Let $f_k = f \wedge k$. By Theorem 1.1, the function

$$v_k(r, x) = -\log P_{\delta_{r,x}} \exp\langle -f_k, X_\tau \rangle \tag{1.22}$$

satisfies the equation (1.16) and the boundary condition

$$v_k = f_k \quad \text{on } \partial_r Q. \tag{1.23}$$

Clearly, $v_1 \le v_2 \le \cdots$ and $v_k \uparrow v$, where v is defined by (1.19). It follows from Theorem 1.2 that v is a solution of (1.16) subject to condition (1.20) on the part of $\partial_r Q$, where $f(t, a) < \infty$. For all k, $\liminf v(r, x) \ge \liminf v_k(r, x) = f_k(t, a)$ as $(r, x) \to (t, a) \in \partial_r Q$, and therefore (1.20) holds if $f(t, a) = \infty$.

For an arbitrary positive solution u of (1.16) subject to condition (1.20), $v_k \le u$ by Lemma A.1. Hence $v \le u$. Formula (1.22) follows from (1.14). □

THEOREM 1.4. *Suppose that Q and τ are the same as in Theorem* 1.3. *Let Γ be a relatively closed subset of ∂Q and let $A = \partial Q \setminus \Gamma \subset \partial_r Q$, $B = \Gamma \cap \bar{A}^c$. Then*

$$v(r,x) = -\log P_{\delta_{r,x}}\{X_\tau(\Gamma) = 0\} \tag{1.24}$$

is a solution of the equation (1.16) *subject to the conditions*

$$v = 0 \quad on\, A, \tag{1.25}$$

$$v = \infty \quad on\, B. \tag{1.26}$$

For every $\mu \in \mathfrak{M}$,

$$P_\mu\{X_\tau(\Gamma) = 0\} = e^{-\langle v, \mu\rangle}. \tag{1.27}$$

REMARK. If $\Gamma = \partial_r Q$, then $A = \phi$ and $B = \partial_r Q$. Hence

$$v(r,x) = -\log P_{\delta_{r,x}}\{X_\tau = 0\} \tag{1.28}$$

is a solution of (1.16) subject to the condition

$$v = \infty \quad \text{on } \partial_r Q. \tag{1.29}$$

PROOF. Consider a monotone decreasing sequence of positive continuous functions f_k: $\partial Q \to [0,\infty]$ such that $\{f_k = 0\} \uparrow A$ and $f_k = \infty$ on Γ. Let v_k correspond to f_k by formula (1.19). Clearly, $v_k \downarrow v$, where v is defined by (1.24). By Theorem 1.2, v is a solution of (1.16) subject to the boundary condition (1.25). For every $(r^0, x^0) \in B$, there exists a continuous function $\tilde{f}$ from ∂Q to $[0,\infty]$ such that $\tilde{f}(r^0,x^0) = \infty$ and $\tilde{f} = 0$ on A. If $\tilde{v}$ corresponds to $\tilde{f}$ by (1.19), then $\tilde{v}(r^0,x^0) = \infty$ by Theorem 1.3. Since $\tilde{v} \le v$ in Q, v satisfies (1.26). □

2. The maximal positive solutions and the graphs.

2.1. Let $\omega \to F(\omega)$ be a mapping from Ω to the space of all closed sets in S. For every $A \subset S$, we put $\Omega^A = \{\omega\colon F(\omega) \cap A \ne \phi\}$. We say that F is a *random closed set* relative to a σ-algebra $\mathscr{F}$ if $\Omega^K \in \mathscr{F}$ for all compact sets K. This is equivalent to the condition $\Omega^U \in \mathscr{F}$ for all open sets U. It is known (see, e.g., [21], Chapter 2) that for every analytic set B in S, Ω^B belongs to the universal completion $\mathscr{F}^*$ of $\mathscr{F}$ and, for every probability measure P on $\mathscr{F}$,

$$P(\Omega^B) = \sup P(\Omega^K) = \inf P(\Omega^U), \tag{2.1}$$

where K runs over all compact subsets of B and U runs over all open sets which contain B.

2.2. The (closed) graph $\mathscr{G}$ of X is the minimal closed subset of S such that, for every $t \in \mathbb{R}$, the measure X_t is concentrated on the t-section $\mathscr{G}_t$ of $\mathscr{G}$. More generally, if $\tilde{X}$ is the part of X on an open set Q, then we denote by

$\mathscr{G}_Q$ the minimal closed subset of S such that, for every $t \in \mathbb{R}$, $\tilde{X}_t$ is concentrated on the t-section of $\mathscr{G}_Q$. (The part $\tilde{X}$ on Q is defined up to indistinguishability and therefore $\mathscr{G}_Q$ is defined up to equivalence).

Note that:

2.2.A. $P_\mu\{\mathscr{G}_Q = \phi\} = 1$ if $\mu(Q) = 0$.

2.2.B. $P_\mu\{\mathscr{G}_Q \subset \bar{Q}\} = 1$ for every $\mu \in \mathfrak{M}$.

2.2.C. For every open set U and every $\mu \in \mathfrak{M}$,

$$\{\mathscr{G}_Q \cap U \neq \phi\} = \{\tilde{X}_t(U_t) > 0 \text{ for some } t \in \mathbb{R}\}$$
$$= \{\tilde{X}_t(U_t) > 0 \text{ for some rational } t\} \; P_\mu\text{-a.s.}$$

2.2.D. $\mathscr{G}_Q$ coincides, P_μ-a.s., with the support of the measure Y_τ.

Properties 2.2.A and 2.2.B are obvious and 2.2.C and 2.2.D follow from 1.1.A (or 1.1.A′).

Let $\mathscr{F}_X$ be the σ-algebra generated by $\{X_t, t \in \mathbb{R}\}$ and let $\hat{\mathscr{F}}_X$ be its completion with respect to the family $\{P_\mu, \mu \in \mathfrak{M}\}$. The graph $\mathscr{G}_Q$ is a random closed set relative to $\hat{\mathscr{F}}_X$.

THEOREM 2.1. *Suppose that $Q \subset \tilde{Q}$ are two open sets, Γ is a closed set such that $\tilde{Q} \cap \partial Q \subset \Gamma \subset Q^c$ and all irregular points of ∂Q belong to Γ. Then*

$$v(r, x) = -\log P_{\delta_{r,x}}\{\mathscr{G}_{\tilde{Q}} \cap \Gamma = \phi\} \tag{2.2}$$

is the maximal positive solution of the problem

$$\dot{v} + Lv - v^\alpha = 0 \quad \text{in } Q, \qquad v = 0 \quad \text{on } \Gamma^c \cap \partial Q. \tag{2.3}$$

For every $\mu \in \mathfrak{M}^0$,

$$\mathscr{G} \text{ is compact } P_\mu\text{-a.s.} \tag{2.4}$$

and, for every $\mu \in \mathfrak{M}^0_Q$,

$$P_\mu\{\mathscr{G}_{\tilde{Q}} \cap \Gamma = \phi\} = \exp\langle -v, \mu\rangle. \tag{2.5}$$

2.3. The proof of Theorem 2.1 is based on two lemmas:

LEMMA 2.1. *Let τ_r be the first exit times from an open set Q. Then, for every closed set $B \subset Q$ and every $\mu \in \mathfrak{M}_q$,*

$$\{\mathscr{G}_Q \subset B\} \subset \{X_\tau(B^c) = 0\} \qquad P_\mu\text{-a.s.} \tag{2.6}$$

PROOF. By Theorem 1 in the Addendum to [**12**] applied to the part $\tilde{X}$ of X on Q,

$$\{\langle f, \tilde{X}_t\rangle = 0 \text{ for all } t\} \subset \{\langle f, X_\sigma\rangle = 0\} \qquad P_\mu\text{-a.s.}$$

for every positive bounded continuous function f and for the first exit time σ from an arbitrary open set U such that $\bar{U} \subset Q$. Let $U^n \uparrow Q$ and $\bar{U}^n \subset Q$. The corresponding first exit times $\sigma^n \uparrow \tau$, which implies: $\langle f, X_{\sigma^n}\rangle \to \langle f, X_\tau \rangle$ P_μ-a.s. (see [**15**], page 351). We conclude that

$$\{\langle f, \tilde{X}_t\rangle = 0 \text{ for all } t\} \subset \{\langle f, X_\tau\rangle = 0\} \qquad P_\mu\text{-a.s.} \tag{2.6a}$$

If $B = \{f = 0\}$, then $\{\mathscr{G}_Q \subset B\} \subset \{\langle f, \tilde{X}_t\rangle = 0$ for all $t\}$ P_μ-a.s. for all $\mu \in \mathscr{M}_Q$. Therefore (2.6a) implies (2.6).

LEMMA 2.2. *Let $U \subset Q$ be open sets and let σ_r be the first exit times from U. Then*

$$\{X_\sigma(Q) = 0\} \subset \{\mathscr{G}_Q \subset \bar{U}\} \qquad P_\mu\text{-a.s.} \tag{2.7}$$

for every $\mu \in \mathfrak{M}$.

PROOF. If $\mu(\bar{U}) = 0$, then P_μ-a.s., $X_\sigma(Q) = \mu(Q)$ and (2.7) follows from 2.2.A. Therefore it is sufficient to prove that, for $\mu \in \mathfrak{M}_{\bar{U}}$ and every t,

$$P_\mu\{X_\sigma(Q) = 0,\ X_{\tau\wedge t}(Q \setminus \bar{U}) > 0\} = 0.$$

The special Markov property implies

$$\{X_\sigma(Q) = 0\} \subset \{X_{\sigma\wedge t}(Q_{<t}) = 0\} \qquad P_\mu\text{-a.s.}$$

Hence

$$\begin{aligned} &P_\mu\{X_\sigma(Q) = 0,\ X_{\tau\wedge t}(Q \setminus \bar{U}) > 0\} \\ &\quad\le P_\mu\{X_{\sigma\wedge t}(Q_{<t}) = 0,\ X_{\tau\wedge t}(Q \setminus \bar{U}) > 0\} \\ &\quad= P_\mu\left\{X_{\sigma\wedge t}(Q_{<t}) = 0,\ P_{X_{\sigma\wedge t}}\left[X_{\tau\wedge t}(Q \setminus \bar{U}) > 0\right]\right\} = 0 \end{aligned}$$

because $X_{\tau\wedge t} = \nu$ P_ν-a.s. if $\nu(Q_{<t}) = 0$ and $\nu(Q \setminus \bar{U}) = 0$ if ν is concentrated on $\bar{U}$. □

PROOF OF THEOREM 2.1. Choose a sequence of bounded regular open sets $Q_n \uparrow Q$ such that

$$\partial Q \cap \partial Q_n \uparrow \partial Q \cap \Gamma^c,$$
$$\overline{Q_{n-1}} \cap Q \subset Q_n.$$

Put

$$B_n = \partial Q_n \cap Q, \qquad \Gamma_n = \bar{B}_n, \qquad A_n = \partial Q_n \setminus \Gamma_n;$$
$$Z_n = X_{\tau_n}(\Gamma_n).$$

By Theorem 1.4,

$$P_\mu\{Z_n = 0\} = \exp\langle -v_n, \mu\rangle, \tag{2.8}$$

where v_n is a solution of (1.16) in Q_n with the boundary conditions $v^n = 0$ on

A_n, $v^n = \infty$ on B_n. If $\mu \in \mathfrak{M}_{Q_n}$, then by Lemma 2.1 ($B = \overline{Q}_{n-1}$),

$$(2.9)\qquad \begin{aligned}\left\{\mathscr{G}_Q \subset \overline{Q}_{n-1}\right\} &\subset \left\{\mathscr{G}_{Q_n} \subset \overline{Q}_{n-1}\right\}\\ &\subset \left\{X_{\tau^n}\left(\overline{Q}_{n-1}^c\right) = 0\right\} \subset \{Z_n = 0\} \qquad P_\mu\text{-a.s.}\end{aligned}$$

By Lemma 2.2 (with $U = Q_n$),

$$(2.10)\quad \{Z_n = 0\} = \{X_{\tau^n} \text{ is concentrated on } A_n\} \subset \left\{\mathscr{G}_{\tilde{Q}} \subset \overline{Q}_n\right\} \qquad P_\mu\text{-a.s.}$$

If $\mu \in \mathfrak{M}_Q^0$, then, by (2.8), (2.9) and (2.10),

$$(2.11)\qquad \begin{aligned}\lim P_\mu\{Z_n = 0\} &= \lim P_\mu\left\{G_{\tilde{Q}} \subset \overline{Q}_n\right\}\\ &= P_\mu\left\{\mathscr{G}_{\tilde{Q}} \text{ is compact and } \mathscr{G}_{\tilde{Q}} \cap \Gamma = \phi\right\}.\end{aligned}$$

It follows from (2.8) and (2.11) that

$$(2.12)\qquad \lim\langle v_n, \mu\rangle = -\log P_\mu\{\mathscr{G}_{\tilde{Q}} \text{ is compact and } \mathscr{G}_{\tilde{Q}} \cap \Gamma = \phi\}$$

and

$$v(r, x) = \lim v_n(r, x) = -\log P_{r,\delta_x}\left\{\mathscr{G}_{\tilde{Q}} \text{ is compact and } \mathscr{G}_{\tilde{Q}} \cap \Gamma = \phi\right\}.$$

Since v_n are uniformly bounded on the support of μ, $\langle v_n, \mu\rangle \to \langle v, \mu\rangle$ and, by (2.12),

$$(2.13)\qquad P_\mu\left\{\mathscr{G}_{\tilde{Q}} \text{ is compact and } \mathscr{G}_{\tilde{Q}} \cap \Gamma = \phi\right\} = e^{-\langle v, \mu\rangle}.$$

It follows from Theorem 1.2 that v satisfies (2.3). By applying (2.13) to $Q = S$, $\Gamma = \phi$, we get that, for every $\mu \in \mathfrak{M}^0$,

$$P_\mu\{\mathscr{G} \text{ is compact}\} = e^{-\langle v, \mu\rangle},$$

where v is a positive solution of (1.16) in S. By Theorem A.1, $v = 0$ and we get (2.4).

If $\tilde{v}$ is an arbitrary solution of (2.3), then, by Lemma A.1, $\tilde{v} \le v^n$ in Q_n. Thus $\tilde{v} \le v$ and v is the maximal solution of (2.3). Clearly, (2.13) and (2.4) imply (2.5). □

2.4. By applying Theorem 2.1 to $\Gamma = Q^c$, we get: For every open sets $Q \subset \tilde{Q}$,

$$v(r, x) = -\log P_{r,\delta_x}\{\mathscr{G}_{\tilde{Q}} \subset Q\}$$

is the maximal positive solution of the equation (1.16) in Q. Formula (2.5) with $\tilde{Q} = S$, $\Gamma = Q^c$ implies that

$$(2.14)\qquad \log P_\mu\{\mathscr{G} \cap \Gamma = \phi\} = \int \mu(dr, dx) \log P_{r,\delta_x}\{\mathscr{G} \cap \Gamma = \phi\}$$

if the support of μ is compact and disjoint from Γ.

Note that, for every open Q and for all $(r, x) \in Q$,

$$(2.15)\qquad P_{r,\delta_x}\{\mathscr{G}_Q \subset Q\} = P_{r,\delta_x}\{\mathscr{G} \subset Q\}.$$

Indeed, both parts are equal to $e^{-v(r,x)}$ where v is the maximal positive solution of (1.16) in Q.

Arguments presented in the proof of Theorem 2.1 show that

$$\{X_{\tau_n} = 0\} \uparrow \{\mathscr{S}_{\tilde{Q}} \subset Q\} \qquad P_\mu\text{-a.s.} \tag{2.16}$$

if τ_n is the first exit time from $Q_n = \{z\colon \operatorname{dist}(z, Q^c) > 1/n\}$ and if $\operatorname{dist}(Q^c, \operatorname{supp}\mu) > 0$.

2.5. We extend formula (2.14) to a more general situation.

LEMMA 2.3. *Let B be an analytic set and let $C = \operatorname{supp}\mu$ be disjoint from B. Then*

$$\log P_\mu\{\mathscr{S} \cap B = \phi\} = \int \mu(dr, dx) \log P_{r,\delta_x}\{\mathscr{S} \cap B = \phi\}. \tag{2.17}$$

PROOF. (i) First we assume that B is compact. Then $\operatorname{dist}(B, C) > 0$ and we can apply (2.16) to $\tilde{Q} = S$ and $Q = B^c$. Therefore

$$\{X_{\tau_n} = 0\} \uparrow W \qquad P_\mu\text{-a.s.},$$

where $W = \{\mathscr{S} \cap B = \phi\}$. Suppose that $C \subset Q_{n_0}$ and put $\sigma = \tau_{n_0}$. By Theorem 2 in the Addendum to [12], $P_\mu(W) = P_\mu P_{X_\sigma}(W)$. Since X_σ is supported by a compact set $\{z\colon \operatorname{dist}(z, B) = 1/n_0\}$ which is disjoint from B, we have by (2.14), $P_{X_\sigma}(W) = \exp\langle -u, X_\sigma\rangle$, where $u(r, x) = -\log P_{r,\delta_x}(W)$. Now (2.17) follows from (1.10).

(ii) Let B be an arbitrary analytic set. Put

$$\tilde{P}_\mu = \int \mu(dr, dx) P_{r,\delta_x}.$$

There exists a monotone increasing sequence of compact subsets B_n of B such that

$$P_\mu\{\mathscr{S} \cap B_n = \phi\} \downarrow P_\mu\{\mathscr{S} \cap B = \phi\} \tag{2.18}$$

and an analogous formula holds for $\tilde{P}_\mu$. By (i),

$$P_\mu\{\mathscr{S} \cap B_n = \phi\} = \exp\langle -v_n, \mu\rangle, \tag{2.19}$$

where

$$v_n(r, x) = -\log P_{r,\delta_x}\{\mathscr{S} \cap B_n = \phi\}.$$

Denote by $\tilde{B}$ the union of B_n. We have

$$\begin{aligned} v_n(r, x) \uparrow \tilde{v}(r, x) &= -\log P_{r,\delta_x}\{\mathscr{S} \cap \tilde{B} = \phi\} \\ &\le -\log P_{r,\delta_x}\{\mathscr{S} \cap B = \phi\} = v(r, x). \end{aligned}$$

On the other hand,

$$\tilde{P}_\mu\{\mathscr{S} \cap B_n = \phi\} = \int \mu(dr, dx) e^{-v_n(r,x)} \downarrow \int \mu(dr, dx) e^{-v(r,x)}$$

and therefore $\exp(-\tilde{v}(r,x))$ and $\exp(-v(r,x))$ have the same integrals with respect to μ. We conclude that $\tilde{v} = v$ μ-a.e. Therefore $\langle v_n, \mu\rangle \uparrow \langle \tilde{v}, \mu\rangle = \langle v, \mu\rangle$. Formula (2.17) follows from (2.18) and (2.19). □

3. $\mathscr{G}$-polar sets.

3.1. Put $B_{\geq t} = B \cap S_{\geq t}$. An analytic set B is called $\mathscr{G}$-polar if

$$P_{r,\delta_x}\{\mathscr{G} \cap B_{\geq t} = \phi\} = 1 \tag{3.1}$$

for all $r < t$ and all $x \in E$. By Lemma 2.3, this condition is equivalent to the condition:

$$P_\mu\{\mathscr{G} \cap B_{\geq t} = \phi\} = 1 \tag{3.2}$$

for every t and μ such that $\operatorname{supp}\mu$ is disjoint from $B_{\geq t}$. It follows from (2.1) that B is $\mathscr{G}$-polar if and only if all compact subsets of B are $\mathscr{G}$-polar.

THEOREM 3.1. *An analytic set B is $\mathscr{G}$-polar if it contains no set $(-\infty, t) \times E$ and if*

$$P_{\delta_{r,x}}\{\mathscr{G} \cap B \neq \phi\} = 0 \quad \textit{for all } (r,x) \notin B. \tag{3.3}$$

A compact set Γ is $\mathscr{G}$-polar if and only if the conditions

$$\dot{v} + Lv - v^\alpha = 0, \quad v \geq 0 \textit{ in } \Gamma^c \tag{3.4}$$

imply that $v = 0$.

REMARK. The set $S_{\leq 0}$ satisfies (3.3). But it is not G-polar.

PROOF. (i) If U is a neighborhood of x and if $T_U = \inf\{t\colon X_t(U^c) > 0\}$, then $q = P_{r,\delta_x}\{T_U > r\} = 1$. Indeed, by Blumenthal's 0–1 law, $q = 0$ or 1 and, by Theorem 2.1, for every $a < r$, $q \geq P_{r,\delta_x}\{\mathscr{G} \subset (a,\infty) \times U\} = e^{-v(r,x)}$, where v is the maximal solution of (1.16) in $Q = (0,\infty) \times U$.

This result implies: $\mathscr{G}$ consists, P_{r,δ_x}-a.s., of the point (r,x) and the union of $\mathscr{G}_{\geq s}$ over all $s > r$. Therefore (3.3) holds for all $\mathscr{G}$-polar sets.

(ii) Suppose that an analytic set B satisfies (3.3). By our assumption, for every r, there exists $r^0 < r$ and x^0 such that $(r^0, x^0) \notin B$. If $s > r$, then

$$0 = P_{r^0,\delta_{x^0}}\{\mathscr{G} \cap B^s \neq \phi\} \geq P_{r^0,x^0}\{X_s(B_s) > 0\}.$$

Hence

$$0 = P_{r^0,\delta_{x^0}} X_s(B_s) = \Pi_{r^0,\delta_{x^0}}\{\xi_s \in B_s\} = \int_{B_s} p(r^0, x^0; s, y)\, dy,$$

where p is the transition density of the diffusion ξ. Since $p > 0$, the Lebesgue measure of B_s is equal to 0. Therefore

$$P_{r,\delta_x} X_s(B_s) = \int_{B_s} p(r,x;s,y)\, dy = 0 \tag{3.5}$$

for all x. If $t > s > r$, then, by (3.5) and by the Markov property,

$$P_{r,\delta_x}\{\mathscr{G} \cap B_{\geq t} \neq \phi\} = P_{r,\delta_x}\{X_s(B_s) = 0, P_{X_s}(\mathscr{G} \cap B_{\geq t} \neq \phi)\}. \tag{3.6}$$

By Lemma 2.3, for every $\nu \in M$,

$$P_{s,\nu}\{\mathscr{G} \cap B_{\geq t} = \phi\} = e^{-\langle v, \nu \rangle}, \tag{3.7}$$

where $v(s,x) = -\log P_{s,\delta_x}\{\mathscr{G} \cap B_{\geq t} = \phi\}$. By (3.3), $P_{s,\delta_x}\{\mathscr{G} \cap B_{\geq t} = \phi\} = 1$ for all $x \notin B_s$. By (3.7), $P_{s,\nu}\{\mathscr{G} \cap B_{\geq t} = \phi\} = 1$ if $\nu(B_s) = 0$ and, by (3.6), $P_{r,\delta_x}\{\mathscr{G} \cap B_{\geq t} \neq \phi\} = 0$.

If Γ is closed and if $Q = \Gamma^c$, then

$$P_{\delta_{r,x}}\{\mathscr{G} \cap \Gamma = \phi\} = P_{\delta_{r,x}}\{\mathscr{G} \subset Q\} = \exp\{-v(r,x)\},$$

where v is the maximal solution of (1.16). This implies the second part of Theorem 3.1.

3.2. Let

$$k(r,x;t,y) = k_{t-r}(y-x), \tag{3.8}$$

where

$$\begin{aligned} k_r(x) &= (2\pi r)^{-d/2} \exp\{-r/2 - |x|^2/2r\} \quad \text{for } r > 0, \\ k_r(x) &= 0 \quad \text{for } r \leq 0 \end{aligned} \tag{3.9}$$

(k is the transition density of the Brownian motion in $\mathbb{R}^d$ with killing rate $\frac{1}{2}$). We put $B \in \mathscr{N}_\alpha$ if B is an analytic set and if, for every measure η on $\mathscr{B}_S$, the function

$$f(r,x) = \int_B k(r,x;t,y)\eta(dt,dy) \tag{3.10}$$

belongs to $L^\alpha(S)$ only if $\eta(B) = 0$. The class $\mathscr{N}_\alpha$ can also be described as the class of all analytic sets of capacity 0 with respect to one of the capacities studied by Meyers in [22]. [In the notation of [22] this is $c_{k,m,\alpha'}$, where k is the kernel (3.8), m is the Lebesgue measure and α' is the conjugate for α, i.e., $(1/\alpha) + (1/\alpha') = 1$.]

THEOREM 3.2. *The class of $\mathscr{G}$-polar sets coincides with $\mathscr{N}_\alpha$.*

PROOF. (i) It is easy to see that $B \in \mathscr{N}_\alpha$ if and only if all compact subsets of B belong to $\mathscr{N}_\alpha$. Therefore it is sufficient to prove that a compact set K is $\mathscr{G}$-polar if and only if it belongs to $\mathscr{N}_\alpha$.

(ii) Let $B = K$ be $\mathscr{G}$-polar and let f given by (3.10) belong to $L^\alpha(S)$. We need to show that $\eta(K) = 0$. We can assume that η is concentrated on K. Clearly, η does not charge any set $B \in \mathscr{N}_\alpha$ and, by Theorem 1.1 in [14], there exists an additive functional I^η of X such that, for every $\mu \in \mathfrak{M}_\alpha$,

$$P_\mu I^\eta = \int_{S \times S} \mu(dr,dx) p(r,x;t,y) \eta(dt,dy). \tag{3.11}$$

Here $I^\eta = I^\eta(\mathbb{R})$, $p(r, x; t, y)$ is the transition density of ξ and $\mathfrak{M}_\alpha$ is the class of all measures $\mu \in \mathfrak{M}$ of the form $\mu(dr, dx) = q(r, x)\, dr\, dx$ with $q \in L^{\alpha'}(S)$. Moreover, by 1.3.F in [14],

$$(3.12) \qquad \{\mathscr{S} \cap K = \phi\} \subset \{I^\eta = 0\} \qquad P_\mu\text{-a.s.}$$

Since K is $\mathscr{S}$-polar, $P_\mu\{\mathscr{S} \cap K_{\geq t} = \phi\} = 1$ for every t and μ such that $\operatorname{supp} \mu \cap B_{\geq t} = \phi$. We can choose $\mu \neq 0$ of class $\mathfrak{M}_\alpha$ to satisfy this condition, and we conclude from (3.11) and (3.12) that $\eta(S_{<t}) = 0$. Since this is true for every t, $\eta = 0$.

(iii) Now let $K \in S_\alpha$ and let Q be a simple rectangle which contains K. By Theorem 4.1 in [2], every positive solution of the equation

$$(3.13) \qquad \dot{v} + Lv - v^\alpha = 0 \quad \text{in } Q \setminus K$$

belongs to $L^\alpha_{\text{loc}}(Q)$ and satisfies the equation

$$(3.14) \qquad \dot{v} + Lv - v^\alpha = 0 \quad \text{in } C_0^\infty(Q).$$

Moreover, by Lemma 3.4 in [2], if

$$(3.15) \qquad \dot{v} + Lv \geq 0 \quad \text{in } C_0^\infty(Q),$$

$$(3.16) \qquad \limsup v \leq 0 \quad \text{on } \partial_r Q,$$

then $v \leq 0$ a.e. on Q. By Theorem 2.1, $v(r, x) = -\log P_{r,\delta_x}\{\mathscr{S}_Q \cap K = \phi\}$ is the maximal solution of (3.13) with the boundary condition

$$(3.17) \qquad v = 0 \quad \text{on } \partial_r Q.$$

(We apply Theorem 2.1 to open sets $Q \setminus K \subset Q$ and to $\Gamma = K \cup (\partial Q \setminus \partial_r Q)$.) Since (3.13) implies (3.14), it implies also (3.15). Clearly, (3.17) implies (3.16) and, since $v \geq 0$, we conclude that $v = 0$ in Q. By (2.15),

$$P_{r,\delta_x}\{\mathscr{S} \subset Q \setminus K\} = P_{r,\delta_x}\{\mathscr{S}_{Q\setminus K} \subset Q \setminus K\} \geq P_{r,\delta_x}\{\mathscr{S}_Q \subset Q \setminus K\} = 1$$

for all $(r, x) \in Q \setminus K$. This is true for an arbitrary Q, and K is $\mathscr{S}$-polar by Theorem 3.1. □

COROLLARY. *A singleton $\{c\}$ is a $\mathscr{S}$-polar set if and only if $d \geq 2/(\alpha - 1)$.*

Indeed, let $c = (0, 0)$. Then f given by (3.7) is proportional to $k_{-r}(x)$ and

$$\int_S k_{-r}(x)^\alpha\, dr\, dx = \text{const.} \int_S 1_{s>0} e^{-\alpha s/2} s^{-\alpha d/2} \exp\left(-\alpha |x|^2/2s\right) ds\, dx$$

$$= \text{const.} \int_0^\infty e^{-\alpha s/2} s^{-(\alpha-1)d/2}\, ds.$$

The right side is finite if and only if $d < 2/(\alpha - 1)$.

3.3. Theorem 3.2 allows us to investigate $\mathscr{S}$-polarity of product sets $\Delta \times A$, where Δ is an interval and A is an analytic set in E. We say that an analytic set $A \subset E$ is S-polar if $\mathbb{R} \times A$ is $\mathscr{S}$-polar and we say that A is H-polar if $\{t\} \times A$ is $\mathscr{S}$-polar for every $t \in \mathbb{R}$. Obviously, all S-polar sets are H-polar.

Criteria of S-polarity and H-polarity can be stated in terms of the Bessel kernels $g_\beta(x,y) = g_\beta(y-x)$, where

$$g_\beta(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i\lambda x}(1+|\lambda|^2)^{-\beta/2}\,d\lambda$$
$$= N_{d,\beta}|x|^{(\beta-d)/2}K_{(\beta-d)/2}(|x|).$$

Here $N_{d,\beta}$ is a constant and K_ν stands for the modified Bessel function of the third kind of order ν. Note that

$$g_2(x) = \int_0^\infty k_r(x)\,dr.$$

To every analytic set A and to every measure η, there corresponds a function

$$f_A^{\beta,\eta}(x) = \int_A g_\beta(x,y)\eta(dy).$$

We put $A \in \mathscr{E}_\alpha^\beta$ if $f_A^{\beta,\eta} \notin L^\alpha(E)$ unless $\eta(A) = 0$. (Note that $\mathscr{E}_\alpha^2$ coincides with the class Q_α in [13].) In general, $A \in \mathscr{E}_\alpha^\beta$ if and only if $C_{\beta,\alpha'}(A) = 0$, where $C_{\beta,p}$ is the Bessel capacity corresponding to (3.17) (see [22], Section 7).

The following result is an immediate implication of Theorem 3.2 and Propositions 2.1 and 2.3 in [2]:

THEOREM 3.3. *The class of S-polar sets coincides with $\mathscr{E}_\alpha^2$ and the class of H-polar sets coincides with $\mathscr{E}_\alpha^{2/\alpha}$. A set A is S-polar if $\Delta \times A$ is $\mathscr{G}$-polar for some interval Δ with the length $|\Delta| > 0$. A set A is H-polar if $\{t\} \times A$ is $\mathscr{G}$-polar for some $t \in \mathbb{R}$.*

THEOREM 3.4. *If $\alpha = 2$, then an analytic set A is H-polar if and only if*

$$\left\{\int_A g_2(x,y)\nu(dy) \text{ is bounded}\right\} \Rightarrow \{\nu(A) = 0\} \tag{3.18}$$

($\Rightarrow$ *means the logical implication*).

PROOF. Since the convolution of g_β and g_γ is equal to $g_{\beta+\gamma}$, the condition

$$\int_A g_1(x,y)\nu(dy) \in L^2(E)$$

is equivalent to the condition

$$\int_{A\times A}\nu(dx)g_2(x,y)\nu(dy) < \infty. \tag{3.19}$$

It is well known (see, e.g., [8], 1.XIII.2) that (3.19) is equivalent to (3.18). □

REMARK. The condition (3.18) means that A is a set of capacity 0 in the sense of classical potential theory.

3.4. Criteria for S-polarity in terms of the Hausdorff measures have been given in [13]. By similar arguments we get the following criteria for H-polarity.

THEOREM 3.5. *Let H_γ and $H_{\gamma,\beta}$ be the Hausdorff measures corresponding to the functions*

$$h_\gamma(r) = r^\gamma, \quad \gamma > 0 \quad \text{and} \quad h_{\gamma,\beta}(r) = r^\gamma\left(\log^+ \frac{1}{r}\right)^{-\beta}, \quad \gamma \geq 0, \beta > 0.$$

Put $\lambda_\alpha = 2/(\alpha - 1)$, $\gamma = d - \lambda_\alpha$, $\beta_0 = 1/(d-1)$. We have:

(i) *If $\gamma < 0$, then ϕ is the only H-polar set.*

(ii) *If $\gamma > 0$, then*

$$\{H_\gamma(A) < \infty\} \Rightarrow \{A \text{ is } H\text{-polar}\} \Rightarrow \{H_{\gamma,\beta}(A) = 0 \text{ for all } \beta > \beta_0\}.$$

(iii) *If $\gamma = 0$, then*

$$\{H_{0,\beta_0}(A) < \infty\} \Rightarrow \{A \text{ is } H\text{-polar}\} \Rightarrow \{H_{0,\beta}(A) = 0 \text{ for all } \beta > \beta_0\}.$$

The Hausdorff dimension H-dim A is defined as the supremum of γ such that $H_\gamma(A) > 0$. The Carleson logarithmic dimension L-dim A is the supremum of β such that $H_{0,\beta}(A) > 0$. The following result is an obvious implication of Theorem 3.5.

THEOREM 3.6. *Let $c(A)$ be the Hausdorff codimension of A [i.e., $c(A) = d - H\text{-dim } A$]. If $\gamma > 0$, then*

$$\{c(A) > \lambda_\alpha\} \Rightarrow \{A \text{ is } H\text{-polar}\} \Rightarrow \{c(A) \geq \lambda_\alpha\}.$$

Let $l(A) = d - L\text{-dim } A$. If $\gamma = 0$, then

$$\{l(A) > \beta_0\} \Rightarrow \{A \text{ is } H\text{-polar}\} \Rightarrow \{l(A) \geq \beta_0\}.$$

REMARK. Criteria for S-polarity proved in [13] can be obtained from Theorems 3.5 and 3.6 by replacing λ_α by $\kappa_\alpha = \lambda_\alpha + 2 = 2\alpha/(\alpha - 1)$ (or by replacing d by $d - 2$).

4. Survey of the literature. Concluding remarks.

4.1. The monographs [17] and [19] are the standard reference books on parabolic linear and semilinear partial differential equations. In the literature, the equation (1.1) is usually written in the form

$$\dot{u}(t,x) - Lu(t,x) + u(t,x)^\alpha = \rho(t,x), \tag{4.1}$$

which is equivalent to (1.1) with $u(t,x) = v(r,x)$, $t = -r$. (The reversed time direction is more natural from probabilistic point of view.) Besides, the term u^α is often replaced by $|u|^{\alpha-1}u$ (which, of course, makes no difference if $u \geq 0$).

A singular solution of the equation

$$\dot{v} + \Delta v - v^\alpha = 0 \quad \text{in } (-\infty, t) \times D \tag{4.2}$$

subject to the boundary condition

$$v \to \delta_y \quad \text{as } r \uparrow t, \tag{4.3}$$

where δ_y is Dirac's delta-function at point y has appeared, first, in [3]. The authors proved that such a solution exists if and only if $\alpha < (d+2)/d$. (A probabilistic implication of this result is stated as a Corollary to Theorem 3.2.) It follows from [14] that the solution discovered by Brezis and Friedman can be represented by the formula

$$v(r,x) = -\log P_{\delta_{r,x}} \exp(-L_{t,y}), \tag{4.4}$$

where $L_{t,y}$ is the local time for X at point (t,y) (the additive functional I^η corresponding to the measure $\eta = \delta_{t,y}$). This result and also a theorem proved in [16] suggest that, for $\alpha < (d+2)/d$, $X_t(dy) = L_{t,y}\, dy$.

By replacing $L_{t,y}$ with $kL_{t,y}$ in (4.4) and by letting k tend to ∞ we arrive at the function

$$v(r,x) = -\log P_{\delta_{r,x}}\{L_{t,y} = 0\}. \tag{4.5}$$

This is the *very singular solution* of (4.2) constructed, first in [4] and investigated in detail in [18]. Apparently, it coincides with the maximal solution

$$u(r,x) = -\log P_{\delta_{r,x}}\{(t,y) \notin \mathscr{G}\} \tag{4.6}$$

in the domain $Q = S \setminus \{(t,y)\}$.

4.2. In [5], [6] and [23] the set valued process $S_t = \{\text{support of } X_t\}$ has been studied in the case $L = \Delta$, $\alpha = 2$. In particular, it was proved that S_t is right continuous with left limits (in the topology induced by the Hausdorff metric). Moreover, $S_{t-} = S_t$ a.s. for every fixed t. Clearly, the graph $\mathscr{G}$ coincides with the union of all $\{t\} \times S_t$ and $\{t\} \times S_{t-}$.

Perkins has shown that if B has a positive classical capacity, then $S_t \cap B \neq \phi$ with positive probability (see [23], Theorem 6.1). Le Gall [20] gave a new proof of this statement based on his trajectorial construction of X. He also proved the converse statement for $d = 2$. Clearly, these results follow from Theorem 3.4. Le Gall has developed his approach further in [20a] where he gave a purely probabilistic proof of a part of Theorems 3.2 and 3.3. Namely, he proved that, in the case $\alpha = 2$, all $\mathscr{G}$-polar sets belong to $\mathscr{S}_\alpha$ and all S-polar sets belong to $\mathscr{E}_\alpha^2$.

4.3. If the coefficients of L are time independent, then the superdiffusion X with parameters (L, α) is a Markov process with a stationary transition function. In [13] we established the connections between X and positive solutions of the elliptic equation $Lv - v^\alpha = -\rho$. In particular, we used the results in [1] to investigate the range $\mathscr{R}$ of X. Since $\mathscr{R}$ coincides P_μ-a.s. for $\mu \in \mathfrak{M}^0$ with the projection of the graph $\mathscr{G}$ on $E = \mathbb{R}^d$, most of the results in

[12] can be deduced (and improved) by using the results in the present paper. Theorem 1.3 implies a result stated without proof at the end of Section 5 in [12].

APPENDIX

A.1. Let $E = \mathbb{R}^d$ and let Q be an open set in $S = \mathbb{R} \times E$. Denote by $\mathbb{C}(Q)$ the set of all continuous functions in Q. Put $u \in \mathbb{C}^1(Q)$ if u and $\partial u/\partial x_i$, $i = 1, \dots, d$, belong to $\mathbb{C}(Q)$; and put $u \in \mathbb{C}^2(Q)$ if u, $\dot{u}$, $\partial u/\partial x_i$, $i = 1, \dots, d$, and $\partial^2 u/\partial x_i\, \partial x_j$, $i, j = 1, \dots, d$, belong to $\mathbb{C}(Q)$.

Let $\xi = (\xi_t, \Pi_{r,x})$ be a diffusion process in E with the generator L given by (1.2). We say that $T \subset \partial Q$ is a total subset of ∂Q if

$$\Pi_{r,x}\{(\tau_r, \xi_{\tau_r}) \in T\} = 1 \quad \text{for } (r,x) \in Q. \tag{A.1}$$

If $Q = (r,t) \times D$ where D is an open set in E, then T, which consists of all $(s,x) \in \partial Q$ such that $s > r$, is total in ∂Q.

A point (t,a) of ∂Q is called *regular* if $\Pi_{t,a}\{\tau_{t+} = t\} = 1$. We say that Q is *regular* if the set $\partial_r Q$ of all regular points is total in ∂Q.

A cell is the product set $[r,t] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$. Every finite union of cells is called a simple compact set and the totality of all its interior points is called a simple open set. A simple rectangle $(r,t) \times (a_1, b_1) \times \cdots \times (a_d, b_d)$ is an example of a simple open set.

We have:

A.1.A. Every simple open set is regular. The intersection of two regular open sets is regular.

A.1.B. Suppose that Q is a bounded open set and that $u \in \mathbb{C}^2(Q)$ is bounded from above and satisfies the conditions

$$\dot{u} + Lu \geq 0 \quad \text{in } Q, \tag{A.2}$$

$$\limsup u(r,x) \leq 0 \quad \text{as } (r,x) \to (t,a) \in \partial' Q. \tag{A.3}$$

for a total subset T of ∂Q. Then $u \leq 0$ in Q.

A.1.C. Let Q be a bounded open set and let τ_r be the first exit times from Q. Then for every bounded Borel function f on ∂Q,

$$h(r,x) = \Pi_{r,x} f(\tau_r, \xi_{\tau_r}) \tag{A.4}$$

belongs to $\mathbb{C}^2(Q)$ and

$$\dot{h} + Lh = 0 \quad \text{in } Q. \tag{A.5}$$

If $(t,a) \in \partial Q$ is regular and if f is continuous at (t,a), then

$$h(r,x) \to f(t,a) \quad \text{as } (r,x) \to (t,a). \tag{A.6}$$

A.1.D. Let Q and τ_r be the same as in A.1.C. If ρ is a bounded Borel function in Q, then

$$F(r,x) = \Pi_{r,x} \int_r^{\tau_r} \rho(s,\xi_s)\,ds \tag{A.7}$$

is bounded, belongs to $\mathbb{C}^1(Q)$ and, for every regular point $(t,a) \in \partial Q$,

$$F(r,x) \to 0 \quad \text{as } (r,x) \to (t,a). \tag{A.8}$$

If, in addition, ρ is continuous and if, for every $(r,x) \in Q$, there exist a neighborhood U and constants $0 < \lambda \le 1$, $C < \infty$ such that

$$|\rho(s,y) - \rho(s,z)| \le C|y-z|^\lambda \quad \text{for all } (s,y),(s,z) \in U, \tag{A.9}$$

then F belongs to $\mathbb{C}^2(Q)$ and

$$\dot{F} + LF = -\rho \quad \text{in } Q. \tag{A.10}$$

Suppose, in addition, that Q is regular. Then, by A.1.B, h is uniquely determined by (A.5) and (A.6) and F is uniquely determined by (A.8) and (A.10).

If L is the Laplacian, then all these properties follow immediately from the results in [7] and [8] (see, in particular, Theorems 3.1 and 3.3 in [7] and Theorem 1.XVII.6 in [8]). The general case can be treated by using Ito's stochastic differential equations for paths of ξ and by applying results on linear parabolic PDE (in particular, A.1.D. follows from Theorem 1.9 in [17]).

LEMMA A.1 (Comparison principle). *Let Q be a bounded domain and let $\Psi: \mathbb{R}^+ \times Q \to \mathbb{R}^+$ satisfy the condition*

$$\psi(r,x;u) \ge \psi(r,x;v) \quad \textit{for all } (r,x) \in Q,\ u \ge v \in \mathbb{R}^+. \tag{A.11}$$

If $u, v \ge 0$ belongs to $\mathbb{C}^2(Q)$ and if

$$\begin{aligned} &\dot{u}(r,x) + Lu(r,x) - \psi(r,x;u(r,x)) \\ &\quad \ge \dot{v}(r,x) + Lv(r,x) - \psi(r,x;v(r,x)) \quad \textit{in } Q; \end{aligned} \tag{A.12}$$

$u - v$ is bounded above,

$$\limsup[u(r,x) - v(r,x)] \le 0 \quad \textit{as } (r,x) \to (t,a) \in T, \tag{A.13}$$

for a total subset T of ∂Q, then $u \le v$ in Q.

PROOF. Let $w = u - v$. If our statement is false, then $\tilde{Q} = \{(r,s): (r,x) \in Q,\ w(r,x) > 0\}$ is not empty. By (A.13) and (A.11), $\dot{w}(r,x) + Lw(r,x) \ge \psi(r,x;u(r,x)) - \psi(r,x;v(r,x)) \ge 0$ in $\tilde{Q}$. Note that $\tilde{T} = \partial\tilde{Q} \cap (Q \cup T)$ is a total subset of $\partial\tilde{Q}$. If $(t,a) \in \partial\tilde{Q} \cap Q$, then $w(t,a) = 0$. If $(t,a) \in \partial\tilde{Q} \cap T$, then

$$\limsup w(r,x) \le 0 \quad \text{as } (r,x) \to (t,a),\ (r,x) \in Q$$

by (A.13). We get a contradiction with A.1.B. □

LEMMA A.2. *Let* $Q = (s, t) \times U$, *where* $U = \{x: |x - x^0| < R\}$ *and let*

$$v(r, x) = \lambda\Big[(t - r)(R^2 - \rho^2)^2\Big]^{-1/(\alpha-1)}, \qquad (r, x) \in Q, \tag{A.14}$$

where λ *is a positive constant and* $\rho = |x - x^0|$. *We have*

$$\lim v(r, x) \to \infty \quad as\ (r, x) \to (t, a) \in \partial_r Q. \tag{A.15}$$

Moreover

$$\dot{v} + Lv - v^\alpha \le v^\alpha\left(\left(\frac{\lambda_0}{\lambda}\right)^{\alpha-1} - 1\right) \quad in\ Q, \tag{A.16}$$

where

$$\lambda_0^{\alpha-1} = a_0 R^4 + (t - s)\big(a_1 R^3 + a_2 R^2\big) \tag{A.17}$$

and $a_0 > 0$, $a_1, a_2 \ge 0$ *are constants which depend only on* α, *the dimension* d *and the upper bounds for* a_{ij} *and* b_i *in* Q.

PROOF. Put $u(x) = (R^2 - \rho^2)^{-2/(\alpha-1)}$. By direct computation we get

$$Lu = \big(R^2 - \rho^2\big)^{-2\alpha/(\alpha-1)}\Big\{c_1 \sum a_{ij} z_i z_j + c_2\big(R^2 - r^2\big)\big(\sum a_{ii} + \sum b_i z_i\big)\Big\},$$

where $z_i = x_i - x_i^0$, $c_1 = 8(\alpha + 1)(\alpha - 1)^{-1}$, $c_2 = 4(\alpha - 1)^{-1}$. Let $\Lambda(r, x)$ be the biggest eigenvalue of the matrix $a_{ij}(r, x)$ and let $B(r, x)^2 = \sum b_i(r, x)^2$. If Λ and B are upper bounds for $\Lambda(r, x)$ and $B(r, x)$ in Q then $\sum a_{ii} \le \Lambda d$ and $\sum b_i z_i \le BR$ in Q. Therefore

$$Lu \le \big(R^2 - \rho^2\big)^{-2\alpha/(\alpha-1)}\big[\Lambda R^2(c_1 + c_2 d) + c_2 BR^3\big]$$

and (A.16)–(A.17) hold with $a_0 = 1/(\alpha - 1)$, $a_1 = c_2 B$, $a_2 = \Lambda(c_1 + c_2 d)$. □

A.2. We apply Lemma A.1 with $\psi(r, x, u) = u^\alpha$ and Lemma A.2 to prove some properties of positive solutions of the equation

$$\dot{u} + Lu - u^\alpha = 0 \quad \text{in } Q. \tag{A.18}$$

We set

$$U_\varepsilon(r^0, x^0) = \{r: |r - r^0| < \varepsilon\} \times \{x: |x - x^0| < \varepsilon\}. \tag{A.19}$$

THEOREM A.1. *The only solution* $u \ge 0$ *of the equation* (A.18) *in the entire space* S *is equal to* 0.

PROOF. Let v be defined by (A.14) with $s < r^0 < t$ and $\lambda = \lambda_0$. Then by Lemma A.1,

$$\begin{aligned} u(r^0, x^0) &\le v(r^0, x^0) = \lambda_0\big[(t - r^0)R^4\big]^{-1/(\alpha-1)} \\ &= \Big[a_0(t - r^0)^{-1} + \big(a_1 R^{-1} + a_2 R^{-2}\big)(t - s)/(t - r^0)\Big]^{1/(\alpha-1)}. \end{aligned}$$

The right side tends to 0 as $t \uparrow \infty$, $R \uparrow \infty$. □

THEOREM A.2. *For an arbitrary open set Q, the class of all positive solutions v of* (A.18) *is locally uniformly bounded.*

PROOF. Let $U_\beta = U_\beta(r^0, x^0)$ and let v_β be defined by (A.14) with $s = r^0 - \beta$, $t = r^0 + \beta$, $R = \beta$ and $\lambda = \lambda_0$. If β is sufficiently small, then $\overline{U}_\beta \subset Q$ and, by Lemma A.1, $v \le v_\beta$ in U_β. Hence

$$v(r,x) \le N \quad \text{for all } v \text{ and all } (r,x) \in \overline{U}_\varepsilon(r^0, x^0) \tag{A.20}$$

for $\varepsilon = \beta/2$ and N equal to the maximum of v_β on $\overline{U}_\varepsilon$. □

THEOREM A.3. *Suppose that B is a relatively open subset of ∂Q. Let $c \in \mathbb{R}$ and let $V(B,c)$ be the class of all positive solutions of* (A.18) *in Q such that*

$$\limsup v(r,x) \le c \quad \text{as } (r,x) \to (t,a) \in B. \tag{A.21}$$

Then, for every $(r^0, x^0) \in B$, there exist $\varepsilon > 0$ and $N < \infty$ such that $\partial Q \cap U_\varepsilon(r^0, x^0) \subset B$ and

$$v(r,x) \le N \quad \text{for all } v \in V(B,c) \text{ and all } (r,x) \in Q \cap \overline{U}_\varepsilon(r^0, x^0). \tag{A.22}$$

PROOF. Let $Q_\beta = Q \cap U_\beta(r^0, x^0)$. Consider v_β introduced in the previous proof and put

$$f_\beta(t,a) = \limsup(v - v_\beta) \quad \text{as } (r,x) \to (t,a),\ (r,x) \in Q_\beta.$$

Put $A_\beta = \partial_r U_\beta \cap \overline{Q}$ and $\Gamma_\beta = U_\beta \cap \partial Q$. Since $\partial_r Q$ is total in ∂U_β, the set $A_\beta \cup \Gamma_\beta$ is total in $\partial Q_\beta = (Q \cap \partial U_\beta) \cup \Gamma_\beta$. Clearly, $f_\beta(t,a) = -\infty$ on A_β. It remains to show $f_\beta \le 0$ on Γ_β for some β and to repeat the arguments in the proof of Theorem A.2.

The minimum of v_β on U_β is attained at (r^0, x^0) and it is equal to

$$\varphi(\beta) = \left[\left(\tfrac{1}{2}a_0 + a_1\right)\beta^{-1} + a_2\beta^{-2}\right]^{1/(\alpha-1)}.$$

If $\Gamma_\beta \subset B$, then $f_\beta \le c - \varphi(\beta)$ on Γ_β and, $c - \varphi(\beta) \le 0$ for sufficiently small β. □

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