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Superfluid signatures in magnetar seismology

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ABSTRACT

We investigate the role of neutron star superfluidity for magnetar oscillations. Using a plane-wave analysis, we estimate the effects of a neutron superfluid in the elastic crust region. We demonstrate that the superfluid imprint is likely to be more significant than the effects of the crustal magnetic field. We also consider the region immediately beneath the crust, where superfluid neutrons are thought to coexist with a type II proton superconductor. Since the magnetic field in the latter is carried by an array of fluxtubes, the dynamics of this region differ from standard magnetohydrodynamics. We show that the presence of the neutron superfluid (again) leaves a clear imprint on the oscillations of the system. Taken together, our estimates show that the superfluid components cannot be ignored in efforts to carry out 'magnetar seismology'. This increases the level of complexity of the modelling problem, but also points to the possibility of using observations to probe the superfluid nature of supranuclear matter.

Key words: dense matter – gravitation – MHD – stars: neutron – stars: oscillations.

1 INTRODUCTION

Anomalous X-ray pulsars (AXPs) and soft gamma-ray repeaters (SGRs) are widely believed to be magnetars; neutron stars powered by an ultrastrong magnetic field (see Woods & Thompson 2004, for a review). Observations (mainly conducted by X-ray satellites) have established basic parameters like the magnetic field intensity, $B \sim 10^{15}$ G, and spin period, $P \sim 10$ s, for this class of objects. They have also revealed a complex emission pattern with alternating periods of burst activity and quiescence. SGRs, which are typically more active than AXPs, are the only ones exhibiting giant flares. These flares, which are believed to be triggered by some sort of instability in the magnetic field (Duncan & Thompson 1992; Thompson & Duncan 1995), are by far the most energetic events associated with magnetars.

An exciting contribution to magnetar phenomenology was provided by the recent discovery of quasi-periodic oscillations (QPOs) in the late tail spectrum of the two giant flares (Israel et al. 2005; Strohmayer & Watts 2005; Watts & Strohmayer 2006). There may also be evidence for a single QPO in the data of the third known flare, observed back in the 1970s (Barat et al. 1983). The frequencies of the most prominent QPOs lie in the range 30–100 Hz, exactly where one would expect to find the seismic oscillation modes of the magnetar's crust (Hansen & Cioffi 1980). This is consistent with the theoretical expectation that the energy released in a giant flare is sufficient to fracture the crust and excite its normal modes (Duncan 1998). If this interpretation of the QPOs is correct then we may have the opportunity to carry out magnetar 'asteroseismology'; a comparison between theoretically predicted mode frequencies and the QPO data, with the ultimate goal of constraining the properties of neutron star matter (Samuelsson & Andersson 2007).

Indeed, several recent papers have attempted to constrain the bulk equation of state of neutron star matter, assuming that the observed QPO frequencies are identified with the first few toroidal seismic modes of the crust (see Strohmayer 2007 for a recent review and references). This is natural as a first step, but in reality the situation is likely to be more complicated. As suggested by Levin (2006) and Glampedakis, Samuelsson & Andersson (2006), the strong magnetic field would likely couple an oscillating crust to the liquid core on a very short time-scale. Then, the observed QPOs would be a manifestation of the coupled crust–core dynamics rather than the dynamics of the crust alone. Possible evidence that the magnetic core plays an active role is given by the presence of a low-frequency QPO in the data of the 2004 December flare in SGR 1806−20. This QPO is difficult to reconcile with the seismic mode interpretation (Israel et al. 2005). It is therefore conceivable that magnetar 'seismology' also probes the (much less well known) properties of the interior magnetic field. This obviously comes at a price. We now have to model global crust–core oscillations, a problem that is considerably more challenging than that of pure seismic crust modes.

Another aspect of neutron star physics, which is directly relevant to the mode interpretation of the QPOs, has received almost no

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attention so far. Young and mature neutron stars (older than a month or so) are sufficiently cold that the bulk of their interior liquid matter is in a superfluid state. In the crust, for densities above the neutron drip density $\rho \approx 4 \times 10^{11}$ g cm^{-3}, the 'dripped' neutrons are expected to form a superfluid below a temperature $\sim 5 \times 10^9$ K. The dynamical role of these 'free' neutrons could be important. After all, they account for ∼80 per cent of the total mass in the crust. Similarly, in the liquid core we expect to find both neutrons and protons in a superfluid state (below a similar threshold temperature). In the outer core, the protons most likely form a type II superconductor, provided that the interior magnetic field does not exceed a critical value, $H_{c2} \sim 10^{16}$ G (Baym, Pethick & Pines 1969). As a consequence, any magnetic field that penetrates the proton plasma will form a large number of quantized magnetic fluxtubes.

It is clearly relevant to ask to what extent the physical components (the crust and the magnetic field) that play the leading role in the magnetar QPO problem are sensitive to neutron and proton superfluidity/superconductivity. The aim of this investigation is to provide an insight into this issue, and improve our understanding of the relative importance of the multifluid aspects of the magnetar oscillation problem. By carrying out a local analysis, i.e. considering uniform parameter model, we learn how the shear waves in the neutron star crust are affected by the presence of a superfluid neutron component. Similarly, a local analysis in the core tells us how the Alfvén waves are altered by the presence of the neutron superfluid (which provides the bulk of the core mass). Not surprisingly, the entrainment between neutrons and protons turns out to be the key parameter in these problems. This initial (order of magnitude) analysis serves as a useful guideline for future (more detailed) work for realistic neutron star models.

2 MULTIFLUID DYNAMICS OF THE CRUST

2.1 Lagrangian perturbation equations

We want to model linear perturbations in a neutron star crust penetrated by a superfluid neutron component. It is natural to use a Lagrangian picture to describe this problem. Hence, we combine the two-fluid Lagrangian perturbation equations (Andersson, Comer & Grosart 2004) with the relevant elastic terms and the magnetic force (Glampedakis & Andersson 2007). Since all known magnetars are slowly rotating, with periods of several seconds, it makes sense to focus on the non-rotating problem. Then, we need an equation for the superfluid neutron displacement which can be written

$$(1-\varepsilon_n)\partial_t^2 \xi_i^n + \varepsilon_n \partial_t^2 \xi_i^c + \nabla_i \delta\Phi + \xi_n^j \nabla_j \nabla_i \Phi - (\nabla_i \xi_n^j)\nabla_j \tilde{\mu}_n + \nabla_i \Delta_n \tilde{\mu}_n = 0. \tag{1}$$

We label the variables associated with the neutrons by a constituent index n. Δ_n represents a Lagrangian variation along the neutron flow (associated with the displacement ξ_n^i). The variable $\delta\Phi$ represents the (Eulerian) perturbation of the gravitational potential Φ and $\tilde{\mu}_n$ is the chemical potential (scaled with the neutron mass) for the neutrons. We have

$$\Delta_n \tilde{\mu}_n = \left(\frac{\partial \tilde{\mu}_n}{\partial n_n}\right)_{n_c} \nabla_i (n_n \xi_n^i) + \left(\frac{\partial \tilde{\mu}_n}{\partial n_c}\right)_{n_n} \nabla_i (n_c \xi_c^i) + \xi_n^i \nabla_i \tilde{\mu}_n, \tag{2}$$

where n_n and n_c are the number densities of the neutrons and the baryons making up the crust nuclei, respectively. The variables associated with the crust nuclei are labelled by the constituent index c.

The parameter ε_n (assumed constant in the following) encodes the entrainment effect. In the crust, the entrainment is due to Bragg scattering of the free neutrons on the crystal lattice (see Chamel & Haensel 2008, for a recent review of the pioneering work in this area). The available results are somewhat uncertain due to, for example, many-body effects. Further work on this problem should be encouraged.

Finally, it should be noted that we are not accounting for effects due to the presence of neutron vortices, e.g. the mutual friction and the vortex tension here. In principle, these effects will be present even in the very slowly rotating magnetars, and it will be interesting to consider them at a later stage (see van Hoven & Levin 2008, for a relevant recent discussion). Our initial aim is to explore the leading order effects of the problem.

The corresponding equation of motion for the crust nuclei can be written

$$(1-\varepsilon_c)\partial_t^2 \xi_i^c + \varepsilon_c \partial_t^2 \xi_i^n + \nabla_i \delta\Phi + \xi_c^j \nabla_j \nabla_i \Phi - \left(\nabla_i \xi_c^j\right) \nabla_j \tilde{\mu}_c + \nabla_i \Delta_c \tilde{\mu}_c = \Delta_c f_i^{el} + \Delta_c f_i^{mag}. \tag{3}$$

Here, Δ_c represents the Lagrangian variation along the crust motion (associated with a displacement ξ_c^i), and

$$\Delta_c \tilde{\mu}_c = \left(\frac{\partial \tilde{\mu}_c}{\partial n_c}\right)_{n_n} \nabla_i (n_c \xi_c^i) + \left(\frac{\partial \tilde{\mu}_c}{\partial n_n}\right)_{n_c} \nabla_i (n_n \xi_n^i) + \xi_c^i \nabla_i \tilde{\mu}_c. \tag{4}$$

It is also worth noting that

$$n_n \varepsilon_n = n_c \varepsilon_c. \tag{5}$$

The charged component equation includes both elastic and magnetic contributions. The former can be written

$$\Delta_c f_i^{el} = \frac{1}{\rho_c} \nabla^j \sigma_{ij}, \tag{6}$$

where

$$\sigma_{ij} = \mu\left(\nabla_i \xi_j^c + \nabla_j \xi_i^c\right) - \frac{2}{3}\mu(\nabla^l \xi_l^c)\delta_{ij} \tag{7}$$

(here one should not confuse the shear modulus μ with the chemical potentials μ_x). Meanwhile, the magnetic term follows from the standard electromagnetic Lorentz force. That is, in this case we have

$$f_i^{mag} = f_i^L = \frac{1}{c\rho_c} \epsilon_{ijk} J^j B^k. \tag{8}$$

Eliminating the total current with the help of Ampére's law, i.e. $J^i = (c/4\pi)\, \epsilon^{ijk}\, \nabla_j\, B_k$, this becomes

$$f_i^L = \frac{B^j}{4\pi\rho_c}(\nabla_j B_i - \nabla_i B_j). \tag{9}$$

Working out the Lagrangian variation using (Glampedakis & Andersson 2007)

$$\Delta_c\left(\frac{B^j}{\rho_c}\right) = 0, \tag{10}$$

we arrive at

$$\Delta_c B^i = -B^i \nabla_j \xi_c^j \tag{11}$$

and

$$\Delta_c B_i = B_j \nabla_i \xi_c^j - B_i \left(\nabla_j \xi_c^j\right) + B^j \nabla_j \xi_i^c. \tag{12}$$

Finally, using

$$\Delta_c(\nabla_j B_i) = \nabla_j(\Delta_c B_i) - B_l \nabla_j \nabla_i \xi_c^l, \tag{13}$$

we obtain from (9)

$$\Delta_c f_i^{\rm mag} = \frac{B^j}{4\pi\rho_c}[\nabla_j(\Delta_c B_i) - \nabla_i(\Delta_c B_j)]. \tag{14}$$

These are all the relations we need to solve the problem. As far as we are aware, this is the first time that the perturbation problem for combined superfluidity, elasticity and magnetic fields has been formulated. The equations we have given can be directly applied to studies of global mode oscillations of a superfluid neutron star with a crust.

For later convenience, it is useful to note that we could equally well have worked with Eulerian variations. In fact, since $f_{\rm L}^i = 0$ in the background configuration we must have $\Delta_c f_i^{\rm mag} = \delta f_i^{\rm mag}$. Moreover, one can show that in the case of an incompressible fluid and a uniform background field (see below) we have

$$\Delta_c f_i^{\rm mag} = \delta f_i^{\rm mag} = v_{\rm A}^2 \left[(\hat{B}^j\nabla_j)^2\xi_i^{\rm c} - \hat{B}^j\hat{B}^l\nabla_i\nabla_l\xi_j^{\rm c}\right], \tag{15}$$

where 'hats' denote unit vectors. We have also defined the Alfvén wave velocity

$$v_{\rm A}^2 = \frac{B^2}{4\pi\rho_c}. \tag{16}$$

It is important to note that in the superfluid system the Alfvén velocity scales with the number density of charged nucleons, not the total baryon number density (Mendell 1998).

2.2 Plane-wave analysis

As a first step towards understanding the problem, let us consider the simple case of a uniform, non-rotating background. For an incompressible model, we have

$$\nabla_i\xi_{\rm x}^i = 0 \longrightarrow \Delta_{\rm x}\tilde{\mu}_{\rm x} = 0. \tag{17}$$

Then, the problem simplifies to (note that we will have $\delta\Phi = \nabla_i\Phi = 0$ for a uniform background)

$$(1-\varepsilon_{\rm n})\partial_t^2\xi_i^{\rm n} + \varepsilon_{\rm n}\partial_t^2\xi_i^{\rm c} = 0 \tag{18}$$

$$(1-\varepsilon_{\rm c})\partial_t^2\xi_i^{\rm c} + \varepsilon_{\rm c}\partial_t^2\xi_i^{\rm n} = \Delta_c f_i^{\rm el} + \Delta_c f_i^{\rm mag}. \tag{19}$$

We also need

$$\Delta_c f_i^{\rm el} = v_s^2\nabla^2\xi_i^{\rm c}, \tag{20}$$

where the shear velocity, v_s, is defined by

$$v_s^2 = \frac{\mu}{\rho_c}. \tag{21}$$

Note that the shear velocity scales with the number density of nucleons locked in the crust lattice, not the total nucleon number density as would be the case if there were no superfluid component. This distinction has not been made in previous work where the crust is modelled as a single component (see e.g. Duncan 1998; Piro 2005).

We now consider short wavelength ($\ll$ the radius of the star) wave propagation in this system. Making the standard plane-wave Ansatz (see Sidery, Andersson & Comer 2008, for a recent analysis of the analogous non-magnetic two-fluid problem)

$$\xi_i^{\rm x} = A_i^{\rm x} e^{i(\omega t + k_j x^j)}, \tag{22}$$

where the index x is either n or c, we have

$$k^i A_i^{\rm x} = 0, \tag{23}$$

i.e. the waves are transverse, and

$$\Delta_c f_i^{\rm mag} = -v_{\rm A}^2 k^2 \left[(\hat{B}^j\hat{k}_j)^2\xi_i^{\rm c} - \hat{k}_i(\hat{B}_j\xi_{\rm c}^j)(\hat{B}^l\hat{k}_l)\right]. \tag{24}$$

From equation (18), we immediately get the relation

$$A_i^{\rm n} = -\frac{\varepsilon_{\rm n}}{1-\varepsilon_{\rm n}}A_i^{\rm c}. \tag{25}$$

Using this in equation (19), we arrive at

$$\left[\frac{\omega^2}{\varepsilon_\star} - v_s^2k^2 - v_{\rm A}^2k^2(\hat{B}_j\hat{k}^j)^2\right]A_i^{\rm c} = -v_{\rm A}^2k^2(\hat{B}_j\hat{k}^j)(\hat{B}^lA_l^{\rm c})\hat{k}_i, \tag{26}$$

where we have introduced

$$\varepsilon_\star = \frac{1-\varepsilon_{\rm n}}{1-\varepsilon_{\rm n}-\varepsilon_{\rm c}}. \tag{27}$$

Defining

$$\omega_0^2 = v_s^2k^2 \tag{28}$$

the frequency of 'pure' elastic waves, and the Alfvén wave frequency

$$\omega_{\rm A}^2 = v_{\rm A}^2k^2, \tag{29}$$

we have an equation for $A_i^{\rm c}$,

$$\left[\frac{\omega^2}{\varepsilon_\star} - \omega_0^2 - (\hat{B}_j\hat{k}^j)^2\omega_{\rm A}^2\right]A_i^{\rm c} = -\omega_{\rm A}^2(\hat{B}_j\hat{k}^j)(\hat{B}^lA_l^{\rm c})\hat{k}_i. \tag{30}$$

In order to arrive at the final dispersion relation, we first note that contracting the above equation with $\hat{k}^i$ leads to the constraint

$$\omega_{\rm A}^2(\hat{B}_j\hat{k}^j)(\hat{B}^lA_l^{\rm c}) = 0. \tag{31}$$

Thus, we can either choose to look for solutions where k^i is orthogonal to the local magnetic field or we see that the polarization $A_i^{\rm c}$, and hence $A_i^{\rm n}$, must be orthogonal to both k^i and B^i. Since the right-hand side of (30) vanishes in both cases we find that all non-trivial solutions must be such that

$$\frac{\omega^2}{\varepsilon_\star} - \omega_0^2 - (\hat{B}_j\hat{k}^j)^2\omega_{\rm A}^2 = 0. \tag{32}$$

That is, we have the general dispersion relation

$$\omega^2 = \omega_0^2\varepsilon_\star\left[1 + (\hat{B}_j\hat{k}^j)^2\frac{\omega_{\rm A}^2}{\omega_0^2}\right]. \tag{33}$$

Note that, in the degenerate case when $B_ik^i = 0$ we cannot uniquely determine the polarization; it can lie in any direction in the plane orthogonal to k^i. Also, it is clear that such waves do not depend on the magnetic field at all. Generically, the polarization is, however, well defined (up to scale since we have a homogeneous system) to be orthogonal to both k^i and B^i.

We need to estimate the magnitude of the various terms. Let us first focus on the entrainment. Using (5), we find that

$$1-\varepsilon_{\rm n}-\varepsilon_{\rm c} = 1 - \frac{\varepsilon_{\rm n}}{x_{\rm c}}, \tag{34}$$

where $x_{\rm c} = \rho_{\rm c}/\rho$. We can express this in terms of the effective mass of the free neutrons, $m_{\rm n}^*$. Then, we have (see Prix, Comer & Andersson 2002, for a discussion of the analogous problem in a superfluid neutron star core)

$$\varepsilon_{\rm n} = 1 - \frac{m_{\rm n}^*}{m_{\rm n}} \tag{35}$$

and it follows that

$$\varepsilon_\star = x_{\rm c}\left(1 - x_{\rm n}\frac{m_{\rm n}}{m_{\rm n}^*}\right)^{-1} = \frac{x_{\rm c}}{\chi}. \tag{36}$$

It is easy to show that $\chi^{-1/2}$ encodes the difference between the superfluid result and the standard result for a single component

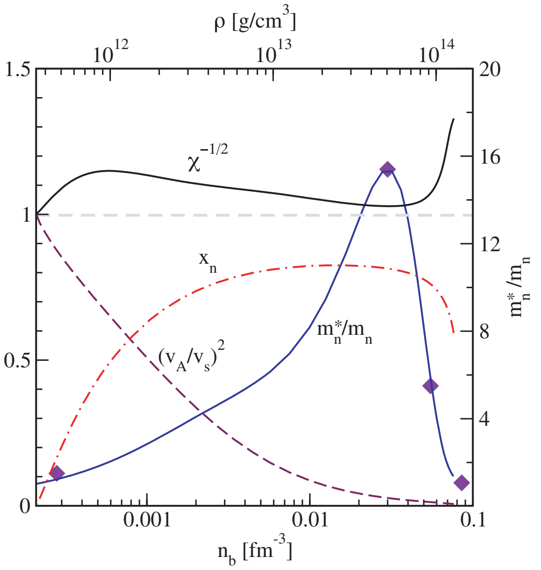

Figure 1. This figure illustrates the density dependence of the different parameters that affect the wave propagation in the crust. We show, as functions of the total baryon number density n_{b}, the superfluid neutron fraction $x_{\mathrm{n}} = n_{\mathrm{n}}/n_{\mathrm{b}}$ (dash–dot, left-hand scale) for the equation of state discussed by Douchin & Haensel (2001), a fit to the effective neutron mass $m_{\mathrm{n}}^*/m_{\mathrm{n}}$ (solid, right-hand scale) based on the numerical results of Chamel (2006) (data points indicated by diamonds), and the ratio between the Alfvén and the shear wave speeds $(v_{\mathrm{A}}/v_s)^2$ (dashed, left-hand scale). The overall effect that the presence of the crust superfluid has on the local wave propagation, compared to the standard single component crust, is (as discussed in the main text) represented by $\chi^{-1/2}$ (solid, left-hand scale). The horizontal dashed grey line indicates unity on the left-hand scale.

crust, i.e. with $\rho_{\mathrm{c}} \to \rho$ in (21). This may be the most meaningful comparison to make, since all previous studies of crust oscillations have assumed the single component model.

What do we learn from these results? First of all, we see that in the limit $m_{\mathrm{n}}^* \to m_{\mathrm{n}}$, when the medium effects that lead to the effective mass differing from the bare mass are not so great, we have $\varepsilon_\star \to 1$ and $\chi \to x_{\mathrm{c}}$. The waves in such a system, cf. (33), are the usual shear waves with a (as we will see later) relatively small magnetic correction. Of course, the results could still differ significantly from the standard single component model. The largest effect that one would expect would be, for $x_{\mathrm{c}} \approx 0.8$ cf. Fig. 1, a frequency increase by about a factor of 2. However, the effective mass is expected to be larger than the bare mass so let us consider the opposite limit, which may well apply in parts of the neutron star crust (see e.g. Chamel 2006). Then we have $m_{\mathrm{n}}^* \gg m_{\mathrm{n}}$. Using also $x_{\mathrm{c}} \leq 1$, we see that $\varepsilon_\star \approx x_{\mathrm{c}}$ or $\chi \to 1$. In this limit, it is very difficult for the free neutrons to move relative to the crust. The upshot of this is that the waves in the system tend to the frequency predicted for a pure elastic crust *without* a superfluid component.

These two extremes show that the presence of the superfluid in the neutron star crust can have a significant effect on the waves in the system. According to the data in Fig. 1, the combined effect is at the 10 per cent level (compared to the single component crust result). The results clearly show that the superfluid component must be considered if we want to develop high-precision magnetar crust seismology. Of course, in reality we are mainly interested in the global oscillations. Then the local effects that we have worked out will be (in some sense) averaged throughout the crust. One may expect this to decrease the role of the superfluid since the effective neutron mass may only be large in parts of the crust. Of course, the real answer requires a detailed mode calculation. This problem remains to be solved. In order to provide reliable results, such an effort should draw on more complete studies of the entrainment for the crust superfluid. One should also worry about the relevance of vortex pinning and the mutual friction.

Finally, let us discuss the relative importance of the magnetic field. Scaling to 'typical' values, we have

$$\mu \approx 10^{13} \left(\frac{v_s}{10^8\,\mathrm{cm\,s^{-1}}} \right)^2 \left(\frac{\rho}{10^{14}\,\mathrm{g\,cm^{-3}}} \right) \mathrm{dyne\,cm^{-2}}. \qquad (37)$$

Then, it follows from (33) that we need to consider

$$\left(\frac{\omega_A}{\omega_0} \right)^2 \approx 0.08 \left(\frac{v_s}{10^8\,\mathrm{cm\,s^{-1}}} \right)^{-2} \left(\frac{\rho}{10^{14}\,\mathrm{g\,cm^{-3}}} \right)^{-1} \left(\frac{B}{10^{15}\,\mathrm{G}} \right)^2. \qquad (38)$$

This shows that we can safely ignore the magnetic effects in the high-density region of the crust, cf. Fig. 1. In order for the magnetic term to dominate at the base of the crust, we need $B \sim 10^{16}$ G, stronger than the field strength inferred for magnetars. Of course, one has to be a little bit careful here. First of all, it is entirely possible that the interior field is stronger than the exterior dipole field which leads to the observed braking of the magnetar spin. Secondly, (38) indicates that the magnetic terms will dominate as we approach the surface of the star. However, our analysis breaks down completely in the surface region. Basically, the ideal magnetohydrodynamics (MHD) approximation is only valid as long as the Alfvén wave speed is significantly below the speed of light. If this is not the case, one cannot neglect the displacement current (see Blaes et al. 1989 for a detailed analysis). Hence, we require

$$\rho \gg 10^8 \left(\frac{B}{10^{15}\,\mathrm{G}} \right)^2 \mathrm{g\,cm^{-3}}. \qquad (39)$$

It is interesting to compare (39) to the estimated density for the top of the crust, cf. for example equation (1) from Piro (2005),

$$\rho_{\mathrm{top}} \approx 2.3 \times 10^9 \left(\frac{T}{3 \times 10^8\,\mathrm{K}} \right)^3 \left(\frac{26}{Z} \right)^6 \left(\frac{A}{56} \right) \mathrm{g\,cm^{-3}}, \qquad (40)$$

where A is the number of nucleons and Z is the charge per ion, respectively. This density is scaled to iron, which means that the fiducial values should be relevant near the top of the crust. This suggests that our approximation remains valid throughout a magnetar crust. The displacement current becomes important in the neutron stars envelope. The nature of this transition is an important problem that deserves more attention.

3 MULTIFLUID DYNAMICS OF THE CORE

3.1 Lagrangian perturbation equations

The magnetohydrodynamics (MHD) problem in the core is, in the simplest case, formulated in terms of three distinct fluids associated with the neutrons, protons and electrons. The former two particle species are expected to be superfluid and superconducting, respectively. Due to the smallness of the electron mass, the electron fluid degree of freedom can be suppressed and the MHD equations effectively lead to a two-fluid model (Mendell 1998). The protons in the outer core are expected to form a type II superconductor (Baym et al. 1969) which means that the magnetic field is carried by a large number of fluxtubes, each with a flux quantum $\phi_0 = hc/2e$. This should be the case provided the magnetic field is below the critical value H_{c2} (Baym et al. 1969). This critical value represents the field strength at which the magnetic fluxtubes overlap and can no longer be treated individually. Above this threshold, the magnetic field behaves 'classically'. It should be noted that, even though the critical field is large, this is not an unrealistic possibility for magnetars given that the magnetic field in the interior could be considerably higher than the exterior dipole field. Hence, one ought to consider both superconducting and normal protons. The latter case is, however, trivial. The desired result follows immediately from the previous section, e.g. (33), if we take the limit $\mu \to 0$.

For a non-rotating star, we again neglect the vortex-mediated mutual friction and the neutron vortex tension. Omitting also a small entrainment induced magnetic term that originates from the London field by means of which the proton superconductor rotates (see Glampedakis et al., in preparation for discussion), the superfluid neutron dynamics is still governed by (1).

The combined proton–electron dynamics is a little bit more complicated. As discussed by Glampedakis et al. (in preparation), the relevant equation of motion takes the form

$$\left(\partial_t + v_{\rm c}^j \nabla_j\right)\left(v_i^{\rm c} + \varepsilon_{\rm c} w_i^{\rm nc}\right) + \nabla_i(\tilde{\mu} + \Phi) + \varepsilon_{\rm c} w_j^{\rm nc} \nabla_i v_{\rm c}^j = \frac{1}{\rho_{\rm c}}\left(f_i^{\rm L} + t_i^{\rm c}\right), \tag{41}$$

where $v_{\rm c}^i$ and $v_{\rm n}^i$ are the velocities, $w_{\rm nc}^i = v_{\rm n}^i - v_{\rm c}^i$ and we have (again) neglected the mutual friction. There are two force terms on the right-hand side of this equation. The first, $f_{\rm L}^i$, is the usual electromagnetic Lorentz force given by (8). The second, $t_{\rm c}^i$, represents the smooth-averaged tension of the magnetic fluxtubes. Remarkably, the Lorentz force does not play a role in the final superconducting MHD equations. As discussed by Glampedakis et al. (in preparation) (see also Mendell 1998), it is exactly cancelled by a term originating from the fluxtube tension. In the case of a non-rotating star, this leads to the magnetic force taking the form

$$f_i^{\rm L} + t_i^{\rm c} = \frac{m_{\rm c} c}{4\pi e} \mathcal{W}_{\rm c}^j \left[\nabla_j (H_{\rm c1} \hat{\mathcal{W}}_i^{\rm c}) - \nabla_i (H_{\rm c1} \hat{\mathcal{W}}_j^{\rm c})\right], \tag{42}$$

where the lower critical magnetic field $H_{\rm c1} = H_{\rm c1}(\rho_{\rm c}) \approx 10^{15}$ G.

We have defined the vector $\mathcal{W}_{\rm c}^i$, representing the (averaged) canonical proton vorticity (Prix 2005). This means that we have[1]

$$\mathcal{W}_{\rm c}^i \approx \frac{e}{m_{\rm c} c} B^i. \tag{43}$$

From this, we see that, as expected, the magnetic force (42) does not have a component along the magnetic field. However, the approximation (43) neglects a small term proportional to the London field. This piece is aligned with the charged component's rotation axis, and if it is misaligned with the magnetic field (the generic situation) then the magnetic force will have a component along B^i. Of course, this contribution is very small and can likely be neglected in most situations of interest.

Comparing the form of the magnetic forces in the normal and superconducting cases, it is easy to see some generic differences, some of which appear due to the density dependence of the critical field $H_{\rm c1}$. It is clear that the terms in the force (42) that contain the gradient of $H_{\rm c1}$ have no correspondence in the standard Lorentz force (8). However, the two forces are different even for the case of a uniform incompressible background. In this case, the perturbed form of the force (42) is

$$\delta f_i^{\rm mag} = \frac{1}{\rho_{\rm c}} \delta(f_i^{\rm L} + t_i^{\rm c}) \approx \frac{H_{\rm c1}}{4\pi\rho_{\rm c}} B^j \left[\nabla_j \delta \hat{B}_i - \nabla_i \delta \hat{B}_j\right]. \tag{44}$$

This expression can be compared to the perturbed Lorentz force (8),

$$\delta f_i^{\rm L} = \frac{B B^j}{4\pi\rho_{\rm c}} \left[\nabla_j \delta \hat{B}_i - \nabla_i \delta \hat{B}_j\right] + \frac{B^j}{4\pi\rho_{\rm c}} \left[\hat{B}_i \nabla_j \delta B - \hat{B}_j \nabla_i \delta B\right]. \tag{45}$$

The two forces will differ unless the second term in (45) vanishes. This requires that the following condition is satisfied (using $\delta B = \hat{B}_j \delta B^j$)

$$(g^{ij} - \hat{B}^i \hat{B}^j) \hat{B}_k \nabla_j \delta B^k = 0. \tag{46}$$

For a generic perturbation, this will *not* be the case. The conclusion of this discussion is that one should be careful before using intuition gained from standard MHD problems in the case of a superconducting core. There is certainly more[2] to the problem than a simple 'replacement' $B^2 \to B H_{\rm c1}$.

The superconducting MHD equations form a closed system once we provide a relation between the magnetic field and the fluid velocity. This relation follows from the magnetic induction equation. Neglecting the coupling forces between the electrons and the neutron and proton fluids (see Glampedakis et al., in preparation for discussion), the induction equation takes the standard form,

$$\partial_t B^i \approx \epsilon^{ijk} \epsilon_{klm} \nabla_j \left(v_{\rm c}^l B^m\right). \tag{47}$$

Its perturbed form is given by (10) from which we obtain the Eulerian perturbation of the magnetic field,

$$\delta B^i = B^j \nabla_j \xi_{\rm c}^i - \nabla_j \left(\xi_{\rm c}^j B^i\right). \tag{48}$$

Using this result, we find (for a uniform background and incompressible perturbations)

$$\delta f_{\rm mag}^i \approx c_{\rm A}^2 \hat{B}^j \hat{B}^l (g^{ik} - \hat{B}^i \hat{B}^k) \nabla_j \nabla_l \xi_k^{\rm c}, \tag{49}$$

where

$$c_{\rm A}^2 = \frac{H_{\rm c1} B}{4\pi\rho_{\rm c}}. \tag{50}$$

We now have all the relations we need to discuss short wavelength waves in the superfluid/superconducting system.

3.2 Plane-wave analysis

Most of the analysis works out exactly as in the crust problem (obviously in the $\mu \to 0$ limit). The only difference is the form of the perturbed magnetic force. Making the plane-wave assumption, we see that

$$\Delta_{\rm c} f_i^{\rm mag} = -c_{\rm A}^2 k^2 (\hat{B}^j \hat{k}_j)^2 \left[\xi_i^{\rm c} - (\hat{B}^l \xi_l^{\rm c}) \hat{B}_i\right]. \tag{51}$$

Comparing the magnetic forces in the normal and superconducting cases (equations 21 and 51, respectively), we see that the characteristic speeds are different, $c_{\rm A}^2 = (H_{\rm c1}/B)\, v_{\rm A}^2$. This is a well-known effect (Easson & Pethick 1977; Mendell 1998). The two velocities would differ by a factor $\sim 10^3$ for a canonical pulsar with

[1] Strictly speaking, this expression is only valid for the non-rotating background. Perturbations of $\mathcal{W}_{\rm c}^i$ will also contain the perturbed fluid velocities. However, as long as we are focusing on the leading order contributions these can be neglected. That this is a legitimate approximation is easy to see since the characteristic frequency

$$\frac{e}{m_{\rm c} c} B \approx 10^{19} \left(\frac{B}{10^{15}\,{\rm G}}\right)\ {\rm s}^{-1}$$

is much higher than any other relevant frequency in the problem.

[2] For an interesting recent discussion on how superconductivity may affect the stability properties of the star, see Akgün & Wasserman (2008).

$B = 10^{12}$ G. However, the effect will not be as dramatic for magnetars for which $B \approx H_{c1} \approx 10^{15}$ G.

Combining the perturbation equations as in the previous section, we readily arrive at

$$\left[\frac{\omega^2}{\varepsilon_\star} - c_A^2 k^2 \left(\hat{k}_j \hat{B}^j\right)^2\right] A^i_c = -c_A^2 k^2 \left(\hat{k}_j \hat{B}^j\right)^2 \left(\hat{B}_j A^j_c\right) \hat{B}^i. \quad (52)$$

Working things out as in the crust case, we project this equation on to $\hat{k}_i$. Since $k_i A^i{}_c = 0$, it then follows that we have the constraint

$$c_A^2 \left(\hat{k}_j \hat{B}^j\right)^3 \left(\hat{B}_j A^j_c\right) = 0. \quad (53)$$

This has the same implications as in the crust problem. It follows that the right-hand side of (52) must vanish. Hence, we have the dispersion relation,

$$\omega^2 = \varepsilon_\star c_A^2 k^2 \left(\hat{k}_j \hat{B}^j\right)^2 = \varepsilon_\star \frac{H_{c1} B}{4\pi\rho_c} k^2 \left(\hat{k}_j \hat{B}^j\right)^2. \quad (54)$$

Our main interest here concerns the role of the superfluid neutron component. Its presence is reflected by the entrainment factor in (54). To quantify its relevance, we express the entrainment in terms of the effective proton mass, i.e. we use $\varepsilon_c = 1 - m_p^*/m_p$. Then it follows, since the proton fraction in the core is small, that

$$\omega^2 \approx \frac{m_p}{m_p^*} c_A^2 k^2. \quad (55)$$

Since it is expected that $0.3 < m_p^*/m_p < 0.7$ (see Prix et al. 2002, for discussion), we see that the presence of the superfluid neutrons will lead to a ~20–80 per cent *increase* in the frequency of the core waves. This effect is large enough that it cannot be neglected. It may, in fact, be observable. If one accepts the argument that the magnetic field couples motion in the crust to the core, and that the core fluid is therefore partaking in the oscillation, then the entrainment will affect the observed frequencies.

We cannot at this point say much about the global oscillations of a magnetic neutron star core; it is a problem that remains to be solved in detail. It is complicated by the likely presence of an 'Alfvén continuum' (Levin 2007). At this point, it is not clear to what extent the continuum prevails in more detailed neutron star models. However, it is easy to see how the presence of the superfluid will manifest itself in the continuum toy model considered by Levin (2007). The frequency range of the continuum will simply scale according to (54).

4 CONCLUDING REMARKS

In this paper, we have investigated the role of neutron star superfluidity for magnetar oscillations. The results impact on attempts to use data from observed QPOs in the tails of magnetar flares to place constraints on neutron star parameters (see e.g. Samuelsson & Andersson 2007). Using a plane-wave analysis, we estimated the effects of the neutron superfluid in the elastic crust region. This, the first ever, analysis of the combined magnetic-elastic-superfluid crust problem demonstrated that the superfluid imprint is likely to be more significant than the effects of the crustal magnetic field. This is, of course, assuming that the SGR flare mechanism does not deposit sufficient heat in the crust to raise the system above the superfluid transition temperature. Available estimates, e.g. Kouveliotou et al. (2003), suggest that this is unlikely. We also considered the region immediately beneath the crust, where superfluid neutrons are thought to coexist with a type II proton superconductor. Since the magnetic field in the latter is carried by an array of fluxtubes (see Glampedakis et al., in preparation, for discussion), the dynamics of this region differ from standard MHD. We showed that the presence of the neutron superfluid (again) affects the oscillations of the system. This accords well with previous results of, in particular, Mendell (1998).

Our estimates show that the superfluid components cannot be ignored in efforts to carry out magnetar seismology. This increases the level of complexity of the modelling problem, but also points to the exciting possibility of using observations to probe the superfluid nature of supranuclear matter. Future work needs to extend our analysis to consider the global oscillations of magnetic-superfluid-elastic neutron stars. This is a very interesting problem because, in addition to enabling a more detailed seismology analysis, it may also provide insight into rotational glitches in magnetars (Dib, Kaspi & Gavriil 1992). It is generally believed that superfluidity plays a key role in radio pulsar glitches. Recent developments in modelling these events are showing some promise (Glampedakis & Andersson 2008), and it would obviously be highly relevant to extend this analysis to strongly magnetized systems.

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