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Superintegrability in three-dimensional Euclidean space

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Potentials for which the corresponding Schrödinger equation is maximally superintegrable in three-dimensional Euclidean space are studied. The quadratic algebra which is associated with each of these potentials is constructed and the bound state wave functions are computed in the separable coordinates. © *1999 American Institute of Physics.* [S0022-2488(99)02602-X]

I. INTRODUCTION

The present paper continues our study of the systems with *hidden symmetry* or so-called *superintegrable* systems in spaces with constant curvature.

The best known systems of this kind in three-dimensional Euclidean space are the harmonic oscillator and Kepler–Coulomb problems, which have many special properties distinct from other spherically symmetric potentials. These include the phenomena of separation of variables for the Hamilton–Jacobi and Schrödinger equations in more than one orthogonal coordinate system and the existence of integrals of motion in addition to the total angular momentum $\mathbf{L}^2$. In particular for the isotropic oscillator there is the Demkov tensor $D_{ik}=p_ip_k+\omega^2x_ix_k$,[1] and, in the case of the Kepler–Coulomb problem, the Pauli–Runge–Lenz vector $\mathbf{A}=1/2([\mathbf{L}\times\mathbf{p}]-[\mathbf{p}\times\mathbf{L}])-\mathbf{r}/|r|$. Both these systems possess five functionally independent integrals of motion.[2,3] The first systematic search for all potentials for which the Schrödinger equation admits separation of variables in two or more coordinate systems was begun by Smorodinsky and Winternitz with co-workers in Refs. 4–6 and continued by Evans in Refs. 3 and 7. They found all such systems in two- and three-dimensional flat space and introduced the notion of *superintegrability*. In general, a physical system in N dimensions is called *minimally* superintegrable if it has $2N-2$ integrals of motion, and *maximally* superintegrable if it has $2N-1$ integral of motions. There are five known maximally (and some minimally) superintegrable potentials listed in Refs. 3, 8, and 10 and investigated from different points of view in the last decade.[8–13] Note also that superintegrable potentials in spaces of constant curvature were introduced in Refs. 14–16.

In previous articles[17–19] we have looked at potentials in two-dimensional Euclidean space and the two-dimensional sphere and hyperboloid, for which the Schrödinger equation is maximally superintegrable. In this article we extend this study to the case of three-dimensional Euclidean space. As previously seen in the case of two dimensions, some of these potentials (see Table I) admit bound state or finite solutions and it is these to which we draw attention in this article.

The basic equation that we investigate is of course the Schrödinger equation ($\hbar=m=1$)

$$\mathcal{H}\Psi=-\frac{1}{2}\Delta\Psi+V(x,y,z)\Psi=-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi+V(x,y,z)\Psi=E\Psi. \tag{1}$$

The idea is to find solutions of this equation via a separation of variables ansatz

TABLE I. The three-dimensional maximally superintegrable potentials.

Potential $V(x,y,z)$	Separating coordinates
$V_1=\frac{\omega^2}{2}(x^2+y^2+z^2)+\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}+\frac{k_3^2-\frac{1}{4}}{z^2}\right)$	Cartesian Spherical Cylindrical polar Cylindrical elliptic Sphero-conical Oblate spheroidal Prolate spheroidal Ellipsoidal
$V_2=\frac{\omega^2}{2}(x^2+y^2+4z^2)+\frac{1}{2}\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)$	Cartesian Cylindrical polar Cylindrical parabolic Cylindrical elliptic Parabolic
$V_3=-\frac{\alpha}{\sqrt{x^2+y^2+z^2}}+\frac{1}{2}\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)$	Spheroidal-conical Spherical Parabolic Prolate spheroidal II

$$\Psi=\prod_{j=1}^{3}\psi_j(u_j)$$

for some suitable orthogonal coordinates u_j (see Table II).

In Secs. II–IV we consider three maximally superintegrable potentials (see Table I) and use the Niven-type (or Bethe[20]) ansatz for constructing the solution of the Schrödinger equation in coordinates such as spheroidal, sphero-conical, and ellipsoidal (see Table II). In addition we discuss the extension to the quadratic algebras that were in evidence in the case of two dimensions and see what their implications may be.

Section V is devoted to the calculation of interbasis expansion coefficients for the V_3 potential between spherical and parabolic bases.

II. GENERALIZED ISOTROPIC OSCILLATOR

The first potential (see Table I) on our list of three is

$$V_1(x,y,z)=\frac{\omega^2}{2}(x^2+y^2+z^2)+\frac{1}{2}\left[\frac{(k_1^2-\frac{1}{4})}{x^2}+\frac{(k_2^2-\frac{1}{4})}{y^2}+\frac{(k_3^2-\frac{1}{4})}{z^2}\right], \tag{2}$$

where the constant $k_i\geqslant\frac{1}{2}$. For $k_i=\frac{1}{2}$ we have the ordinary isotropic oscillator potential. The corresponding Schrödinger equation admits solutions via a separation of variables in eight coordinate systems: Cartesian, spherical, sphero-conical, cylindrical polar, cylindrical elliptic, prolate and oblate spheroidal, and ellipsoidal. We summarize the bound state solutions in each case.

Before considering various coordinate systems we note that a basis for the symmetries of Schrödinger's equation with the potential (2) consists of the six operators:

TABLE II. Systems of coordinate in three-dimensional Euclidean space.

Coordinate system	Coordinates
I. Cartesian $x,y,z\in\mathbf{R}$	x,y,z
II. Cylindrical polar $\rho>0,\ \varphi\in[0,2\pi)$	$x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z$
III. Cylindrical elliptic $z\in\mathbf{R},\ e_1<\mu_1<e_2<\mu_2$	$x^2=\dfrac{(\mu_1-e_1)(\mu_2-e_1)}{(e_2-e_1)},\quad y^2=\dfrac{(\mu_1-e_2)(\mu_2-e_2)}{(e_1-e_2)},\ z$
IV. Cylindrical parabolic $\xi,x\in\mathbf{R},\ \eta\geqslant 0$	$x,y=\xi\eta,\ z=\frac{1}{2}(\xi^2-\eta^2)$
V. Spherical $r>0,\ \theta\in[0,\pi],\ \varphi\in[0,2\pi)$	$x=r\cos\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\theta$
VI. Prolate spheroidal $e_1<u_1<e_2<u_2,\ \varphi\in[0,2\pi)$	$x^2=\dfrac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}\cos^2\varphi,$ $y^2=\dfrac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}\sin^2\varphi,$ $z^2=\dfrac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}$
VII. Oblate spheroidal $e_1<u_1<e_2<u_2,\ \varphi\in[0,2\pi)$	$x^2=\dfrac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}\cos^2\varphi,$ $y^2=\dfrac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}\sin^2\varphi,$ $z^2=\dfrac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}$
VIII. Sphero-conical $r\geqslant 0,\ e_1<\rho_1<e_2<\rho_2<e_3$	$x^2=r^2\dfrac{(\rho_1-e_1)(\rho_2-e_1)}{(e_1-e_2)(e_1-e_3)},$ $y^2=r^2\dfrac{(\rho_1-e_2)(\rho_2-e_2)}{(e_2-e_1)(e_2-e_3)},$ $z^2=r^2\dfrac{(\rho_1-e_3)(\rho_2-e_3)}{(e_3-e_2)(e_3-e_1)}$
IX. Parabolic $\xi,\eta\geqslant 0,\ \varphi\in[0,2\pi)$	$x=\xi\eta\cos\varphi,\ y=\xi\eta\sin\varphi,\ z=\frac{1}{2}(\xi^2-\eta^2)$
X. Ellipsoidal $a_1<u_1<a_2<u_2<a_3<u_3$	$x^2=\dfrac{(u_1-a_1)(u_2-a_1)(u_3-a_1)}{(a_3-a_1)(a_2-a_1)},$ $y^2=\dfrac{(u_1-a_2)(u_2-a_2)(u_3-a_2)}{(a_1-a_2)(a_3-a_2)},$ $z^2=\dfrac{(u_1-a_3)(u_2-a_3)(u_3-a_3)}{(a_1-a_3)(a_2-a_3)}$
XI. Paraboloidal $0<\eta_1<a_2<\eta_2<a_3<\eta_3$	$x^2=\dfrac{(\eta_1-a_3)(\eta_2-a_3)(\eta_3-a_3)}{(a_3-a_2)},$ $y^2=\dfrac{(\eta_1-a_2)(\eta_2-a_2)(\eta_3-a_2)}{(a_2-a_3)},$ $z^2=\frac{1}{2}(\eta_1+\eta_2+\eta_3-a_2-a_3)$

$$M_i = -D_{ii} - \frac{k_i^2 - \frac{1}{4}}{x_i^2}, \quad -\mathcal{H} = M_1 + M_2 + M_3, \tag{3}$$

$$J_{ij} = L_{ij}^2 - \left(k_i^2 - \frac{1}{4}\right)\frac{x_j^2}{x_i^2} - \left(k_j^2 - \frac{1}{4}\right)\frac{x_i^2}{x_j^2} - \frac{1}{2}, \quad i,j = 1,2,3, \tag{4}$$

where $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$, $D_{ii} = -\partial_{x_i}^2 + \omega^2 x_i^2$ is a diagonal components of the Demkov tensor[1] and we have the notation $x_1 = x$, $x_2 = y$, $x_3 = z$.

The commutators of the operators (3) and (4) can be closed to form a quadratic algebra as follows:

$$[M_i, M_j] = 0, \quad [M_i, J_{jk}] = 0, \quad [M_i, J_{ij}] = Q_{ij} = Q_{[ij]}, \quad [J_{ij}, J_{ik}] = R_{[ijk]} = R,$$

where Q_{ij} is totally antisymmetric and the totally antisymmetric quantity $R_{[ijk]}$ is denoted by R. Further commutators are calculated to be

$$[M_i, Q_{jk}] = 0, \quad [M_i, Q_{ij}] = 4\{M_i, M_j\} + 16J_{ij}, \quad [M_i, R] = 4\{M_k, J_{ij}\} - 4\{M_j, J_{ik}\},$$

$$[J_{ij}, Q_{ij}] = 4\{M_i, J_{ij}\} - 4\{M_j, J_{ij}\} - 8(k_j^2 - 1)M_i + 8(k_i^2 - 1)M_j,$$

$$[J_{ij}, Q_{ik}] = 4\{M_i, J_{jk}\} - 4\{M_j, J_{ik}\},$$

$$[J_{ij}, R] = 4\{J_{ij}, J_{jk}\} - 4\{J_{ij}, J_{ik}\} - 8(k_i^2 - 1)J_{jk} + 8(k_j^2 - 1)J_{ik},$$

where $\{A,B\} = AB + BA$. The expression for the commutators of the Q and R are

$$[Q_{ij}, Q_{ik}] = 4\{M_i, Q_{jk}\}, \quad [Q_{ij}, R] = -4\{J_{ij}, Q_{ik}\} - 4\{J_{ij}, Q_{jk}\}.$$

All the commutators of the operators M_i, J_{mn}, Q_{pq}, and R can be expressed in terms of quadratic symmetric products of themselves. The algebra, therefore, is closed quadratically. There are relations between the symmetric products of the generators of this algebra. The exhaustive list of these is as follows:

$$\begin{aligned} Q_{ij}^2 = {} & \tfrac{8}{3}\{J_{ij}, M_i, M_j\} + \tfrac{64}{3}\{M_i, M_j\} + 16\omega^2 J_{ij}^2 - 16(1 - k_j^2)M_i^2 \\ & - 16(1 - k_i^2)M_j^2 - \tfrac{128}{3}\omega^2 J_{ij} - 64\omega^2(1 - k_i^2)(1 - k_j^2), \end{aligned}$$

$$\begin{aligned} \{Q_{ij}, Q_{ik}\} = {} & \tfrac{8}{3}\{J_{ij}, M_i, M_k\} + \tfrac{8}{3}\{J_{ik}, M_i, M_j\} - \tfrac{8}{3}\{J_{jk}, M_i, M_i\} \\ & + 32\omega^2(1 - k_i^2)\{J_{ij}, J_{ik}\} - 32(1 - k_i^2)M_j M_k - 64\omega^2(1 - k_i^2)J_{jk}, \end{aligned}$$

$$\begin{aligned} \{Q_{ij}, R\} = {} & \tfrac{8}{3}\{J_{ij}, J_{ij}, M_k\} - \tfrac{8}{3}\{J_{ij}, J_{ik}, M_j\} - \tfrac{8}{3}\{J_{ij}, J_{jk}, M_i\} - \tfrac{64}{3}\{J_{ij}, M_k\} - \tfrac{64}{3}\{J_{ik}, M_j\} \\ & - \tfrac{64}{3}\{J_{jk}, M_i\} + 16(1 - k_i^2)\{J_{jk}, M_j\} + 16(1 - k_j^2)\{J_{ik}, M_i\} - 64(1 - k_i^2)(1 - k_j^2)M_k, \end{aligned}$$

$$\begin{aligned} R^2 = {} & -\tfrac{4}{3}\{J_{ij}, J_{ik}, J_{jk}\} + \tfrac{64}{3}\{J_{ij}, J_{ik}\} + \tfrac{64}{3}\{J_{ij}, J_{jk}\} + \tfrac{64}{3}\{J_{ik}, J_{jk}\} - 16(1 - k_k^2)J_{ij}^2 \\ & - 16(1 - k_j^2)J_{ik}^2 - 16(1 - k_i^2)J_{jk}^2 + \tfrac{128}{3}(1 - k_k^2)J_{ij} \\ & + \tfrac{128}{3}(1 - k_j^2)J_{ik} + \tfrac{128}{3}(1 - k_i^2)J_{jk} + 64(1 - k_i^2)(1 - k_j^2)(1 - k_k^2), \end{aligned}$$

where $\{A,B,C\} = ABC + CAB + BCA$. Note that only five operators from (3) and (4) are functionally independent,[7] and for all the coordinate systems that provide separable solutions for the Schrödinger equation the operators characterizing the separation are always combinations of the M_i and J_{ij}.

In the limiting case $k_i=\frac{1}{2}$, we obtain a quadratic algebra, too. In this case

$$Q_{ij}=2(L_{ij}D_{ij}+D_{ij}L_{ij}), \quad R=\{L_{ik},\{L_{ij},L_{kj}\}\},$$

and instead of operators $\{M_i,J_{ij},Q_{ij},R\}$ we can consider as a basis for the symmetries the Demkov tensor D_{ij}, and the components of orbital momentum, L_{ij}. In this regard we arrive at the Lie algebra corresponding to the symmetries of the isotropic oscillator.[1]

Of all the coordinate systems for which separation is possible, in the case of this potential there are only five which are not essentially a Euclidean two-space coordinate system supplemented by an additional Cartesian coordinate z. Such coordinate systems we do not consider further here and the corresponding solutions of the Schrödinger equation and invariant algebra are given in our previous paper[17] (see also Refs. 3 and 8). For the remaining systems we now work out bound state solutions and their corresponding symmetry characterization.

A. Oblate spheroidal basis

Let us consider what we call oblate spheroidal coordinates (see Table II). If we write these coordinates in the form

$$x=x'\cos\varphi, \quad y=x'\sin\varphi, \quad z=y', \tag{5}$$

and put $\Psi=(x')^{-1/2}\Phi$, the Schrödinger equation (1) with potential (2) assumes the form

$$\frac{\partial^2\Phi}{\partial x'^2}+\frac{\partial^2\Phi}{\partial y'^2}+\left[2E-\omega^2(x'^2+y'^2)+\frac{1}{x'^2}\left(\frac{\partial^2}{\partial\varphi^2}-\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}-\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right)+\frac{1}{4x'^2}-\frac{k_2^3-\frac{1}{4}}{y'^2}\right]\Phi=0.$$

If we now write

$$\Phi=\Lambda(x',y')Y(\varphi),$$

the φ dependence can be extracted by requiring that

$$\left[\frac{\partial^2}{\partial\varphi^2}-\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}-\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right]Y(\varphi)=-M^2Y(\varphi). \tag{6}$$

The orthonormal solution of Eq. (6) for $\varphi\in[0,\pi/2]$ has the following form:

$$Y_m^{(k_1,k_2)}(\varphi)=\sqrt{\frac{2(2m+k_1+k_2+1)m!\Gamma(m+k_1+k_2+1)}{\Gamma(m+k_1+1)\Gamma(m+k_2+1)}}\times(\cos\varphi)^{k_1+1/2}(\sin\varphi)^{k_2+1/2}P_m^{(k_1,k_2)}(\cos 2\varphi), \tag{7}$$

where $P_n^{(\alpha,\beta)}(z)$ is a Jacobi polynomial and the separation constant quantizes as

$$M=2m+k_1+k_2+1, \quad m=0,1,2,\ldots. \tag{8}$$

The remaining equation for the function $\Lambda(x',y')$ is

$$\left\{\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\left[2E-\omega^2(x'^2+y'^2)-\frac{k_3^2-\frac{1}{4}}{y'^2}-\frac{M^2-\frac{1}{4}}{x'^2}\right]\right\}\Lambda(x',y')=0.$$

This is exactly the equation we have already found (see Ref. 17) in the case of two-dimensional Euclidean space in elliptic coordinates. In terms of the original Cartesian coordinates the bound state solutions have the form

$$\Lambda_{nm}^{k_3}(x,y,z)=e^{-(\omega/2)(x^2+y^2+z^2)}(x^2+y^2)^{m+(k_1+k_2+1)/2}z^{k_3+1/2}\prod_{i=1}^{n}\left(\frac{x^2+y^2}{\theta_i-e_1}+\frac{z^2}{\theta_i-e_2}-1\right), \quad (9)$$

where the θ_i satisfy of the system of n nonlinear equations

$$\frac{M+1}{\theta_i-e_1}+\frac{k_3+1}{\theta_i-e_2}+\sum_{j\neq i}^{n}\frac{2}{\theta_i-\theta_j}-\omega=0.$$

We note that this prescription does correctly give a separable solution by noting the identity

$$\frac{x^2+y^2}{\theta-e_1}+\frac{z^2}{\theta-e_2}-1=-\frac{(u_1-\theta)(u_2-\theta)}{(\theta-e_1)(\theta-e_2)}.$$

The energy E is quantized according to

$$E=\omega(2n+M+k_3+3)=\omega(2N+k_1+k_2+k_3+3), \quad (10)$$

where $N=n+m$ is the principal quantum number.

Consider the Schrödinger equation in the spheroidal separable coordinates (u_1,u_2,φ). After the substitution $\Psi=\psi_1(u_1)\psi_2(u_2)Y(\varphi)$, the separation equations are

$$\frac{d^2\psi(u)}{du^2}+\frac{1}{2}\left(\frac{2}{u-e_1}+\frac{1}{u-e_2}\right)\frac{d\psi(u)}{du}+\frac{1}{4}\left\{\frac{2Eu-\omega^2(u-e_1)(u-e_2)+\lambda}{(u-e_1)(u-e_2)}+\frac{(e_2-e_1)M^2}{(u-e_1)^2(u-e_2)}\right.$$
$$\left.+\frac{(e_1-e_2)(k_3^2-\frac{1}{4})}{(u-e_1)(u-e_2)^2}\right\}\psi(u)=0, \quad (11)$$

where $u=u_1,u_2$ and λ is the oblate spheroidal separation constant. The operator whose eigenvalue is λ is

$$\mathcal{L}_1=\frac{u_2(u_1-e_1)(u_1-e_2)}{u_1-u_2}\left\{\frac{\partial^2}{\partial u_1^2}+\frac{1}{2}\left[\frac{2}{u_1-e_1}+\frac{1}{u_1-e_2}\right]\frac{\partial}{\partial u_1}\right\}-\frac{u_1(u_2-e_1)(u_2-e_2)}{u_1-u_2}$$
$$\times\left\{\frac{\partial^2}{\partial u_2^2}+\frac{1}{2}\left[\frac{2}{u_2-e_1}+\frac{1}{u_2-e_2}\right]\frac{\partial}{\partial u_2}\right\}+\frac{1}{4}\left[\omega^2(e_2e_2-u_1u_2)+\frac{M^2(e_1-e_2)}{(u_1-e_1)(u_2-e_1)}\right.$$
$$\left.\times(u_1+u_2-e_1)+\frac{(k_3^2-\frac{1}{4})(e_2-e_1)}{(u_1-e_2)(u_2-e_2)}(u_1+u_2-e_2)\right]$$
$$=J_{13}+J_{23}+J_{12}+(e_1-e_2)M_3-e_2\mathcal{H}-(k_1^2+k_2^2+k_3^2)+\tfrac{3}{4} \quad (12)$$

with eigenvalues

$$\lambda=-4e_2\sum_{i}^{n}\frac{M+1}{\theta_i-e_1}-4e_1\sum_{i}^{n}\frac{k_3+1}{\theta_i-e_2}-2[(e_1+e_2)+(e_1k_3+e_2M)]\omega-(k_1^2+k_2^2+k_3^2)+\frac{3}{4}, \quad (13)$$

and the second operator which characterizes the separation of variables in these coordinates is

$$\mathcal{L}_2\Psi=(J_{12}-k_1^2-k_2^2+1)\Psi=-M^2\Psi. \quad (14)$$

To close this paragraph let us note that in the limit $(e_2-e_1)\to 0$ and $(e_2-e_1)\to\infty$ the oblate spheroidal coordinates are changed into spherical and cylindrical polar coordinates, respectively.[8] Correspondingly, the oblate spheroidal bases transform to spherical and cylindrical polar ones.

B. Prolate spheroidal basis

For prolate coordinates the description is almost exactly the same. All that is essentially involved is the interchange of e_1 and e_2.

C. Ellipsoidal basis

For ellipsoidal coordinates (see Table II) we proceed as follows. We consider the Schrödinger equation in Cartesian coordinates,

$$\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+\left[2E-\omega^2(x^2+y^2+z^2)-\frac{(k_1^2-\frac{1}{4})}{x^2}-\frac{(k_2^2-\frac{1}{4})}{y^2}+\frac{(k_3^2-\frac{1}{4})}{z^2}\right]\Psi=0.$$

If we now write

$$\Psi(x,y,z)=e^{-\omega(x^2+y^2+z^2)}x^{2k_1+1}y^{2k_2+1}z^{2k_3+1}\Phi(x,y,z),$$

the equation for Φ becomes

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{2k_1+1}{x}\frac{\partial}{\partial x}+\frac{2k_2+1}{y}\frac{\partial}{\partial y}+\frac{2k_3+1}{z}\partial_z-2\omega\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\right.$$
$$\left.-2\omega(k_1+k_2+k_3+3)\right]\Psi=-2E\Psi.$$

To obtain the appropriate finite solutions we can make use of the identity

$$\frac{x^2}{\theta-e_1}+\frac{y^2}{\theta-e_2}+\frac{z^2}{\theta-e_3}-1=-\frac{(u_1-\theta)(u_2-\theta)(u_3-\theta)}{(\theta-e_1)(\theta-e_2)(\theta-e_3)}$$

and write

$$\Phi(x,y,z)=\prod_{j=1}^{N}\left(\frac{x^2}{\theta_j-e_1}+\frac{y^2}{\theta_j-e_2}+\frac{z^2}{\theta_j-e_3}-1\right), \tag{15}$$

where

$$\frac{k_1+1}{\theta_i-e_1}+\frac{k_2+1}{\theta_i-e_2}+\frac{k_3+1}{\theta_i-e_3}+\sum_{j\neq i}^{N}\frac{2}{\theta_i-\theta_j}-\omega=0$$

and the energy level E is given by Eq. (10).

Writing the Schrödinger equation in terms of the ellipsoidal coordinates u_i and using the identities

$$E\equiv\sum_{i=1}^{3}\frac{E_i^2}{\Pi_{i\neq j}(u_i-u_j)},\quad (x^2+y^2+z^2)\equiv\sum_{i=1}^{3}\frac{P(u_i)}{\Pi_{i\neq j}(u_i-u_j)},$$

$$\left[\frac{(k_1^2-\frac{1}{4})}{x^2}+\frac{(k_2^2-\frac{1}{4})}{y^2}+\frac{(k_3^2-\frac{1}{4})}{z^2}\right]\equiv\sum_{i=1}^{3}\frac{A(u_i)}{\Pi_{i\neq j}(u_i-u_j)},$$

where $P(u_i)=(u_i-a_1)(u_i-a_2)(u_i-a_3)$ and $[a_{ik}\equiv(a_i-a_k)]$ and

$$A(u)=a_{31}a_{21}\frac{(k_1^2-\frac{1}{4})}{u-a_1}+a_{12}a_{32}\frac{(k_2^2-\frac{1}{4})}{u-a_2}+a_{13}a_{23}\frac{(k_1^2-\frac{1}{4})}{u-a_3},$$

we arrive at the following equation:

$$\sum_{i=1}^{3}\frac{1}{\Pi_{i\neq j}(u_i-u_j)}\left\{4\sqrt{P(u_i)}\frac{\partial}{\partial u_i}\sqrt{P(u_i)}\frac{\partial}{\partial u_i}+2Eu_i^2-\omega^2P(u_i)-A(u_i)\right\}\Psi=0,$$

which, after the substitution $\Psi=\psi_1(u_1)\psi_2(u_2)\psi_3(u_3)$ and the introduction of the ellipsoidal constants λ_1 and λ_2, is divided into three identical differential equations

$$\sqrt{P(u)}\frac{d}{du}\sqrt{P(u)}\frac{d\psi}{du}+\frac{1}{4}\left[2Eu^2-\omega^2P(u)+\lambda_1u-\lambda_2-\frac{(k_1^2-\frac{1}{4})}{(u-a_1)}(a_3-a_1)(a_2-a_1)\right.$$
$$\left.-\frac{(k_2^2-\frac{1}{4})}{(u-a_2)}(a_1-a_2)(a_3-a_2)-\frac{(k_3^2-\frac{1}{4})}{(u-a_3)}(a_1-a_3)(a_2-a_3)\right]\psi=0,$$

where $u=u_1,u_2,u_3$. The operators that specify the eigenvalues λ_1 and λ_2 are

$$\Lambda_1=J_{12}+J_{23}+J_{23}+(a_2+a_3)M_1+(a_2+a_1)M_3+(a_1+a_3)M_2-(k_1^2+k_2^2+k_3^2)+\tfrac{3}{4}$$

and

$$\Lambda_2=a_3J_{12}+a_2J_{13}+a_1J_{23}+a_2a_3M_1+a_2a_1M_3+a_1a_3M_2$$
$$-k_1^2(a_3+a_2-a_1)-k_2^2(a_1+a_3-a_2)-k_3^2(a_1+a_2-a_3)+\tfrac{3}{4}(a_1+a_2+a_3),$$

respectively. In terms of the zeros θ_j the eigenvalues of these operators are

$$\lambda_1=-2[k_2(a_1+a_3)+k_1(a_2+a_3)+k_3(a_1+a_2)-4(a_1+a_2+a_3)]\omega-2(k_1+k_2+k_3)$$
$$\times(k_1+k_2+k_3+1)-\tfrac{9}{2}+4\left[\sum_{i=1}^{N}a_2\frac{(k_2+1)}{(\theta_i-a_2)}+\sum_{i=1}^{N}a_1\frac{(k_1+1)}{(\theta_i-a_1)}+\sum_{i=1}^{N}a_3\frac{(k_3+1)}{(\theta_i-a_3)}\right] \quad (16)$$

and

$$\lambda_2=-\tfrac{1}{2}(a_1+a_2+a_3)-2\omega[a_2a_3(k_1+1)+a_2a_1(k_3+1)+a_1a_3(k_2+1)]-a_1(k_2+k_3+1)^2$$
$$-a_2(k_1+k_3+1)^2-a_3(k_2+k_1+1)^2$$
$$-4\left[\sum_{i=1}^{N}a_3a_1\frac{(k_2+1)}{(\theta_i-a_2)}+\sum_{i=1}^{N}a_3a_2\frac{(k_1+1)}{(\theta_i-a_1)}+\sum_{i=1}^{N}a_2a_1\frac{(k_3+1)}{(\theta_i-a_3)}\right]. \quad (17)$$

D. Spherical and sphero-conical bases

For spherical-type coordinates there are two possibilities.
If we choose coordinates in Euclidean space accordingly,

$$x=rs_1,\quad y=rs_2,\quad z=rs_3, \quad (18)$$

where $s_1^2+s_2^2+s_3^2=1$, and put the wave function in the form

$$\Psi=R(r)S(\rho_1,\rho_2), \quad (19)$$

where ρ_1,ρ_2 are the spherical or sphero-conical coordinates, after separation of variables, we arrive at two equations,

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left[2E-\omega^2r^2-\frac{J(J+1)}{r^2}\right]R=0, \tag{20}$$

$$\left\{L_{12}+L_{23}+L_{13}+\left[J(J+1)-\frac{(k_1^2-\frac{1}{4})}{s_1^2}-\frac{(k_2^2-\frac{1}{4})}{s_2^2}-\frac{(k_3^2-\frac{1}{4})}{s_3^2}\right]\right\}S=0, \tag{21}$$

where J is the spherical separation constant.

(1) In the spherical coordinates (see Table II) the wave function $S(\rho_1,\rho_2)$ has the separable form

$$S(\vartheta,\varphi)=Z(\vartheta)Y_m^{(k_1,k_2)}(\varphi),$$

where $Y_m^{(k_1,k_2)}(\varphi)$ is given by formula (7). This leads to the equation for Z:

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\sin\vartheta\frac{dZ}{d\vartheta}+\left[J(J+1)-\frac{M^2}{\sin^2\vartheta}-\frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta}\right]Z=0,\quad M=2m+k_1+k_2+1.$$

The solution of the above equation is (see Ref. 8)

$$Z(\theta)=\sqrt{\frac{2[2(m+l+1)+k_1+k_2+k_3]l!\Gamma(l+2m+k_1+k_2+k_3+2)}{\Gamma(l+k_3+1)\Gamma(l+2m+2+k_1+k_2)}}$$
$$\times(\cos\theta)^{1/2+k_3}(\sin\theta)^M P_l^{(M,k_3)}(\cos 2\theta),\quad l\in\mathbf{N}, \tag{22}$$

and for a spherical separation constant we get

$$J=2l+M+k_3+\tfrac{1}{2}=2l+2m+k_1+k_2+k_3+\tfrac{3}{2}. \tag{23}$$

(2) If we choose the sphero-conical coordinates on the sphere (see Table II), the solution of the equation (21) has the form

$$S(\rho_1,\rho_2)=\prod_{\ell=1}^{3}s_\ell^{k_\ell+1/2}\prod_{j=1}^{n}\left(\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}\right), \tag{24}$$

and the spherical separation constant is quantized according to Eq. (23) where $n=l+m$. This achieves a separation of variables solution because of the identity

$$\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}=\frac{(\rho_1-\theta_j)(\rho_2-\theta_j)}{(\theta_j-e_1)(\theta_j-e_2)(\theta_j-e_3)}$$

and the Niven equations

$$\frac{k_1+1}{\theta_i-e_1}+\frac{k_2+1}{\theta_i-e_2}+\frac{k_3+1}{\theta_i-e_3}+\sum_{j\neq i}^{q}\frac{2}{\theta_i-\theta_j}=0.$$

The functions $S(\rho_1,\rho_2)$ have the separable form

$$S(\rho_1,\rho_2)=B_1(\rho_1)B_2(\rho_2), \tag{25}$$

and the separation equations are $[P(\rho)=(\rho-e_1)(\rho-e_2)(\rho-e_3)]$

$$\sqrt{P(\rho)}\frac{d}{d\rho}\sqrt{P(\rho)}\frac{dB}{d\rho}+\frac{1}{4}\left[\lambda-J(J+1)-\frac{(k_1^2-\frac{1}{4})}{(\rho-e_1)}(e_1-e_2)(e_1-e_3)-\frac{(k_2^2-\frac{1}{4})}{(\rho-e_2)}(e_2-e_1)(e_2-e_3)\right.$$
$$\left.-\frac{(k_3^2-\frac{1}{4})}{(\rho-e_3)}(e_3-e_2)(e_3-e_1)\right]B=0, \tag{26}$$

where $B=B_1,B_2$ according as $\rho=\rho_1,\rho_2$, respectively. The sphero-conical wave functions satisfy the eigenfunction equations

$$(J_{12}+J_{13}+J_{23})S=[(k_1^2+k_2^2+k_3^2)-(2q+2+k_1+k_2+k_3)^2-\tfrac{1}{2}]S \tag{27}$$

$$(e_1J_{23}+e_2J_{13}+e_3J_{12})S$$
$$=[k_1^2(e_2+e_3-e_1)+k_2^2(e_1+e_3-e_2)+k_3^2(e_1+e_2-e_3)-\tfrac{3}{4}(e_1+e_2+e_3)-\lambda]S, \tag{28}$$

where

$$\lambda=2[k_1(e_2+e_3)+k_2(e_1+e_3)+k_3(e_2+e_1)+e_3k_1k_2+e_2k_1k_3+e_1k_3k_2]+\tfrac{3}{2}(e_1+e_2+e_3)$$
$$-4\left[e_2e_3\sum_{i=1}^{n}\frac{k_1+1}{\theta_i-e_1}+e_1e_3\sum_{i=1}^{n}\frac{k_2+1}{\theta_i-e_2}+e_2e_1\sum_{i=1}^{n}\frac{k_3+1}{\theta_i-e_3}\right]. \tag{29}$$

Let us now go to the radial equation (20). This equation is very reminiscent of the radial equation for the three-dimensional harmonic oscillator except that the orbital quantum number l is replaced by $2l+2m+k_1+k_2+k_3+\frac{3}{2}$. The orthonormal solution of the radial equation (20), in terms of Laguerre polynomials $L_n^{\alpha}(x)$, is

$$R_{n_rJ}(r)=\sqrt{\frac{2\omega^{3/2}n_r!}{\Gamma(n_r+2l+2m+k_1+k_2+k_3+3)}}(\sqrt{\omega}r)^J\exp\left(-\frac{\omega}{2}r^2\right)L_{n_r}^{J+1/2}(\omega r^2), \tag{30}$$

and the energy spectrum is given by formula (10) where the $n_r=0,1,2,...$ is the radial quantum number and the principal quantum number now is $N=(n_r+n)=(n_r+l+m)$.

III. GENERALIZED ANISOTROPIC OSCILLATOR

The second potential (see Table I) is

$$V_2(x,y,z)=\frac{\omega^2}{2}(x^2+y^2+4z^2)+\frac{1}{2}\left[\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right]. \tag{31}$$

The corresponding Schrödinger equation has separable solutions in five coordinate systems: Cartesian coordinates, cylindrical polar coordinates, cylindrical elliptic coordinates, cylindrical parabolic coordinates, and parabolic coordinates. It is the last of these that gives interesting new solutions. The first four coordinate systems are of cylindrical type and can be deduced from what we already know for Euclidean two-dimensional space (see Refs. 8 and 17). Before considering the bound state solutions in the case of the parabolic coordinate system we consider the quadratic algebra of second-order symmetry operators which are associated with this potential. A basis for these operators is

$$M_1=\partial_x^2-\omega^2x^2+\frac{k_1^2-\frac{1}{4}}{x^2},\quad M_2=\partial_y^2-\omega^2y^2-\frac{k_2^2-\frac{1}{4}}{y^2},\quad P=\partial_z^2-4\omega^2z^2, \tag{32}$$

$$L=L_{12}^2-\left(k_1^2-\frac{1}{4}\right)\frac{y^2}{x^2}-\left(k_2^2-\frac{1}{4}\right)\frac{x^2}{y^2}-\frac{1}{2}, \tag{33}$$

$$S_1=-\frac{1}{2}(p_xL_{13}+L_{13}p_x)+z\left(\omega^2x^2-\frac{k_1^2-\frac{1}{4}}{x^2}\right),$$

$$S_2=-\frac{1}{2}(p_yL_{23}+L_{23}p_y)+z\left(\omega^2y^2-\frac{k_2^2-\frac{1}{4}}{x^2}\right), \tag{34}$$

where $p_{x,y}=\partial_{x,y}$.

The relations that define the quadratic algebra are obtained by exhaustive commutation. The nonzero commutators of the above basis are

$$[M_1,L]=[L,M_2]=Q,\quad [L,S_1]=[S_2,L]=B,\quad [M_i,S_i]=A_i,\quad [P,S_i]=-A_i.$$

Further nonzero commutators with Q are

$$[M_i,Q]=[Q,M_2]=4\{M_1,M_2\}+16\omega^2L,\quad [S_1,Q]=[Q,S_2]=4\{M_1,M_2\},$$

$$[L,Q]=4\{M_1,L\}-4\{M_2,L\}+16(1-k_1^2)M_1-16(1-k_2^2)M_2;$$

nonzero commutators with A_i are

$$[M_i,A_i]=16\omega^2S_i,\quad [L,A_1]=[A_2,L]=4\{M_1,S_2\}-4\{M_2,S_1\},\quad [P,A_i]=-16\omega^2S_i,$$

$$[S_i,A_i]=\{M_i,M_i\}-2\{M_i,P\}+8\omega^2(1-k_i^2),\quad [S_i,A_j]=\{M_i,M_j\}+4\omega^2L;$$

and nonzero commutators with B are

$$[M_1,B]=-4\{M_2,S_1\},\quad [M_2,B]=-4\{M_1,B\},\quad [P,B]=4\{M_2,S_1\}-4\{M_1,S_2\},$$

$$[L,B]=-4\{L,S_1\}+4\{L,S_2\}-16(1-k_2^2)S_1+16(1-k_1^2)S_2,$$

$$[S_1,B]=\{L,M_1\}-2\{L,P\}-4\{S_1,S_2\}-4(1-k_1^2)M_2,$$

$$[B,S_2]=\{L,M_2\}-2\{L,P\}-4\{S_1,S_2\}-4(1-k_2^2)M_1.$$

The remaining nonzero commutators are

$$[Q,A_i]=-4\{M_i,A_j\},\quad [Q,B]=-4\{L,A_1\}-4\{L,A_2\},\quad [A_1,A_2]=4\omega^2Q,$$

$$[A_1,B]=\{M_1,Q\}-4\{S_1,A_2\},\quad [B,A_2]=\{M_2,Q\}-4\{S_2,A_1\}.$$

There are also various relations among the generators of our quadratic algebra:

$$\{M_1,B\}=\{L,A_1\}-\{S_1,Q\}-4(1-k_1^2)A_2,$$

$$\{M_2,B\}=-\{L,A_2\}-\{S_2,Q\}+4(1-k_2^2)A_1,$$

$$\{P,Q\}=2\{S_1,A_2\}-2\{S_2,A_1\},\quad \{M_1,A_2\}-\{M_2,A_1\}-4\omega^2B=0,$$

$$Q^2=\tfrac{8}{3}\{L,M_1,M_2\}+8\omega^2\{L,L\}-16(1-k_1^2)M_1^2-16(1-k_2^2)M_2^2$$
$$+\tfrac{64}{3}\{M_1,M_2\}-\tfrac{128}{3}\omega^2L-128\omega^2(1-k_1^2)(1-k_2^2),$$

$$\{Q,A_1\}=\tfrac{8}{3}\{M_1,M_2,S_1\}-\tfrac{8}{3}\{M_1,M_1,S_2\}+16\omega^2\{L,S_1\}-64(1-k_1^2)S_2,$$

$$\{Q,A_2\}=-\tfrac{8}{3}\{M_1,M_2,S_2\}+\tfrac{8}{3}\{M_2,M_2,S_1\}-16\omega^2\{L,S_2\}+64(1-k_2^2)S_1,$$

$$\{Q,B\}=-\tfrac{8}{3}\{M_2,L,S_1\}-\tfrac{8}{3}\{M_1,L,S_2\}+16(1-k_1^2)\{M_2,S_2\}$$
$$+16(1-k_2^2)\{M_1,S_1\}-\tfrac{64}{3}\{M_1,S_2\}-\tfrac{64}{3}\{M_2,S_1\},$$

$$A_1^2=\tfrac{2}{3}\{M_1,M_1,P\}+8\omega^2\{S_1,S_1\}+16\omega^2(1-k_1^2)P-32\omega^2M_1,$$

$$\{A_1,A_2\}=\tfrac{4}{3}\{M_1,M_2,P\}+16\omega^2\{S_1,S_2\}+8\omega^2\{L,P\},$$

$$\{A_1,B\}=\tfrac{8}{3}\{M_1,S_1,S_2\}-\tfrac{8}{3}\{M_2,S_1,S_1\}+\tfrac{4}{3}\{M_1,L,P\}+\tfrac{32}{3}\{M_1,M_2\}-8(1-k_1^2)\{M_1,P\}-\tfrac{64}{3}\omega^2L,$$

$$\{A_2,B\}=-\tfrac{8}{3}\{M_2,S_2,S_1\}+\tfrac{8}{3}\{M_1,S_2,S_2\}-\tfrac{4}{3}\{M_2,L,P\}$$
$$-\tfrac{32}{3}\{M_1,M_2\}+8(1-k_2^2)\{M_2,P\}+\tfrac{64}{3}\omega^2L,$$

$$B^2=\tfrac{8}{3}\{L,S_1,S_2\}+\tfrac{2}{3}\{L,L,P\}+\tfrac{64}{3}\{S_1,S_2\}-16(1-k_1^2)S_2^2-16(1-k_2^2)S_1^2$$
$$+\tfrac{16}{3}\{L,M_1\}-\tfrac{16}{3}\{L,P\}+\tfrac{32}{3}(1-k_2^2)M_1+\tfrac{32}{3}(1-k_1^2)M_2-16(1-k_1^2)(1-k_2^2)P.$$

This completes the nonzero relations for the quadratic algebra and the associated relations among the generators. For the last coordinate system in our list we develop the bound state solutions.

Parabolic basis

The Schrödinger equation in Cartesian coordinates with this potential has the form

$$\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+\left[2E-\omega^2(x^2+y^2+4z^2)-\frac{(k_1^2-\frac{1}{4})}{x^2}+\frac{(k_2^2-\frac{1}{4})}{y^2}\right]\Psi=0.$$

If we choose the coordinates (x',y',φ) according formula (6) and the wave function Ψ in the form

$$\Psi(x',y',\varphi)=(x')^{-1/2}\Lambda(x',y')Y_m^{(k_1,k_2)}(\varphi),\quad m=0,1,...,$$

where $Y_m^{(k_1,k_2)}(\varphi)$ is given by (7), then the equation for the function $\Lambda(x',y')$ is

$$\left[\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}-\omega^2(x'^2+4y'^2)-\frac{(M^2-\frac{1}{4})}{x'^2}+2E\right]\Lambda(x',y')=0.$$

This just the problem whose solution has been found (see Ref. 1) in the case of two-dimensional Euclidean space. If now we write

$$\Lambda(x,y,z)=e^{-(\omega/2)(x^2+y^2)-\omega z^2}(x^2+y^2)^{(k_2/2+1/4)}P(x,y),$$

where

$$P(x,y)=\prod_{j=1}^{n}\left(\frac{x^2+y^2}{\theta_j^2}+2z-\theta_j^2\right),$$

then the λ_j satisfy

$$\frac{4(\lambda+1)}{\theta_{\not l}^2}+\sum_{i\neq l}^{n}\frac{4}{\theta_{\not l}^2-\theta_i^2}-2\omega\theta_{\not l}^2=0$$

and energy E quantizes according to

$$E=\omega(2n+M+2)=\omega(2N+k_1+k_2+3), \tag{35}$$

where the principal quantum number $N=n+m$. This method of solution is based on the identity

$$\frac{x^2+y^2}{\theta^2}+2z-\theta^2=\frac{(\xi^2-\theta^2)(\eta^2+\theta^2)}{\theta^2}.$$

In fact, the separation equations in ξ and η for solution of the Schrödinger equation

$$\Psi(\xi,\eta,\varphi)=X_1(\xi)X_2(\eta)Y_m^{(k_1,k_2)}(\varphi)$$

have the form

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\left(2E\rho^2-\omega^2\rho^6-\frac{M^2}{\rho^2}+\epsilon\beta\right)\right]X(\rho)=0, \tag{36}$$

where $\epsilon=1$ if $\rho=\xi$ and -1 if $\rho=\eta$, and β is the parabolic separation constant. By eliminating the energy E from Eqs. (36) we produce the operator, the eigenvalues of which is β:

$$\mathcal{L}=\frac{1}{\xi^2+\eta^2}\left(\frac{\xi^2}{\eta}\frac{\partial}{\partial\eta}\eta\frac{\partial}{\partial\eta}-\frac{\eta^2}{\xi}\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi}\right)+\omega^2\xi^2\eta^2(\xi^2-\eta^2)-\frac{\xi^2-\eta^2}{\xi^2\eta^2}\left(\frac{\partial^2}{\partial\varphi^2}-\frac{k_1^2-\frac{1}{4}}{\cos^2\varphi}-\frac{k_2^2-\frac{1}{4}}{\sin^2\varphi}\right). \tag{37}$$

In Cartesian coordinates the operator $\mathcal{L}$ can be written as

$$\mathcal{L}=z\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\omega^2(x^2+y^2)-\frac{k_1^2-\frac{1}{4}}{x^2}-\frac{k_2^2-\frac{1}{4}}{y^2}\right]-\frac{\partial}{\partial z}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-1\right),$$

and thus the parabolic basis satisfies two eigenvalue equations

$$L\Psi=(k_1^2+k_2^2-M^2-1)\Psi,\quad \mathcal{L}\Psi=2(S_1+S_2)\Psi=\beta\Psi,$$

where operators L, $S_{1,2}$ are given by formulas (33) and (34) and the eigenvalue β is

$$\beta=-2(M-1)\prod_{j=1}^{n}\theta_j^2\left(\sum_{k=1}^{n}\theta_k^{-2}\right). \tag{38}$$

IV. GENERALIZED KEPLER–COULOMB SYSTEM

The third potential we consider is

$$V_3(x,y,z)=-\frac{\alpha}{\sqrt{x^2+y^2+z^2}}+\frac{1}{2}\left[\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right]. \tag{39}$$

The corresponding Schrödinger equation has the form

$$\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+\left[2E+\frac{2\alpha}{\sqrt{x^2+y^2+z^2}}-\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right)\right]\Psi=0.$$

This equation admits separable solutions in the four coordinate systems: spherical, spheroconical, prolate spheroidal, and parabolic.

The second-order symmetries of the corresponding Schrödinger equation are

$$J_{23}=L_{23}^2-\left(k_2^2-\frac{1}{4}\right)\frac{z^2}{y^2}-\frac{1}{2},\quad J_{13}=L_{13}^2-\left(k_1^2-\frac{1}{4}\right)\frac{z^2}{x^2}-\frac{1}{2},$$

$$J_{12}=L_{12}^2-\left(k_2^2-\frac{1}{4}\right)\frac{x^2}{y^2}-\left(k_1^2-\frac{1}{4}\right)\frac{y^2}{x^2}-\frac{1}{2}, \tag{40}$$

$$L=-\frac{1}{2}[\{p_x,L_{13}\}+\{p_y,L_{23}\}]+\frac{\alpha z}{\sqrt{x^2+y^2+z^2}}-z\left(\frac{k_1^2-\frac{1}{4}}{x^2}+\frac{k_2^2-\frac{1}{4}}{y^2}\right). \tag{41}$$

These symmetry operators do not appear to close under repeated commutation. One obvious subalgebra that is quadratically closed is that generated by the elements J_{12}, J_{13}, and J_{23}. The closure relations can be readily deduced from the algebra given for the first potential with the proviso that $k_3=\frac{1}{2}$.

A. Spherical and sphero-conical bases

If we use polar coordinates according to (18) and write the wave function Ψ in the separable form $\Psi=R(r)S(\rho_1,\rho_2)$, then the separable equations are

$$\frac{d^2R}{dr}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2\alpha}{r}-\frac{J(J+1)}{r^2}\right]R=0, \tag{42}$$

$$\left\{L_{12}+L_{23}+L_{13}+\left[J(J+1)-\frac{(k_1^2-\frac{1}{4})}{s_1^2}-\frac{(k_2^2-\frac{1}{4})}{s_2^2}\right]\right\}S=0. \tag{43}$$

(1) In the spherical coordinates, choosing the wave function $S(\rho_1,\rho_2)$ according to

$$S(\vartheta,\varphi)=Z(\vartheta)Y_m^{(k_1,k_2)}(\varphi),\quad m=0,1,2,...,$$

where $Y_m^{(k_1,k_2)}(\varphi)$ is given by formula (7), we go to the equation for Z:

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\sin\vartheta\frac{dZ}{d\vartheta}+\left[J(J+1)-\frac{M^2}{\sin^2\vartheta}\right]Z=0,\quad M=2m+k_1+k_2+1.$$

The orthonormal solution of the above equation for $\vartheta\in[0,\pi]$ is

$$Z(\vartheta)=\frac{2^M}{\sqrt{\pi}}\sqrt{\frac{(2l+2M+1)l!}{2\Gamma(l+2M+1)}}\Gamma\left(M+\frac{1}{2}\right)(\sin\vartheta)^M C_l^{M+1/2}(\cos\vartheta), \tag{44}$$

where $l\in\mathbf{N}$ and $C_n^\lambda(x)$ is a Gegenbauer polynomial.[21] The spherical separation constant is given by

$$J=l+M=l+2m+k_1+k_2+1. \tag{45}$$

(2) The solution of the Schrödinger equation (43) in sphero-conical coordinates follows from what we have done before in Sec. II D, part (2). If we write $S(\rho_1,\rho_2)$ as

$$S(\rho_1,\rho_2)=s_3^\epsilon\prod_{\ell=1}^{2}s_\ell^{k_\ell+1/2}\prod_{j=1}^{n}\left(\frac{s_1^2}{\theta_j-e_1}+\frac{s_2^2}{\theta_j-e_2}+\frac{s_3^2}{\theta_j-e_3}\right),$$

where $\epsilon=0,1$ then the zeros satisfy the Niven equations

$$\frac{k_1+1}{\theta_i-e_1}+\frac{k_2+1}{\theta_i-e_2}+\frac{\epsilon+\frac{1}{2}}{\theta_i-e_3}+\sum_{j\neq i}^{n}\frac{2}{\theta_i-\theta_j}=0.$$

The functions $S(\rho_1,\rho_2)$ satisfy the eigenfunction equations

$$(J_{12}+J_{13}+J_{23})S=[(k_1^2+k_2^2)-(2q+\tfrac{3}{2}+k_1+k_2+\epsilon)^2-\tfrac{7}{4}]S, \tag{46}$$

$$(e_1J_{23}+e_2J_{13}+e_3J_{12})S=[(e_2-e_1)(k_1^2-k_2^2)+e_3(k_1^2+k_2^2-1)-\tfrac{1}{2}(e_1+e_2)-\lambda]S, \tag{47}$$

where the sphero-conical separation constant λ is

$$\lambda=-2[k_1(e_2+e_3)+k_2(e_1+e_3)+(\epsilon-\tfrac{1}{2})(e_2+e_1)+e_3k_1k_2+(e_2k_1+e_1k_2)(\epsilon-\tfrac{1}{2})]$$
$$-\frac{3}{2}(e_1+e_2+e_3)-4\left[e_2e_3\sum_{i=1}^{n}\frac{k_1+1}{\theta_i-e_1}+e_1e_3\sum_{i=1}^{n}\frac{k_2+1}{\theta_i-e_2}+e_2e_1\sum_{i=1}^{n}\frac{\epsilon+\frac{1}{2}}{\theta_i-e_3}\right]. \tag{48}$$

Finally, let us consider the radial equation (42). The introduction of (45) into (42) leads to

$$\frac{d^2R}{dr}+\frac{2}{r}\frac{dR}{dr}+\left[2E+\frac{2\alpha}{r}-\frac{(l+2m+k_1+k_2+1)(l+2m+k_1+k_2+2)}{r^2}\right]R=0,$$

which is the radial equation for the Coulomb problem, except the orbital quantum number l is replaced here by $(l+2m+k_1+k_2+1)$. The bound state solution of Eq. (43) is

$$R_{NJ}(r)=\frac{2(\alpha)^{3/2}}{N^2}\sqrt{\frac{\Gamma(N+J+1)}{(N-J-1)!}}\left(\frac{2\alpha r}{N}\right)^J\frac{e^{-\alpha r/N}}{\Gamma(2J+1)}{}_1F_1\left(-N+J+1;2J+2;\frac{2\alpha r}{N}\right)$$

and the energy spectrum given by

$$E=-\frac{\alpha^2}{N^2},\quad N=n_r+J+1=2m+n_r+l+k_1+k_2+2,\quad n_r=0,1,2,\ldots.$$

B. Parabolic and prolate spheroidal bases

The remaining solutions for which separation of variables is possible can be best observed by writing the Schrödinger equation in parabolic coordinates. If we do this and choose solutions of the form

$$\Psi=S(\xi,\eta)(\xi\eta)^{-1/2}Y_m^{(k_1,k_2)}(\varphi),\quad m=0,1,2,..., \tag{49}$$

where $Y_m^{(k_1,k_2)}(\varphi)$ is given by formula (7), we find that the Schrödinger equation has the reduced form

$$\frac{\partial^2S}{\partial\xi^2}+\frac{\partial^2S}{\partial\eta^2}+\left[2E(\xi^2+\eta^2)+\left(M^2-\frac{1}{4}\right)\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)+4\alpha\right]S=0.$$

This is clearly recognizable as solvable via separation of variables in parabolic coordinates ξ and η. The separable solution for the wave function $S(\xi,\eta)$ is

$$S(\xi,\eta)=\frac{\sqrt{2}(\alpha)^{3/2}}{N^2}(\xi\eta)^{-1/2}f_{n_1}^M(\xi)f_{n_2}^M(\eta),\quad n_{1,2}\in\mathbf{N},\quad M=2m+k_1+k_2+1, \tag{50}$$

where

$$f_n^M(x)=\sqrt{\frac{\Gamma(n_1+M+1)}{n_1!\quad\Gamma(M+1)}}\,e^{-(\alpha/2N)x^2}\left(\frac{\alpha x^2}{N}\right)^{M/2}{}_1F_1\left(-n_1;M+1;\frac{\alpha x^2}{N}\right) \tag{51}$$

and $N=n_1+n_2+M+1=n_1+n_2+2m+k_1+k_2+2$.

It is also interesting to observe that we could contemplate an E-dependent algebra of second-order symmetries acting on the functions $H(\xi,\eta)$. Indeed, a basis for such symmetries is

$$P_1=\partial_\xi^2+\left(M^2-\frac{1}{4}\right)\frac{1}{\xi^2}+2E\xi^2,\quad P_2=\partial_n^2+\left(M^2-\frac{1}{4}\right)\frac{1}{\eta^2}+2E\eta^2,$$

$$M=(\xi\partial_\eta-\eta\partial_\xi)^2-\left(M^2-\frac{1}{4}\right)\left(\frac{\xi^2}{\eta^2}+\frac{\eta^2}{\xi^2}\right)-\frac{1}{2}.$$

The corresponding closure relations can be deduced from those given for the first potential.

Apart from the symbols this has the same form as was dealt with in two dimensions. If we now regard ξ and η as Cartesian coordinates, separation is also possible in polar and elliptical coordinates. The case of polar coordinates has essentially been done above. The case of elliptic coordinates can be done by the standard prescription. This is achieved by looking for solutions of the form

$$S(\xi,\eta)=e^{-\sqrt{-2E(x^2+y^2+z^2)}}(x^2+y^2)^{(1/2)(M+1/2)}\prod_{j=1}^{s}\left(\frac{\sqrt{x^2+y^2+z^2}+z}{\theta_m-e_1}+\frac{\sqrt{x^2+y^2+z^2}-z}{\theta_m-e_1}-1\right),$$

where we have written the solutions in the coordinate representation. (Recall that $\xi^2=\sqrt{x^2+y^2+z^2}+z$ and $\eta^2=\sqrt{x^2+y^2+z^2}-z$.) With

$$\xi=\sqrt{\frac{(u_1-e_1)(u_2-e_1)}{(e_2-e_1)}},\quad \eta=\sqrt{\frac{(u_1-e_2)(u_2-e_2)}{(e_1-e_2)}},$$

where $e_1<u_1<e_2<u_2$, the choice of Cartesian coordinates that is appropriate in this case is

$$x=\frac{1}{e_2-e_1}\sqrt{\left[\left(\frac{e_2-e_1}{2}\right)^2-\left(u_1-\frac{e_2+e_1}{2}\right)^2\right]\left[\left(u_2-\frac{e_2+e_1}{2}\right)^2-\left(\frac{e_2-e_1}{2}\right)^2\right]}\cos\varphi,$$

$$y=\frac{1}{e_2-e_1}\sqrt{\left[\left(\frac{e_2-e_1}{2}\right)^2-\left(u_1-\frac{e_2+e_1}{2}\right)^2\right]\left[\left(u_2-\frac{e_2+e_1}{2}\right)^2-\left(\frac{e_2-e_1}{2}\right)^2\right]}\sin\varphi,$$

$$z=\frac{1}{e_2-e_1}\left[\left(u_1-\frac{e_2+e_1}{2}\right)\left(u_2-\frac{e_2+e_1}{2}\right)+\left(\frac{e_2-e_1}{2}\right)^2\right].$$

This corresponds to the choice of prolate spheroidal coordinates of type II.[22,8]

V. INTERBASIS EXPANSION

According to the principles of quantum mechanics the solutions of the same Schrödinger equation in the different separable coordinate systems for a given value of energy E are connected by unitary transformations or interbasis expansions. For example, we examine here the direct calculation of the interbasis expansion between the spherical and parabolic wave functions for potential V_3. We have

$$\Psi_{n_1,n_2,m}(\xi,\eta,\varphi)=\sum_{l=0}^{n_1+n_2} W^l_{n_1n_2m}(k_1,k_2)\Psi_{n_rlm}(r,\vartheta,\varphi). \tag{52}$$

where $n_r+l=n_1+n_2$. For calculation of the coefficients of interbasis expansion in (52) we may use the ''asymptotic method,'' [18,22] which is the following. Writing the parabolic wave function on the left-hand side of (52) in spherical coordinates (r,ϑ,φ) accordingly,

$$\xi^2=r(1+\cos\vartheta),\quad \eta^2=r(1-\cos\vartheta),$$

eliminating the function $Y_m^{(k_1,k_2)}(\varphi)$ on both sides of (52), and using the formula

$${}_1F_1(-n;\alpha;x)\sim\frac{\Gamma(\alpha)}{\Gamma(\alpha+n)}(-x)^n$$

for x arbitrary large, we see that the expansion (52) yields an equation which depends only on the variable ϑ. Then, by using the orthogonality relations for the functions $Z_{lm}(\vartheta)$ in the quantum number l, we arrive at the following expression for interbasis expansions coefficients:

$$\begin{aligned}W^l_{n_1n_2m}(k_1,k_2)=&(-1)^l\frac{\Gamma(M+1/2)}{2^{n_1+n_2+1}\sqrt{\pi}}\\&\times\sqrt{\frac{(2l+2M+1)\Gamma(n_1+n_2+l+2M+2)(n_1+n_2-l)!l!}{\Gamma(l+2M+1)\Gamma(n_1+M+1)\Gamma(n_1+M+1)(n_1)!(n_2)!}}\\&\times\int_0^\pi(1+\cos\vartheta)^{n_1+M}(1-\cos\vartheta)^{n_2+M}C_l^{M+1/2}(\vartheta)\sin\vartheta\,d\vartheta,\\M=&2m+k_1+k_2+1.\end{aligned}\tag{53}$$

By using the Rodrigues formula for the Gegenbauer polynomials[21]

$$C_n^\lambda(x)=\frac{(-1)^l}{l!}\frac{\sqrt{\pi}\Gamma(l+2\lambda)}{2^{l+2\lambda-1}\Gamma(\lambda)\Gamma(l+\lambda+1/2)}(1-x^2)^{-\lambda+1/2}\frac{d^l}{dx^l}(1-x^2)^{l+\lambda-1/2}$$

and comparing (53) with the integral representation for the Clebsch–Gordan coefficients of the Lie group SU(2) (Ref. 23),

$$\begin{aligned}C^{c\gamma}_{a\alpha;b\beta}=&\delta_{\alpha+\beta,\gamma}\sqrt{\frac{(2c+1)(J+1)!(J-2c)!(c+\gamma)!}{(J-2a)!(J-2b)!(a-\alpha)!(a+\alpha)!(b-\beta)!(b+\beta)!(c-\gamma)!}}\frac{(-1)^{a-c+\beta}}{2^{J+1}}\\&\times\int_{-1}^1(1-x)^{a-\alpha}(1+x)^{b-\beta}\frac{d^{c-\gamma}}{dx^{c-\gamma}}[(1-x)^{J-2a}(1+x)^{J-2b}]dx\end{aligned}$$

with $J=a+b+c$, we obtain

$$\begin{aligned}&W^l_{n_1n_2m}(k_1,k_2)=(-1)^{n_2}C^{c,\alpha+\beta}_{a\alpha;a\beta},\\&a=\frac{n_1+n_2+2m+k_1+k_2+1}{2},\quad c=l+2m+k_1+k_2+1,\\&\alpha=\frac{n_1-n_2+2m+k_1+k_2+1}{2},\quad \beta=\frac{n_2-n_1+2m+k_1+k_2+1}{2}.\end{aligned}\tag{54}$$

Since the parameters in (54) in general are not integers or half-integers, the coefficients of interbasis expansion (51) may be considered as analytic continuation, for real values of their arguments, of the SU(2) Clebsch–Gordan coefficients. Note also that the inverse expansion of (52) follows from the orthonormality of SU(2) Clebsch–Gordan coefficients.

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