Supergravity duals of supersymmetric four dimensional gauge theories

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Abstract

This article contains an overview of some recent attempts at understanding supergravity and string duals of four dimensional gauge theories using the AdS/CFT correspondence. We discuss the general philosophy underlying the various ways to realize Super Yang-Mills theories in terms of systems of branes. We then review some of the existing duals for $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories. We also discuss differences and similarities with realistic theories.

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Contents

In recent years, a great deal of attention has been attracted by a new kind of duality between gauge theories and string theories, known as the AdS/CFT correspondence. According to it, certain superconformal gauge theories have a dual description in terms of critical string backgrounds. This provides the first explicit realization of the old idea that the strongly coupled dynamics of a gauge theory has a description in terms of an effective theory of strings. The correspondence also naturally implements the 't Hooft large N expansion, thus providing a verification of many ideas about gauge theories at large N. In addition to these qualitative successes, AdS/CFT also provides quantitative tools for understanding gauge theories. For example, correlation functions of the conformal gauge theory in the strongly coupled regime at large N, which cannot be computed in perturbative quantum field theory, can be reduced to a classical computation in supergravity. Originally formulated as a duality between $\mathcal{N} = 4$ super Yang-Mills (SYM) and Type IIB string theory on $AdS_5 \times S^5$ [1, 2, 3], the correspondence can also be extended to conformal gauge theories with less supersymmetry and in different dimensions and, nowadays, there are few doubts about its correctness. It is somewhat ironic that the first successful example of a stringy description of gauge theories deals with conformal theories and not with confining ones, where the string would be naturally identified with the color flux tubes of confinement, or, briefly, the QCD strings.

In this review, we will give an overview of some recent attempts to extend the AdS/CFT ideas to non conformal theories. Considered the huge literature on the subject, we decided to discuss only four-dimensional gauge theories with unitary groups. We will focus, in particular, on two specific ways of generalizing the correspondence to other pairs of gauge/gravity duals. The first one consists in deforming a conformal theory for which we possess a well defined supergravity dual. The gauge theory obtained in this way is non conformal at energies below the scale set by the deformation. The second method uses wrapped and fractional branes engineering theories that are non-conformal at all scales.

The extension of the AdS/CFT ideas to non-conformal theories is not straightforward. From a technical point of view, it is difficult to avoid singularities in the solutions. No regular solutions dual to $\mathcal{N} = 2$ gauge theories are indeed known. The $\mathcal{N} = 1$ case is more successful: two completely regular supergravity solutions describing $\mathcal{N} = 1$ gauge theories have been found [4, 5]. The road to realistic theories, like QCD, is still long. Classical supergravity solutions give a quite accurate description of theories that are not pure YM theories, but contain infinite additional fields. It is a general expectation that classical supergravity alone cannot describe realistic gauge theories, which contain higher spin glueballs. The dual of pure QCD is therefore expected to be a strongly coupled string model. The AdS-inspired solutions that we will describe are nevertheless interesting. Firstly, the possibility of re-summing all string world-sheet corrections

for a string background is not unconceivable. The inclusion of these corrections would give a good description of pure gauge theories in the large N limit. The computation of world-sheet corrections, which is relatively easy in flat space-time, here is complicated by the presence of RR-fields, but some progress in this direction has been recently made. Secondly, the supergravity duals provide many exactly solvable models exhibiting confinement and other phenomena typical of the pure gauge theory. Thus, even if not quantitatively relevant for QCD, they provide a good laboratory for studying the mechanism of confinement and the qualitative properties of QCD.

The purpose of this article is to provide a general overview of the literature, to describe the main features of the various methods to realize interesting gauge theories and to give a unifying picture for various models. We do not plan to be exhaustive. Since there exist many good reviews in the literature covering some of the models we will discuss, we will sometimes refer to them for the details and the proof of specific results. Inevitably, many methods to extend the correspondence are not covered here, including some that were largely discussed in the literature and came first historically. Two basic subjects are not discussed here at all: the introduction of finite temperature and Type 0 theories. Even in the context of deformations and fractional/wrapped branes we will make several omissions.

The review is organized as follows. In Section 1 we briefly review the AdS/CFT correspondence for the $\mathcal{N} = 4$ and $\mathcal{N} < 4$ cases. In Section 2 we discuss the general aspects of the deformation method while in Section 3 we discuss fractional and wrapped branes. In Section 4 we give an overview of the known supergravity solutions with $\mathcal{N} = 2$ supersymmetry. In Section 5, we discuss supergravity solutions with $\mathcal{N} = 1$ supersymmetry. The case of softly broken theories is discussed in the last part of Section 5. In each Section, we chose to cover in more detail the case of wrapped branes, which therefore forms the backbone of this review.

1 Basic dictionary of the AdS/CFT correspondence

A throughful introduction to the AdS/CFT correspondence [1, 2, 3] would itself require a whole review. This Section has been inserted for completeness, to recall the basic facts we will use and generalize in the following Sections. Therefore we suggest the reader who already has a certain knowledge of the correspondence to start with Section 2 and come back to Section 1 when necessary. On the contrary, for a more complete discussion of AdS/CFT we refer the reader to the very good reviews in the literature [6, 7, 8]. Here we will first focus on the best known example of the AdS/CFT correspondence, which involves $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions. Then we will examine some extensions to less supersymmetric models and six-dimensional theories.

1.1 The AdS/CFT correspondence: Motivations

The AdS/CFT correspondence [1] derives from the observation that systems of D-branes in Type II string theory (or systems of M-branes in M theory) admit a complementary description in terms of gauge theories on their world-volume on one side and curved supergravity backgrounds on the other side. Consider the simplest system of branes that realize on the world-volume a four-dimensional gauge theory: a stack of N parallel D3-branes in Type IIB. Since the D-branes preserve half of the space-time supersymmetry, this configuration has $\mathcal{N} = 4$ conformal supersymmetry in four dimensions. The massless fields on the branes form a $\mathcal{N} = 4$ multiplet containing a $U(N)$ gauge field A_μ, four Weyl fermions λ_a and six scalars ϕ_i, all transforming in the adjoint representation of $U(N)$. The $SO(6) \sim SU(4)$ symmetry of the space transverse to the branes is realized as the field theory R-symmetry, under which the fermions transform in the representation **4** and the scalars in the representation **6**. The theory has a $6N$ dimensional moduli space of vacua labeled by the Cartan values of the adjoint scalar VEVs. In a generic vacuum, the gauge group is broken to its maximal abelian subgroup $U(1)^N$. Being BPS objects, the D-branes can be separated with no cost in energy. The generic vacuum of the gauge theory is then represented by a string configuration where the branes have arbitrary positions in the transverse space $\mathbb{R}^6$. Notice that a typical massive W-boson in a generic vacuum is represented by an open string connecting two branes and its mass is given by $m = \Delta r / \alpha'$, where Δr is the brane separation.

At low energies, the system is conveniently described by the $\mathcal{N} = 4$ massless fields on the branes coupled to the massless fields of Type IIB supergravity in the bulk. The low energy Lagrangian for the coupled brane/bulk system reads

$$-\frac{1}{8\pi g_s} \int d^4x \sqrt{g}\, Tr(F^2) + \frac{1}{(2\pi)^7 \alpha'^4 g_s^2} \int d^{10}x \sqrt{g} R + \cdots \tag{1}$$

In this expression, we integrated out all the open and closed string oscillator modes. In the low energy limit $E \ll 1/\sqrt{\alpha'}$ the gauge theory on the branes decouples from the bulk and we recover $4d$ $\mathcal{N} = 4$ SYM theory with gauge coupling g_{YM} determined by the string coupling: $g_{YM}^2 = 4\pi g_s$[1]. Since the mass of the generic gauge excitation in a broken vacuum is of order $m = \Delta r / \alpha'$, we can still detect the existence of a moduli space by focusing on the region very close to the branes, or equivalently by rescaling the distances Δr.

The stack of N D3-branes has an equivalent description in terms of a 3-brane extremal solution in IIB supergravity. This solution contains a constant dilaton, a RR four-form

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[1]In this review we use the conventions (see for example [9]) $\mathcal{L} = -\frac{1}{4g_{YM}^2} F_{\mu\nu}^a F^{a\mu\nu} + \frac{\theta_{YM}}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu} = -\frac{1}{2g_{YM}^2} Tr(F_{\mu\nu} F^{\mu\nu}) + \frac{\theta_{YM}}{32\pi^2} Tr F_{\mu\nu} \tilde{F}^{\mu\nu}$. The complex coupling is $\tau = \frac{\theta_{YM}}{2\pi} + i\frac{4\pi}{g_{YM}^2}$.

potential and a metric given (in the string frame) by

$$
\begin{aligned}
ds^2 &= Z(r)^{-1/2}dx_\mu dx^\mu + Z(r)^{1/2}(dr^2 + r^2 d\Omega_5^2), \\
C_{(4)} &= Z(r)^{-1}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \\
Z(r) &= 1 + \frac{4\pi g_s N \alpha'^2}{r^4}.
\end{aligned} \tag{2}
$$

In this description the decoupling limit can be realized by sending $\alpha' \to 0$ while keeping the parameters of the gauge theory fixed. As we saw, we should also rescale distances to preserve the existence of a moduli space. We then send $\alpha' \to 0$ keeping g_s and $r/\alpha' \equiv U$ fixed. In the limit we have just described, we can discard the 1 in the expression (2) for Z. This is equivalent to focusing on the near-horizon geometry

$$
ds^2 = \alpha' \left\{ R^2 \frac{dU^2}{U^2} + \frac{U^2}{R^2}dx_\mu dx^\mu + R^2 d\Omega_5^2 \right\}. \tag{3}
$$

The metric is the direct product of two spaces of constant curvature, $AdS_5 \times S^5$, with the same radius $R^2 = \sqrt{g_{YM}^2 N}\alpha'$.

This is the observation that led Maldacena [1] to conjecture that four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM in $3 + 1$ dimensions is equivalent to Type IIB string theory on $AdS_5 \times S^5$. This is the content of the so-called AdS/CFT correspondence. The matching of the parameters on the two sides of the correspondence reads

$$
\begin{aligned}
4\pi g_s &= g_{YM}^2 = \frac{x}{N}, \\
\frac{R^2}{\alpha'} &= \sqrt{g_{YM}^2 N} = \sqrt{x},
\end{aligned} \tag{4}
$$

where $x = g_{YM}^2 N$ is the 't Hooft coupling. The string theory is weakly coupled if we first send $N \to \infty$ at fixed x (thus suppressing string loops), and then we take $x \gg 1$ (thus suppressing world-sheet corrections). From eq. (4) we see that the latter condition means that the large-N gauge theory is strongly coupled. We can therefore think in terms of a duality: strongly coupled phenomena in the large N limit of a gauge theory are described by a dual weakly coupled string background. The double perturbative expansion of string theory, in powers of g_s (string loop) and α' (higher derivative terms) is associated respectively with the $1/N$ expansion (at fixed x) and the $1/x$ expansion at each order in N. The old proposal that gauge theories at large N have a dual description in terms of a string theory is explicitly realized.

It is also instructive to compare the symmetries of the two theories. $\mathcal{N} = 4$ SYM is invariant under the conformal group $SO(4,2)$, has $\mathcal{N} = 4$ supersymmetry that is doubled with the addition of the superconformal generators, and a $SO(6)$ R-symmetry. In the dual theory $SO(4,2)$ is the isometry group of AdS_5, the $\mathcal{N} = 8$ supersymmetries are

those of Type IIB supergravity compactified on $AdS_5 \times S^5$, and $SO(6)$ is the isometry group of S^5. In a word, the symmetries on both sides form the superconformal group $SU(2,2|4)$.

1.2 Precise definition of the AdS/CFT correspondence

The AdS/CFT correspondence, in a little more general form than the one introduced in the previous Section, relates a $4d$ CFT to a critical string in $10d$ on $AdS_5 \times H$. If H is compact, the string theory is effectively five-dimensional. The AdS_5 factor guarantees that the dual theory is conformal, since its isometry group $SO(4,2)$ is the same as the group of conformal transformations of a four-dimensional quantum field theory.

To define the correspondence, we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. The correspondence is via holography [2, 3]. Let us start by writing the AdS metric as

$$ds^2 = dy^2 + e^{2y/R}dx_\mu dx^\mu, \tag{5}$$

where the radial coordinate y is related to that in eq. (3) by $y = R \log(U/R)$. We see that the metric has a conformal boundary at $y = \infty$ isomorphic to Minkowski spacetime and this will play an important role in the following. The CFT is specified by a complete set of conformal operators. In a gauge theory at large N, a distinguished role will be played by single-trace operators[2]. The fields in AdS, on the other hand, are the excitations of the string background. They certainly contain the metric and many other fields. We may assume that, when a semi-classical description is applicable, their interaction is described by an effective action $S_{AdS_5}(g_{\mu\nu}, A_\mu, \phi, ...)$. Suppose that we have a map between observables in the two theories. We can formulate a prescription to relate correlation functions in the CFT with scattering amplitudes in AdS_5. In CFT we can define the functional generator $W(h)$ for the connected Green functions for a given operator O. $h(x)$ is a source, depending on 4 coordinates, which is coupled to the operator O through

$$L_{CFT} + \int d^4x h O. \tag{6}$$

O is associated with a scalar field $\hat{h}$ in AdS, which, for simplicity, we assume to be a canonically normalized scalar: $S_{AdS} = \int d^4x dy \sqrt{g}[(\partial\hat{h})^2 - m^2\hat{h}^2 + ...]$. The solution of the equation of motion of $\hat{h}(x,y)$ for large y is

$$\hat{h}(x,y) \rightarrow e^{(4-\Delta)y/R}\hat{h}_\infty(x), \tag{7}$$

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[2]Multiple trace operators are usually associated with multi-particles states in AdS.

where

$$m^2 = \frac{\Delta(\Delta - 4)}{R^2}.$$

$$(8)$$

Since we expect that the large y behavior of $\hat{h}$ reflects the conformal scaling of the field we identify Δ with the dimension of the dual operator O. The prescription for identifying correlation functions with scattering amplitudes is the following: given a solution $\hat{h}$ of the equations of motion derived from S_{AdS} that reduces to $\hat{h}_{\infty}(x) \equiv h(x)$ at the boundary, we claim that [2, 3][3]

$$e^{W(h)} = \left\langle e^{\int hO} \right\rangle = e^{-S_{AdS_5}(\hat{h})}.$$

$$(9)$$

This prescription is valid in the low energy limit where supergravity is valid. In full string theory, the right-hand side of the last equation should be replaced by some S-matrix element for the state $\hat{h}$. Notice that we used equations of motion in AdS: an off-shell theory in four dimensions corresponds to an on-shell theory in $5d$. This is a generic feature of all the AdS-inspired correspondences. The previous prescription allows to compute Green functions for a strongly coupled gauge theory at large N using classical supergravity. In all the computations done up to now, there is an amazing agreement between the CFT and the supergravity predictions, whenever a comparison can be made. This leaves very few doubts about the validity of the AdS/CFT correspondence. For more details on the subject, the reader is referred to [6].

The map between CFT operators and AdS fields should be worked out case by case. For specific operators the dual field can be found using symmetries. For example, the natural couplings

$$L_{CFT} + \int d^4x \sqrt{g}(g_{\mu\nu}T_{\mu\nu} + A_\mu J_\mu + \phi F_{\mu\nu}^2 + \cdots)$$

$$(10)$$

suggest that the operator associated with the graviton is the stress-energy tensor and the operator associated to a gauge fields in AdS is a CFT global current. In general, global symmetries in CFT correspond to gauge symmetries in AdS. In the previous formula, we also included a coupling that is very natural in string theory. Since $g_s \sim g_{YM}^2$, the operator associated to the dilaton is the derivative of the classical Lagrangian with respect to $1/g_{YM}^2$.

We are mainly interested in the limit where string theory is weakly coupled, and reduces to Type IIB supergravity. Since H is compact, the bosonic massless modes in $10d$ can be expanded in a set of Kaluza-Klein (KK) modes on H with masses of order $1/R^2$. We see that all operators with finite dimension for $x \to \infty$ (supergravity limit) should correspond to Kaluza-Klein (KK) modes on $AdS \times H$. We can explicitly describe

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[3]The equations of motion in AdS are second order equations, but the extension of the boundary value inside the space is unique. What we implicitly impose is regularity in the interior of AdS.

the relation for $\mathcal{N} = 4$ SYM, where the large amount of supersymmetry allows for a complete classification. All KK modes on $AdS_5 \times S^5$ were computed in the eighties [10]. They are organized in $\mathcal{N} = 8$ multiplets [11]. The difference in spin in a generic $\mathcal{N} = 8$ multiplet may reach four units. The KK multiplets, with maximum spin 2, should correspond to short (and therefore protected) multiplets. $\mathcal{N} = 8$ short multiplets A_k are labeled by integers $k \geq 2$, and their lowest state is a scalar in the k-fold symmetric representation of $SO(6)$ with mass $m^2 = k(k-4)/R^2$. The KK spectrum contains each A_k for $k \geq 2$ exactly once. The corresponding multiplets on the CFT side are obtained by applying the supersymmetry charges to the operator [3, 12]

$$Tr\phi_{\{i_1} \cdots \phi_{i_k\}} - \text{traces} \tag{11}$$

of dimension k. One can prove that these multiplets are short and therefore have protected dimensions. There is a complete correspondence with the KK spectrum. It is believed that the previously defined CFT multiplets exhaust the (single trace) short multiplets of $\mathcal{N} = 4$ SYM. A special role is played by A_2, which is the supergravity massless multiplet (in five-dimensional sense) containing the graviton and the 15 gauge fields of $SU(4)$. It corresponds to the supermultiplet of currents in the CFT side.

For further reference, we list the lowest fields/operators appearing in the KK spectrum:

SU(4) rep.	operator	multiplet/dim.	mass
20	$Tr\phi_{\{i}\phi_{j\}} - \text{traces}$	$A_2 \ \Delta = 2$	$m^2 = -4$
50	$Tr\phi_{\{i}\phi_j\phi_{k\}} - \text{traces}$	$A_3 \ \Delta = 3$	$m^2 = -3$
10$_\mathbf{c}$	$Tr\lambda_a\lambda_b + \phi^3$	$A_2 \ \Delta = 3$	$m^2 = -3$
105	$Tr\phi_{\{i}\phi_j\phi_k\phi_{p\}} - \text{traces}$	$A_4 \ \Delta = 4$	$m^2 = 0$
45$_\mathbf{c}$	$Tr\lambda_a\lambda_b\phi_i + \phi^4$	$A_4 \ \Delta = 4$	$m^2 = 0$
1$_\mathbf{c}$	$\text{on} - \text{shell Lagrangian}$	$A_2 \ \Delta = 4$	$m^2 = 0$

Some of the masses of AdS fields are negative but this does not represent an instability. Due to the negative curvature, a mode is stable if $m^2 R^2 \geq -4$ (Breitenlohner-Freedman bound [13]). Using formula (8), we see that a scalar field has negative, null or positive mass when it corresponds to a relevant, marginal or irrelevant CFT operator, respectively. In the previous table we listed all the relevant scalar operators appearing in the KK spectrum.

Let us also briefly consider the stringy states. In the supergravity limit, all stringy states are very massive and should decouple. In the CFT these states correspond to operators with large anomalous dimension[4], which, for consistency, decouple from all the OPEs and Green functions. It is then a prediction of the AdS/CFT correspondence that,

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[4]Using formula (8) with $m^2 = \text{integer}/\alpha'$ we predict $\Delta \sim (x)^{1/4}$.

at large N and at large 't Hooft coupling the only single trace $\mathcal{N} = 4$ SYM operators with finite dimensions are the protected ones, which have been classified above. The simplest example of an operator dual to a string state is $Tr(\phi_i\phi_i)$ (missing in the previous classification). Some progress in the understanding of a certain class of stringy states has been made in [14, 15].

We finish with one particularly important comment. Notice that the theory realized on the world-volume of D3-branes in Type IIB is $\mathcal{N} = 4$ SYM with gauge group $U(N)$. It is believed that the $U(1)$ factor is not described by the correspondence. Some evidence for the disappearing of the $U(1)$ factor comes from the analysis of the spectrum of $AdS_5 \times S^5$. Indeed, the KK multiplet A_1, which would be associated with the CFT multiplet with lowest component $\mathrm{Tr}\phi_i$, is not present in the supergravity spectrum. This is a strong evidence that the SYM gauge group is $SU(N)$ and not $U(N)$, as one could naively expect. Another strong evidence comes from the existence of a baryonic vertex (obtained as a wrapped brane in the bulk [16]), which can only exist in a $SU(N)$ theory. The gauge $U(1)$ factor on the D3-brane theory is frozen in the holographic dual, meaning that it reduces to a global symmetry. In this particular case, the $U(1)$ is completely decoupled.

1.3 The correspondence for NS5-branes

A generalization of the AdS/CFT correspondence that we will need in the following deals with 5-branes in Type II. Consider a stack of $N > 1$ coincident Type IIB NS5-branes in flat space-time. The background they generate is

$$
\begin{aligned}
ds^2 &= dx_\mu dx^\mu + Z(r)\left(dr^2 + r^2 d\Omega_3^2\right), \\
e^{2\Phi} &= g_s^2 Z = g_s^2\left(1 + \frac{\alpha' N}{r^2}\right), \\
B_{(6)} &= (Z^{-1} - 1)g_s dx^0 \wedge \cdots \wedge dx^5.
\end{aligned}
\tag{12}
$$

Via S-duality, one can write an analogous solution for a set of D5-branes. Both NS5 and D5-branes are BPS objects, so they preserve $(1,1)$ supersymmetry on the six dimensional world-volume. At very low energies the world volume theory is a $6d$ $(1,1)$ supersymmetric gauge theory with coupling $(m_s^2 = 1/\alpha')$

$$
\frac{1}{g_D^2} \sim \frac{m_s^2}{g_s}, \qquad\qquad \frac{1}{g_{NS}^2} \sim m_s^2.
\tag{13}
$$

This result is easily deduced from the Born-Infeld action for the D5-branes. An S-duality gives then the result for the NS5-branes. Now, in the limit $g_s \to 0$ with m_s fixed the bulk modes which interact with a NS5 brane through the string coupling g_s would decouple. We are thus left with a six dimensional, non gravitational theory with sixteen

supercharges and a mass scale m_s [17]. Since $m_s \neq 0$, the theory still contains strings (they emerge, for example, as instantons of the low energy gauge theory), from which the name "Little String Theory" (LST). It is a non-local theory of strings, exhibiting a form of T-duality. We refer the interested reader to [18] and references therein for a comprehensive review of the subject. At very low energy, the LST reduces to $(1,1)$ SYM in six dimensions. The massless fields of the $6d$ theory are: one gauge vector field, 4 scalars parameterizing the directions transverse to the branes and two symplectic Majorana fermions, all in the adjoint of the gauge group. The scalars and the spinors transform as the $(\mathbf{1}, \mathbf{4})$ and $(\mathbf{4}_+, \mathbf{2}_+) + (\mathbf{4}_-, \mathbf{2}_-)$ of $SO(1,5) \times SO(4)$, respectively, where the isometries of the transverse $\mathbb{R}^4$ become the $SO(4)$ R-symmetry group of the world volume theory.

What is important for the gauge/gravity correspondence is that LST has a holographic dual [19]. Send $r \to 0$ at the same rate as g_s in the background (12). Defining $r = g_s e^\rho$, we find

$$ds^2 = dx_\mu dx^\mu + N\alpha'(d\rho^2 + d\Omega_3^2), \qquad \Phi = -\rho + const. \tag{14}$$

This is the so-called "linear dilaton background". String theory on this space-time has an exact conformal field theory description in terms of six free coordinates, a Liouville field for the radius and an $SU(2)$ WZW model at level N. We will be mostly satisfied with the supergravity approximation. This is reliable in the large N limit, as usual, and far away from the branes, where the string coupling is vanishing as can be seen from (14). As one approaches the branes, the string coupling diverges and one has to go to the S-dual D5-brane description.

A discussion of the observable mapping can be found in [18]. Obviously, the non-locality of the theory makes the mapping difficult. However in the low energy limit, where the LST reduces to SYM, the operators become local and we can still make some natural identifications. In particular we can use the $SO(4)$ global symmetry to classify the operators. On the supergravity side, we will perform a dimensional reduction on the S^3 transverse to the branes, obtaining a tower of KK states. As we will see, the massless multiplet will be described by an $SO(4)$ gauged supergravity in seven dimensions, where we will be able to identify the dual operators. In the KK spectrum, in analogy with $\mathcal{N} = 4$ SYM, we expect to find scalars dual to the operators $\text{Tr} X_{\{i_1}...X_{i_k\}}$-traces, where X_i are the four massless scalars in the (1,1) SYM theory.

1.4   Conformal field theories with $\mathcal{N} < 4$: Orbifolds

In general, in the gauge/gravity correspondence the amount of supersymmetry can be reduced by placing the branes in curved geometries. Since the AdS/CFT correspondence involves the near-brane region and every smooth manifold is locally flat, we may expect

to find new models only when the branes are placed at a singular point of the transverse space [20, 21, 22, 23]. There is no general method for determining the gauge theory living on the world-volume of branes placed at generic singularities. For orbifold singularities, however, such a method exists and we will start reviewing it.

Consider N D3-branes sitting at the singularity of the orbifold $\mathbb{R}^6/\Gamma$, where Γ is a discrete group $\Gamma \subset SO(6) \sim SU(4)$. The supergravity solution in this case reads

$$ds^2 = Z^{-1/2}(r)dx_\mu dx^\mu + Z^{1/2}(r)(dr^2 + r^2 ds^2_{S^5/\Gamma}), \tag{15}$$

where Z was given in eq. (2). We also used the fact that the radial coordinate $r^2 = \sum_{i=1}^{6} x_i^2$ is unaffected by the projection since $\Gamma \subset SO(6)$. We see from eq. (15) that the near-horizon geometry is $AdS_5 \times (S^5/\Gamma)$. The presence of an AdS factor predicts that the field theory on the world-volume of the D3-branes is conformal, at least at large N [20].

The supersymmetries preserved by the orbifold projection are determined by considering the action of Γ on the holonomy group of the transverse space and are summarized as follows:

- Γ subgroup of $SU(2)$: $SU(4) \rightarrow SU(2)_R \times U(1)_R \rightarrow \mathcal{N} = 2$ supersymmetry.

- Γ subgroup of $SU(3)$: $SU(4) \rightarrow U(1)_R \rightarrow \mathcal{N} = 1$ supersymmetry.

- Γ subgroup of $SU(4) \rightarrow \mathcal{N} = 0$ supersymmetry.

In the previous list we also reported the subgroup of $SU(4) \sim SO(6)$ that survives the projection and appears as the R-symmetry of the brane world-volume theory.

Similarly, to determine the gauge theory living on branes on $\mathbb{R}^6/\Gamma$ [24], we have to study the action of the orbifold projection on the world volume fields. For simplicity, we consider only abelian groups $\mathbb{Z}_k$. In the covering space $\mathbb{R}^6$, a D3-brane has $k-1$ images under $\mathbb{Z}_k$. We can think of a collection of k D3-branes as making a physical D3-brane. $\mathbb{Z}_k$ acts on the set of k branes by a cyclic permutation: this is called the regular representation of Γ. Before the projection, a set of kN branes realizes a $U(kN)$ gauge theory. Let each element $\alpha \in \Gamma$ act on the Chan-Paton factors with a matrix γ_α in the regular representation of Γ. The projected theory is then obtained by

$$\begin{aligned} A_\mu &= \gamma_\alpha A_\mu \gamma_\alpha^{-1}, \\ \lambda_a &= R(\alpha)_{ab} \gamma_\alpha \lambda_b \gamma_\alpha^{-1}, \\ \phi_i &= R(\alpha)_{ij} \gamma_\alpha \phi_j \gamma_\alpha^{-1}, \end{aligned} \tag{16}$$

where $i, j = 1, \ldots, 6$ and $a, b = 1, \ldots, 4$. The matrices $R(\alpha)$ take into account that the original $\mathcal{N} = 4$ scalars and fermions transform non trivially under $SO(6) \sim SU(4)$ (in the **6** and the **4**, respectively) and therefore under its subgroup Γ.

As an example, consider an $\mathcal{N} = 2$ theory: the orbifold $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^2$. Representing $\mathbb{R}^6$ with three complex coordinates z_i , the action of $\mathbb{Z}_2$ is given by

$$z_1 \rightarrow -z_1, z_2 \rightarrow -z_2, z_3 \rightarrow z_3. \tag{17}$$

There is only one non trivial matrix γ_α corresponding to the generator of $\mathbb{Z}_2$ and it can be chosen as $\gamma_\alpha = \mathrm{diag}\{I_N, -I_N\}$. A simple application of the previous rules shows that the gauge group is $U(N) \times U(N)$, with adjoint $\mathcal{N} = 2$ vector multiplets and two bi-fundamental hypermultiplets.

The gauge theories obtained as projections have a characteristic quiver (or moose) form. In the $\mathcal{N} = 2$ case, a complete classification exists [24] based on the Mc Kay correspondence [25]. The discrete subgroups of $SU(2)$ are in one-to-one correspondence with the simply-laced Lie algebras A_k, D_k and E_6, E_7, E_8. The gauge theory on N physical branes at a singularity $\mathbb{R}^4/\Gamma \times \mathbb{R}^2$ is associated with the affine Dynkin diagram of the Lie algebra corresponding to Γ. A $U(n_i N)$ vector multiplet is associated with each node with Dynkin label n_i and a bi-fundamental hypermultiplet is associated with each link connecting two different nodes. The $\mathcal{N} = 1$ case is considerably more complicated. We refer to [26] for a detailed discussion. We just notice that, in the $\mathcal{N} = 1$ case, the quiver theory inherits a superpotential from the projection of the $\mathcal{N} = 4$ one.

The Green functions for Γ-invariant operators are also obtained by projection from $\mathcal{N} = 4$ SYM: they are identical to those of the parent theory in the large N limit [20, 27]. Notice, however, that the orbifold may have extra fields and operators that are not invariant under Γ. In string theory, they come from twisted sectors. Their Green functions are obviously not determined by those of $\mathcal{N} = 4$.

Finally, the AdS/CFT correspondence predicts that all the orbifold theories constructed as above are CFT at large N. It is easy to check that the one-loop beta function is zero in all these theories [20, 27] .

1.5   Conformal field theories with $\mathcal{N} < 4$: Conifolds

Another efficient way of obtaining CFT's makes use of conifold singularities. We place branes at the singularity of a six-dimensional Ricci-flat manifold C_6 whose metric has the conical form

$$ds^2_{C_6} = dr^2 + r^2 ds^2_{H_5}. \tag{18}$$

The supergravity solution for N branes is of the form

$$ds^2 = Z^{-1/2}(r)dx_\mu dx^\mu + Z^{1/2}(r)(dr^2 + r^2 ds^2_{H_5}), \tag{19}$$

with Z given in eq. (2). One can prove that C_6 is a Calabi-Yau if H_5 is a five-dimensional Einstein manifold [21, 22, 23]. The AdS/CFT correspondence then applies for the background $AdS_5 \times H_5$, which is the near horizon limit of the previous metric.

Useful and simple Einstein manifolds are the cosets G/K, where G and K are Lie groups. There are only two supersymmetric examples in five dimensions: $S^5 = SO(6)/SO(5)$ with $\mathcal{N} = 8$ supersymmetry, corresponding to $\mathcal{N} = 4$ SYM, and $T^{1,1} = (SU(2) \times SU(2))/U(1)$ with $\mathcal{N} = 2$ supersymmetry. In this Section we discuss the solution corresponding to $AdS_5 \times T^{1,1}$ [22].

The manifold C_6 relevant for this example can be written as a singular quadric in $\mathbb{C}^4$, $\sum_{a=1}^4 w_a^2 = 0$ [28], or equivalently

$$\mathrm{det} W = 0, \quad (W \equiv \sigma^a w^a, \ \sigma = (\sigma^i, \ i\mathbf{1})), \tag{20}$$

σ^i being Pauli matrices. This equation is invariant under $SO(4) \times U(1)_R \sim SU(2) \times SU(2) \times U(1)_R$. The constraint (20) can be solved in terms of complex doublets A_i, B_j ($W_{ij} \sim A_i B_j$) satisfying

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2, \quad A_i \sim e^{i\alpha} A_i, \ B_i \sim e^{-i\alpha} B_i. \tag{21}$$

C_6 is a cone over $T^{1,1}$. The base of the cone is obtained by intersecting C_6 with the sphere $\sum_{a=1}^4 |w_a|^2 = 1$, or, equivalently, by restricting $\sum |A_i|^2 = \sum |B_i|^2 = 1$ in eq. (21). In this way we obtain an equation for $(S^3 \times S^3)/U(1) = (SU(2) \times SU(2))/U(1) = T^{1,1}$. To write a metric on $T^{1,1}$ we can introduce the following basis of one forms

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}}, \quad g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad g^5 = e^5, \tag{22}$$

with

$$e^1 = -\sin\theta_1 d\phi_1, \quad e^2 = d\theta_1, \quad e^3 = \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2,$$
$$e^4 = \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \quad e^5 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \tag{23}$$

An Einstein metric on $T^{1,1}$ is

$$ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^4 (g^i)^2 = \frac{1}{9}(d\psi + \sum_{i=1}^2 \cos\theta_i d\phi_i)^2 + \frac{1}{6}\sum_{i=1}^2 (d\theta_i^2 + \sin^2\theta_i d\phi_i^2). \tag{24}$$

The angular variable ψ ranges from 0 to 4π, while (θ_1, ϕ_1) and (θ_2, ϕ_2) parameterize two S^2's in the standard way. The expression above shows that $T^{1,1}$ is an S^1 bundle over $S^2 \times S^2$. The metric is invariant under $SU(2) \times SU(2) \times U(1)_R$, where the $SU(2)$ factors act on the two S^2 and $U(1)_R$ shifts the angle ψ. By forgetting an $SU(2)$, $T^{1,1}$ can be also written as an S^3 bundle over S^2. It can be proved that such bundle is topologically trivial (see for instance [22]), so that $T^{1,1}$ is isomorphic to $S^3 \times S^2$. In particular, $T^{1,1}$ has non-trivial two and three cycles where we could wrap D-branes. In Sections 3.3 and

5.3 we will need to wrap D5-branes on a two cycle; for the metric (24) a minimal volume S^2 is parameterized by $\theta_1 = \theta_2, \phi_1 = -\phi_2$.

It is difficult, in general, to determine the world-volume theory of branes sitting at singularities different from the orbifold ones. A powerful hint in this direction is provided by the observation that the space transverse to the branes should describe the moduli space of the gauge theory. In our case, equations (21) can be viewed as the D-terms of an $\mathcal{N} = 1$ abelian gauge theory [22] $U(1) \times U(1)$ with two sets of chiral multiplets A_i and B_i with charges $(\mathbf{1}, -\mathbf{1})$ and $(-\mathbf{1}, \mathbf{1})$, respectively. Here the diagonal $U(1)$ factor is decoupled while the other linearly independent combination of the $U(1)$'s acts as in eq. (21). We identify this theory with that living on the world-volume of a brane placed at the conifold singularity. The moduli space of vacua of such abelian $\mathcal{N} = 1$ theory is in fact identical to C_6. When we consider a stack of N parallel D3-branes at the singularity, we have to extrapolate this result to the non-abelian case. We then consider a $U(N) \times U(N)$ theory with two sets of chiral fields A_i, B_i transforming in the representations $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$. We must also add to the theory the superpotential

$$W = h\, \epsilon_{ij}\, \epsilon_{pq} Tr(A_i B_p A_j B_q). \tag{25}$$

Such superpotential respects all the symmetries of the model and is crucial for avoiding a proliferation of geometrically-redundant non-abelian modes [22]. The global symmetry of the CFT is $SU(2) \times SU(2) \times U(1)_R$, which corresponds to the isometry of $T^{1,1}$.

There are various strong checks that the identification is correct. First of all, the theory has to be conformal. Using the results of [22] it can be rigorously proved that this non-abelian gauge theory flows at low energies to an interacting conformal field theory. Indeed, even though the theory depends on various parameters, the couplings $g_{YM,i}$ and h, the conditions for conformal invariance [29] impose a single relation among them [22]. For both groups, the vanishing of the exact NSVZ beta functions [30][5] gives the relation

$$\mu \frac{d}{d\mu} \frac{8\pi^2}{g_{YM,i}^2} \sim 3C_2(G) - \sum T(R_a)(3 - 2\Delta_a) = N(2(\Delta_A + \Delta_B) - 3) = 0, \tag{26}$$

where $\Delta_{A,B}(g_{YM,i}, h)$ are the dimensions of the fields A_i and B_j. These dimensions do not depend on the indices i, j due to the $SU(2) \times SU(2)$ invariance. When (26) is satisfied, the last condition, which requires that the superpotential has scaling dimension three

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[5]In $\mathcal{N} = 1$ gauge theories, if we use a holomorphic scheme, the beta function is completely determined at 1-loop. From this result one can then deduce the following beta function for the 1PI coupling $\mu \frac{d}{d\mu} \frac{8\pi^2}{g_{YM}^2} = f(g_{YM})(3C_2(G) - \sum T(R_a)(3 - 2\Delta_a))$ where C_2 is the second Casimir of the group G, $T(R_a)$ are the dimensions of the representations R_a of the matter fields, and $f(g_{YM})$ is a positive scheme dependent function of the coupling. With a Pauli-Villars regularization $f(g_{YM}) = 1/(1 - Ng_{YM}^2/8\pi^2)$. The knowledge of $f(g_{YM})$ is not necessary when imposing the scheme independent condition $\beta(g_{YM}) = 0$.

[29], is automatically satisfied. We are thus left with a manifold of fixed points, defined implicitly by the requirement that the dimension of the gauge invariant operator $Tr(AB)$ is 3/2. As a further check, the complete KK spectrum of Type IIB compactified on $T^{1,1}$ has been computed [31], finding a complete agreement with CFT expectations. Let us recall that all $U(1)$ factors are not described by the AdS dual. In this case one of them is decoupled while the other reduces to a global baryonic symmetry. The existence in this model of solitonic objects dual to baryons (obtained as D3-branes wrapped on S^3) [32] is the best evidence that the $U(1)$'s are not dynamical.

2 Breaking conformal invariance I

There are various ways to construct string duals of non-conformal gauge theories. Since the conformal group is equivalent on the supergravity side of the correspondence to the isometry group of AdS_5, one can for instance consider small deformations of the AdS background. In this case the background still asymptotes to $AdS_5 \times S^5$ and we can still easily apply the AdS/CFT dictionary. Alternatively, one can consider completely different geometries generated by fractional and wrapped branes in singular spaces. Another possibility, which we will not discuss here, is to consider theories at finite temperature. Historically this was the first example of non-conformal gauge/gravity duals. We refer the reader to [33, 6] for a discussion of the subject. We will describe the systems of fractional and wrapped branes in the next Section, while here we will focus on the deformations of Anti-de Sitter.

Notice that in all such constructions we will be eventually able to perform predictive calculations only in the limit where the supergravity approximation is valid. It is then difficult to study the complete dynamics of "realistic" theories such as pure Yang-Mills in this context. To understand this point, consider a specific example. We can obtain pure YM by adding a mass deformation M to a CFT that possesses a holographic dual, for example $\mathcal{N} = 4$ SYM. The mass parameter induces a dimensionful scale $\Lambda \sim M e^{-1/Ng_{YM}^2}$. The limit where the low energy theory decouples from the CFT is $M \to \infty$, $x = Ng_{YM}^2 \to 0$, with Λ fixed. However, we can trust supergravity in the opposite limit $x \gg 1$. Thus the description of the low energy pure YM theory requires the knowledge of the full string theory. Similar arguments apply to all the non-conformal models constructed so far. The expectation that the spectrum of bound states in any realistic model should contain higher spin glue-balls suggests that more than supergravity is required to describe the pure YM theory. In the previous example, it would be sufficient to re-sum all world-sheet α' corrections in the string background to correctly describe pure YM in the large N limit. World-sheet corrections are, in principle, more tractable than loop corrections. In flat space, for example, all the α' corrections are computable. In the AdS case, the analogous

computation is made difficult by the presence of RR-fields. In this review, we will mostly remain in the supergravity regime. We may take various attitudes towards the solutions we will find. In the previous example, we may consider the supergravity solution as a description of pure YM with a finite cut-off $\Lambda \sim M$. The situation is similar, in spirit, to a lattice computation at strong coupling. In general, in all the models discussed so far, the supergravity solution describes a YM theory with many non-decoupled massive modes. These theories can be considered as cousins of pure YM, and they have often the same qualitative behavior. At present we have many examples of theories that are, in a certain sense, generalization of pure glue theories. They are interesting as exactly solvable toy models. Moreover, it is interesting to investigate which properties of pure YM, that are not consequences of symmetries, are also realized in these generalized models.

2.1 The radius/energy relation

A crucial ingredient in all the models obtained by the AdS/CFT correspondence is the identification of the radial coordinate in the supergravity solution with an energy scale in the dual field theory.

Let us first consider a conformal field theory and its AdS dual. The identification between radius and energy follows from the form (5) of the AdS metric. A dilatation $x_\mu \to \lambda x_\mu$ in the boundary CFT corresponds in AdS to the $SO(4,2)$ isometry

$$x_\mu \to \lambda x_\mu, \qquad y \to y - R \log \lambda. \tag{27}$$

We see that we can roughly identify $e^{y/R}$ with an energy scale μ. The boundary region of AdS ($y \gg 1$) is associated with the UV regime in the CFT, while the horizon region ($y \ll 1$) is associated with the IR. This is more than a formal identification: holographic calculations of Green functions or Wilson loops associated with a specific reference scale μ are dominated by bulk contributions from the region $y = R \log \mu$. Examples and further references can be found in [6].

Obviously, a change of scale in a CFT has little physical meaning. In a non conformal theory, however, the quantum field theory couplings run with the scale. This suggests we can interpret the running couplings in terms of a specific radial dependence of the fields in the supergravity solution. Moreover, we are also lead to interpret solutions interpolating between different backgrounds as an holographic realization of the Renormalization Group (RG) flow between the dual QFTs. This interpretation works very well at the qualitative level and we will see many explicit examples in this review. As in the AdS case, the region with large (small) radius will be associated with the UV (IR) dynamics of the gauge theory. However, the quantitative identification of the radius with the scale can be difficult to find. For non-conformal theories the precise form of the relation depends

on the physical process we use to determine it [34]. The radius/energy relation can be found for instance by considering the warp factor multiplying the flat four-dimensional part of the metric $ds^2 = Z(y)dx_\mu dx^\mu + \ldots$, since $Z(y)$ is a redshift factor connecting the energies of observers at different points in the bulk: $Z(y')^{-1/2}E' = Z(y)^{-1/2}E$. Alternatively, we can compute a Wilson loop in supergravity [6]: the energy of a string stretched between the boundary and a fixed IR reference radius represents in the gauge theory the self-energy of a quark. Finally, one can also extract the radius/energy relation by analyzing the equation of motion of a supergravity mode with fixed four-dimensional momentum. While for conformal theories all the different methods give the same result, this is no longer true for gravity duals of non-conformal theories. In particular, it is known [34] that for the non-conformal six-dimensional theories living on D5-branes, the radius/energy identification can be ambiguous. This will make the extension of the AdS/CFT dictionary to systems with wrapped branes somehow less clear. Also in the relatively well understood case of the Klebanov-Strassler solution, the different prescriptions give different results [35]. We will be more fortunate in the case of $\mathcal{N} = 2$ theories, where supersymmetry and the existence of a moduli space will give a natural method for determining the radius/energy relation.

2.2 Deformations of $\mathcal{N} = 4$ Super Yang-Mills

To break the conformal invariance of $\mathcal{N} = 4$ Super Yang-Mills we introduce a scale in the theory. This can be done either by deforming the action with gauge invariant operators

$$S \rightarrow S + \int d^4x \, h_i \, O_i(x), \tag{28}$$

or by considering the theory for non-zero VEV of some operators $< O_i(x) >$. For energies lower than the deformation scale, the coupling will run and we expect a Renormalization Group flow to the IR. Depending on the deformation the theory could flow to an IR fixed point, or develop a non trivial IR non-conformal dynamic, like confinement. The choice of the deformation also determines the amount of preserved supersymmetry.

Typically one considers relevant or marginal deformations, i.e. operators with classical conformal dimension $\Delta \leq 4$. This is because we want the deformation to affect the IR dynamic of the theory, being negligible in the UV. As we saw in Section 1.2, almost all the mass terms for scalars and fermions have duals in the KK tower and can be described in the supergravity approximation. The only exception is a diagonal mass term for the scalars, $Tr\phi_i\phi_i$, whose dual operator is a genuine string state. A mass term generically breaks all supersymmetries, but we can also easily consider supersymmetric

mass deformations. The classical examples are

$$\delta\mathcal{L} = \int d\theta^2 \left(\sum_{i,j=1}^{3} m_{ij} Tr(\Phi_i \Phi_j) + h.c \right) = \sum_{i,j=1}^{3} \left(m_{ij}\lambda_i\lambda_j + m_{ik}m_{kj}^* \phi_i^\dagger \phi_j \right), \qquad (29)$$

where Φ_i are the three chiral multiplets of $\mathcal{N} = 4$ in $\mathcal{N} = 1$ notation. The IR theory is generically non-conformal: $m_{ij} = \delta_{ij}$ breaks to $\mathcal{N} = 1$ SYM, $m_{11} = m_{22} = m$ and $m_{33} = 0$ gives $\mathcal{N} = 2$ SYM, and finally $m_{11} = m_{22} = m$ and $m_{33} = M$ with $M \leq m$ gives the soft breaking from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. Since we will always consider those theories in regimes where the massive modes are not decoupled, we will denote these theories with a star, for example: $\mathcal{N} = 2^*$. In some case we can get an IR fixed point. It can be shown that the deformation $m_{11} = m_{22} = 0$ and $m_{33} = m$ flows to the conformal Leigh and Strassler fixed point [29].

Similarly, a simple example of spontaneous symmetry breaking by non-zero VEVs is provided by the Coulomb branch of $\mathcal{N} = 4$ SYM, where the operators $Tr\Phi^k$ acquire a VEV.

2.3   The dual supergravity solutions

The construction of the supergravity duals of deformed $\mathcal{N} = 4$ SYM relies on a simple application of the AdS/CFT dictionary [36, 37, 38], using the map between gauge invariant operators and supergravity states, and the radius/energy relation. The idea is to look for IIB solutions with a non trivial radial dependence and interpret them as RG flows in the dual gauge theory. The candidate backgrounds will be of the form

$$\begin{aligned} ds^2 &= F\left(dy^2 + e^{2Y(y)}dx^\mu dx_\mu\right) + Gds_H^2, \\ \varphi &= \varphi(y), \end{aligned} \qquad (30)$$

where H is the internal $5d$ manifold, F, G are generic warp factors and φ is the supergravity field dual to the operator $O(x)$. Notice that the $5d$ space-time part of the metric is not any longer AdS, consistently with the fact that the field theory is not conformal; the ansatz is dictated by the requirement of Poincaré invariance of the dual field theory, which only leaves undetermined a single function, the $5d$ warp factor Y. For large values of y, interpreted as the UV region, the solutions are asymptotic to $AdS_5 \times S^5$ with the field dual to the gauge theory deformation turned on. This translates into boundary conditions for the $4d$ dimensional warp factor Y and the field φ: $Y \to y/R$ and $\varphi(y) \to 0$ for $y \to \infty$. For small values of y, corresponding to the IR region, the geometry of the solution can be completely different. If the dual gauge theory has an IR fixed point, we expect the background to be of the form $AdS_5 \times H_{IR}$, where the AdS factor reflects the

restoration of conformal invariance at the fixed point. Usually the IR AdS_5 has a different cosmological constant (and a different radius R) from the UV one, corresponding to a different number of degrees of freedom in the dual gauge theory. Alternatively, a non conformal gauge theory should correspond to a geometry with a horizon or a singularity. All along the flow the isometries of the internal part of the metric determine the global symmetries of the dual gauge theory.

An important point in the identification of the gauge and gravity sides is the fact that supergravity solutions can represent both deformations of a CFT and different vacua of the same theory [39, 40]. The asymptotic UV behavior of the solutions discriminates between the two options. To this extent it is enough to look at the $5d$ space-time part of the solution. In the asymptotic AdS region, we just need a linearized analysis. The fluctuation $\varphi(y)$ for a minimally coupled scalar field with mass m in the asymptotically AdS background satisfies

$$\varphi'' + \frac{4}{R}\varphi' = m^2\varphi, \tag{31}$$

where primes denote derivatives with respect to y. The previous equation has a solution depending on two arbitrary parameters

$$\varphi(y) = Ae^{-(4-\Delta)\frac{y}{R}} + Be^{-\Delta\frac{y}{R}}, \tag{32}$$

where Δ (see also Section 1.2) is the dimension of the dual operator, $m^2 = \Delta(\Delta-4)/R^2$ [2, 3]. We are interested in the case of relevant operators, where $\Delta \leq 4$. We associate solutions behaving as $e^{-(4-\Delta)\frac{y}{R}}$ with deformations of the $\mathcal{N}=4$ theory with the operator O. On the other hand, solutions asymptotic to $e^{-\Delta\frac{y}{R}}$ (the subset with $A = 0$) are associated with a different vacuum of the UV theory, where the operator O has a non-zero VEV[6] [39, 40].

Solutions of Type IIB equations of motion with the above properties are difficult to find, even at the perturbative level. However for many of the cases at hand, it is sufficient to consider a lower dimensional truncation of the theory, namely $5d$ $\mathcal{N}=8$ gauged supergravity, with gauge group $SO(6)$ [41]. This is the low energy effective theory for the "massless" modes of the compactification of Type IIB on $AdS_5 \times S^5$. It is believed to be a consistent truncation of Type IIB on S^5 in the sense that every solution of the $5d$ theory can be lifted to a consistent $10d$ Type IIB solution. $5d$ $\mathcal{N}=8$ gauged supergravity has 42 scalars, which transform as the $\mathbf{1_c}$, $\mathbf{20}$, and $\mathbf{10_c}$ of $SO(6)$ (the $\mathcal{N}=4$ SYM R-symmetry $SU(4)$). The singlet is associated with the marginal deformation corresponding to a shift in the complex coupling constant of the $\mathcal{N}=4$ theory. The mode in the $\mathbf{20}$ has mass square $m^2 = -4$ and is associated with a symmetric traceless mass term for the scalars $Tr(\phi_i\phi_j)$, $(i,j = 1,...,6)$ with $\Delta = 2$. The $\mathbf{10_c}$ has mass

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[6]We are not careful about subtleties for particular values of Δ [40].

square $m^2 = -3$ and corresponds to the fermion mass term $Tr(\lambda_a \lambda_b)$, $(a, b = 1, ..., 4)$ of dimension 3. Thus the scalar sector of $\mathcal{N} = 8$ gauged supergravity is enough to discuss at least all mass deformations that have a supergravity description.

The five-dimensional Lagrangian for the scalars of $\mathcal{N} = 8$ gauged supergravity [42]

$$\mathcal{L} = \sqrt{-g} \left[-\frac{\mathcal{R}}{4} - \frac{1}{24} Tr(U^{-1} \partial U)^2 + V(U) \right] \tag{33}$$

is written in terms of a 27×27 matrix U, transforming in the fundamental representation of E_6 and parameterizing the coset $E_6/USp(8)$. In a unitary gauge, U can be written as $U = e^X, X = \sum_A \varphi_A T_A$, where T_A are the generators of E_6 that do not belong to $USp(8)$. This matrix has exactly 42 real independent parameters, which are the scalars of the supergravity theory. Typically the solutions we are looking for only involve a small subset of the 42 scalars, those dual to the gauge theory deformation. Thus by a suitable truncation and parameterization of the coset element U, eq. (33) can be reduced to the Lagrangian for some scalars minimally coupled to gravity. The non trivial scalar potential V is typical of gauged supergravities and has only isolated minima (apart from one flat direction, corresponding to the dilaton). There is a central critical point with $SO(6)$ symmetry and with all the scalars φ vanishing: it corresponds to the unperturbed $\mathcal{N} = 4$ SYM theory. Non-zero VEVs of some of the scalars characterize minima where part of the gauge group is spontaneously broken. Those other minima should correspond to IR conformal field theories.

With a metric of the form $ds^2 = dy^2 + e^{2Y(y)} dx_\mu dx^\mu$, a standard computation shows that the Einstein and scalar equations of motion following from eq. (33) can be deduced from the effective Lagrangian

$$\mathcal{L} = e^{4Y} \left[3 \left(\frac{dY}{dy} \right)^2 - \frac{1}{2} G_{ab} \frac{d\varphi^i}{dy} \frac{d\varphi^j}{dy} - V(\varphi) \right], \tag{34}$$

supported by the zero energy constraint $3(Y')^2 - \frac{1}{2} G_{ab} (\varphi^i)'(\varphi^j)' + V(\varphi) = 0$. The independent equations of motion and constraints read

$$\frac{d}{dy} \left(G_{ij} \frac{d\varphi^j}{dy} \right) + 4 G_{ij} \frac{dY}{dy} \frac{d\varphi^j}{dy} = \frac{\partial V}{\partial \varphi^i},$$

$$6 \left(\frac{dY}{dy} \right)^2 = G_{ij} \frac{d\varphi^i}{dy} \frac{d\varphi^j}{dy} - 2V. \tag{35}$$

Thus the problem of finding interpolating solutions of IIB supergravity reduces to finding solutions of the above equations that for large values of y tend to the maximally symmetric vacuum (on the gauge theory side the UV theory is $\mathcal{N} = 4$ SYM). However, the presence of the potential V, which generally is an exponential in the scalar fields, makes such

solutions not very easy to find. For most of the flows interpolating between two fixed points, the best one can do is to prove that such solutions exist. Things are simpler when some supersymmetry is preserved. In these cases, one can look for solutions for which the fermionic shifts vanish, thus reducing the second order equations to first order ones.

In refs. [38, 43] the conditions for a supersymmetric flow were found[7]. For a supersymmetric solution, the potential V can be written in terms of a superpotential W as

$$V = \frac{1}{8} G^{ij} \frac{\partial W}{\partial \varphi^i} \frac{\partial W}{\partial \varphi^j} - \frac{1}{3} |W|^2. \tag{36}$$

The equations of motion reduce to

$$\begin{aligned} \frac{d\varphi^i}{dy} &= \frac{1}{2} G^{ij} \frac{\partial W}{\partial \varphi^j}, \\ \frac{dY}{dy} &= -\frac{1}{3} W. \end{aligned} \tag{37}$$

It is easy to check that a solution of eqs. (37) satisfies also the second order equations (35). Supersymmetry also helps in unambiguously identifying the UV behavior of the solutions. Close to the boundary, we can always find a basis where the scalar fields are canonically normalized and the superpotential W has the expansion (this is actually possible around any minimum of the potential)

$$W = -\frac{3}{R} + \frac{1}{2} \left. \frac{\partial^2 W}{\partial \varphi^i \partial \varphi^j} \right|_{\varphi_i = 0} \varphi^i \varphi^j + \dots \tag{38}$$

Notice that the value of the superpotential for zero VEV is related to the cosmological constant of AdS_5. From the mass matrix $W_{ij} = \frac{\partial^2 W}{\partial \varphi^i \partial \varphi^j}$ we can read the UV asymptotics of the fields dual to the gauge theory deformations. More precisely, for a diagonal $W_{ij} = \frac{2w_i}{R} \delta_{ij}$[8], we have

$$w_i = \begin{cases} -\Delta & \Rightarrow \text{ VEV}, \\ \Delta - 4 & \Rightarrow \text{ deformation}. \end{cases} \tag{39}$$

The last step in the construction of the supergravity solutions is the lift to ten dimensions. This is necessary for a correct holographic interpretation of the flows, since the $5d$ solutions encode in a very complicated way the gauge theory information. A typical example is the identification of the gauge coupling constant with the dilaton, which is correct only in $10d$, since the $5d$ dilaton is always constant in these solutions. The knowledge of the ten-dimensional solutions is also needed to address another common problem

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[7]Analogous BPS domain wall solutions were originally found for four-dimensional supergravity (see [44]).

[8]For the non diagonal case the same reasoning applies after diagonalization of the mass matrix W_{ij}.

of such supergravity solutions, namely the presence of a naked IR singularity. Usually the $10d$ geometries are still singular but the singularities are milder and may have a physical interpretation as distributions of D-branes or other extended objects [45, 46, 47, 48].

The procedure for the lift to ten dimension is known in principle [49, 50]. The $10d$ metric is expected to be

$$ds^2 = \Omega^{-2/3}ds_5^2 + ds_{\tilde{S}^5}^2, \tag{40}$$

where ds_5^2 and $ds_{\tilde{S}^5}^2$ are the metric of the $5d$ solution and the deformed five-sphere, respectively. The warp factor Ω is a function of the deformed five-sphere metric and it is usually responsible for the mildening of the IR singularity. The ansatz for the dilaton and the metric of the deformed $\tilde{S}^5$ are given in full generality in terms of the scalar coset element U [49, 50]. Then the only difficulty relies in the explicit computation which can be quite awkward depending on the scalars involved in the solution. More complicated is the ansatz for the RR forms, which has to be guessed for every solution on the basis of the symmetries of the problem [49, 50].

We end this Section with a short list of the known five-dimensional solutions and their lifts. For deformations flowing to IR fixed points, the following CFT theories can be obtained:

- Three $\mathcal{N} = 0$ theories with symmetry $SU(3) \times U(1)$, $SO(5)$ and $SU(2) \times U(1)^2$ [51, 36, 37]. All these theories are unstable and correspond to non-unitary CFTs.

- A stable $\mathcal{N} = 1$ theory with symmetry $SU(2) \times U(1)$. It corresponds to the $\mathcal{N} = 4$ theory deformed with a mass for one of the three $\mathcal{N} = 1$ chiral superfields. The results and the supergravity description [52, 38] are almost identical to the $T^{1,1}$ case discussed in Section 1.5, which is a sort of $\mathbb{Z}_2$ projection of this example. The $10d$ lift can be found in [49].

The solutions dual to non-conformal gauge theories are:

- An $\mathcal{N} = 1$ solution with residual symmetry $SU(3)$ [53]. It is dual to the flow from $\mathcal{N} = 4$ to $\mathcal{N} = 1$, after soft breaking with a mass term for the chiral multiplets. It has mass gap and gaugino condensates, and is one of the few solutions known analytically. The $10d$ solution has still a mild singularity [50].

- Solutions corresponding to the Coulomb branch of $\mathcal{N} = 4$ or $\mathcal{N} = 2$ theories. The solutions for the Coulomb branch of $\mathcal{N} = 4$ SYM [45] have various residual symmetries. The $10d$ lifts correspond to distributions of branes. The family of $\mathcal{N} = 2$ solutions [54, 49, 55] has residual symmetry $SU(2) \times U(1)$ and corresponds to points on the moduli space of $\mathcal{N} = 4$ broken to $\mathcal{N} = 2$ by a mass term for two chiral multiplets. In this case the lift is completely known [49] and presents an enhançon type of singularity [56, 46, 47]. It will be discussed in Section 4.

- Solutions describing other patterns of supersymmetry breaking, including a subsequent breaking $\mathcal{N} = 4 \to \mathcal{N} = 2 \to \mathcal{N} = 1$, by giving equal masses to two chiral multiplets and a smaller one to the third [57], and examples of $\mathcal{N} = 0$ solutions [58].

In [48] another $\mathcal{N} = 1$ solution has been constructed directly in $10d$ using configurations of polarized D3-branes. We will give a brief description of such a solution in Section 5.4.

2.4   Holographic RG flow and the c-function

The identification of the radial coordinate of AdS_5 with the energy of the dual gauge theory motivates the interpretation of the supergravity solutions as Renormalization Group flows. The radial profile of the scalars can be associated with the running of the coupling constants in the gauge theory. However, as already mentioned, the precise identification of the couplings in the five-dimensional solutions is often ambiguous and only in the ten-dimensional solution the dictionary can be reliably applied. Here we want to stress that in spite of the above limits, it is possible to extract interesting results already in the five-dimensional approach. One general result about these classes of solutions is the existence of a c-theorem. For the class of field theories that have a supergravity dual one can define a c-function. In a CFT, the central charge c is defined via the OPE of two stress-energy tensors. On the supergravity side, it corresponds to the cosmological constant at the critical points of the potential [59, 60]. In fact, from eq. (33), we can see by a simple scaling that, at a fixed point,

$$\langle T(x)T(0)\rangle = \frac{c}{|x|^8} \;\; \Rightarrow c \sim R^3 \sim (\Lambda)^{-3/2}. \tag{41}$$

More interestingly, all along the flow it is possible to define a c-function that is monotonically decreasing [36, 38] $c(y) \sim (Y')^{-3}$ and reduces to the previous result at the fixed points. The monotonicity of c can be easily checked from the equations of motion (35) and the boundary conditions of the flow [36]. It can be also related to the weak positive energy condition [38] that is expected to hold in all physically relevant supergravity solutions. Let us stress that the value of c is well defined only at a fixed point, where it represents a central charge. In QFT, the value of c along the flow is scheme dependent. Similarly, in supergravity there are several possible definitions of monotonic functions interpolating between the central charges at the fixed points [36, 38, 61, 62].

To strengthen the holographic RG flow interpretation, some attempts to identify more precisely the five-dimensional equations of motion in supergravity with the renormalization group equations have been made in [63, 61]. Also, correlation functions along some of the supersymmetric flows have been explicitly computed using the AdS/CFT prescription [64].

3 Breaking conformal invariance II

In this Section we will discuss wrapped and fractional branes. We will only consider the engineering of the gauge theory in terms of systems of branes. The holographic duals will be discussed in the next Sections. Here and in Section 4 we will mainly work in units $2\pi\alpha' = 1$.

3.1 Fractional and wrapped branes: General observations

Most of the recently proposed duals of non-conformal theories are based on wrapped and fractional branes. The philosophy may be exemplified in the $4d$ case as follows. Consider a geometry with a non-trivial two-cycle S^2 on which we wrap a D5-brane. The world-volume of the brane is thus of the form $\mathbb{R}^4 \times S^2$, and at energies lower than the inverse radius of S^2 the theory living on the world-volume is effectively four dimensional. String theory has many moduli, some geometrical in nature and some related to the bundles of antisymmetric forms which are always present in string theory. For simplicity, we focus on two specific moduli associated with S^2: the volume of S^2 and the integral of the B-field over the cycle. Only the first modulus has a geometrical meaning. These moduli appear in the Born-Infeld action for the D-brane[9]

$$-\frac{1}{(2\pi)^2} \int dx^6 \, e^{-\Phi} \sqrt{G + F + B} =$$

$$= -\frac{1}{(2\pi)^2} \int dx^4 \left[e^{-\Phi} \int_{S^2} d\Omega_2 \sqrt{(G+B)_{S^2}} \right] \sqrt{(G + F + B)_{\mathbb{R}^4}}. \tag{42}$$

We see, by expanding the last square root, that the four dimensional gauge theory has an effective coupling which reads

$$\frac{1}{g^2} \sim e^{-\Phi} \int_{S^2} d\Omega_2 \sqrt{(G+B)_{S^2}}. \tag{43}$$

Whenever the quantity on the r.h.s. of this equation runs, also the coupling does, and the resulting theory is non-conformal. We can then have two basic different models:

- Wrapped branes: configurations of D5-branes wrapped in a supersymmetric fashion on a non-vanishing two-cycle $Vol(S^2) \neq 0$. There is no need to introduce a B-field.

- Fractional branes: configurations of D5-branes wrapped on collapsed cycles. If $\int_{S^2} B \neq 0$, the corresponding four-dimensional theory has still a non-vanishing well-defined coupling constant. Manifolds with collapsed cycles are singular, and

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[9]Our conventions for the BI action for a Dp-brane are: $\frac{-1}{\alpha'^{(p+1)/2}(2\pi)^p} \int dx^{p+1} e^{-\Phi} \sqrt{G + (2\pi\alpha'F + B)}$, $G + (2\pi\alpha'F + B) \equiv -det(G_{ab} + (2\pi\alpha'F_{ab} + B_{ab}))$.

fractional branes must live at the singularity. We discussed some examples of singular manifolds in Section 1.

The amount of supersymmetry preserved in these kinds of model depends on how the S^2 is embedded in the background geometry.

3.2   Wrapped branes

Wrapping a brane on a generic cycle breaks supersymmetry. It turns out that the conditions for a cycle to be supersymmetric are equivalent to the partial twist of the brane theory [65]. To understand what this means consider the case of N IIB D5-branes wrapped on a two-sphere. Using S-duality, we can equivalently think in terms of NS5-branes. A $4d$ supersymmetry is preserved if and only if there exists a covariantly constant spinor on the sphere

$$(\partial_\mu + \omega_\mu)\epsilon = 0, \tag{44}$$

where ω_μ is the spin connection. It is well known that the sphere admit no covariantly constant spinors. However, the theory contains other fields: for example, the external gauge fields A_μ, which couple to the $SO(4)$ R-symmetry currents. One can then redefine the covariant derivative as to include a gauge connection A_μ in a $U(1)$ subgroup of the R-symmetry group

$$D_\mu = \partial_\mu + \omega_\mu + A_\mu. \tag{45}$$

This operation is called a *twist* of the original theory. The new theory obtained this way can be shown to be topological. In our case the twist is made only in the directions tangent to the sphere, so that the remaining flat four dimensions will support an ordinary field theory.

We can preserve supersymmetry by taking the gauge connection to be opposite to the spin connection [66], so that

$$(\partial_\mu + \omega_\mu + A_\mu)\epsilon = \partial_\mu\epsilon \tag{46}$$

admits now solutions, the constant spinors. The number of surviving supersymmetries depends on the way the $U(1)$ gauge connection is embedded in $SO(4)$. The $6d$ theory on the 5-brane world-volume has $\mathcal{N} = (1, 1)$ supersymmetry generated by two symplectic Majorana fermions, η_+ and η_-, transforming as $(\mathbf{4}_+, \mathbf{2}_+)$ and $(\mathbf{4}_-, \mathbf{2}_-)$ of the unbroken $SO(1,5) \times SO(4)$ subgroup of $SO(1,9)$. Wrapping the NS5-branes on S^2 further breaks the isometries of the world-volume as $SO(1,5) \to SO(1,3) \times SO(2)$. Then, imposing the chirality and symplectic conditions, one finds that each $6d$ supersymmetry generator contributes two Weyl fermions in four dimensions with the following $SO(2) \times U(1)^+ \times$

$U(1)^-$ charges

$$\begin{aligned}
\eta_+ &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{0})_+ + (\mathbf{1}, -\mathbf{1}, \mathbf{0})_+ \equiv p + q, \\
\eta_- &\rightarrow (\mathbf{1}, \mathbf{0}, \mathbf{1})_- + (\mathbf{1}, \mathbf{0}, -\mathbf{1})_- \equiv \tilde{p} + \tilde{q},
\end{aligned} \tag{47}$$

where the subscripts $\pm$ indicate the $4d$ chirality, $SO(2)$ is the connection on S^2 and $U(1)^+$, $U(1)^-$ are the abelian factors in $SO(4) \sim SU(2)^+ \times SU(2)^-$. Thus, if we want to preserve 8 supercharges we have to identify the gauge connection with (the opposite of) the diagonal of the two abelian subgroups $U(1)_D = \frac{1}{2}(U(1)^+ + U(1)^-)$. Similarly, $\mathcal{N} = 1$ supersymmetry is obtained with the choice of the gauge connection in (the opposite of) $U(1)^+$.

In all our examples, the sphere will be a non-trivial two-cycle in a Calabi-Yau. The R-symmetry group is the structure group of the bundle normal to the branes. Thus, the twist condition is the requirement that the tangent space group of the two-cycle is identified with a $U(1)$ subgroup of the structure group of the normal bundle. If the six dimensional manifold is a (non-compact) CY_3 the gauge theory will be $\mathcal{N} = 1$ and if it is a non-compact version of K3$\times\mathbb{R}^2$ (an ALE space times $\mathbb{R}^2$) the gauge theory will be $\mathcal{N} = 2$; in all other cases no supersymmetry survives.

The $SO(4)$ subgroup left unbroken by the twist provides the R-symmetries of the $4d$ theory. For $\mathcal{N} = 2$ these are $SU(2)_R \times U(1)_R^{\mathcal{N}=2}$, where $U(1)_R^{\mathcal{N}=2}$ corresponds to the untwisted $U(1) = \frac{1}{2}(U(1)^+ - U(1)^-)$ and the action of the abelian subgroup $U(1)_J \subset SU(2)_R$ on the massless modes can be identified with both the $SO(2)$ spin connection and $U(1)_D$. In the $\mathcal{N} = 1$ case the R-symmetry $U(1)_R^{\mathcal{N}=1}$ is the twisted one, $U(1)^+$.

The twist also determines the field content of the $4d$ theory. In particular, the massless states consist of the zero modes of the compactification on the two-sphere. The two fermions of $6d$ SYM have the same decomposition as the SUSY generators, thus giving four Weyl fermions with the following charge assignments

	$p = \lambda$	$\tilde{p} = \bar{\psi}$	q	$\tilde{q}$
$U(1)_R^{\mathcal{N}=2} = \frac{1}{2}(U(1)^+ - U(1)^-)$	1	-1	-1	1
$U(1)_D = \frac{1}{2}(U(1)^+ + U(1)^-)$	1	1	-1	-1
$U(1)_R^{\mathcal{N}=1} = U(1)^+$	1	0	-1	0

Table 1: Charge assignment of the spinors.

The spinors p, q ($\tilde{p}$, $\tilde{q}$) have positive (negative) chirality. Thus the $\mathcal{N} = 2$ twist

gives mass to the q spinors, while the other two have the right quantum numbers to be identified with the two spinors of $4d$ $\mathcal{N} = 2$ SYM: $p \sim \lambda$ and $\tilde{p} \sim \bar{\psi}$, where λ is the gaugino. On the contrary $p \sim \lambda$ is the only massless spinor in the $\mathcal{N} = 1$ case. The $6d$ theory on the brane also contains four scalars, X_i, $i = 1, \ldots, 4$, transforming as $(\mathbf{2}, \mathbf{2})$ under $SU(2)^+ \times SU(2)^-$. Only those neutral under the twisted gauge field give zero modes. Since all the scalars are charged under $U(1)^+$, none of them survives the $\mathcal{N} = 1$ twist. On the contrary, in the $\mathcal{N} = 2$ case there are two neutral scalars. These are the two scalars, say X_3 and X_4, parameterizing the motion of the brane in the two flat directions transverse to the ALE space. They combine to give the complex scalar of $\mathcal{N} = 2$ SYM (with charges 0 and 2 under $U(1)_J \times U(1)_R^{\mathcal{N}=2}$). Finally, in both cases the gauge field has no zero modes on the two sphere. In summary, the massless states of the $\mathcal{N} = 2$ twisted theory form a four dimensional $\mathcal{N} = 2$ vector multiplet, namely a gauge vector, two Weyl fermions and a complex scalar. Similarly, the vector and the Weyl spinor surviving the $\mathcal{N} = 1$ twist form an $\mathcal{N} = 1$ vector multiplet.

It is important to notice that in the $\mathcal{N} = 2$ case the associated $U(N)$ gauge theory has a moduli space of vacua, since the adjoint scalar fields can acquire a VEV. The moduli space is labeled by the N Cartan values of the scalars and it is represented in the string construction by the possibility of placing the branes in arbitrary positions in the two flat directions.

In this paper we will only discuss geometries with a single 2-cycle, which give rise to gauge theories with a single gauge factor $U(N)$. More complicated models can be realized by considering geometries with several 2-cycles[10].

The holographic duals of $\mathcal{N} = 2$ models with wrapped branes are discussed in Section 4 and those of $\mathcal{N} = 1$ models in Section 5. As mentioned in Section 1.3, the natural setting to study the systems with a single set of NS5-branes is seven dimensional gauged supergravity, which is a consistent truncation of the ten dimensional $\mathcal{N} = 1$ sector of Type IIB supergravity. As usual, all the $U(1)$ factors are not described by the holographic duals.

3.3 Fractional branes

Fractional branes exist both at orbifold and conifold singularities. Let us consider the orbifold case first [67, 68, 69]. In Section 1.4 we have seen that projecting the $\mathcal{N} = 4$ theory with the regular representation of the orbifold discrete group Γ on the Chan-Paton factors gives a conformal theory. We can also use a representation that is not

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[10]Considering geometries with multiple cycles, we obtain gauge theories with gauge factors associated with the cycles and bi-fundamental fields associated with all pairs of intersecting cycles. The $\mathcal{N} = 2$ theories we can construct in this way are then very similar to the ones obtained by placing fractional branes at $\mathcal{N} = 2$ orbifolds.

the regular one. In this way we can obtain non-conformal theories. When taking a non-regular representation of the orbifold group, we obtain fractional branes in Type II. As example we consider again Type IIB string theory on $\mathbb{R}^4/\mathbb{Z}_2$. Choose the coordinates (x_6, x_7, x_8, x_9) for $\mathbb{R}^4$. String theory on $\mathbb{R}^4/\mathbb{Z}_2$ can be defined with an orbifold construction and possesses a twisted sector localized at $x_i = 0, i = 6, ..., 9$. The massless fields in the twisted sector form a tensor multiplet of $(2, 0)$ $6d$ supersymmetry, containing 5 scalars χ_I. $\mathbb{R}^4/\mathbb{Z}_2$ is thus the singular point of a family of regular backgrounds parameterized by the VEVs of the five scalar moduli. Three χ_I, let's say $I = 1, 2, 3$, correspond to the geometrical moduli of a two-sphere replacing the singular point. The geometry of the background with $\chi_I \neq 0, I = 1, 2, 3$ is that of an ALE space. Since fractional branes are associated with singular geometries, in this Section we are particularly interested in the other two scalars $(b, c) \equiv (\chi_4, \chi_5)$. They correspond to the flux of the NSNS and RR 2-form along the 2-cycle: $2\pi b = \int_{S_2} B_{(2)}$, $2\pi c = \int_{S_2} C_{(2)}$. One can show that these fields are periodic (in our conventions $b, c \in [0, 1)$). b and c are non vanishing and well-defined even for singular geometries where the 2-cycle should be thought of as hidden in the orbifold singularity[11].

Add now D3-branes with world-volume (0123) at the point $x_6 = x_7 = x_8 = x_9 = 0$. There are two different RR 4-forms in the orbifold theory. One is the untwisted RR form $C_{(4)}$ and the second one, $C_{(4)}^T$, comes from the twisted sector. $C_{(4)}^T$ is a six-dimensional field localized at the fixed point and it can be dualized to give a scalar field which, as one can show [24], we can identify with c. Consequently, there are two basic types of D3-branes in this theory, which we call fractional and anti-fractional D3-branes[12]. Fractional branes have charges $(b, 1/2)$ with respect to the RR forms $C_{(4)}$ and $C_{(4)}^T \sim c$, respectively; anti-fractional branes have charges $(1 - b, -1/2)$. With a fractional and an anti-fractional D3-brane we can make a physical D3-brane, whose charge is $(1, 0)$. There are several complementary descriptions for fractional branes:

(i)   Consider the perturbative construction of the orbifold. Each brane at $x_i^{(0)}, i = 6, 7, 8, 9$ has an image in $-x_i^{(0)}$. A brane and its image make up a physical brane, which can be moved to an arbitrary point in $\mathbb{R}^4/\mathbb{Z}_2$. For $x_i^{(0)} = 0$, a physical brane appears as a composite object and can be split in the plane (x_4, x_5). The constituents of a physical brane are the two types of fractional branes. It is clear that they can only live at the singular point. The $\mathbb{Z}_2$ action on the Chan-Paton factors

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[11]It is known that the standard orbifold construction of string theory is perturbative in nature and corresponds to a regular world-sheet CFT; the non-zero value of the B-field flux is $b = 1/2$ [70]. String theory develops a singularity and becomes really non-perturbative only when all the $\chi_I = 0$, modulo periodicities. At these points we expect non-perturbative phases of the theory with tensionless strings [71].

[12]The two types of D3-branes are mutually BPS. We use the name anti-fractional with an abuse of language, following the interpretation as wrapped D5-branes.

on n_1 fractional and n_2 anti-fractional branes can be represented with the matrix $\gamma_\alpha = \mathrm{diag}\{I_{n_1}, -I_{n_2}\}$. That charges and tensions of these objects agree with the mentioned value follows from a direct computation in the orbifold construction [24] or in the boundary state formalism [72].

(ii) We can make contact with the discussion in Section 3.1 using the following observation. A fractional brane can be represented as a D5-brane wrapped on the collapsed two-cycle of $\mathbb{R}^4/\mathbb{Z}_2$ [73, 74]. This object appears as a 3-brane and, as we will see shortly, it carries D3-charge. Similarly an anti-fractional brane is an anti-D5-brane with one unit of flux for the gauge field living on it: $\int_{S_2} F = -2\pi$ [73, 74, 72]. This representation is particularly useful when $b \neq 1/2$ and the perturbative description of the orbifold is not adequate. In this representation, $C^T_{(4)}$ is the reduction of $C_{(6)}$ on the two-cycle and the corresponding charge is just the D5-charge. D3-charges and tensions can be read from the action for a D5- or an anti-D5-brane

$$-\frac{1}{(2\pi)^2}\left[\int dx^6 e^{-\Phi}\sqrt{G+F+B}\right. \tag{48}$$
$$\left.\pm \int \left(C_{(6)} + C_{(4)} \wedge (F+B) + \frac{1}{2}C_{(2)} \wedge (F+B)^2 + \frac{1}{6}C_{(0)} \wedge (F+B)^3\right)\right].$$

The induced D3-charges are b and $(1-b)$, while the tensions are proportional to $|b|$ and $|1-b|$. For $b \in [0,1)$, these values satisfy the BPS condition.

(iii) For readers familiar with the Hanany-Witten construction [75, 76], we mention that the same system is T-dual to a set of D4-branes stretched between NS5-branes in Type IIA. The D4-branes have world-volume in the space-time directions $(0,1,2,3,6)$. The direction x_6 is compactified on a circle of radius L. The two NS5-branes have world-volume $(0,1,2,3,4,5)$ and sit at $x_6 = 0$ and $x_6 = 2\pi bL$ respectively, with $x_7 = x_8 = x_9 = 0$. The fractional branes can be identified with the D4-branes stretched from the first to the second NS5-brane, the anti-fractional branes with the D4-branes stretched from the second to the first. A fractional and an anti-fractional brane can join and give a physical D4-brane, which can move away in (x_6, x_7, x_8, x_9).

Applying the rules discussed in Section 1.4 to the representation $\gamma_\alpha = \mathrm{diag}\{I_{n_1}, -I_{n_2}\}$ shows that the gauge theory corresponding to n_1 fractional and n_2 anti-fractional branes is $U(n_1) \times U(n_2)$ with two bi-fundamental hypermultiplets. It is instructive to identify the field theory R-symmetry $SU(2)_R \times U(1)_R$ in terms of the symmetries of the string construction. $SU(2)_R$ is identified with the subgroup of the $SO(4)$ rotating the coordinates $(6,7,8,9)$ that is left unbroken by the orbifold projection. $U(1)_R$ is instead identified with the rotations in the plane $(4,5)$. The theory has a moduli space of vacua
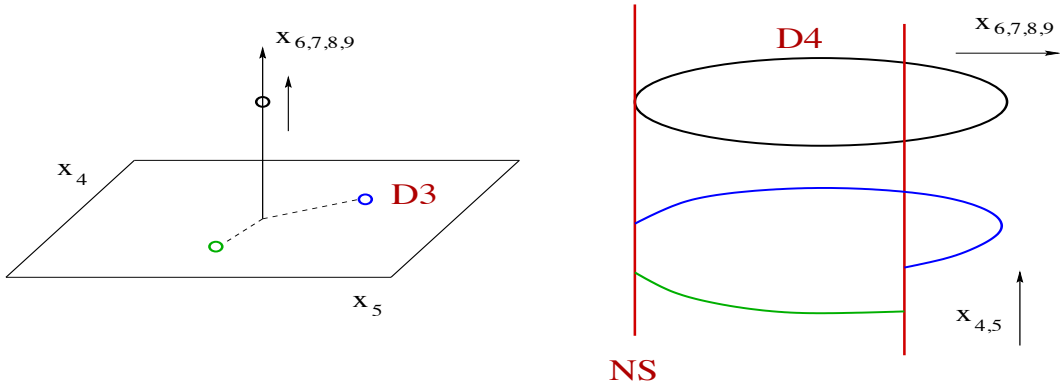
Figure 1: Type IIB and IIA picture for physical and fractional branes.

which consists in a Higgs and a Coulomb branch. In this review, we will only consider the Coulomb branch, which is labeled by the Cartan values of the adjoint scalar fields in the vector multiplets, consisting in $n_1 + n_2$ complex VEVs. These moduli have an obvious interpretation as the positions of the fractional branes in the plane $(4,5)$. This is consistent with the fact that the scalars in the vector multiplets are rotated by $U(1)_R$. For completeness, we notice that the Higgs branch is instead parameterized by the VEVs of the hypermultiplet scalars, rotated by $SU(2)_R$, and it corresponds to the motion of physical branes in the directions $(6,7,8,9)$.

The gauge couplings of the two groups, τ_1, τ_2, are determined (for $b \in [0,1)$) in terms of the space-time fields by equation (49)

$$\tau_1 = (b\tau + c), \qquad \tau_2 = (1-b)\tau - c, \qquad (49)$$

where $\tau = C_{(0)} + ie^{-\Phi}$ is the complex dilaton of Type IIB. As we have already discussed, the case $n_1 = n_2$ corresponds to a conformal field theory. The complex coupling constants of the two groups are exactly marginal parameters and the theory has an AdS dual: $AdS_5 \times S^5/\mathbb{Z}_2$ [20]. When $n_1 = N + M$ and $n_2 = N$, the theory is no longer conformal and the coupling constants run at all scales. One of the two gauge factors is not asymptotically free and it is ill-defined in the UV. We can nevertheless make sense of these theories by finding an $\mathcal{N} = 2$ UV completion that is a CFT. For example, the $\mathcal{N} = 2$ theory $U(n_1) \times U(n_2)$ can be considered as the low energy limit of a broken phase of the CFT $U(N) \times U(N)$, $N > \max(n_1, n_2)$ where some scalar fields developed a vacuum expectation value. The case of a pure $SU(M)$ $\mathcal{N} = 2$ gauge theory can be realized by setting $n_1 = M$, $n_2 = 0$.

In general, in orbifold theories there are as many types of fractional branes as there are nodes of the quiver diagram. We can therefore construct non-conformal $\mathcal{N} = 2$ gauge theories which are products of groups with bi-fundamental hypermultiplets. At least one

gauge group is not asymptotically free in this construction, but the theory can be safely embedded in a UV $\mathcal{N} = 2\ CFT$.

Fractional branes exist in all backgrounds with collapsed 2-cycles. In particular, we may define fractional branes in the conifold geometry defined in Section 1.5 [67, 77, 4]. $T^{1,1}$ has the topology of $S^2 \times S^3$ and at the tip of the cone over $T^{1,1}$ both the 2-cycle and the 3-cycle are vanishing. In view of our previous discussion we are mostly interested in the 2-cycle where we can wrap a D5-brane of Type IIB. While the description (i) for fractional branes given above in the orbifold case is no more applicable, the description (ii) in terms of D5-branes wrapped on 2-cycles can be repeated almost verbatim. We obtain an $\mathcal{N} = 1$ gauge theory of the form $U(n_1) \times U(n_2)$ with bi-fundamental chiral fields and a superpotential inherited from the conformal case. The theory has a Higgs branch where the bi-fundamental fields acquire VEVs. The coupling constants of the two groups can be determined using eq. (43) and read

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} = \frac{1}{4\pi g_s}, \qquad \frac{1}{g_1^2} - \frac{1}{g_2^2} = \frac{1}{4\pi^2 g_s}\left(\int_{S^2} B - \pi\right). \qquad (50)$$

For $n_1 = n_2$ the theory is conformal and the two coupling constants correspond to two exactly marginal parameters in $AdS_5 \times T^{1,1}$: the dilaton and the value of the B-field on S^2. For $n_1 = N + M$ and $n_2 = M$ the theory is no longer conformal. One of the two gauge factor is not asymptotically free in the UV. There is a curious UV completion of this theory in terms of $U(\infty) \times U(\infty)$, based on Seiberg duality; the details will be discussed in Section 5.

The holographic duals for $\mathcal{N} = 2$ theories with fractional branes are discussed in Section 4 and those for $\mathcal{N} = 1$ theories in Section 5. Fractional branes act as sources for closed string states. We will find holographic duals where the corresponding fields depend on the radial coordinate. In particular, since the gauge theory coupling constants run with the scale, we expect to find duals where the twisted fields b and c run with the radial coordinate in the $\mathbb{R}^4/\mathbb{Z}_2$ case (see eq. (49)) and $\int B$ runs in the conifold example (see eq. (50)). Notice, however, that the diagonal coupling $\tau_1 + \tau_2 = \tau$ does not run in every background where the Type IIB dilaton remains constant. As usual, all the $U(1)$ factors are not described by the holographic duals.

4 Supergravity duals of $\mathcal{N} = 2$ gauge theories

In the first part of the present Section we review some basic properties of the $\mathcal{N} = 2$ supersymmetric gauge theories that will be used in the following. Then we discuss some of the corresponding string/gravity duals available in literature. They can be obtained as mass deformations of $\mathcal{N} = 4$ SYM [54, 49, 55, 46, 47], using fractional branes at

orbifold singularities [67, 78, 68, 69] or five-branes wrapped on two-cycles [80, 81, 79]. As already mentioned in the introduction, we will discuss in detail the case of wrapped branes, while for the other examples we will simply review the results. We will see that the conjectured dual supergravity backgrounds are in general plagued by singularities that can be resolved by stringy effects, such as the so-called enhançon mechanism [56].

4.1 Some remarks on $\mathcal{N} = 2$ SYM

We briefly review what is known about the $\mathcal{N} = 2$ physics of the theories we are interested in. For a general discussion about $\mathcal{N} = 2$ supersymmetry, Seiberg-Witten theories and more information about the material in this Section we refer the reader to the many good reviews in the literature [9, 82, 83, 84].

As already mentioned, one important property of $\mathcal{N} = 2$ theories is that they possess a moduli space of vacua. The scalars in the $\mathcal{N} = 2$ multiplets can have a vacuum expectation value. In the moduli space we can distinguish a Higgs branch, where we give VEV to hypermultiplet scalars, and a Coulomb branch, where we give VEV to the complex scalars in the vector multiplets. We will be mainly interested in the Coulomb branch. In a generic Coulomb branch vacuum, the gauge group is broken to the maximal abelian subgroup and the only massless fields are n abelian vector multiplets, where n is the rank of the gauge group. There are correspondingly n massless complex scalar fields u_i, whose VEVs parameterize the Coulomb branch. At low energies, we can write an effective Lagrangian for the massless fields. It is a consequence of $\mathcal{N} = 2$ supersymmetry that the effective Lagrangian is completely determined in terms of a single holomorphic function $\mathcal{F}$ of the u_i, called the prepotential,

$$\mathcal{L} \sim Im(\tau_{ij})F^i_{\mu\nu}F^{\mu\nu j} + Re(\tau_{ij})F^i_{\mu\nu}\tilde{F}^{\mu\nu j} + Im\left(\partial_\mu u_i \partial^\mu \frac{\partial \mathcal{F}}{\partial u_i}\right) + \text{fermions}, \qquad (51)$$

where $\tau_{ij} = \partial^2 \mathcal{F}/\partial u_i \partial u_j$. This effective Lagrangian is a good description of the physics except for certain values of u_i, which correspond to singularities in the moduli space associated with new physical massless particles.

The perturbative contribution to $\mathcal{F}$ is exhausted at 1-loop, all other corrections being given by instantons. To fix the ideas, we discuss the case of the simplest $\mathcal{N} = 2$ gauge theory, with gauge group $SU(N)$ and no flavors. The 1-loop prepotential reads

$$2\pi i \mathcal{F}^{(1)} = -\frac{1}{4}\sum_{j \neq i}(u_i - u_j)^2 \log \frac{(u_i - u_j)^2}{\Lambda^2}, \qquad (52)$$

where $\{u_i\}$ are the N eigenvalues of the $SU(N)$ adjoint scalar field satisfying $\sum_i^N u_i = 0$. The expressions for product groups or adjoint massive hypermultiplets will be written

when needed. Formula (52) fails at scales of order $\Delta u = \Lambda$, where the instantonic contributions to the prepotential become relevant. The apparent singularity in the 1-loop formula is resolved by adding an infinite series of instantonic contributions. The full prepotential can be determined using the Seiberg-Witten curve. This is a family of Riemann surfaces $\Gamma(u_1, \cdots, u_N)$, parameterized by the N complex parameters $u_1, \cdots, u_N$, which label the flat directions of the $\mathcal{N} = 2$ vacua. For the derivation of $\mathcal{F}$ from the SW curve the reader is referred to [9, 82, 83, 84]. Here we only give the form of the curve since it will be used in Section 4.5. The curve for pure $SU(N)$ is given by the genus $(N-1)$ hyperelliptic Riemann surface

$$y^2 = P(x)^2 - \Lambda^{2N}, \tag{53}$$

where the polynomial $P(x)$ is expressed in terms of the moduli

$$P(x) = \prod_{i=1}^{N}(x - u_i). \tag{54}$$

The importance of this curve is that it has a clear interpretation in terms of branes, which will be discussed in Section 4.5.

The existence of a moduli space allows us to probe the theory. Consider just a single modulus z of the pure $SU(N+1)$ theory. A non-zero VEV for z corresponds to the breaking $SU(N+1) \to SU(N) \times U(1)$. Under certain conditions, namely when z is sufficiently large and N big enough, we can study the physics of $SU(N)$ by looking at the effective action for the $U(1)$ factor. We then consider a point in the Coulomb branch where the moduli read $(\bar{u}_1 - z/N, ..., \bar{u}_N - z/N, z)$. The $\bar{u}_i$'s indicate the point in the moduli space of $SU(N)$ that we would like to investigate. The coupling constant associated with z can de determined from

$$\tau(z) = \frac{\partial^2 \mathcal{F}(z, \bar{u}_i)}{\partial z^2} \tag{55}$$

at fixed $\bar{u}_i$. For example, the 1-loop contribution reads

$$\tau(z) = \frac{i}{\pi} \sum_{i=1}^{N} \log \frac{(z - \bar{u}_i)}{\Lambda}. \tag{56}$$

For large N, we may expect that the introduction of a probe would not seriously alter the physics of $SU(N)$; we may also be tempted to send the probe very close to the other moduli, to investigate the non-perturbative dynamics of the gauge theory. For comparison with holographic duals we need to consider the large N limit of the gauge theory. One should not make the mistake of neglecting instantonic contributions for

$N \gg 1$ due to the naive estimate $e^{-1/g_{YM}^2} = e^{-N/x}$: differences of VEVs, which appear in $\mathcal{F}$, can be so small to compensate this exponential factor and this typically happens in strongly coupled vacua [85]. The effective action for a probe is accurate at 1-loop for z greater than the dynamically generated scale of the theory Λ. Instantonic corrections rise up very sharply (at large N) near Λ and dominate the IR physics.

The probe computation allows to compare directly the gauge theory results with a calculation in the holographic dual. When we engineer a system of branes corresponding to an $\mathcal{N} = 2$ gauge theory with a Coulomb branch, we expect that the constituent branes possess a moduli space of vacua isomorphic to that of the gauge theory. We typically have a set of branes that can be arbitrarily distributed in a plane in space-time, as we explicitly saw in Sections 3.2 and 3.3 for the $\mathcal{N} = 2$ theories with wrapped and fractional branes. If we have an holographic dual for our gauge theory, obtained as the near horizon geometry of the system of branes, we may think of studying it by sending in a probe. The probe is represented in the string theory construction by a physical, fractional or wrapped brane which is sent in the background of a large number N of other branes. If the theory is $\mathcal{N} = 2$, such a brane is a BPS object which can freely (that is without feeling any force) move in the moduli space. The effective action on the probe can be rigorously written using the Born-Infeld action for branes in a given background. This result, which is greatly constrained by $\mathcal{N} = 2$ supersymmetry, must agree with the gauge theory result computed via formula (55). It is important to stress that the $\mathcal{N} = 2$ effective action is determined by holomorphicity. Holomorphic (or BPS) quantities are protected and can be often computed in the supergravity regime, despite the presence of many un-decoupled modes. The possibility of comparing the probe action with the $\mathcal{N} = 2$ effective action in the dual field theory also provides an unambiguous way of determining the radius/energy relation in $\mathcal{N} = 2$ solutions: the modulus z, which represents the energy scale we are probing, can be identified with the space-time position of the probe.

The supergravity solutions that we are going to discuss, correctly capture the one loop contribution in field theory but are plagued by singularities at the scale where, in field theory, instantons become important. This is a situation where, as we will discuss in Section 4.5, one can learn from field theory, specifically from the SW curve, how the supergravity singularity is possibly resolved. For completeness, the SW curve associated with the $\mathcal{N} = 2$ theories that will be considered in this review are explicitly discussed in Appendix A.

4.2 $\mathcal{N} = 2$ SYM from wrapped five-branes

As observed in Section 3.2, one way to realize pure $U(N)$ $\mathcal{N} = 2$ SYM in $(1 + 3)$-dimensions is to consider the low energy theory on the world volume of N NS5-branes wrapped on a non-trivial cycle in a geometry of the form $\mathbb{R}^2 \times$ALE [80, 81]. Let us

summarize the basic ingredients in this construction. There is a massless complex scalar field ϕ on the world-volume of the branes that parameterizes their motions on the $\mathbb{R}^2$ plane. The generic vacuum in the Coulomb branch of the gauge theory is labeled by the N eigenvalues of ϕ. They are given by the arbitrary positions of the N branes on the $\mathbb{R}^2$ plane. Moreover, for the compactification to preserve $\mathcal{N} = 2$ supersymmetry, the theory has to be twisted: the spin connection on S^2 has to be identified with a background $U(1)$ field in the $SO(4)$ R-symmetry group [65, 5], which corresponds to the diagonal subgroup $U(1)_D = \frac{1}{2}(U(1)^- + U(1)^+)$ in the decomposition $SO(4) \to SU(2)^+ \times SU(2)^-$. The $U(1)_R$ symmetry of the gauge theory corresponds instead to a rotation in the plane $\mathbb{R}^2$.

To construct the dual supergravity solutions, consider first a set of flat NS5-branes. Such a configuration admits a holographic description in terms of the linear dilaton background [19], with a ten dimensional metric of the form $\mathbb{R}^{5,1} \times \mathbb{R} \times S^3$ (see eq. (14)). It is then natural to associate the solutions for wrapped branes to deformations of the linear dilaton background where the flat six-dimensional part of the metric is replaced by a metric of the form $\mathbb{R}^{3,1} \times S^2$, the S^3 transverse geometry is possibly deformed and a background abelian field in $SO(4)$ is turned on. Since the ultraviolet gauge theory is six dimensional, for large ρ the solutions must asymptote the linear dilaton background.

The actual computation of the solution using the ten-dimensional Type IIB equations is usually quite awkward. As in Section 2, we can try to consider compactifications to lower dimensions, find solutions in the lower dimensional supergravity and then lift them to get the full $10d$ solutions. In the present case, where we deal with deformations of an $\mathbb{R}^{5,1} \times \mathbb{R} \times S^3$ metric, the theory we need is provided by the seven-dimensional $SO(4)$ gauged supergravity, corresponding to the truncation of the $\mathcal{N} = 1$ sector of Type IIB on the 3-sphere transverse to the NS5-branes. This is a consistent choice since the NS5-branes only couple to the NS sector of Type IIB supergravity [5].

The bosonic sector of seven dimensional $SO(4)$ gauged supergravity [86] consists of the metric, $SO(4)$ gauge fields, a three-form and ten scalar fields. The Lagrangian for these fields (which can be obtained as a suitable singular limit [87, 88] of the maximally supersymmetric $SO(5)$ Lagrangian [89]) reads (we use the conventions of [90])

$$2\kappa^2 e^{-1}\mathcal{L} = R + \frac{1}{2}m^2(T^2 - 2T_{ij}T^{ij}) - Tr(P_\mu P^\mu) - \frac{1}{2}(V_I{}^i V_J{}^j F_{\mu\nu}^{IJ})^2, \qquad (57)$$

where I, i are the gauge and composite $SO(4)$ indices. $F_{\mu\nu}^{IJ}$ is the gauge field strength. T_{ij} is a symmetric matrix parameterized by the ten scalar fields and is defined in terms of the $SL(4, R)/SO(4)$ coset element V_i^I as $T_{ij} = V_i^{-1\,I} V_j^{-1\,J} \delta_{IJ}$, $T = T_{ij}\delta_{ij}$. The kinetic term for the scalars, P_μ, is the symmetric part of $V_i^{-1\,I} \mathcal{D}_\mu V_I{}^j = (Q_\mu)_{[ij]} + (P_\mu)_{(ij)}$, where the covariant derivatives are defined as $\mathcal{D}_\mu V_I{}^j = \partial_\mu V_I{}^j + 2m A_{\mu I}^J V_J{}^j$ on the scalars and $\mathcal{D}_\mu \psi = (\partial_\mu + \frac{1}{4}Q_{\mu ij}\Gamma^{ij} + \frac{1}{4}\omega_\mu^{\nu\lambda}\gamma^{\nu\lambda})\psi$ on the spinors; m is the mass parameter (set to one

in our conventions) which by supersymmetry is equal to one half of the gauge coupling constant.

As standard in the AdS/CFT correspondence, the $SO(4)$ gauge fields correspond to the isometries of the 3-sphere and are dual to the R-symmetry fields. Because of the twist condition, we set the $U(1)_D$ gauge field equal to minus the spin connection on S^2. The scalar matrix T_{ij} can always be brought to diagonal form with an $SO(4)$ gauge rotation

$$T_{ij} = \text{diag}(e^{2\lambda_1}, e^{2\lambda_1}, e^{2\lambda_2}, e^{2\lambda_3}). \tag{58}$$

Before the twist, it is natural to associate the matrix T_{ij} with the dual operator $\text{Tr}X_{\{i}X_{j\}}$ constructed with the four scalars living on the NS5-branes. In the representation for the $SO(4)$ we use, the twisted $U(1)_D$ corresponds to a rotation of the first two entries $i = 1, 2$, while the R-symmetry $U(1)_R$ corresponds to a rotation of the last two $i = 3, 4$. This makes it clear that the first two entries of the previous matrix correspond to the scalars that become massive upon twist. Their equality is required by the $\mathcal{N} = 2$ twist. The last two entries correspond to bilinear operators in the scalar field ϕ parameterizing the $\mathcal{N} = 2$ moduli space. In particular, the $U(1)_R$ charges suggest that $\lambda_2 + \lambda_3$ and $\lambda_2 - \lambda_3$ are dual to $\text{Tr}\phi\bar{\phi}$ and $\text{Tr}\phi^2$ respectively.

Thus the general seven dimensional solutions we are interested in, involve a non trivial profile for the $U(1)$ gauge field and some of the above scalars

$$
\begin{aligned}
ds_7^2 &= e^{2f}(dx_4^2 + N\alpha'd\rho^2) + e^{2g}(d\theta^2 + \sin^2\theta d\phi^2), \\
A_3 &= \frac{1}{2}\cos\theta d\phi, \\
T_{ij} &= \text{diag}(e^{2\lambda_1}, e^{2\lambda_1}e^{2\lambda_2}, e^{2\lambda_3}),
\end{aligned}
\tag{59}
$$

where the fields depend only on ρ. The seven dimensional three-form is set to zero.

As long as we are interested in supersymmetric solutions, it is not necessary to look at the equations of motion. It is indeed possible to reduce the problem to the solution of a set of first order equations. This can be done in various way. One can explicitly solve the fermionic shifts [81], as reviewed in Appendix B. Alternatively, one can first write an effective Lagrangian for the radial dependence of the scalars, by substituting the ansatz (59) in the Lagrangian (57) and integrating over S^2 [80, 91]. Imposing $f = -\lambda_1 - (\lambda_2 + \lambda_3)/2$ [13] we obtain

$$
\begin{aligned}
\mathcal{L} = &\frac{3}{16}e^{4Y}[16Y'^2 - 2h'^2 - \frac{1}{4}(2\lambda_1' - \lambda_2' - \lambda_3')^2 - \frac{1}{2}(\lambda_2' - \lambda_3')^2 + 2e^{-2h} + \\
&- \frac{1}{2}e^{-4h - 2\lambda_1 + \lambda_2 + \lambda_3} + 4\cosh(\lambda_2 - \lambda_3) - 2e^{-2\lambda_1 + \lambda_2 + \lambda_3}\sinh^2(\lambda_2 - \lambda_3)], \quad (60)
\end{aligned}
$$

[13]This is allowed by the equations of motion and permits to set the warp factor for the world-volume part of the string frame metric equal to one, as in the linear dilaton background [80].

with $4Y = 2h + 5f + \log(16/3)$, $h = g - f$ and the prime denotes derivation with respect to ρ. Then, one should find a superpotential for this system, in the sense discussed in Section 2.3. With the conventions used there, the superpotential is

$$W = -\frac{3}{8}[2e^{\lambda_1 - (\lambda_2 + \lambda_3)/2} + 2e^{-\lambda_1 + (\lambda_2 + \lambda_3)/2}\cosh(\lambda_2 - \lambda_3) + e^{-2h - \lambda_1 + (\lambda_2 + \lambda_3)/2}]. \quad (61)$$

There are two families of solutions corresponding to $\lambda_2 = \lambda_3$ (solution A) [80, 81] and $\lambda_2 \neq \lambda_3$ (solution B), respectively [81]. Here we will only give the solution B, solution A being a particular case of the former (see also Appendix B.2)

$$
\begin{aligned}
e^{2h} &= u, \\
e^{\frac{\lambda_2 + \lambda_3}{2} - \lambda_1} &= \sqrt{\frac{e^{4u} + b^4}{e^{4u} - b^4} - \frac{1}{2u} + \frac{2Ke^{2u}}{u(e^{4u} - b^4)}}, \\
e^{\frac{\lambda_2 + \lambda_3}{2} + \lambda_1} &= \left(\frac{e^{2u}}{e^{4u} - b^4}\right)^{1/5}\left[\frac{e^{4u} + b^4}{e^{4u} - b^4} - \frac{1}{2u} + \frac{2Ke^{2u}}{u(e^{4u} - b^4)}\right]^{-\frac{1}{10}}, \\
e^{\lambda_2 - \lambda_3} &= \frac{e^{2u} - b^2}{e^{2u} + b^2},
\end{aligned} \quad (62)
$$

with $\frac{du}{d\rho} \equiv e^{(\lambda_2 + \lambda_3)/2 - \lambda_1}$.

Compared with the deformations of AdS_5, the lift to ten dimensions is much simpler and it is known for a generic seven-dimensional solution [87, 88]. The $10d$ solutions contain the metric, the dilaton and the NSNS two-form. In the Einstein frame the solution is

$$
\begin{aligned}
ds_{10}^2 &= X^{\frac{1}{8}}\left(\Delta^{\frac{1}{4}}ds_7^2 + \Delta^{\frac{-3}{4}}T_{ij}^{-1}\mathcal{D}\mu^i\mathcal{D}\mu^j\right), \\
e^{-\Phi} \star H_{(3)} &= -U\epsilon_7 + T_{ij}^{-1}\star\mathcal{D}T_{jk}\wedge(\mu^k\mathcal{D}\mu^i) - \frac{1}{2}T_{ik}^{-1}T_{jl}^{-1}\star F^{ij}\wedge\mathcal{D}\mu^k\wedge\mathcal{D}\mu^l, \\
e^{2\Phi} &= \Delta^{-1}X^{\frac{3}{2}}.
\end{aligned} \quad (63)
$$

where μ^i $(\mu^i\mu^i = 1)$ are S^3 angular variables, $\Delta = T_{ij}\mu^i\mu^j$, $U = 2T_{ik}T_{jk}\mu^i\mu^j - \Delta T_{ii}$, and $X = det(T_{ij})$. Moreover $\mathcal{D}\mu^i = d\mu^i + A^{ij}\mu^j$, $\mathcal{D}T_{ij} = dT_{ij} + A_i^k T_{kj} + A_j^k T_{ik}$. Applying the lift formulae to our case and passing to the string frame, we have

$$
\begin{aligned}
ds^2 &= e^{(2\lambda_1 + \lambda_2 + \lambda_3)}(ds_7^2 + \frac{1}{\Delta}\{e^{-2\lambda_1}[d\mu_1^2 + d\mu_2^2 + \cos^2\theta(\mu_1^2 + \mu_2^2)d\phi^2 \\
&\quad -2\cos\theta(\mu_1 d\mu_2 - \mu_2 d\mu_1)d\phi] + e^{-2\lambda_2}d\mu_3^2 + e^{-2\lambda_3}d\mu_4^2\}), \\
e^{2\Phi} &= e^{(3\lambda_2 + 3\lambda_3 + 6\lambda_1)}\Delta^{-1}
\end{aligned} \quad (64)
$$

and $H_{(3)}$ can be deduced from (63), provided we identify $\Delta = e^{2\lambda_1}(\mu_1^2 + \mu_2^2) + e^{2\lambda_2}\mu_3^2 + e^{2\lambda_3}\mu_4^2$, $\mu_{1,2} = \cos\theta'(\cos\phi_1, \sin\phi_1)$ and $\mu_{3,4} = \sin\theta'(\cos\phi_2, \sin\phi_2)$ $(0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi; 0 \leq \theta' \leq \pi/2, 0 \leq \phi_1, \phi_2 \leq 2\pi)$. For $b = 0$ one recovers solution A with $\lambda_2 = \lambda_3$.

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Solution A has two $U(1)$ isometries corresponding to shifts in ϕ_1 and ϕ_2 which are easily identified with the R-symmetries. ϕ_1 rotations correspond to $U(1)_D$, which coincides with $U(1)_J$ on the massless fields, while shifts in ϕ_2 are associated to $U(1)_R$. On the contrary, in solution B, the scalar $\lambda_3 \neq \lambda_2$ explicitly breaks $U(1)_R$, so that the only isometry is the other $U(1)$.

In the UV, $\rho \to \infty$, the two solutions are asymptotic to the linear dilaton background with the radius of S^2 going to infinity. These are exactly the boundary conditions the two solutions have to satisfy.

For both solutions the metric is singular and the nature of the singularity depends on the value of the integration constant K in (62). For $K \leq (1 - b^4)/4$, $u \in [u_0, \infty)$, where $u_0 \geq 0$ is determined by $e^{-\lambda_1 + (\lambda_2 + \lambda_3)/2} = 0$, and the solutions are singular for $u \to u_0$ and $\theta' = \pi/2$ ($u_0 = 0$ for $K = (1 - b^4)/4$). For $K > (1 - b^4)/4$ the singularity is at $u = 0$ and it seems to be of the bad type according to the criterium in [66][14]. Therefore we will not discuss the $K > (1 - b^4)/4$ solutions in the following.

Close to the singularity, the dilaton becomes very large and, to avoid string corrections, we have to pass to the S-dual solution for the D5-branes. The two solutions are related by the standard S-duality transformation (in string frame)

$$
\begin{aligned}
\Phi_D &= -\Phi, \\
ds_D^2 &= e^{\Phi_D} ds_{NS}^2, \\
dC_{(6)} &= \star F_{(3)} = e^{-2\Phi} \star_{NS} H_{(3)}.
\end{aligned}
\tag{65}
$$

The presence of a (naked) singularity seems to be a common feature of all the supergravity solutions describing $\mathcal{N} = 2$ gauge theories. The problem is then to understand whether the singular behavior is an artifact of the supergravity approximation, which can be resolved in the full string theory, or it signals a pathological behavior of the solution. A standard technique consists in studying the low energy effective action of a single brane probe in the geometry. As already mentioned, the result has a two-fold interpretation: on the supergravity side it describes the effective geometry seen by the probe, thus shading light on the nature of the the singularity, while on the field theory side it helps in identifying the vacuum of the gauge theory.

It is clear from Section 3.1 that, for the D5 solution, our probe will be a D5 brane wrapped on S^2, whose low energy effective action is

$$
(2\pi)^2 S = -\int d^6\xi \, e^{-\Phi_D} \sqrt{G + F} + \int C_{(6)} + \frac{1}{2}\int C_{(2)} \wedge F \wedge F,
\tag{66}
$$

[14]This states that the (Einstein frame) g_{00} component of a metric conjectured to be dual, in some region containing the singularity, to the low energy regime of a field theory, cannot increase while approaching the singularity. This is because fixed proper energy excitations should correspond to low energy ones as measured by an observer at infinity.

where

$$G_{\alpha\beta} = \partial_\alpha x^M \partial_\beta x^N g_{MN} \tag{67}$$

is the induced metric on the probe world-volume ($\alpha, \beta = 0, 1, ..., 5$ label the world-volume coordinates, while $M, N = 0, 1, ..., 9$ are space-time indices), and F is the gauge field strength on the brane (the B field is zero for the D5 solution).

We are interested in the low energy action describing the slow motion of the probe in the directions transverse to the background branes. To this purpose we can expand the action (66) up to quadratic order in the derivatives of the transverse scalars. We work in the static gauge ($\xi^\alpha = x^\alpha$, $\alpha = 0, ..., 5$) and consider slow varying scalar fields in the transverse directions $x^m = x^m(x^\mu)$, ($m = 6, ..., 9$, $\mu = 0, ..., 3$). The contribution from the DBI part has the general expression

$$S_{DBI} \sim \int dx_4^2 \, d\Omega_2 \, e^{-\Phi_D} \sqrt{g} \left[1 + \frac{1}{2} g^{\mu\nu} g_{mn} \partial_\mu x^m \partial_\nu x^n - \frac{1}{2} g^{\mu\tau} g^{\nu\rho} F_{\mu\nu} F_{\tau\rho} \right]. \tag{68}$$

The $\sqrt{g}$ part in eq. (68) combines with $\int C_{(6)}$ to give the potential term in the low energy action, while the rest provides the kinetic term for the scalars and the gauge fields on the brane. The kinetic term for the scalars gives the metric on the moduli space of the gauge theory. Finally the remaining CS term will give the $F\tilde{F}$ part of the SYM action. Notice that this is a very general pattern appearing in all the various examples of probe computations.

In our case, the potential for the probe reads

$$V = \int dx_4^2 \, d\Omega_2 \, e^{2h + 2\Phi_D} \left(1 - \sqrt{1 + \frac{e^{\lambda_2 + \lambda_3 - 2h} \cos^2 \theta'}{\Delta \tan^2 \theta}} \right). \tag{69}$$

There is a region where the potential term vanishes and the probe can move freely. We will focus on solution A, solution B being a lengthy but straightforward generalization. For all values of the parameter K, the potential vanishes for $\theta' = \pi/2$, corresponding to a motion of the probe in the plane (u, ϕ_2). This is naturally identified with the moduli space of the gauge theory, since ϕ_2 generates the $U(1)$ in the R-symmetry group. A bit more surprising is the fact that for $K < \frac{1}{4}$ the probe is BPS also outside the (u, ϕ_2) plane, namely on the spherical disk defined by $u = u_0$, $0 \leq \theta' \leq \pi/2$.

The next step is to look at the kinetic terms for the scalars and the gauge fields to extract the probe tension, i.e. the gauge coupling τ. Note however that in order to be able to identify τ we need to recover the standard structure of the $\mathcal{N} = 2$ effective Lagrangian with

$$Im \, (\tau(z)) \, F^2 + Re \, (\tau(z)) \, F\tilde{F} + Im \, (\tau(z)) \, \partial z \partial \bar{z} \tag{70}$$

for the gauge and scalar kinetic terms. It is important to notice that the same function τ appears both in the moduli space metric and in the gauge kinetic term. The problem is then to find the appropriate change of coordinates that brings the effective action to the above form.

Let us first consider the solution $K = 1/4$. In this case the moduli space is given by the plane (u, ϕ_2), and with the coordinate choice $z = e^{u+i\phi_2}$ the gauge coupling reads

$$\tau(z) = \frac{iN}{\pi} \log \frac{z}{\Lambda}. \tag{71}$$

For the solutions with $K < 1/4$, the probe can move on the (u, ϕ_2) plane down to the radius u_0 and then it starts moving on the spherical disk. The moduli space metric can be computed in both loci: on the (u, ϕ_2) plane we obtain the same result for τ as in (71), while on the disk τ assumes the constant value

$$\tau(z) = \frac{iN}{\pi} \log \frac{z_0}{\Lambda}, \tag{72}$$

where $|z_0| = e^{u_0}$. By comparing the supergravity results for τ with the gauge theory expectation, we can determine the distribution of branes that generate the solutions. At a generic point on the Coulomb branch of $\mathcal{N} = 2$ $SU(N)$ SYM, the one loop expression for τ as a function of the VEVs is (see (56))

$$\tau(z) = \frac{i}{\pi} \sum_i \log(z - a_i) \sim i \int da \mu(a) \log(z - a), \tag{73}$$

where a_i are the classical[15] VEVs and $\mu(a)$ is the VEV distribution in the continuum limit. By equating it with the supergravity expression, eqs. (71), (72), we find

$$\mu(a) = \frac{N}{2\pi z_0} \delta(|a| - z_0), \tag{74}$$

which corresponds to circular $U(1)_R$ invariant distributions of VEVs, with radius z_0. This fits with the fact that the gauge theory we consider only contains the operator $Tr\phi\bar{\phi}$.

For $K < 1/4$ the probe sees a completely smooth moduli space, indicating that the one loop approximation is always valid. Those solutions should then be dual to weakly coupled vacua. Indeed we see from equations (71), (72) that the VEVs are distributed on a circle with radius $z_0 > \Lambda$. In the large N limit, all instantonic corrections are suppressed by factors $(\Lambda/z_0)^N \ll 1$. The probe thus sees the one loop results for $z > z_0$ and a constant coupling at scales below that set by the VEV distribution.

---

[15]In the SW solution one distinguishes between classical VEVs, a_i, and quantum VEVs, u_i. At one loop however there is no difference between the two and we use the classical expression.

For $K = 1/4$, $z = \Lambda$ ($u = 0$ in the natural coordinates) is a singularity for the probe action where it becomes tensionless ($\tau = 0$). On the field theory side, the gauge coupling g_{YM} diverges. One is then tempted to associate the corresponding supergravity solution to a strongly coupled vacuum of the gauge theory. Every vacuum where the moduli u_i in the SW curve are distributed on a circle with radius $r \leq \Lambda$ would reproduce the result seen by the probe[16]. The singularity in τ is interpreted as the point on the moduli space where the perturbative approximation breaks down and instanton corrections come into play. The region at which the probe becomes tensionless is usually called enhançon [56] and it is identified with a quantum distribution of VEVs, where the constituent branes have expanded to form a shell. We will discuss the issue of singularities in more details in Section 4.5. Here we only want to stress that for this particular solution, the singular region is actually point-like in the original coordinates, making the interpretation as the enhançon less clear.

For solution B the probe computation goes on as before, but the identification of the appropriate $\mathcal{N} = 2$ coordinates is more difficult. The choice of coordinates $w = z + b^2/z$ brings the effective action in the form (70) predicted by $\mathcal{N} = 2$ supersymmetry. In solution B the $U(1)_R$ symmetry is spontaneously broken and both the operators $\mathrm{Tr}\phi\bar{\phi}$ and $\mathrm{Tr}\phi^2$ have a VEV. For $K = (1 - b^4)/4$, the probe tension is given by

$$\tau(w) = \frac{iN}{\pi} \left(\mathrm{arcosh}(\frac{w}{2b}) + const \right), \tag{75}$$

corresponding to the linear distribution of VEVs $\mu(a) = N/(\pi\sqrt{4b^2 - a^2})$. Again one can interpret it as a strong coupling vacuum. Curiously, this distribution of VEVs is of the same type as the one appearing for the $\mathcal{N} = 2$ moduli space region where all types of monopoles become massless [85], which is the relevant one for the $\mathcal{N} = 2 \to \mathcal{N} = 1$ breaking. The situation for $K < (1 - b^4)/4$ is similar to the analogous one in solution A, with two loci meeting along an ellipse and a probe that sees a smooth moduli space.

For further works on the subject we refer to [93, 94].

4.3   A supergravity dual of $\mathcal{N} = 2^*$

Pure $\mathcal{N} = 2$ SYM in four dimensions can also be obtained as a deformation of $\mathcal{N} = 4$ SYM with an equal mass for two of the chiral multiplets

$$\delta\mathcal{L} = \int d\theta^2 m Tr(\Phi_1^2 + \Phi_2^2) = \sum_{i=1,2} \left(m\lambda_i\lambda_i + |m|^2|\phi_i|^2 \right).$$

---

[16]The case $r = \Lambda$ is particularly intriguing because it corresponds to an Argyres-Douglas point [92], where the $\mathcal{N} = 2$ theory becomes conformally invariant.

At energies below the mass scale the theory should flow to pure $\mathcal{N} = 2$ SYM. Strictly speaking, to obtain pure SYM we should be able to decouple the massive modes while keeping the low energy scale $\Lambda = me^{-8\pi^2/2Ng_{YM}^2}$ fixed. This requires a fine tuning of the UV parameters which is outside the validity range of supergravity. Thus, via gauge/gravity duality, we can only study theories with the massless content of pure $\mathcal{N} = 2$ SYM but with additional massive states: $\mathcal{N} = 2^*$ theories. Notice that the effective Lagrangian will depend on the holomorphic quantity m.

The general method for finding solutions corresponding to deformations of $\mathcal{N} = 4$ was discussed in Section 2. The solution can be found by using five dimensional gauged supergravity and then lifting to ten dimensions. In our case we are interested in the two fields

$$
\begin{aligned}
\alpha &\rightarrow \sum_{i=1}^{4} Tr(\phi_i\phi_i) - 2\sum_{i=5}^{6} Tr(\phi_i\phi_i), \\
\chi &\rightarrow Tr(\lambda_1\lambda_1 + \lambda_2\lambda_2) + h.c.
\end{aligned}
\tag{76}
$$

The ansatz for the five-dimensional solutions is of the form

$$
\begin{aligned}
ds_5^2 &= dy^2 + e^{2Y(y)}dx_\mu dx^\mu, \\
\alpha &= \alpha(y), \qquad \chi = \chi(y),
\end{aligned}
$$

with the boundary condition that the solution tends in the UV to AdS_5 ($\alpha, \chi \rightarrow 0$, $Y \rightarrow y/R$). In terms of ten dimensional fields, α corresponds to the first KK mode of the complex two form with both indices on S^5, and χ to the linear combination of the internal part of the metric and of the four form potential.

The five dimensional solution was found in [54, 49, 55]. The presence of RR fields makes the lift to ten dimensions more complicated than for the wrapped brane case [49]. In addition to the metric, the ten dimensional Einstein-frame solution contains the self-dual five-form $F_{(5)} = \mathcal{F} + \star\mathcal{F}$, a complex combination of the NSNS and RR two-forms[17], the dilaton and the axion [49]

$$
\begin{aligned}
ds^2 &= \frac{(cX_1X_2)^{1/4}}{\rho^3}\left\{\frac{k^2\rho^6}{c^2-1}(dx_\mu)^2 + \frac{R^2}{\rho^6(c^2-1)^2}dc^2 + \right. \\
&+ \left. R^2\left[\frac{d\theta^2}{c} + \frac{\sin^2\theta}{X_2}d\phi^2 + \rho^6\cos^2\theta\left(\frac{\sigma_3^2}{cX_2} + \frac{\sigma_1^2+\sigma_2^2}{X_1}\right)\right]\right\}, \\
\mathcal{F} &= 4dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dw(r,\theta), \\
\tau &= \frac{\tau_0 - \bar{\tau}_0 B}{1 - B},
\end{aligned}
\tag{77}
$$

---

[17]Since the 2-form is not needed in what follows, we refer the reader to [49] for its explicit expression.

where the radial coordinate y has been traded for $c = \cosh(2\chi)$ using the χ equation of motion[18].

It is important to notice that, in these new variables, the boundary corresponds to $c = 1$ and the IR region to $c \to \infty$. As usual $R^2 = \alpha'\sqrt{g_{YM}^2 N}$, $\tau_0 = \theta_s/2\pi + i/g_s$ is the asymptotic value of the complex dilaton and σ_i are the left invariant one forms parameterizing the three-sphere. All the other functions in (77) are determined in terms of the five-dimensional solution [49]

$$X_1 = \cos^2\theta + c\rho^6\sin^2\theta, \qquad\qquad B = e^{2i\phi}\frac{\sqrt{cX_1} - \sqrt{X_2}}{\sqrt{cX_1} + \sqrt{X_2}},$$

$$X_2 = c\cos^2\theta + \rho^6\sin^2\theta, \qquad\qquad w(r,\theta) = \frac{k^4\rho^6 X_1}{4g_s(c^2-1)^2}$$

$$\rho^6 = e^{6\alpha} = c + (c^2-1)\left[\gamma + \tfrac{1}{2}\log\left(\tfrac{c-1}{c+1}\right)\right]. \tag{78}$$

The solution contains two parameters k and γ. The parameter k can be identified with the mass perturbation m, while γ parameterizes a family of different solutions that should represent different flows to the IR $\mathcal{N} = 2$ theory.

As for the wrapped brane case, the solutions have a naked singularity in the IR and the value of γ distinguishes among bad and good ones. For $\gamma < 0$ the metric and the dilaton become singular for $\rho \to 0$, which corresponds to a finite value $c = c_0$, while for $\gamma = 0$ the singularity is on a ring at $\rho \to 0$, $c \to \infty$ and $\theta = \pi/2$. In these two cases the singularity has a physical interpretation. On the contrary the $\gamma > 0$ solutions turn out to be unphysical. Before discussing the probe results, notice that near the boundary the solutions are asymptotic to $AdS_5 \times S^5$ for every value of γ, as expected.

The background geometry is generated by a stack of flat D3-branes, so that it is natural to use as a probe another D3-brane moving in the transverse directions [46, 47]. As discussed in Section 4.2, one has to expand the probe action for small velocities of the probe in the transverse directions, thus obtaining a potential and a kinetic term for the transverse scalars plus the usual F^2 and $F\tilde{F}$ terms for the gauge fields. We find again that there are two loci where the potential vanishes: the plane (c, ϕ), $\theta = \pi/2$ (I) and the region where $\rho = 0$ (II), which corresponds to a fixed c_0 and can be parameterized by (θ, ϕ). The second region exists only for $\gamma < 0$. For the $\gamma < 0$ solutions the two loci join to give a completely smooth moduli space [46]. For $\gamma = 0$, the gauge coupling is

$$\tau(z) = \frac{i}{g_s}\left(\frac{z^2}{z^2 - k^2R^2}\right)^{1/2} + \frac{\theta_s}{2\pi}, \tag{79}$$

where we define the complex coordinate, $z = kR(c\cos\phi - i\sin\phi)/\sqrt{c^2-1}$, in such a way that the scalar and gauge kinetic term have the standard $\mathcal{N} = 2$ form (see eq.

---

[18]The equations relevant for the change of coordinates are $\frac{d\chi}{dy} = -\rho^4\sinh(2\chi)/2R$, $e^Y = k\rho^2/\sinh(2\chi)$.

(70)). The function $\tau(z)$ is singular for $z = \pm kR$ and has a branch cut on the segment $-kR \leq z \leq kR$. In the original coordinates this corresponds to $c = \infty$. For $\gamma = 0$ the probe tension $\tau(z)$, expressed in the original coordinates, vanishes at the singularities. This is the enhançon locus. We can determine the corresponding brane distribution by comparing our result for the gauge coupling to field theory expectations. We need the one loop coupling constant of $\mathcal{N} = 2$ SYM with massive matter fields in the adjoint [82]

$$\tau(z) = \frac{i}{g_s} + \frac{\theta_s}{2\pi} + \frac{i}{2\pi} \sum_i \log \left(\frac{(z - a_i)^2}{(z - a_i)^2 - m^2} \right) . \tag{80}$$

It will be sufficient to consider values of the moduli larger than the mass deformation

$$\tau(z) \sim \frac{i}{g_s} + \frac{\theta_s}{2\pi} + \frac{i}{2\pi} \sum_i \frac{m^2}{(z - a_i)^2} . \tag{81}$$

As in the previous section, by equating the continuum limit of eq. (81) to the supergravity result (79) we obtain again a linear distribution for the VEVs [46]

$$\mu(a) = \frac{2}{m^2 g_s} \sqrt{a_0^2 - a^2} , \tag{82}$$

with $a_0^2 = m^2 g_s N / \pi$. In the supergravity limit, the size of the VEV distribution is much larger than the adjoint mass. This justifies a posteriori the use of the one-loop approximation in quantum field theory.

4.4 $\mathcal{N} = 2$ SYM from fractional branes

In all the previous examples, we were forced to investigate particular points in moduli space we could not select. The introduction of operators driving the theory to different vacua often induces other severe singularities. This is a general characteristic of models obtained with the dimensional reduction to gauged supergravity. We now show that the use of fractional branes allows, in principle, to study a generic point in moduli space. The moduli indeed appear as free parameters in the solution.

The supergravity solutions corresponding to $\mathcal{N} = 2$ fractional branes have been extensively discussed in [67, 68, 69, 78, 95, 96]. Here we review the solution for our favorite example, the orbifold singularity $\mathbb{R}^4 / \mathbb{Z}_2$. As we saw in Section 3.3, with n_1 fractional and n_2 anti-fractional branes at the orbifold singularity $\mathbb{R}^4 / \mathbb{Z}_2$ we can realize the theory $U(n_1) \times U(n_2)$. As usual, the $U(1)$ factors are not described by the holographic dual. The branes live at $x_6 = x_7 = x_8 = x_9 = 0$ and are arbitrarily distributed in the (x_4, x_5) plane. It is convenient to introduce the complex variable $z = x_4 + ix_5$ and to denote the positions of the fractional and anti-fractional branes by $a_i^{(1)}, a_i^{(2)}$, respectively. In

the gauge theory these correspond to VEVs of the Cartan values of the adjoint scalars parameterizing the generic vacuum.

Following [67] we define

$$\gamma = 2\pi\frac{(\tau_1 - \tau_2)}{2} = 2\pi(c + \tau(b - \frac{1}{2})). \tag{83}$$

Notice that $\gamma(x_0, x_1, x_2, x_3, z)$ is a six-dimensional field living at the fixed plane $x_6 = x_7 = x_8 = x_9 = 0$.

Fractional and anti-fractional branes are sources for the RR fields $C_{(4)}$ and $C_{(4)}^T \sim c$. Since they are oppositely charged under the twisted field γ we can immediately write the linearized result [67]

$$\gamma(z) = \gamma^{(0)} + 2i\left(\sum_{i=1}^{n_1}\log(z - a_i^{(1)}) - \sum_{i=1}^{n_2}\log(z - a_i^{(2)})\right), \tag{84}$$

where the logarithms appear because the 3-brane is an extended source of real codimension two for the localized six-dimensional field γ. Remarkably, γ does not receive any further correction. In fact, the supergravity equations only require γ to be holomorphic. Every holomorphic γ, combined with a black D3-brane ansatz [78, 68]

$$
\begin{aligned}
ds^2 &= Z^{-1/2}dx_\mu dx^\mu + Z^{1/2}ds_K^2, \\
F_5 &= dC_{(4)} + \star dC_{(4)}, \qquad C_{(4)} = \frac{1}{Z}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,
\end{aligned} \tag{85}
$$

is a solution of Type IIB equations of motion provided that

$$-\Box_K Z = \rho_{D3}(x) + \text{const } |\partial\gamma(z)|^2\delta^{(4)}(x_6, x_7, x_8, x_9). \tag{86}$$

Here $\rho(x)$ is an arbitrary density of physical D3-branes [78, 68]. The general solution of this equation is

$$
\begin{aligned}
Z(x_T, z) &= \sum_{i=1}^{n_1}\frac{b^{(0)}}{(x_T^2 + |z - a_i^{(1)}|^2)^2} + \sum_{i=1}^{n_2}\frac{1 - b^{(0)}}{(x_T^2 + |z - a_i^{(2)}|^2)^2} \\
&+ \text{const}\int d^2w\frac{|\partial\gamma(w)|^2}{(x_T^2 + |z - w|^2)^2}.
\end{aligned} \tag{87}
$$

We see that, by taking γ as in eq. (84), we obtain a solution depending on $n_1 + n_2$ parameters representing the moduli of the $\mathcal{N} = 2$ gauge theory.

The logarithmic behavior in (84) reproduces the one-loop beta function of the $\mathcal{N} = 2$ gauge theory [67]. To see this more clearly, we can introduce a probe in the system. Just send in an extra fractional brane represented as a D5-brane wrapped on the vanishing

cycle and positioned at z. One can immediately see that all factors of the warp function Z cancel in the Born-Infeld action. Indeed, the very same argument that led to eq. (49), tells us that the effective coupling constant on the probe is

$$\tau(z) = \frac{\gamma(z)}{2\pi} + \frac{\tau}{2} = \tau^{(0)} + \frac{i}{\pi}\left(\sum_{i=1}^{n_1}\log(z - a_i^{(1)}) - \sum_{i=1}^{n_2}\log(z - a_i^{(2)})\right). \qquad (88)$$

This result obviously agrees with the 1-loop coupling constant of the $U(1)$ factor as predicted by gauge theory[19].

The solution (87) presents various kinds of singularity. The supergravity background only reproduces the 1-loop result in the gauge theory and presents a singularity in the IR region, where the physics becomes non-perturbative. We can consider, for example, the case of pure $SU(N)$ gauge group, $n_1 = N$ and $n_2 = 0$, and a typical strongly coupled vacuum, $a_i^{(1)} = 0$. Equation (88) becomes $\tau(z) = \frac{i}{\pi}\log z/\Lambda$. The supergravity solution has an IR singularity at $z = 0$. The probe, on the other hand, becomes tensionless at the scale Λ before reaching the singularity. From eq. (83) we see that, at the same scale, the space-time fields b and c vanish. In string theory new massless fields (in this case tensionless strings) are expected. The singularity is thus surrounded by a spherical shell, impermeable to the probe, which is characterized by new space-time states becoming massless. All these phenomena are usually associated with the enhançon mechanism, that will be discussed in Section 4.5, and suggests a possible resolution of the IR singularity. In a general model, we may expect other singularities at the positions of the constituent branes and a break down of the supergravity approximation near the orbifold fixed planes.

A general discussion of the interpretation of the supergravity solution can be found in [69, 97, 98]. More general solutions for systems of fractional branes at orbifold singularities with $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry can be found in [99]. Models with fundamental matter fields can be obtained by adding D7-branes and the corresponding solution, which involves a non-trivial holomorphic dilaton, were discussed in [78, 95].

4.5 The issue of singularity

In all previous examples, we have seen that the supergravity solution only captures the 1-loop result in the gauge theory and it is plagued by IR singularities. It seems widely believed that the resolution of singularities in the $\mathcal{N} = 2$ models is obtained with the mechanism known as the enhançon [56], where the constituent branes reach

---

[19]As far as the 1-loop result is concerned, the anti-fractional gauge group can be considered as inert. We are left with an $SU(N)$ theory with $2n_2$ fundamental hypermultiplets with masses $m_i \equiv a_i^{(2)}$. The one loop result for an $SU(N)$ theory with N_f fundamentals hypermultiplets reads [82] $\tau(z) = \tau^{(0)} + \frac{i}{\pi}\left(\sum_{i=1}^{N}\log(z - a_i) - \frac{1}{2}\sum_{i=1}^{N_f}\log(z - m_i)\right)$.

an equilibrium configuration by forming shells. Such a behavior is suggested by the SW solution of $\mathcal{N} = 2$ gauge theories. In the M theory approach to solving $\mathcal{N} = 2$ gauge theories, the SW curve actually describes what a system of branes looks like after quantum corrections are taken into account. Consider, for example, pure $SU(N)$ gauge theory. In a generic strongly coupled vacuum, the 1-loop result for a probe is valid until the scale Λ where, extremely suddenly at large N, instantonic corrections start to dominate the physics. What happens below the scale Λ is accurately described by the SW curve $y^2 = P(x)^2 - \Lambda^{2N}, P(x) = \prod(x - u_i)$. The importance of the curve for our purposes is that, in the M theory description [75], the moduli u_i describe the positions of the constituent branes. To exemplify the general situation, let us consider a circular distribution of VEVs $|u_i| = r$, as that encountered in Section 4.2. The curve reads $y^2 = (x^N - r^N)^2 - \Lambda^{2N}$. We can try to engineer this system by forcing N branes on a circle of radius r. At large N, we can get a hint of what the quantum distribution of branes looks like by taking a symmetric section of the curve at $y = 0$ (the set of branch points of the curve)

$$(x^N - r^N)^2 - \Lambda^{2N} = 0. \tag{89}$$

In a strongly coupled vacuum $r < \Lambda$, the solutions of this equation are distributed on a circle of radius Λ, since all powers of $(r/\Lambda)^N$ are negligible for large N. We see that the distribution of branes, at the quantum level, expand to form a spherical shell of radius Λ, that we will call enhançon. One can also see that a probe cannot move beyond $x = \Lambda$[20]. A similar analysis can be repeated for all points in moduli space showing that this phenomenon occurs for all strongly coupled vacua, with an enhançon that may change shape, and even degenerate (into a segment, for example) in particular situations. In weakly coupled vacua, on the other hand, one can check that a probe can move freely everywhere. The reader can easily check it in the case of a circular distribution of radius $r > \Lambda$. The case $r = \Lambda$ is special since it corresponds to an Argyres-Douglas conformal fixed point [92]. These results are quite general for $\mathcal{N} = 2$ theories: using the much more complicated curve discussed in Appendix A, one can also check for instance that the same phenomena occur for $SU(N + M) \times SU(N)$ groups [98].

We would like to use this information from quantum field theory to learn about the holographic dual. We identify solutions where a probe can move freely everywhere with weakly coupled vacua. We already made this identification in the previous Sections. On the other hand, we identify solutions where the probe encounters a barrier with strongly

---

[20]A probe can be indeed introduced considering the $SU(N+1)$ theory in the vacuum $(u_1 - z/N, ..., u_N - z/N, z)$. Eq. (89) is replaced, for large N, by $(x^N - r^N)^2(x - z)^2 - \Lambda^{2N+2} = 0$. We see that, for $z > \Lambda$ we have a pair of branch-points $x \sim z$, corresponding to the probe moving outside the enhançon. For $z < \Lambda$, instead, the $2N + 2$ solutions of the equation are distributed on the circle $x \sim \Lambda$. The probe cannot enter the enhançon and instead dissolves in the spherical shell.

coupled vacua. The SW curve then suggests that the constituent branes form shells. We may expect, using the Gauss law, that the space-time field becomes constant inside the shell thus resolving the singularity. The supergravity result is then an accurate description of the physics only outside the enhançon. The name enhançon is used in the literature associated to the following features:

- It is the natural boundary for the motion of a probe. At the enhançon the probe stops being a BPS object or stops being an elementary object. It is supposed to dissolve in the enhançon.

- It is a gravitational shell where gravity stops being attractive. It is a shell around a repulson singularity.

- It is a locus where extra massless string states become important.

- It is a locus where there is an enhanced symmetry in space-time; from this the name!

Actually, all the previous four features were realized only in the original example with D2 and D6 branes on $K3$ in Type IIA [56]. Since then, some of these features are randomly realized in $\mathcal{N} = 2$ models discussed in the literature. The enhançon picture seems a good description for the fractional brane system and a slightly less good description for the wrapped brane system, as discussed in Section 4.2. In the latter case, one can also study the system using world-sheet methods. A world-sheet CFT describing a T-dual of the supergravity solution A described in Section 4.2 has been written in [100] and seems a good description for the weakly coupled vacua $K < 1/4$, but not for $K = 1/4$ where it becomes singular.

5   Supergravity duals of $\mathcal{N} = 1$ gauge theories

In the first part of this Section we review the basic features of four dimensional $\mathcal{N} = 1$ gauge theories that will be used in the following. We will then focus on two string-supergravity duals obtained with wrapped and fractional branes. These models are known as the Maldacena-Nuñez (MN) [5] and Klebanov-Strassler (KS) [4] solutions, respectively. The basic difference with respect to the corresponding $\mathcal{N} = 2$ solutions is that they are completely regular. Both theories exhibit confinement and spontaneous breaking of the chiral symmetry. We will also briefly describe the Polchinski-Strassler (PS) solution [48], describing the $\mathcal{N} = 1^*$ theory, obtained as a deformation of the $\mathcal{N} = 4$ CFT. We conclude with some comments about $\mathcal{N} = 0$ models obtained as soft breaking of $\mathcal{N} = 1$.

5.1 Some remarks on $\mathcal{N} = 1$ SYM

In this Section we review some of the basic features of $\mathcal{N} = 1$ supersymmetric gauge theories, focusing on the aspects that are relevant for the comprehension of string duals. For a general and more complete discussion about $\mathcal{N} = 1$ gauge theories we refer the reader to good reviews such as [9, 101]. In recent years, several string models generalizing pure $\mathcal{N} = 1$ SYM have appeared in the literature, from MCQD [102] to the many *AdS*-inspired models. Here we review pure $\mathcal{N} = 1$ SYM and discuss how this results can be adapted to its generalisation encountered in string duals.

It is widely believed that pure $\mathcal{N} = 1$ SYM confines and has a mass gap. The characteristic scale of the theory Λ is set by the tension of the color flux tubes, or briefly QCD strings. They are not BPS objects and the value of their tensions cannot be fixed in terms of central charges or symmetries. Strings connecting external sources in different representations of the gauge group are, in general, different physical objects. They are classified by the center of the gauge group. In a confining $\mathcal{N} = 1$ $SU(N)$ SYM theory, we can define $N - 1$ different types of QCD strings, since there are exactly $N - 1$ representations of the gauge group that are not screened by gluons. A k-string, $k = 1, ..., N - 1$, connects external sources in the k-fold antisymmetric representation of $SU(N)$. It is then interesting to ask what is the ratio of the tensions for k-strings. In many stringy-inspired models one can derive the sine formula

$$\frac{T_k}{T_{k'}} = \frac{\sin k\pi/N}{\sin k'\pi/N}. \tag{90}$$

This formula, or mild modifications of it, is valid in a variety of toy models exhibiting confinement, from softly broken $\mathcal{N} = 2$ SYM [85] to MQCD [103]. As we will see, it is also realized in the MN solution (and, with a small correction, in the KS model). It is certainly not an universal formula. There are many quantum field theory counterexamples showing that it can have corrections [104]. It would be quite interesting to understand if this formula is valid in pure YM theories. Unfortunately, since the QCD strings are not BPS, there is no known method of performing a rigorous computation in $\mathcal{N} = 1$ SYM. Interestingly, the sine formula has been supported by recent lattice computations for pure non supersymmetric YM [105].

Another common feature of $\mathcal{N} = 1$ theories is spontaneous breaking of chiral symmetry. The $\mathcal{N} = 1$ Lagrangian can be written in superfield notation as

$$L = -\frac{i}{16\pi} \int d^2\theta \tau W_\alpha^2 + \text{h.c.} \tag{91}$$

There is a classical $U(1)_R$ symmetry, rotating the gaugino, which is broken to a discrete $\mathbb{Z}_{2N}$ subgroup by instantons. This theory has N vacua associated with the spontaneous

breaking of the $\mathbb{Z}_{2N}$ symmetry to $\mathbb{Z}_2$ by gaugino condensation,

$$< \lambda\lambda > \sim N\Lambda^3 e^{2\pi i n/N}, \tag{92}$$

where Λ is the physical, RG invariant mass scale, and may be written in terms of the bare coupling τ at some UV scale as $\Lambda = \Lambda_{UV} e^{2\pi i \tau/3N}$. The integer $n = 0,..,N-1$ in (92) labels the different vacua. The gaugino condensate is an operator with protected dimension three, since it is part of a chiral multiplet. All information about the vacuum can be conveniently described by a non-perturbative holomorphic superpotential

$$W = N^2 \Lambda^3 e^{2\pi i n/N}. \tag{93}$$

Indeed from eq. (91) we see that the vacuum expectation value of the superfield W_α^2, whose lowest component is the bilinear $\lambda\lambda$, can be obtained by differentiating the effective superpotential with respect to τ. The result (92) then follows from eq. (93). In presence of a spontaneous breaking of the $\mathbb{Z}_{2N}$ symmetry, we expect the existence of domain walls (classical field configurations of codimension one) separating different vacua [106, 102]. The domain walls in $\mathcal{N} = 1$ gauge theories are BPS saturated and their tension is determined by a central charge of the supersymmetry algebra [106, 102], in terms of holomorphic data. The tension of a domain wall connecting the vacua i and j is determined by the difference of the superpotential

$$T_{DW} \sim |W(i) - W(j)|, \tag{94}$$

which for pure $\mathcal{N} = 1$ explicitly reads

$$T_{DW} \sim N|(\lambda\lambda)_i - (\lambda\lambda)_j| \sim N^2 \Lambda^3 \sin\frac{(i-j)\pi}{N}. \tag{95}$$

In the large N limit the tension is then linear in N. By analogy with D-branes, it was conjectured that the QCD strings can end on $\mathcal{N} = 1$ domain walls[102]. This typically happens in all stringy-inspired generalization of pure $\mathcal{N} = 1$ SYM.

The properties that are constrained by holomorphicity and symmetries are also valid in many generalization of the pure $\mathcal{N} = 1$ SYM. In theories with spontaneous breaking of the $\mathbb{Z}_{2N}$ symmetry, we may expect N vacua, a vacuum superpotential determining the condensates and domain walls, whose tensions are fixed by eq. (94). This happens indeed in MQCD, and in the MN and KS solutions. In pure $\mathcal{N} = 1$ SYM, the characteristic scales of chiral symmetry breaking and of QCD strings are fixed in terms of a single scale: $T_{DW} \sim N\Lambda^3$ and $T_s \sim \Lambda^2$. In more general models, T_{DW} and T_s can be distinct. While T_s is not protected and computable only in the semiclassical approximation, T_{DW} is BPS-protected. Its explicit value may nevertheless depend on the extra parameters in the theory, as it happens in MQCD, for example [102], or in the MN and KS solutions,

as we will see. Notice that some of the previous results are not applicable to the PS model [48], which describes $\mathcal{N} = 1^*$, because the chiral symmetry is not visible in the supergravity approximation.

5.2 $\mathcal{N} = 1$ SYM from wrapped five-branes

In this Section we will review the solution corresponding to $\mathcal{N} = 1$ SYM that can be constructed with wrapped five-branes [5]. The set up is similar to the $\mathcal{N} = 2$ case of Section 4.2. In order to have a four dimensional world-volume theory, we wrap N Type IIB NS5-branes on a two-cycle. This will be a gauge field theory, since at low energies the LST on the world-volume of flat NS5-branes reduces to six dimensional SYM. The difference with the $\mathcal{N} = 2$ case comes from the ambient geometry for the two-sphere. In order to preserve only four supercharges the manifold in which the two-sphere is embedded must be a Calabi-Yau threefold. We refer to Section 3.2 for conventions about charges and a detailed discussion of the twist. Recall that, with the transverse group of symmetries of a 5-brane written as $SO(4) \sim SU(2)^+ \times SU(2)^-$, the abelian field responsible for the twist is now $U(1)^+ \subset SU(2)^+$. It was shown in Section 3.2 that with this choice of twist the brane field content at low energies is simply an $\mathcal{N} = 1$ vector multiplet. $U(1)^+$ appears as the surviving $U(1)_R$ symmetry of the $\mathcal{N} = 1$ theory.

The supergravity solution will be a deformation of the linear dilaton background, dual to the LST. As in the $\mathcal{N} = 2$ case, the solution can be found using seven dimensional gauged supergravity. We can consistently truncate this theory to the sector invariant under $SU(2)^-$. Only one scalar field, the dilaton $\phi = 5\lambda$ ($\lambda_i = \lambda$, $i = 1, 2, 3$ in eq. (58)), survives this truncation. In addition to the dilaton, we expect that at least the metric warp factors and the $U(1)_R$ abelian field should be turned on. A supersymmetric solution with these fields exists but it is singular. It turns out that a regular solution can be found by turning on also the non Abelian part of the $SU(2)^+$ gauge connection. As we will show below, the new field is dual to the gaugino condensate. It is remarkable that the space-time field de-singularizing the solution is associated with the non-trivial IR dynamic of the $\mathcal{N} = 1$ SYM theory.

We are interested in solutions of the form

$$
\begin{aligned}
ds_7^2 &= e^{2f}(dx_4^2 + N\alpha' d\rho^2) + e^{2g}(d\theta^2 + \sin^2\theta d\phi^2), \\
A &= \frac{1}{2}\left[\sigma^3 \cos\theta d\phi + \frac{a}{2}(\sigma^1 + i\sigma^2)(d\theta - i\sin\theta d\phi) + \text{c.c}\right], \\
T_{ij} &= e^{2\lambda}\delta_{ij},
\end{aligned}
\tag{96}
$$

where all the fields f, g, λ and a only depend on ρ. The BPS equations for supersymmetric solutions are considerably more complicated than in the $\mathcal{N} = 2$ case. They can be

found in Appendix B.3. Alternatively, we can write an effective action allowing for a superpotential and solve the corresponding first order equations as we did in Section 4.2.

Substituting the ansatz (96) in the seven dimensional supergravity Lagrangian and integrating over S^2, we obtain the following one dimensional effective Lagrangian

$$\mathcal{L} = \frac{3}{16}e^{4Y}[16(Y')^2 - 2(h')^2 - \frac{1}{2}e^{-2h}|a'|^2 + 2e^{-2h} - \frac{1}{4}e^{-4h}(|a|^2 - 1)^2 + 4], \qquad (97)$$

where $h = f - g$, $4Y = 2h - 2\phi + \log(16/3)$. Analogously to what we did for the $\mathcal{N} = 2$ case, we set by hand $f = -2\lambda$. The associated superpotential is [107, 108, 93]

$$W = -\frac{3}{8}e^{-2h}\sqrt{(1 + 4e^{2h})^2 + 2(-1 + 4e^{2h})|a|^2 + |a|^4}. \qquad (98)$$

The supersymmetric solution was first found in [109] and subsequently reinterpreted in the context of gauge/gravity duals in [5]. It reads

$$e^{2h} = \rho\coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \qquad a = \frac{2\rho}{\sinh 2\rho}, \qquad e^{2\phi} = \frac{2e^h}{\sinh 2\rho}. \qquad (99)$$

The ten-dimensional solution is obtained using formulae (63) for the lift and . We write the solution using the Euler angles on the three-sphere

$$\begin{aligned}
g &= e^{\frac{i\psi\sigma^3}{2}}e^{\frac{i\tilde{\theta}\sigma^1}{2}}e^{\frac{i\tilde{\phi}\sigma^3}{2}}, \\
\frac{i}{2}w^a\sigma^a &= dgg^{-1}, \\
w^1 + iw^2 &= e^{-i\psi}(d\tilde{\theta} + i\sin\tilde{\theta}d\tilde{\phi}), \\
w^3 &= d\psi + \cos\tilde{\theta}d\tilde{\phi}, \qquad (100)
\end{aligned}$$

with $\psi \in [0, 4\pi]$.

In the string frame, the $10d$ solution for the wrapped NS5-branes is $(A = \frac{1}{2}A^a\sigma^a)$ [5]

$$\begin{aligned}
ds^2_{str} &= dx_4^2 + N\alpha'[d\rho^2 + e^{2h(\rho)}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{4}\sum_a(w^a - A^a)^2], \\
H^{NS}_{(3)} &= \frac{N\alpha'}{4}[-(w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \sum_a F^a \wedge (w^a - A^a)], \\
e^{2\Phi_{NS}} &= e^{2\Phi_{NS,0}}\frac{2e^{h(\rho)}}{\sinh 2\rho}. \qquad (101)
\end{aligned}$$

This is the Maldacena-Nuñez solution. The metric is completely regular in the IR, where it is of the form $\mathbb{R}^7 \times S^3$. Indeed, $a \to 1$ in the IR and A is a pure gauge which can be reabsorbed by a coordinate transformation on S^3. Moreover, since $e^{2h} \to \rho^2$, the original S^2 is now contractible and combines with ρ to give $\mathbb{R}^3$. The (squared) radius of the

three-sphere is of order $N\alpha'$ and the supergravity approximation is valid when $N \gg 1$. The string coupling is vanishing for large ρ and reaches its maximum value, $e^{\Phi_{NS,0}}$, at $\rho = 0$. For $e^{\Phi_{NS,0}} \ll 1$ the string coupling is everywhere small and all loop corrections are suppressed. However from the gauge theory point of view, we would like to decouple the KK modes in order to get pure SYM. As we will see in detail below, the ratio between the scales of the QCD strings and the KK modes is of order $e^{-\Phi_{NS,0}} N$ so that to decouple to scales we need to send $e^{\Phi_{NS,0}} \to \infty$ [5, 110, 111]. This forces us to use the S-dual D5 solution (see eq. (65))

$$
\begin{aligned}
ds^2_{str} &= e^{\Phi}\left[dx_4^2 + N\alpha'[d\rho^2 + e^{2h(\rho)}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{4}\sum_a (w^a - A^a)^2]\right], \\
F_{(3)} &= \frac{N\alpha'}{4}[-(w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \sum_a F^a \wedge (w^a - A^a)], \\
e^{2\Phi} &= e^{2\Phi_0}\frac{\sinh 2\rho}{2e^{h(\rho)}}.
\end{aligned}
\tag{102}
$$

Notice that for the D5 solution the (squared) radius of the IR three-sphere is $e^{\Phi_0} N\alpha'$ and the smallest value of the string coupling is $e^{\Phi_0} = e^{-\Phi_{NS,0}}$, reached for $\rho = 0$. The string coupling grows with ρ and eventually will diverge in the UV. We would also like to stress that the solution in the IR is very similar to the Klebanov-Strassler one [4], which will be discussed in Section 5.3, since both models involve an IR geometry that corresponds to a deformed conifold.

The rest of this section is devoted to the analysis of the properties of the MN solution. In this discussion we will always use the D5 solution.

We first show how confinement is realized in the MN model. The natural candidate for a QCD string is a fundamental string. Using a standard argument in AdS/CFT, we can see that there is confinement and compute the QCD string tension. Confinement is expected because the space-time components of the metric at $\rho = 0$ are non-vanishing. The value of a Wilson loop, indeed, can be computed using a fundamental string coming from infinity, with endpoints on the boundary at $\rho = \infty$. The string will minimize its energy by reaching $\rho = 0$ where the metric components $\sqrt{g_{xx}g_{tt}}$ have a minimum. All the relevant contribution to the energy between two external sources is then due to a string sitting at $\rho = 0$ and stretched in the x direction. The estimate for the string tension is then easily obtained [5, 112]

$$
T_s = \frac{e^{\Phi_0}}{2\pi\alpha'}.
\tag{103}
$$

As reminded in Section 5.1, in a confining $\mathcal{N} = 1$ $SU(N)$ SYM theory, we can define $N - 1$ different type of strings connecting different external sources. It has been shown

in [113] that the ratio of the tensions in the MN solution follows the sine formula

$$\frac{T_k}{T_{k'}} = \frac{\sin k\pi/N}{\sin k'\pi/N}.$$

(104)

This formula can be obtained by considering the IR metric $\mathbb{R}^7 \times S^3$. The QCD string is described by a bound state of k fundamental strings that minimize its energy by expanding in a D3-brane with k units of flux wrapping an S^2 inside S^3. The tension of the brane is balanced by the space-time three-form and a stable configuration is obtained for a specific, k-dependent, S^2 inside S^3. Formula (104) then follows from a ratio of volumes [113]. The ratio of k-tensions is a genuine, non BPS, prediction of the MN solution.

Let us now discuss the spontaneous breaking of the chiral symmetry. From our general discussion in Section 5.1 we expect various phenomena:

- The anomaly. In quantum field theory, $U(1)_R$ is anomalous and broken to $\mathbb{Z}_{2N}$ by instantonic effects. These non-perturbative effects in the field theory are already captured at the supergravity level. The existence of an anomaly can be detected with an UV computation in quantum field theory and therefore should be already visible in the UV region of the solution. In the MN solution the $U(1)_R$ symmetry acts as a shift of the angle ψ. The UV form of the metric is invariant under such shift, but this is not the case for the RR two-form $C_{(2)} \sim -\frac{N\alpha'}{2}\psi \sin\theta d\theta \wedge d\phi$. In particular, the flux $\frac{1}{2\pi\alpha'}\int_{S^2} C_{(2)}$ varies by $-N\delta\psi$ under a shift of ψ. Since, as we discussed in Section 3.3, the flux is periodic with period 2π, the only allowed transformations are those with $\delta\psi = \frac{2\pi n}{N}$ [114]: the R-symmetry is then broken to the $\mathbb{Z}_{2N}$ subgroup $\psi \to \psi + 2\pi k/N$ (ψ has period 4π). This is a purely supergravity result. Nevertheless, we can explicitly see the role of instantons in the anomaly by considering an instantonic probe. In the D5 solution, instantons are identified with an Euclidean D1-brane wrapping the same S^2 as the D5-branes. An important point is that the S^2 where the branes can be wrapped in a stable way is not the original S^2 parameterized by (θ, ϕ), but it is mixed with an S^2 contained in the transverse S^3 [21]. Indeed, since in the UV (where a is vanishing) $d\psi$ only appears in the combination $w^3 - A^3 = d\psi + \cos\tilde{\theta}d\tilde{\phi} - \cos\theta d\phi$, a stable D1-brane can live at a fixed ψ only when it wraps the sphere $\theta = \tilde{\theta}, \phi = \tilde{\phi}$. The action for a D1-brane should reproduce the coupling constant and theta angle of the gauge theory,

$$-\frac{1}{2\pi\alpha'}\int_{S^2} e^{-\Phi}\sqrt{G} + \frac{i}{2\pi\alpha'}\int_{S^2} C_{(2)} = -\frac{8\pi^2}{g_{YM}^2} + i\theta_{YM}.$$

(105)

[21]"For a detailed discussion of the geometry of these cycles and their relation to the anomaly see [115].

In particular, the theta angle is $\theta_{YM} \sim -N\psi$. The anomaly of $U(1)_R$ and its breaking to $\mathbb{Z}_{2N}$ are evident in the shift of the theta angle. Only those transformations that shift θ_{YM} by a multiple of 2π remain good symmetries.

- The N vacua. The spontaneous breaking $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$ is manifest in eq. (102). $\mathbb{Z}_{2N}$ indeed is a good symmetry only in the UV. It is broken to the $\mathbb{Z}_2$ symmetry $\psi \rightarrow \psi + 2\pi$ by the explicit form of the supergravity solution, due to the presence of a non-zero a. In general, different vacua of the theory correspond to different regular solutions with the same asymptotic behavior. In our case, one can show that precisely N solutions are nonsingular, corresponding to the N vacua of $\mathcal{N} = 1$ SYM [5]. The vacua are permuted by the elements of $\mathbb{Z}_{2N}$, which multiply a by a phase. The N regular solutions are then as in eq. (102) with a replaced by $a = e^{2\pi n i / N} \frac{2\rho}{\sinh 2\rho}$, $n = 0, ..., N - 1$.

- The gaugino condensate. We expect that each vacuum is associated with a non-zero gaugino condensate. We can show this [116] by using the AdS/CFT philosophy of Section 2.3. Recall that we can determine if a given operator has a VEV by looking at the asymptotic UV behavior of the dual supergravity field. One solve the asymptotic second order equations of motion for the field, and associates a non-normalizable solution with a deformation by the dual operator and a normalizable one with a VEV. In our case, the background is not asymptotically AdS but we can still try to see what this philosophy suggests. The natural candidate for the field dual to the gaugino condensate is a, since it has the right $U(1)_R$ charge and its phase distinguishes among the various vacua. From the reduction on S^2 of the 5-brane coupling

$$A_{ij}^\mu \bar{\Psi} \gamma^\mu \Gamma^{ij} \Psi, \tag{106}$$

one can indeed show that a couples to the fermionic bilinear $a\bar{\lambda}^c\lambda$, corresponding to the gaugino condensate [116]. The asymptotic solutions of the second order equations for a can be derived from the Lagrangian (97) [108]

$$a \sim \frac{Y}{\sqrt{2\rho}} + 2C\rho e^{-2\rho}. \tag{107}$$

In the BPS MN solution $Y = 0$ and $C \neq 0$, so the AdS/CFT dictionary adapted to our case suggests that a behaves as appropriate for a condensate. The quantity C determines the value of the condensate in a given vacuum. We can also strengthen this interpretation by studying the UV radial dependence of the field a. For this we have to choose an energy/radius relation. As recalled in Section 2.1, this relation is ambiguous for D5-branes [34]. We can choose to determine the relation by looking

at the coupling constant behavior[22]. An estimate for the UV behavior of the gauge coupling follows from our general discussion in Section 3.1

$$\frac{1}{g_{YM}^2} = \frac{1}{2(2\pi)^3\alpha'} \int_{S^2} e^{-\Phi}\sqrt{G} \to \frac{N}{4\pi^2}\rho. \tag{108}$$

This result has been obtained by using a stack of D5-branes, but can be equivalently derived by using the action for an instanton, eq. (105). If we try to enforce the one-loop gauge theory result $\frac{1}{g_{YM}^2} \sim \frac{3N}{8\pi^2}\log\mu$, we obtain the asymptotic radius/energy relation $\rho \sim \frac{3}{2}\log\mu$. From $a \sim \rho e^{-2\rho}$ we see that a scales as $1/\mu^3$, as appropriate for a protected dimension three operator [116].

- Domain walls. In a theory with spontaneous breaking of $\mathbb{Z}_{2N}$ and multiple vacua, we expect the existence of domain walls. In the string solution, they correspond to D5-branes wrapped on S^3, located at $\rho = 0$ in order to minimize the energy. One can estimate the tension T_{DW} of the domain wall from the fact that the metric in the IR is approximately of the form $\mathbb{R}^7 \times S^3$ and the radius of the three-sphere goes as $\sqrt{e^{\Phi_0}N\alpha'}$. We have

$$T_{DW} \sim \frac{1}{\alpha'^3}\int_{S^3} e^{-\Phi}\sqrt{G} = \frac{e^{2\Phi_0}N^{3/2}}{\alpha'^{3/2}}. \tag{109}$$

Since a fundamental string can end on a D5-brane and the QCD string is a fundamental string, we see that a QCD string can end on a domain wall.

We can also estimate in supergravity the masses of glueballs and Kaluza-Klein states. These can be determined by studying the equations of motion for supergravity fields in the background (102). The masses of the lightest glueballs are given by the lower value of the gravitational redshift. The order of magnitude for the KK masses can be deduced from the inverse radius of the three-sphere. Both of them are proportional to $1/\sqrt{\alpha'N}$. We see from formula (103) that the ratio T_s/m_{KK}^2 is of order Ne^{Φ_0}. In order to decouple the KK and gauge theory scales we need $Ne^{\Phi_0} \ll 1$, a condition that requires large curvatures in the IR and cannot be obtained in the supergravity approximation. Nevertheless, as we have seen, the theory described by the supergravity solution exhibits confinement and chiral symmetry breaking and shares many properties of its cousin, pure $\mathcal{N} = 1$ SYM. We then have a family of string backgrounds dual to gauge theories that are a one-parameter generalization of ordinary SYM. The extra parameter can be identified with Ne^{Φ_0}. Notice that the scales determined by the string and domain-wall tensions

---

[22]The following identification can be motivated by the fact that, below the compactification scale, the one-loop β-function is fixed by the chiral anomaly and can be extrapolated from the weak coupling result.

are not equal and explicitly depend on the extra parameter. In this family, pure SYM corresponds to a strongly coupled string background.

We end this Section with another quantitative property that is not determined by the symmetries of the problem. As noticed in [93], by using the full background and making some assumptions, one can make more precise predictions on the behavior of the beta function. Since there is no moduli space in $\mathcal{N} = 1$, there is no intrinsic prescription for computing the behavior of the gauge coupling with the scale; both definitions of coupling and scale are ambiguous. In order to fix the radius/energy relation, the authors in [93] proposed to enforce the equation $a = \frac{\Lambda^3}{\mu^3}$ that defines the gaugino condensate. They also proposed to define the coupling constant as that seen by a stack of D5-branes for all values of ρ. Using (108) with S^2 being the two sphere at $(\theta = \tilde{\theta}, \phi = \tilde{\phi})^{23}$ [117], one obtains the same result as in the UV limit (108), up to exponentially suppressed terms[24]. These two pieces of information uniquely determine the β-function [93, 117]

$$\beta = -\frac{3Ng_{YM}^3}{16\pi^2}\left(1 - \frac{Ng_{YM}^2}{8\pi^2}\right)^{-1}. \tag{110}$$

This formula coincides with the NSVZ β−function [30]. In field theory it gives the full perturbative result and it is not corrected by instanton contributions. In supergravity it is exact up to exponential terms, which can be interpreted as fractional instantonic corrections. The non-trivial content of this formula is the analyticity in g_{YM} and the one and two-loop coefficients, that are the only scheme-independent objects. It is not clear why this result is captured by the supergravity approximation which only describes a cousin of the ordinary pure $SU(N)$ gauge theory.

For other works on the MN solution we refer to [118].

5.3 $\mathcal{N} = 1$ SYM from fractional D-branes on the conifold

As discussed in Sections 1.5 and 3.3, N physical and M fractional D3-branes placed at the apex of a conifold realize on their world-volume a four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(N + M) \times SU(N)$. There is also an $SU(2) \times SU(2) \times U(1)_R$ global symmetry inherited from the isometries of $T^{1,1}$. The gauge theory is coupled to bi-fundamental chiral multiplets A and B, interacting through the superpotential W given in eq. (25). A and B transform in the $(\boldsymbol{N + M}, \bar{\boldsymbol{N}})$ and in the $(\overline{\boldsymbol{N + M}}, \boldsymbol{N})$ representation of the gauge group, respectively, and are a doublet of one of the global $SU(2)$'s each. In this Section we will describe the corresponding Type IIB solution. On the supergravity side, we expect to find a metric of warped form

$$ds_{10}^2 = h^{-1/2}(\tau)dx_4^2 + h^{1/2}(\tau)ds_6^2 \,, \tag{111}$$

---

[23]It is the same cycle we introduced to discuss instantons in the MN background.

[24]Some mistakes in [93] were eventually corrected in [117].

with non-trivial $F_{(3)}$ and $F_{(5)}$ RR-fields induced by the D5 and D3 sources. Since, as discussed in Section 3.3, the integral $\int_{S^2} B_{(2)}$ determines the difference of the gauge couplings, we also expect a non-zero $H_{(3)}$ reflecting the running of the couplings in the non-conformal theory. A supersymmetric solution with this minimal set of fields and internal metric given by the conifold one, was found in [77], but it has a naked singularity in the IR. In [4], a regular solution was found by considering a deformed conifold instead of the original singular one. We will see that the deformation of the conifold corresponds to the requirement that the supergravity background knows of the gaugino condensation in the dual field theory. This situation is then similar to that occurring in the MN solution.

In terms of complex geometry, the deformation of the singular conifold $\sum w_a^2 = 0$ is described by the equation in $\mathbb{C}^4$

$$\sum w_a^2 = \varepsilon^2. \tag{112}$$

The deformation consists in blowing-up an S^3 at the apex of the conifold, so to obtain a smooth manifold. The deformed conifold metric can be written as

$$ds_6^2 = \frac{\varepsilon^{4/3}}{2} K(\tau) \left[\frac{(d\tau^2 + g_5^2)}{3K^3(\tau)} + \cosh^2 \frac{\tau}{2} (g_3^2 + g_4^2) + \sinh^2 \frac{\tau}{2} (g_1^2 + g_2^2) \right], \tag{113}$$

where $K(\tau) = (2^{1/3} \sinh \tau)^{-1} (\sinh(2\tau) - 2\tau)^{1/3}$ and the g_i are as in (22).

The regular solution is known as the Klebanov Strassler solution [4]. It consists of a metric of the form (111), with ds_6 as in (113), warp factor given by

$$h(\tau) = (g_s M \alpha')^2 2^{2/3} \varepsilon^{-8/3} \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh 2x - 2x)^{1/3}, \tag{114}$$

and antisymmetric fields

$$
\begin{aligned}
B_{(2)} &= \frac{g_s M \alpha'}{2} [f(\tau) g^1 \wedge g^2 + k(\tau) g^3 \wedge g^4] , \\
F_{(3)} &= \frac{M \alpha'}{2} \left[(1 - F) g^5 \wedge g^3 \wedge g^4 + F g^5 \wedge g^1 \wedge g^2 + F' d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \right] , \\
F_{(5)} &= \mathcal{F}_{(5)} + \star \mathcal{F}_{(5)}, \\
\mathcal{F}_{(5)} &= B_{(2)} \wedge F_{(3)} = \frac{g_s M^2 (\alpha')^2}{4} [f(1 - F) + kF] g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5.
\end{aligned}
\tag{115}
$$

The functions of τ appearing in the previous formulae read

$$
\begin{aligned}
F(\tau) &= \frac{\sinh \tau - \tau}{2 \sinh \tau}, \\
f(\tau) &= \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau - 1), \\
k(\tau) &= \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1).
\end{aligned}
\tag{116}
$$

The complex dilaton of Type IIB is constant and this allows for a small string coupling everywhere.

Let us examine the asymptotic behavior of the solution. For large values of τ (which correspond to the UV limit of the dual gauge theory) it is convenient to introduce the radial coordinate $r \sim \varepsilon^{2/3} e^{\tau/3}$. The metric thus reduces to

$$ds_{10}^2 \to h^{-1/2}(r)dx_4^2 + h^{1/2}(r)(dr^2 + r^2 ds_{T^{1,1}}^2), \qquad (117)$$

with $r_s \sim \varepsilon^{2/3}$ and $h(r) = \frac{81(\alpha' g_s M)^2}{8r^4}\log(r/r_s)$. It can be viewed, in some sense, as a logarithmic deformation of $AdS_5 \times T^{1,1}$. This was the solution first found in [77]. If we would allow r to range in $[0,\infty)$ it would be singular for $r = r_s$. In this limit, the RR and NSNS forms reduce to

$$F_{(3)} \to \frac{M\alpha'}{4}g_5 \wedge (g_1 \wedge g_2 + g_3 \wedge g_4), \quad B_{(2)} \to \frac{3g_s M\alpha'}{4}\log(r/r_s)(\, g_1 \wedge g_2 + \, g_3 \wedge g_4),$$

$$\mathcal{F}_{(5)} \to \frac{3g_s M^2(\alpha')^2}{8}\log(r/r_s)g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5. \qquad (118)$$

For small τ the metric instead approximates to

$$ds_{10}^2 \to \frac{\varepsilon^{4/3}}{bg_s M\alpha'}dx_\mu dx^\mu + c(g_s M\alpha')\left[\frac{1}{2}(d\tau^2 + g_5^2) + (g_3^2 + g_4^2) + \frac{1}{4}\tau^2(g_1^2 + g_2^2)\right], \quad (119)$$

where b and c are numerical constants. From the definition of the forms g_i (eq.(22)), it is easy to see that the angular part splits in a non-vanishing S^3 and a shrinking S^2 fibered over it, just as in the MN case. The curvature is controlled by the value of $g_s M$, and it is small when this parameter is large. The antisymmetric fields $B_{(2)}, F_{(5)}$ go to zero in the limit, while $F_3 \to (M\alpha'/2)g_5 \wedge g_3 \wedge g_4$ is supported only by the non-vanishing S^3.

Let us now see how the KS solution encodes the properties of the dual gauge theory. Since there exist many good reviews in the literature, we will just sketch the basic results referring the reader to [4, 119] for more details. It is believed that the $SU(N+M) \times SU(N)$ theory exhibits a series of Seiberg dualities until it eventually reduces in the deep IR to pure $SU(M)$. At each step of the cascade, the group is $SU(N + M - kM) \times SU(N - kM)$. The strongly coupled factor $SU(N + M - kM)$ undergoes a Seiberg duality to $SU(N - M - kM)$, while the other factor remains inert[25]. As a result, k is increased by one unit. In the KS solution, this can be seen from the UV limit of the RR five-form field strength (118) which can be rewritten as

$$\mathcal{F}_{(5)} \sim N_{eff}(r)vol(T^{1,1}), \qquad N_{eff}(r) = N + \frac{3}{2\pi}g_s M^2 \log(r/r_0). \qquad (120)$$

---

[25]We refer to [9, 101] for a detailed discussion of Seiberg duality. We simply mention that this duality occur for a $SU(N)$ theory with $N_f > N + 1$ flavors of quark chiral superfields A_i, $\bar{A}_i$, $i = 1, ..., N_f$, in the $\mathbf{N}$, $\overline{\mathbf{N}}$ representations. In this case the theory is dual to another $\mathcal{N} = 1$ SYM with $SU(N_f - N)$ gauge group, N_f flavors C_i, $\bar{C}_i$, and an extra gauge singlet chiral superfield N^{ij} interacting by the superpotential $W = CN\bar{C}$.

We introduced a convenient reference scale r_0 defined so that the effective D3-charge $N_{eff}(r_0) = N$. The logarithmic decreasing of N_{eff} with the radius was interpreted in [77] as a decreasing in the rank of the dual gauge theory group as the theory flows to the IR. At the UV scale $r = r_0$, $N_{eff} = N$ and the dual field theory has $SU(N+M) \times SU(N)$ as gauge group. At $r_k = r_0 \exp(-2\pi k/3g_s M)$, with k integer, the dual gauge group is $SU(N-kM+M) \times SU(N-kM)$. If $N = kM$, we thus find that after k cascade steps the gauge group flows to $SU(M)$. The UV completion of the theory is somewhat peculiar. The inverse cascade never stops. In a sense, the UV limit is a $SU(\infty) \times SU(\infty)$ gauge theory.

The metric in eq. (117) can be used to study the UV properties of the $SU(N + M) \times SU(N)$ gauge theory when $M \ll N$. Indeed, the curvature, which is determined by Ng_s at the reference scale r_0, decreases for larger values of r. Moreover, if Mg_s is sufficiently small the cascade steps will be well separated. In these conditions, the singular metric (117), which is a logarithmic deformation of $AdS_5 \times T^{1,1}$, will give a convenient description of the almost conformal theory $SU(N + M) \times SU(N)$. As shown in Section 3.3, the gauge couplings are related to some of the supergravity moduli

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} = \frac{1}{4\pi g_s}; \qquad \frac{1}{g_1^2} - \frac{1}{g_2^2} = \frac{1}{4\pi^2 g_s}\left(\frac{1}{2\pi\alpha'}\int_{S^2} B_{(2)} - \pi\right). \tag{121}$$

In order to use this formula, we must identify the cycle S^2 where the D5 branes are wrapped. As discussed in Section 1.5, we can identify the S^2 with $\theta_1 = \theta_2, \phi_1 = -\phi_2$. In the large r limit, we thus find that the sum of the gauge couplings is constant while (see (118)) the difference runs as

$$\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} = 3M\log(r/r_s) = 3M\log(\mu/\Lambda). \tag{122}$$

The last equality in the above equation requires a specific choice of how to relate the radial coordinate to the energy scale of the field theory. Also for the KS solution there are ambiguities in determining the radius/energy relation[26]. We use the same relation as for the conformal $AdS_5 \times T^{1,1}$ solution, $r/r_s = \mu/\Lambda$, Λ being the IR scale. This is obtained by considering the energy of a string stretched in the background. One can show that eq. (122) reproduces, up to orders M/N, the UV gauge theory result obtained from the exact NSVZ beta function [30] for $\mathcal{N} = 1$ gauge theories. Using formula (26)

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[26]The different methods for computing the radius/energy relation give different results [119]. However, if we are only interested in the leading logarithmic UV behavior all methods agree. The comments on the holographic dual of the gaugino condensate we will give in the following also confirm the above identification.

we can indeed write

$$\frac{4\pi^2}{g_1^2} = \frac{1}{2}(3(N+M) - N(6 - 2\Delta_A - 2\Delta_B))\ln(\mu/\Lambda),$$

$$\frac{4\pi^2}{g_2^2} = \frac{1}{2}(3N - (N+M)(6 - 2\Delta_A - 2\Delta_B))\ln(\mu/\Lambda). \tag{123}$$

At leading order in M/N, $\Delta_A + \Delta_B = 3/2$, which is the result for the conformal case. The difference of the two equations in (123) then reproduces the supergravity result. We refer the reader interested in a more detailed comparison to [4, 119].

The IR region of the KS solution should describe a pure $SU(M)$ SYM. We notice however that, in the supergravity approximation, $g_s M \to \infty$ and so the cascade steps are not well separated and the additional massive fields of the original theory $SU(N + M) \times SU(M)$ are not decoupled. As usual, we can get a pure SYM theory only beyond the supergravity regime. The supergravity solution is dual to a four-dimensional gauge theory with a large number of massive matter fields. As its cousin, pure SYM, the theory confines and presents the standard pattern of chiral symmetry breaking. The analysis is similar to that for the MN solution. The $U(1)_R$ symmetry of the theory corresponds to shifts of the transverse angular variable ψ. The transformation law for the RR field $C_{(2)}$ (or the analysis of an instantonic probe) shows that $U(1)_R$ is anomalous and broken to $\mathbb{Z}_{2M}$ [114, 119]. The spontaneous symmetry breaking $\mathbb{Z}_{2M} \to \mathbb{Z}_2$ is manifest in the full KS solution. The IR expression (119), which depends on ψ through $\cos\psi$ and $\sin\psi$, has in fact only a $\mathbb{Z}_2$ invariance under $\psi \to \psi + 2\pi$. The breaking is also evident in eq. (112) that is not invariant under arbitrary phase shifts of the w_a, but only under $w_a \to -w_a$. The IR limit of the KS background also shows that the dual field theory confines. This is due to the fact that the warp factor approaches a constant value $h \sim (g_s M \alpha')^2 \varepsilon^{-8/3}$ when $\tau \to 0$. The tension for confining strings $\alpha' T_s \sim h^{-1/2}$ is thus of the order $\varepsilon^{4/3}/g_s M \alpha'$, and the glueball masses scale as $\varepsilon^{2/3}/g_s M \alpha'$. The deformation parameter thus gives the fundamental scale of the dual field theory. The KS background also allows to extract information about other field theory features like baryons, domain walls, etc. (we refer the interested reader to [112, 119]).

In the final part of this Section we derive the effective Lagrangian for the KS solution. This will fill a gap in the previous discussion and show how the KS solution can be derived from a set of first order equations. This will also allow us to give the map between at least some of the fields appearing in the KS solution and the gauge invariant operators in the dual gauge theory. In particular we will identify the holographic duals of the gaugino bilinears, and check that their behavior is consistent with the existence of a condensate in the dual field theory. To this purpose it is convenient to write the ansatz for the solution

in the form [107][27]

$$
\begin{aligned}
ds^2 &= 2^{1/2}3^{3/4}\left[e^{-5q(u)}(du^2 + e^{2Y(u)}dx_\mu dx^\mu) + ds_{int}^2\right], \\
ds_{int}^2 &= e^{3q(u)}\left[\frac{e^{-8p(u)}}{9}g_5^2 + \frac{e^{2p(u)+y(u)}}{6}(g_1^2 + g_2^2) + \frac{e^{2p(u)-y(u)}}{6}(g_3^2 + g_4^2)\right], \\
B_{(2)} &= -(\tilde{f}(u)\, g_1 \wedge g_2 + \tilde{k}(u)\, g_3 \wedge g_4), \quad \Phi = \Phi(u), \\
F_{(3)} &= 2Pg_5 \wedge g_3 \wedge g_4 + d[\tilde{F}(u)(g_1 \wedge g_3 + g_2 \wedge g_4)], \\
F_{(5)} &= \mathcal{F}_{(5)} + \star\mathcal{F}_{(5)}, \quad \mathcal{F}_{(5)} = -L(u)g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5.
\end{aligned}
\tag{124}
$$

This is a quite general ansatz since it includes the conformal case $AdS_5 \times T^{1,1}$ as well as the singular and regular non-conformal solutions. In particular, the fields $y, \tilde{f} - \tilde{k}$ distinguish between the singular ($y = \tilde{f} - \tilde{k} = 0$) and regular ($y, \tilde{f} - \tilde{k} \neq 0$) conifold geometries. For convenience, we have rescaled the functions in the KS solution as follows: $\tilde{f} = -2g_sPf$, $\tilde{k} = -2g_sPk$, $\tilde{F} = 2PF$. The supergravity equations for $F_{(5)}$ set $L(u) = Q + (\tilde{k} - \tilde{f})\tilde{F} + 2P\tilde{f}$, where Q and P are constants related to the number of physical and fractional branes. More precisely $P = \frac{M\alpha'}{4}$; for $P = 0$, Q is proportional to N, while for $P \neq 0$ it can be reabsorbed in a redefinition of $\tilde{f}, \tilde{k}$.

Using the ansatz (124) and integrating over the $10d$ coordinates, the Type IIB Lagrangian reduces to the following effective action [107, 120]

$$
S = \int du e^{4Y}\left(3(\dot{Y})^2 - \frac{1}{2}G_{ab}\dot{\varphi}^a\dot{\varphi}^b - V(\varphi)\right),
\tag{125}
$$

supported by the constraint $3\dot{Y}^2 - \frac{1}{2}G_{ab}\dot{\varphi}^a\dot{\varphi}^b + V(\varphi) = 0$, with

$$
\begin{aligned}
G_{ab}\dot{\varphi}^a\dot{\varphi}^b &= 15\dot{q}^2 + 10\dot{p}^2 + \frac{\dot{y}^2}{2} + \frac{\dot{\Phi}^2}{4} \\
&+ e^{-\Phi-6q-4p}(e^{-2y}\frac{\sqrt{3}\dot{\tilde{f}}^2}{2} + e^{2y}\frac{\sqrt{3}\dot{\tilde{k}}^2}{2}) + \sqrt{3}e^{\Phi-6q-4p}\dot{\tilde{F}}^2, \\
V(\varphi) &= e^{-8q}[e^{-12p} - 6e^{-2p}\cosh y + \frac{9}{4}e^{8p}(\sinh y)^2] + \frac{9\sqrt{3}}{8}e^{4p-14q}e^{-\Phi}(\tilde{f} - \tilde{k})^2 \\
&+ \frac{9\sqrt{3}}{4}e^{4p-14q+\Phi}[e^{-2y}\tilde{F}^2 + e^{2y}(2P - \tilde{F})^2] + \frac{27}{2}e^{-20q}L^2.
\end{aligned}
\tag{126}
$$

For supersymmetric solutions, the second order equations of motion from eq. (125) can be reduced to a set of first order ones, since the previous Lagrangian admits a superpotential (in the sense of Section 2.3)

$$
W = -3e^{4p-4q}\cosh y - 2e^{-6p-4q} - 3\sqrt{3}e^{-10q}\left(Q + \tilde{F}(\tilde{k} - \tilde{f}) + 2P\tilde{f}\right).
\tag{127}
$$

---

[27]Since we want to use the methods of Section 2, we introduce a radial coordinate u analogous to that of Section 2.

One can check that the KS solution satisfies the first order equations following from this superpotential (the KS radial coordinate is related to u by $d\tau = 3e^{-4q+4p}du$ and $e^y = \tanh\tau/2$). The dilaton does not appear in W and so it is constant for all the possible solutions.

In absence of fractional branes, $P = 0$, the potential V in (126) has an $\mathcal{N} = 1$ critical point, corresponding to the conformal background $AdS_5 \times T^{1,1}$ generated by a stack of physical D3-branes. If we choose to rescale $Q = -2/3\sqrt{3}$, the critical point is at $q = p = y = \tilde{F} = 0$ (with arbitrary Φ and $\tilde{k} = \tilde{f}$). With this conventions the AdS radius is $R = 1$. If we consider P (and so M/N) as a small deformation of the conformal background, we can still relate the supergravity fields we turn on to deformations or changes in the vacuum of the theory, using the rules of the AdS/CFT correspondence [120]. For $P = 0$, the potential and superpotential around the critical point read

$$
\begin{aligned}
V &\approx -3 + 32q^2 + 12p^2 - 3y^2 + 21\xi_1^2 - 3\xi_2^2, \\
W &\approx -3 + 4q^2 - 6p^2 - 3y^2 + 3\xi_1^2 - 3\xi_2^2,
\end{aligned}
\tag{128}
$$

where all the scalars have been redefined in order to have diagonal and canonically normalized kinetic terms. The fields p, q, y have only been rescaled while the ξ have been defined as $^4\sqrt{12}\tilde{F} = \xi_1 - \xi_2$, $^4\sqrt{3}(\tilde{f} - \tilde{k}) = \sqrt{2}(\xi_1 + \xi_2)$; we also define $2s = ^4\sqrt{3}(\tilde{f} + \tilde{k})$. For canonically normalized scalars, the quadratic terms in the expansion of the potential give the masses of the supergravity fields. As discussed in Section 1.2, with the normalization chosen, the mass/dimension relation is $\Delta = 2 + \sqrt{4 + m^2}$ and we thus find that q, p, y, ξ_1, ξ_2 correspond to operators of dimension $\Delta = 8, 6, 3, 7, 3$ respectively. The fields Φ and s do not appear in the superpotential: their mass squared is zero and thus they correspond to operators of dimension $\Delta = 4$. Using the results in [31], we can tentatively identify the fields with gauge theory operators in the following multiplets [120]

$$
\begin{array}{llll}
q, p \to \mathrm{Tr}(W^2 \bar{W}^2), & \Delta = 8, 6; & \xi_1 \to \mathrm{Tr}(A\bar{A} + B\bar{B})W^2, & \Delta = 7; \\
\xi_2 \to \mathrm{Tr}(W_1^2 + W_2^2), & \Delta = 3; & y \to \mathrm{Tr}(W_1^2 - W_2^2), & \Delta = 3.
\end{array}
\tag{129}
$$

In particular we see that the fields y, ξ_2 can be read as holographic duals of the gaugino bilinears [120]. The field s is the massless field $\int_{S^2} B_{(2)}$ associated with a marginal direction in the CFT [22, 40, 67]. The corresponding operator is $\mathrm{Tr}(F_1^2 - F_2^2)$. Finally, the dilaton Φ corresponds to $\mathrm{Tr}(F_1^2 + F_2^2)$.

When $P \neq 0$ a tadpole term for $s \sim \tilde{f} + \tilde{k}$ is introduced in the effective potential which makes the coupling constant run as in (122). In the limit where the solution is a small deformation of the conformal case, we can still reasonably use the identifications made above. From the quadratic terms in the W expansion one can read the leading asymptotic behavior of the fields near the critical point and, applying the rules of Section 2.3, tell whether they correspond to a deformation $(\Delta - 4)$ or a choice of a different vacuum $(-\Delta)$.

As the reader can see, q and ξ_1 correspond to deformations, while the other fields are related to VEVs. The fields y, ξ_2 can thus be related to field theory vacua with a non zero gaugino condensate [120]. As a check one can explore their asymptotic behavior using the full KS solution. In the UV the fields ξ_2, y go like ε^2/r^3 and this is indeed appropriate for a protected dimension 3 operator [112]. As mentioned before, ξ_2 and y are precisely the fields that control the deformation of the conifold. Thus, their asymptotic behavior confirms the relation between the deformation of the geometry and the chiral symmetry breaking on the field theory side.

For further works on the KS model we refer to [121].

5.4 Supergravity duals of $\mathcal{N} = 1^*$

In this Section we will briefly describe the known supergravity solutions dual to $\mathcal{N} = 1^*$, referring the reader to the original papers for the details of the computations [53, 48, 50]. $\mathcal{N} = 1^*$ theories are obtained deforming $\mathcal{N} = 4$ with a supersymmetric mass term for the three chiral superfields. The potential then reads (here we consider for simplicity the case of equal masses, the generalization being straightforward)

$$\int d\theta^2 \left(2\sqrt{2}\, Tr(\Phi_1[\Phi_2, \Phi_3]) + m \sum_{i=1}^{3} (\Phi_i)^2 \right). \tag{130}$$

The theory possesses a very rich vacuum structure, parameterized by the N dimensional, generally reducible, representations of $SU(2)$. These are indeed the solutions of the F-term equations for supersymmetric vacua $[\Phi_i, \Phi_j] = -\frac{m}{\sqrt{2}}\epsilon_{ijk}\Phi_k$. For a generic vacuum the matrices Φ will have a block diagonal structure, where the blocks represent irreducible $SU(2)$ representations of different dimension n_i (including dimension 1) such that $\sum_i n_i = N$. A detailed discussion of the classical and quantum properties of the various vacua can be found in [48, 122]. Here we only focus on the two cases that have the simplest interpretation on the supergravity side. One is the Higgs vacuum, corresponding to the N dimensional irreducible representation of $SU(2)$. In this case the gauge group is completely broken and there is a mass gap already at the classical level. The other vacuum is characterized by zero VEVs for the scalar fields, $< \Phi >= 0$ (i.e. N copies of the trivial representation). $SU(N)$ is unbroken and the theory is expected to confine and to have N distinct vacua parameterized by the gaugino condensate $< \lambda\lambda >$.

Following the philosophy of Section 2, the supergravity duals of $\mathcal{N} = 1^*$ should be given by Type IIB solutions with non-zero profile for the modes corresponding to the mass deformation (130). From the perspective of 5d $\mathcal{N} = 8$ supergravity, fermion bilinears are dual to scalars in the $\mathbf{10}$ of $SU(4)$. The supersymmetric mass term for the chiral multiplets, m_{ij}, transforms as the $\mathbf{6}$ of $SU(3) \subset SU(4)$, and the corresponding

supergravity mode appears in the decomposition of the $\mathbf{10} \to \mathbf{1} + \mathbf{6} + \mathbf{3}$ of $SU(4)$ under $SU(3) \times U(1)$. The term $\mathbf{1}$ in this decomposition corresponds instead to the scalar σ dual to the gaugino condensate in $\mathcal{N} = 1$ SYM. If we further require an $SO(3)$ symmetry, by taking equal masses $m_{ij} = m\,\delta_{ij}$, one can show that the Lagrangian can be consistently truncated to the fields m and σ. In units $R = 1$ it reads [53]

$$
\begin{aligned}
\mathcal{L} \;=\; \sqrt{-g}\bigg\{ &-\frac{\mathcal{R}}{4} + \frac{1}{2}(\partial m)^2 + \frac{1}{2}(\partial \sigma)^2 + \\
&-\frac{3}{8}\left[\left(\cosh \frac{2m}{\sqrt{3}}\right)^2 + 4\cosh \frac{2m}{\sqrt{3}}\cosh 2\sigma - (\cosh 2\sigma)^2 + 4\right]\bigg\},
\end{aligned}
\tag{131}
$$

and admits the superpotential $W = -\frac{3}{2}\left(\cosh\frac{2m}{\sqrt{3}} + \cosh 2\sigma\right)$. The explicit supersymmetric solution of the $5d$ equations of motion was found in [53], and consists of a family of solutions depending on two independent parameters. Using the rules discussed in Section 2.3 we can show that these solutions correspond to vacua with a non-zero gaugino condensate. Indeed, by expanding the superpotential around the UV fixed point $W \sim -3 - m^2 - 3\sigma^2$ we see that m and σ have the right asymptotic behavior to be identified with a mass term for the matter superfields and the gaugino condensate, respectively. One can further check that the theory has a mass gap. It is then natural to identify these solutions with the confining vacua of $\mathcal{N} = 1^*$. However, the gaugino condensate in the solution is a continuous parameter rather than a discrete one as expected from field theory. Moreover, the solutions have a naked singularity, which is still present in the ten dimensional lift [50], thus making the physical interpretation not very clear. Nevertheless, the five-dimensional solution, which is one of the few analytically known, has proved to be useful for computing Green functions along the RG flow [62, 64]. Interestingly, sensible results have been obtained despite the presence of a singularity.

The analysis of the five dimensional flow seems to suggest that the supergravity approximation is not sufficient to describe duals of $\mathcal{N} = 1^*$ theories and that a stringy mechanism is required to resolve the singularity. One possibility is to introduce D-brane sources in the $AdS_5 \times S^5$ background: this is the idea behind the Polchinski Strassler solution [48]. In this case, the sources are D3-branes polarized via Myers' effect [123] into five-branes with world-volume $\mathbb{R}^4 \times S^2$ (S^2 is an equator of S^5 and $\mathbb{R}^4$ is a slice of AdS_5 at fixed radius).

To see how this works consider first the Higgs vacuum. In the field theory we have scalar VEVs in the N dimensional irreducible representation of $SU(2)$ which are distributed on a non-commutative sphere

$$
\sum_{i=1}^{3} |\phi_i|^2 \sim m^2 N^2 I_N.
\tag{132}
$$

On the string theory side, $\mathcal{N} = 1^*$ is realized as the world volume theory on a set of N D3-branes, the scalar fields being the transverse coordinates of the branes ($x_i = 2\pi\alpha'\phi_i$). Equation (132) then corresponds to a configuration of D3-branes non-commutatively expanded into an S^2. From the non-abelian generalization of the CS action [123]

$$S \sim \mu_3 \int C_{(4)} + \mu_3(2\pi\alpha')^2 \int F^{(7)}_{0123ijk}[x_i, x_j]x_k, \tag{133}$$

we see that the expanded D3-branes have an additional electric coupling to the RR 6-form (equivalently a magnetic coupling to the RR 2-form, $C_{(2)}$) and are therefore equivalent to a single D5-brane with world volume $\mathbb{R}^4 \times S^2$, N units of D3-brane charge and zero net D5-brane charge. It is then possible to identify the Higgs vacuum with such a single D5-brane. Notice that this interpretation also fits with the standard AdS/CFT dictionary. Indeed the mass deformation $m\lambda\lambda$ corresponds in $10d$ to the linear combination of the NSNS and RR two-forms, which are the fields to which the D5-brane is coupled. The full supergravity solution corresponding to the wrapped D5 in the $AdS_5 \times S_5$ background is not known. In [48], the asymptotic solutions near the boundary and near the D5 were given. The solution is stable due to the balance between the 2-form potential and the energy of the non-commutative expansion. As a result the D5-brane sits at a fixed radius $z \sim \alpha'mN$. By analyzing fundamental strings in this background, one can check that electric charges are screened, as appropriate for an Higgs vacuum [48].

The basic idea behind the derivation of the solution is that the D5-brane can be seen as a small perturbation of the AdS background. This is however not the case for the confining vacuum which should correspond to N coincident D5-branes. Instead one can use S-duality, since it can be proved that in $\mathcal{N} = 4$ SYM it maps the Higgs vacuum to one of the N confining vacua. On the supergravity side S-duality corresponds to the $SL(2, \mathbb{Z})$ symmetry of Type IIB. Hence each of the confining vacua should be described by an NS5-brane. By analyzing fundamental strings, one can check that, in this vacuum, the electric charges confine. From the equation of motion for the 2-form field, one can also check the existence of a subleading solution corresponding to a gaugino condensate in field theory, according to the prescription of Section 2.

We refer to [48] for more information about the rich physics of this model and for a study of the dual solutions to the other vacua of $\mathcal{N} = 1^*$.

5.5   A $\mathcal{N} = 0$ solution

We will only marginally discuss four-dimensional non-supersymmetric theories in this review. Non supersymmetric theory can be obtained introducing a finite temperature in a higher dimensional theory that possesses a holographic dual, or starting directly

with non-supersymmetric string theories, like Type 0. Alternatively, we can also study non-supersymmetric deformations of four dimensional gauge theories.

Here we will only consider models that can be obtained as soft breakings of the solutions discussed in this and the previous Sections. The general strategy is to consider solutions with the same fields as the supersymmetric ones. Some of the fields discussed in the MN or KS solutions, or in the $\mathcal{N} = 2$ solutions in Section 4, are dual to scalar or fermionic bilinears and they can be used to introduce mass terms in the theories. Known examples of supersymmetry breaking in the literature are a massive sector of $\mathcal{N} = 1$ [124], soft deformations of the MN solution [108, 125, 126, 91] and of the KS one [127, 91].

For simplicity, we will only consider the MN solution. We can introduce a gaugino mass term in the system, leaving us with pure YM in the far IR. We will need a non-supersymmetric solution of the Lagrangian (97). Of course, we cannot use the BPS equations any longer and the equations of motions are needed. As already mentioned, a solution with an asymptotic behavior as in (107) with $Y \neq 0$ represents a deformation of the MN solution with a mass term for the gaugino. The general analytical solution of the equations of motion corresponding to the Lagrangian (97) is not known. The asymptotics for large and small ρ, together with a numerical interpolation between the two, which proves that the solutions actually exist, have been discussed in [108]. In the UV the solution is

$$
\begin{aligned}
a &= \frac{Y}{\sqrt{2\rho}}(1 + \frac{1 - |Y|^2/2}{2\rho} + ...) + 2C\rho e^{-2\rho}(1 + \frac{\gamma}{2\rho} + ...), \\
h &= \frac{1}{2}\log\rho - \frac{|Y|^2}{8\rho^2}(1 + ...) + P\sqrt{2\rho}e^{-2\rho}(1 + \frac{\alpha}{2\rho} + ...), \\
\Phi &= \Phi_0 + \rho - 1/4\log\rho + \frac{5|Y|^2}{64\rho^2}(1 + ...) - P'\sqrt{2\rho}e^{-2\rho}(1 + \frac{\beta}{2\rho} + ...),
\end{aligned}
\tag{134}
$$

where dots stand for corrections in $1/\rho$. Using the equations of motion and choosing a convenient parameterization we find $P = P' = kRe(\bar{C}Y)$ where k is a free parameter. The other parameters α, β, γ are uniquely determined as functions of k and Y. We work in the D5-brane setup.

The striking point is that there exists a family of *regular* solutions whose expansion in the IR is

$$
\begin{aligned}
a &= 1 - b\rho^2 + ..., \\
e^h &= \rho - (\frac{b^2}{4} + \frac{1}{9})\rho^3 + ..., \\
\Phi &= \Phi(0) + (\frac{b^2}{4} + \frac{1}{3})\rho^2 + ...,
\end{aligned}
\tag{135}
$$

where $b \in (0, 2/3]$. The full solutions can be found by numerically integrating the IR solutions to the UV and solving for the UV parameters as functions of b and $\Phi(0)$. $\Phi(0)$

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matches with g_{YM}^2 and b with the gaugino mass term. The other UV parameters can be expressed in terms of these two. The $\mathcal{N} = 1$ solution corresponds to $b = 2/3$ and, of course, $Y = 0$. The other values of the parameter b correspond to non supersymmetric solutions.

In $\mathcal{N} = 1$ $SU(N)$ SYM, soft breaking terms may be introduced into supersymmetric theories by promoting the parameters of the theory to background superfields. A non-zero F-component for τ introduces in the bare Lagrangian a gaugino mass. If the mass is small compared to Λ, we can treat the supersymmetry breaking term as a perturbation. We see from formula (93) in Section 5.1 that the vacuum energy is no longer zero but, at leading order in the mass, it is given by (for $\theta_{YM} = 0$ and m_λ real)

$$\Delta V \sim m_\lambda \Lambda^3 \cos\left[\frac{2\pi n}{N}\right]. \tag{136}$$

The degeneracy of the vacua is removed and there is a single unique vacuum ($n = 0$). The differences in energies between the N different vacua, after supersymmetry breaking, can be computed also using the $\mathcal{N} = 1$ supergravity solution [126]. The computation of the free-energy in the supergravity solutions was done, for somehow different purposes, in [108]. Here we just quote the result: the difference in energy between the non-supersymmetric solution and the reference BPS one is

$$\Delta I \sim e^{2\Phi_0} P \sim e^{2\Phi_0} k \, Re(\bar{C}Y). \tag{137}$$

In the supersymmetric limit where Y, the gaugino mass, is zero, the solutions with different phases of the condensate C are degenerate. We can compute from this formula the energy of the vacua when a gaugino mass term $Y = m_\lambda$ is introduced. At leading order in m_λ, we write $C = \Lambda^3 e^{2\pi i n/N} + O(m_\lambda)$ and, as can be shown numerically, $k = $ constant $+ O(m_\lambda)$. It is important that k is a $U(1)_R$ invariant quantity not depending on the phase of C. The vacuum energy then reads

$$E \sim Re(\bar{C}Y) \sim Re(m_\lambda \Lambda^3 e^{-2\pi i n/N}), \tag{138}$$

reproducing the field theory result (136).

It is interesting to notice that the deep IR form of the metric is exactly the same for all solutions, $\mathcal{N} = 1$ supersymmetric and not. Furthermore, all the features of the $\mathcal{N} = 1$ solution which depend only on the far IR form of the metric, such as the Wilson loop and the string tension, are similarly realized in the non supersymmetric case. The presence of the mass gap can be taken as an argument for the classical stability of this background: even if not supersymmetric, the gap should prevent any mode to become tachyonic if the mass deformation is small enough. This way of reasoning is due to the authors of [124].

A similar analysis was performed in [127, 91] in the case of the KS solution.

Acknowledgments

A SW curves for $\mathcal{N} = 2$ systems of branes

Here we list the SW curves associated with the $\mathcal{N} = 2$ theories that are discussed in this review. For the majority of $\mathcal{N} = 2$ models obtained with fractional and wrapped branes the curve can be determined using the M-theory lifting of the Hanany-Witten set-up for NS and D4-branes [75]:

(i) The curve for $SU(N)$ with N_f flavors is given by the genus-$(N-1)$ hyperelliptic Riemann surface

$$y^2 = P(x)^2 - \Lambda^{2N-N_f} P_f(x), \tag{139}$$

where the polynomials $P(x)$ and $P_f(x)$ are expressed in terms of the moduli and hypermultiplet masses

$$P(x) = \prod_{i=1}^{N}(x - u_i), \qquad P_f(k) = \prod_{\alpha=1}^{N_f}(x - m_\alpha). \tag{140}$$

This theory can be realized with N D4-branes stretched between two NS-branes and N_f semi-infinite D4-branes. The zeros of the polynomials $P(x)$ and $P_f(x)$ represent the positions of the finite and semi-infinite D4-branes, respectively.

(ii) The curve for $\prod_{i=1}^{k} SU(N_i)$ with bi-fundamental hypermultiplets in the $(\boldsymbol{N_i}, \boldsymbol{N_{i+1}})$ and fundamental hypermultiplets for $SU(N_1)$ and $SU(N_k)$ is given by the polynomial in t and x

$$P_{k+1}(x)t^{k+1} + P_k(x)t^k + ... + P_1(x)t + P_0(x) = 0, \tag{141}$$

where $P_s(x) = \prod_{i=1}^{N_s}(x - u_i^s), s = 1, ..., k$ are degree-N_s polynomials containing the information about the moduli for the sth-group and $P_{0,k+1} = \prod_i(x - m_i^{0,k+1})$ are polynomials containing the information about the masses of the fundamental hypermultiplets. The curve is not hyperelliptic. The system can be realized with $k + 1$ NS-branes, N_i D4-branes stretching between the $i - (i + 1)$ pair of NS-branes and two set of semi-infinite D4-branes. The zeros of the polynomials $P_i(x)$ represent the positions of the D4-branes. All the dependence on the dynamically generated scales have been suppressed for simplicity. The case $k = 1$ with $P_0 \equiv \Lambda^{2N-N_f}P_f$, $P_1 \equiv 2P$, $P_2 \equiv 1$, reduces to the curve given above for $SU(N)$ with N_f flavors with the redefinition $y = t + P(x)$.

(iii) The $\mathcal{N} = 2 \, CFT$ associated to the affine A_{k-1} Dynkin diagram is the cyclic quiver $\prod_{i=1}^k SU(N)$ with bi-fundamentals in the $(\boldsymbol{N_i}, \boldsymbol{N_{i+1}})$, $i = 1, ..., N$ where $i = N+1 \equiv 1$. The theory is conformal. The SW curve is not hyperelliptic and it is an N-sheeted covering of a torus with modular parameter τ corresponding to the diagonal coupling constant in the CFT $\tau = \sum_i \tau_i$. The curve can be written in terms of a meromorphic section of an Higgs bundle on the torus (locally an hermitian $N \times N$ matrix Φ)

$$\det(xI - \Phi)|_{\text{locally}} = \prod_{i=1}^N(x - u_i) = 0. \tag{142}$$

Φ is meromorphic on the torus with exactly k simple poles whose residues determine the hypermultiplet masses [75]. The systems can be realized with N D4-branes wrapped on a circle in the presence of k NS-branes. The example for the A_1 case, corresponding to the $\mathbb{R}^4/\mathbb{Z}_2$ singularity, was pictured in Section 3.3. The poles of Φ correspond to the positions of the NS-branes upon lifting to M-theory where the circle combines with the M-theory circle in giving a torus[28]. The case $k = 1$ corresponds to $SU(N)$ with a massless adjoint, i.e. the $N = 4$ theory. $SU(N)$ with a massive adjoint can be described with a suitable twist along the circle [122, 75].

The latter case actually contains all the previous ones and all the models we are interested in. Indeed the non-conformal cases (i) and (ii) can be obtained from case (iii) by a combination of the following two operations. Firstly, we can freeze some of the coupling constants τ_i, thus obtaining non-cyclic models with fundamental hypermultiplets

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[28]An explicit representation of the meromorphic functions can be given in terms of theta functions; if u is the standard coordinate on the torus represented as a parallelogram in $\mathbb{C}$, the curve can be represented in a form similar to the previous cases as an infinite polynomial in $t = e^{iu}$ with coefficients that are degree-N polynomials in x. Only k polynomials are independent and give the position of the D4-branes. The circle has been lifted to its universal covering $\mathbb{R}$; there is accordingly a precise pattern of repetition in the x-polynomials.

as in (i) or (ii). Secondly, theories with generic group $\prod_{i=1}^{k} SU(N_i)$ can be obtained by considering suitable corners in the moduli space of the original CFT. $SU(N) \times SU(N+M)$ with two bi-fundamentals, for example, can be obtained from the A_1 theory $SU(N+M) \times SU(N+M)$ at low energies by giving large VEVs to some of the moduli of the first group. We just saw that the $SU(N+M) \times SU(N+M)$ curve can be written as a meromorphic function on the torus with two poles. The curve depends on two polynomials $P_i(x)$ representing the positions of the D4-branes. We can take, for example, $P_2 = \prod_{i=1}^{N}(x - u_i^{(1)})$ and $P_1(x) = \prod_{i=1}^{N}(x - u_i^{(2)})(x^M - Y^M)$, with Y much bigger than the surviving moduli $u^{(i)}$ for $SU(N) \times SU(N+M)$. The curve for $SU(N) \times SU(N+M)$ is then obtained for $Y \to \infty$ with τ suitably scaled.

B   Fermionic shifts for systems with wrapped branes

Here we will review in details the seven dimensional gauged supergravity setup used in the text for the wrapped brane system. We will derive the relevant BPS equations and solve them for the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ duals.

B.1   Supergravity equations

The strategy to obtain supersymmetric solutions in the $SO(4)$ $7d$ gauged sugra for the systems of wrapped five-branes, is setting to zero the supersymmetry variations, which gives first order equations, and then check if the second order equations of motion are satisfied by the solutions. We will write the general formulae [116] for the supersymmetry variations of fermions with only three diagonal scalars[29]

$$V_I{}^i = \text{diag}(e^{-\lambda_1}, e^{-\lambda_1}, e^{-\lambda_2}, e^{-\lambda_3}). \tag{143}$$

We take the $SO(4) = SU(2)^+ \times SU(2)^-$ gauge fields of the form

$$
\begin{aligned}
A &= \alpha \left[\cos\theta \, d\phi \, \eta_1^+ + a(\rho) \, d\theta \, \eta_2^+ + b(\rho) \sin\theta \, d\phi \, \eta_3^+ \right] + \\
&\quad \beta \left[\cos\theta \, d\phi \, \eta_1^- + \tilde{a}(\rho) \, d\theta \, \eta_2^- + \tilde{b}(\rho) \sin\theta \, d\phi \, \eta_3^- \right],
\end{aligned} \tag{144}
$$

where α, β are constants and the η matrices are the generators of the $SU(2)^{\pm}$ in the $SO(4)$ notation and take the form

$$
\eta_1^{\pm} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \mp 1 & 0 \end{pmatrix} \quad
\eta_2^{\pm} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & 1 \\ \pm 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad
\eta_3^{\pm} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \pm 1 & 0 \\ 0 & \mp 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
$$
$$\tag{145}$$

---

[29]This and the following ones turn out to be sufficiently general ansatz for the dual theories considered here and probably for some generalization.

The field strength is normalized as $F = dA + 2m[A, A]$. The ansatz for the metric (in the Einstein frame) is

$$ds_7^2 = e^{2f}(dx_4^2 + d\rho^2) + e^{2g}(d\theta^2 + \sin^2\theta d\phi^2). \tag{146}$$

It has to be thought as the seven dimensional part of a "warped" linear dilaton metric with the two-sphere compactification. We do not use different notations for curved and flat indices. To pass from the former to the latter one must multiply $\gamma^\phi, \gamma^\theta, \gamma^{\rho,\chi}$ by the inverse vielbein ($\chi = 0, 1, 2, 3$ labels the four dimensional coordinates). From (146) it follows that the non trivial components of the spin connection are

$$\omega_\chi^{\chi\rho} = f', \quad \omega_\theta^{\theta\rho} = g'e^{g-f}, \quad \omega_\phi^{\phi\rho} = g'e^{g-f}\sin\theta, \quad \omega_\phi^{\phi\theta} = \cos\theta, \tag{147}$$

where the prime denotes differentiation with respect to ρ. The general form of the supersymmetry variations can be obtained as a singular limit of the ones in [90] and reads

$$\delta\psi_\mu = \left[\mathcal{D}_\mu + \frac{1}{4}\gamma_\mu\gamma^\nu V_i^{-1\,I}\partial_\nu V_I^{\ i} + \frac{1}{4}\Gamma^{ij}F_{\mu\lambda}^{ij}\gamma^\lambda\right]\epsilon, \tag{148}$$

$$\delta(\Gamma^{\hat{i}}\lambda_{\hat{i}}) = \left[\frac{m}{2}(T_{\hat{i}j} - \frac{1}{5}T\delta_{\hat{i}j})\Gamma^{\hat{i}j} + \frac{1}{2}\gamma_\mu P_{\hat{i}j}^\mu\Gamma^{\hat{i}j} + \frac{1}{16}\gamma^{\mu\nu}(\Gamma^{\hat{i}}\Gamma^{kl}\Gamma^{\hat{i}} - \frac{1}{5}\Gamma^{kl})F_{\mu\nu}^{kl}\right]\epsilon, \tag{149}$$

with $F_{\mu\lambda}^{ij} = V_I^{\ i}V_J^{\ j}F_{\mu\lambda}^{IJ}$. Notice that the index $\hat{i}$ is not summed over.

Now, let's impose $\partial_\theta\epsilon = \partial_\phi\epsilon = \partial_x\epsilon = 0$ and concentrate our attention on the gravitino, for example on its θ component

$$\begin{aligned}
\delta\psi_\theta = \Bigg[&\frac{1}{2}(g + \lambda_1 + \frac{\lambda_2 + \lambda_3}{2})'e^{g-f}\gamma^{\theta\rho} + \frac{1}{4}m[(e^{\lambda_1-\lambda_2} + e^{\lambda_2-\lambda_1})(-\alpha\,a + \beta\,\tilde{a})\Gamma^{13} \\
&+ (e^{\lambda_1-\lambda_3} + e^{\lambda_3-\lambda_1})(\alpha\,a + \beta\,\tilde{a})\Gamma^{24}] + \frac{1}{2}\Gamma^{12}e^{-2\lambda_1}\frac{1}{2}[\alpha\sin\theta(2\,m\,\alpha\,a\,b - 1) \\
&+ \beta\sin\theta(2\,m\,\beta\,\tilde{a}\,\tilde{b} - 1)]\gamma^\phi\frac{e^{-g}}{\sin\theta} + \Gamma^{34}e^{-\lambda_2-\lambda_3}\frac{1}{2}[\alpha\sin\theta(2\,m\,\alpha\,a\,b - 1) \\
&- \beta\sin\theta(2\,m\,\beta\,\tilde{a}\,\tilde{b} - 1)]\gamma^\phi\frac{e^{-g}}{\sin\theta} + \Gamma^{14}e^{-\lambda_1-\lambda_3}\frac{1}{2}[\alpha\cos\theta(b - 2\,m\,\alpha\,a) \\
&+ \beta\cos\theta(\tilde{b} - 2\,m\,\beta\,\tilde{a})]\gamma^\phi\frac{e^{-g}}{\sin\theta} + \Gamma^{23}e^{-\lambda_1-\lambda_2}\frac{1}{2}[\alpha\cos\theta(b - 2\,m\,\alpha\,a) \\
&- \beta\cos\theta(\tilde{b} - 2\,m\,\beta\,\tilde{a})]\gamma^\phi\frac{e^{-g}}{\sin\theta} + \Gamma^{13}e^{-\lambda_1-\lambda_2}\frac{1}{2}[\alpha\,a' - \beta\,\tilde{a}']\gamma^\rho e^{-f} \\
&+ \Gamma^{24}e^{-\lambda_1-\lambda_3}\frac{1}{2}[-\alpha\,a' - \beta\,\tilde{a}']\gamma^\rho e^{-f}\Bigg]\epsilon = 0.
\end{aligned} \tag{150}$$

Since the spinor ϵ is charged under $SU(2)^+ \times SU(2)^-$, let's separate the two components letting $\epsilon = \epsilon^+ \oplus \epsilon^-$. Now take the following basis of sigma matrices for $\epsilon^\pm$,

$$\Gamma^{12} \pm \Gamma^{34} = 2i\sigma_3^\pm, \quad \Gamma^{24} \pm \Gamma^{31} = 2i\sigma_1^\pm, \quad \Gamma^{14} \pm \Gamma^{23} = -2i\sigma_2^\pm, \tag{151}$$

and the following basis for the seven dimensional gamma matrices

$$\gamma^\mu = \gamma^\mu_{(4)} \otimes 1, \qquad \gamma^{\theta,\phi,\rho} = \gamma^5_{(4)} \otimes \sigma^{1,2,3}. \tag{152}$$

In these notations the spinor ϵ can be seen as a 2×2 matrix, with each entry a five dimensional spinor

$$\epsilon = \begin{pmatrix} p & q \\ iq^c & -ip^c \end{pmatrix}. \tag{153}$$

This form is dictated by the symplectic-Majorana condition to be imposed in seven dimensions. The space-time σ's act on the matrix from the left, while the (transposed) gauge ones from the right.

Let's concentrate only on ϵ^+; ϵ^-, which will be present in the $\mathcal{N} = 2$ solutions, behaves in an analogous way. There are some constraints which come from the dependence on θ in the fermionic variation (150). From the $\frac{\cos\theta}{\sin\theta}$ terms one gets $b = 2\,m\,\alpha\,a$ and $\tilde{b} = 2\,m\,\beta\,\tilde{a}$. The contribution in $\cos\theta$ gives instead the twist condition

$$\left[\gamma^{\phi\theta} + m[(\alpha + \beta) + \frac{1}{2}(\alpha - \beta)(e^{\lambda_2 - \lambda_3} + e^{\lambda_3 - \lambda_2})]i\sigma_3^+ \right] \epsilon^+ = 0, \tag{154}$$

which will set $2m\alpha = 1$ in the $\mathcal{N} = 1$ case and $2m\alpha = 2m\beta = 1$ in the $\mathcal{N} = 2$ solutions. The gauge field A_μ was taken in (144) to have a component, the $\cos\theta$ one, proportional to the sphere spin connection, and formula (154) gives the complete twist condition. After these relations are imposed, the remaining part of (150) gives the actual first order differential equation to be solved.

The ψ_ϕ component of the gravitino variation is very similar to the ψ_θ one and it ultimately has two contributions that have to vanish separately. The $\cos\theta$ part gives again the twist condition, while the $\sin\theta$ one gives the very same equation of the ψ_θ variation once the twist is imposed. In an analogous way one can deduce the equations following from the ψ_χ and ψ_ρ components of the gravitino, as well as the ones coming from the gaugino variations [116].

The full set of BPS equations then reads

$$\delta\psi_\chi \quad\to\quad f' + x' = 0,$$

$$\delta\psi_\rho \quad\to\quad \left[\partial_\rho + \frac{1}{2}x' + \frac{1}{2}e^{-h}\gamma^\theta i\sigma_1^+(a'\cosh z + \tilde{a}'\sinh z)\right]\epsilon^+ = 0,$$

$$\delta\psi_\phi \quad\to\quad \Big[h'e^h + \gamma^{\rho\theta}i\sigma_1^+(a\cosh z\cosh y + \tilde{a}\sinh z\sinh y) +$$
$$+\frac{1}{2}\gamma^\theta i\sigma_1^+(a'\cosh z + \tilde{a}'\sinh z) +$$
$$-\frac{1}{2}e^{-h}\gamma^\rho[(a^2-1)\cosh y - (\tilde{a}^2-1)\sinh y]\Big]\epsilon^+ = 0,$$

$$\delta\lambda_i \quad\to\quad \Big[e^{-y}\sinh 2z - z'\gamma^\rho + \gamma^\theta i\sigma_1^+ e^{-h}(a\sinh z\cosh y + \tilde{a}\cosh z\sinh y) +$$

$$+\frac{1}{2}\gamma^{\rho\theta}i\sigma_1^+ e^{-h}(a'\sinh z + \tilde{a}'\cosh z)\Big]\epsilon^+ = 0,$$

$$\Big[\frac{1}{5}(e^y + e^{-y}\cosh 2z) + \frac{1}{10}e^{-2h}[(a^2-1)\cosh y - (\tilde{a}^2-1)\sinh y] +$$

$$-x'\gamma^\rho - \frac{1}{5}\gamma^{\rho\theta}i\sigma_1^+ e^{-h}(a'\cosh z + \tilde{a}'\sinh z)\Big]\epsilon^+ = 0,$$

$$\Big[(e^y - e^{-y}\cosh 2z) - \frac{1}{2}e^{-2h}[(a^2-1)\sinh y - (\tilde{a}^2-1)\cosh y] +$$

$$-y'\gamma^\rho + 2\gamma^\theta i\sigma_1^+ e^{-h}(a\cosh z\sinh y + \tilde{a}\sinh z\cosh y)\Big]\epsilon^+ = 0, \qquad (155)$$

where $x = \lambda_1 + \frac{\lambda_2+\lambda_3}{2}$, $y = \lambda_1 - \frac{\lambda_2+\lambda_3}{2}$, $z = \frac{\lambda_2-\lambda_3}{2}$ and $h = g - f$.

To proceed, the ansatz for the fields must be refined. The way to do it depends primarily on how many supersymmetries are to be preserved. In terms of the fields, one must embed the $U(1)_s$ spin connection on the sphere in the $SO(4)$ normal bundle. There are essentially two ways to do it. One can break $SO(4) \to U(1)_{(1)} \times U(1)_{(2)}$ and embed $U(1)_s$ in, say, $U(1)_{(1)}$; this will preserve $\mathcal{N} = 2$ supersymmetry. Or one can view $SO(4) = SU(2)^+ \times SU(2)^-$ and embed $U(1)_s$ in, say, $U(1)^+ \subset SU(2)^+$; this leads to $\mathcal{N} = 1$ theories.

B.2 $\mathcal{N} = 2$ solutions

Let us begin with the first case, in which $U(1)_s \sim U(1)_{(1)}$. The latter is the diagonal of the two $U(1)$ factors in $SU(2)^+ \times SU(2)^-$. Moreover, in this case we want no condensate or mass term for the $\mathcal{N} = 2$ fermions, so the relevant equations follow from (155) putting

$a = \tilde{a} = 0$. They read

$$
\begin{aligned}
f' &= -(\lambda_1' + \frac{\lambda_2' + \lambda_3'}{2}), \\
g' &= -(\lambda_1' + \frac{\lambda_2' + \lambda_3'}{2}) + \frac{1}{2}e^{f-2g-2\lambda_1}, \\
\lambda_2' + 2\lambda_3' + 2\lambda_1' &= -e^{f+2\lambda_3}, \\
2\lambda_2' + \lambda_3' + 2\lambda_1' &= -e^{f+2\lambda_2}, \\
3\lambda_1' + \lambda_2' + \lambda_3' &= -e^{f+2\lambda_1} + \frac{1}{2}e^{f-2g-2\lambda_1}.
\end{aligned}
\tag{156}
$$

Note that we have linear differential equations. The solutions then read

$$
\begin{aligned}
f &= -(\frac{\lambda_2 + \lambda_3}{2} + \lambda_1), \\
e^{2g-2f} &= u, \\
e^{\frac{\lambda_2+\lambda_3}{2}-\lambda_1} &= \sqrt{\frac{e^{4u}+b^4}{e^{4u}-b^4} - \frac{1}{2u} + \frac{2Ke^{2u}}{u(e^{4u}-b^4)}}, \\
e^{\frac{\lambda_2+\lambda_3}{2}+\lambda_1} &= \left(\frac{e^{2u}}{e^{4u}-b^4}\right)^{1/5} \left[\frac{e^{4u}+b^4}{e^{4u}-b^4} - \frac{1}{2u} + \frac{2Ke^{2u}}{u(e^{4u}-b^4)}\right]^{-\frac{1}{10}}, \\
e^{\lambda_2-\lambda_3} &= \frac{e^{2u}-b^2}{e^{2u}+b^2},
\end{aligned}
\tag{157}
$$

with

$$
\frac{du}{d\rho} \equiv e^{\frac{\lambda_2+\lambda_3}{2}-\lambda_1}.
\tag{158}
$$

These solutions reduce to the two-scalars ones when $\lambda_2 = \lambda_3$, i.e. when $b = 0$.

B.3 $\mathcal{N} = 1$ solutions

As explained in section 5.2, the $\mathcal{N} = 1$ dual solution corresponds to the identification $U(1)_s \sim U(1)^+ \subset SU(2)^+$. This twist amounts then on taking only the $SU(2)^+$ part of the connection in equations (155). Together with a single scalar in the matrix T_{ij}, call it λ_1, this provides a consistent truncation of the gauged supergravity [87]. In general, care must be taken in that the spinor ϵ^+ is a two component $SU(2)$ vector and not only σ_3 is present the equations, which will thus retain their nonlinear structure. There is an exception to this statement, namely if one considers the $U(1)^+$ case $a = 0$, which reduces immediately the equations (155) to

$$
h' = \frac{1}{2}e^{-2h}, \qquad \lambda' = -\frac{1}{5} + \frac{1}{20}e^{-2h},
\tag{159}
$$

where $\lambda_i = \lambda$, with $i = 1, 2, 3$. The solution is

$$e^{2h} = \rho, \qquad 5\lambda = \frac{1}{4}\log\rho - \rho. \qquad (160)$$

This solution has a bad type singularity, so it's not interesting. As argued in [5], the way around this problem is turning on the non Abelian part of the connection, i.e. a. This is not only a technical trick to get a nonsingular solution, but it is the right physical answer to the problem, as a is dual to the gaugino condensate of the $\mathcal{N} = 1$ theory.

Now the equations are truly non-linear, and defining

$$A = \frac{1}{2}h'e^h, \quad B = \frac{1}{2}a, \quad C = \frac{1}{4}e^{-h}(a^2 - 1), \quad D = -\frac{1}{4}a', \qquad (161)$$

the ψ_θ equation becomes

$$[\gamma^{\theta\rho}A + iB\sigma_1^+ + i\gamma^\phi C\sigma_3^+ + i\gamma^\rho D\sigma_1^+]\epsilon^+ = 0, \qquad (162)$$

that can be rewritten as

$$i\sigma_1^+\gamma^{\rho\theta}\epsilon^+ = (\Delta + \Pi\gamma^\rho)\epsilon^+, \qquad (163)$$

with

$$\Delta = -\frac{AB - CD}{A^2 - C^2}, \qquad \Pi = -\frac{AD - BC}{A^2 - C^2}. \qquad (164)$$

Multiplying (163) by $i\sigma_1^+\gamma^{\rho\theta}$, one obtains the consistency relation

$$\Delta^2 - \Pi^2 = 1. \qquad (165)$$

The gaugino variation (only the second gaugino variation in (155) survives), in the notation

$$E = \frac{2}{5} + \frac{1}{10}e^{-2h}(a^2 - 1), \quad F = -2\lambda', \quad G = \frac{1}{5}e^{-h}a', \qquad (166)$$

reads

$$[E + \gamma^\rho F + i\sigma_1^+\gamma^{\rho\theta}G]\epsilon^+ = 0, \qquad (167)$$

that again can be rewritten in the form (163) but now with

$$\Delta = \frac{4e^h + e^{-h}(a^2 - 1)}{2a'}, \qquad \Pi = -\frac{10e^h\lambda'}{a'}, \qquad (168)$$

so that from consistency of (164) with (168) one obtains the other two equations

$$\frac{AB - CD}{A^2 - C^2} = \frac{E}{G}, \qquad \frac{AD - BC}{A^2 - C^2} = \frac{F}{G}. \qquad (169)$$

Finally, there is the ψ_ρ variation, giving the ρ dependence of the spinor and another equation

$$[2 + \frac{1}{2}e^{-2h}(a^2 - 1)]\Delta - \partial_\rho\Pi = 0. \qquad (170)$$

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One can verify that all these equations are solved by the functions in [109, 5] (the seven dimensional dilaton in the text is $\phi = 5\lambda$)

$$e^{2h} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \qquad a = \frac{2\rho}{\sinh 2\rho}, \qquad e^{10\lambda} = \frac{2e^h}{\sinh 2\rho}. \tag{171}$$

As a final remark, note that the system of equations (155) should have the right ingredients to provide also a solution corresponding to the breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$. The $\mathcal{N} = 2 \to \mathcal{N} = 1$ field theory contains the gaugino condensate, which is dual to a, and a mass term for the ψ fermion, which is dual to $\tilde{a}$. Then the full $SU(2)^+ \times SU(2)^-$ group is needed. Moreover, one can expect that both operators $\text{Tr}\phi\bar{\phi}$ and $\text{Tr}\phi^2$, which are dual to $(\lambda_2 \pm \lambda_3)/2$, have a VEV in QFT. All in all, on the supergravity side this should then correspond to a solution where all the fields $(\lambda_1, \lambda_2, \lambda_3, a, \tilde{a})$ are turned on. The BPS equations for this case are nothing else than (155). The solution is still lacking, due to the technical difficulty in solving the equations. However, a simplified model with $\lambda_2 = \lambda_3$ and only the $SU(2)^+$ group admits a solution [116], despite the fact that the number of equations is redundant. The ten dimensional solution has a good type singularity and its UV normalizable behavior indicates that it corresponds to the attempt of giving a VEV to scalar fields. Since these scalars are massive to begin with, one expects an instability in QFT that may explain the singular behavior of the supergravity solution.

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