SUPERPOSITION OF RENEWAL PROCESSES

C.Y. Teresa Lam
Department of Industrial & Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117

John P. Lehoczky
Department of Statistics
Carnegie Mellon University
Pittsburgh, PA 15213

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C. Y. Teresa Lam[*]

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Abstract

This paper extends the asymptotic results for ordinary renewal processes to the superposition of independent renewal processes. In particular, the ordinary renewal functions, renewal equations, and the key Renewal Theorem are extended to the superposition of independent renewal processes. We fix the number of renewal processes, p, and study the asymptotic behavior of the superposition process when time, t, is large. The Key Superposition Renewal Theorem is applied to the study of $(\sum_{i=1}^{p} GI_i)/M/1/1$ queueing systems.

S-RENEWAL FUNCTIONS; S-RENEWAL EQUATIONS; KEY SUPERPOSITION RENEWAL THEOREM; $(\sum_{i=1}^{p} GI_i)/M/1/1$ QUEUEING SYSTEMS

1 Introduction

In queueing or production networks, an individual service facility may receive inputs from many different sources, some of which are outputs from other servers. To model the queue at some server, it may, therefore, be reasonable to postulate that the arrival process to that server is a superposition of (nearly) statistically independent component processes.

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As pointed out by Albin (1982), if the arrival process to a queue is a superposition of many independent, relatively sparse component processes, one can invoke the theorem that a superposition process converges to a Poisson process as the number of component processes tend to infinity, and approximately represent the analytically difficult superposition process by the simple Poisson process [Cinlar (1972)]. For example, the stream of calls arriving at a telephone exchange is the superposition of thousands of streams of calls generated by the individual customers. Albin (1982) did extensive simulations for a single server with exponentially distributed service times and with an arrival process which is the superposition of p independent renewal processes (the $(\sum_{i=1}^{p} GI_i)/M/1$ system). She showed that as p increases the average queue length approaches that of an $M/M/1$ system, but that, for fixed p, the difference in average queue length between the $(\sum_{i=1}^{p} GI_i)/M/1$ and $M/M/1$ systems dramatically increases as the traffic intensity increases from 0.5 to 0.9.

In 1982, Whitt used the stationary-interval method and the asymptotic method to approximate the superposition arrival process of the $(\sum_{i=1}^{p} GI_i)/M/1$ queue by a single renewal process. It is then easy to describe the steady-state distribution of the number of customers in the resulting $GI/M/1$ system. Whitt (1982) then compared the two approximating steady-state distributions with the actual distribution as estimated by computer simulation. Albin (1984) showed that the mean queue length for the $(\sum_{i=1}^{p} GI_i)/G/1$ system can be approximated by that of a suitably chosen $GI/G/1$ system. The appropriate independent interarrival time distribution is found as a function of ρ, the traffic intensity, and p by fitting curves of various forms to simulation results. Newell (1984) described in detail some of the qualitative properties of $(\sum_{i=1}^{p} GI_i)/G/1$ systems and, in particular, showed that they approach limiting $M/G/1$ behavior when $p(1 - \rho)^2$ is large. Suppose there is only a small number of arrival streams, say 2 to 10, and the traffic intensity is close to 1, so that the expected queue size is large. Newell (1984) showed that the Poisson approximation can be disastrous, and the queueing system with a superposition arrival process should be analyzed more carefully.

The superposition of point processes has been widely discussed since the original investigation by Cox and Smith (1954). Surveys or reviews have appeared by Cox (1962) and Cinlar (1972). The emphasis of much of the early work has been either on the Poisson nature of a large number

of superpositions, or on the distribution of counts of events in superpositions. Lawrance (1973) studied the distributions and the dependency of the intervals between events. Cherry (1972) considered the problem of superposing two independent renewal processes and two Markov renewal processes. He applied his results to describe the joint departure process of two independent $M/G/1$ queues and a classic problem in machine repair and maintenance. For more details, see Cherry and Disney (1973, 1983) and Disney (1975). Kshirsagar and Becker (1981) generalized some of the results in Cox and Smith (1954) to the superposition of Markov Renewal processes. In particular, the interval between two successive visits to a particular state and the asymptotic variance of the number of visits to a state in the superposed process are considered. However, none of these authors have dealt with the extension of the fundamental concepts of renewal processes, namely renewal functions, renewal equations, and the Key Renewal Theorem to the superposition of renewal processes. The Key Renewal Theorem for ordinary renewal processes is useful in studying the behavior of the process remote from the time origin or when it is in equilibrium. A generalization of the Key Renewal Theorem is therefore useful in studying the asymptotic behavior of the process formed by superposing independent renewal processes. A detailed study of the extension of ordinary renewal functions, renewal equations, and the Key Renewal Theorem to the superposition of independent renewal processes is given in this paper.

In this paper, we fix the number of renewal processes, p, and study the asymptotic behavior of the superposition process when the time t is large. In particular, we define the *S-renewal functions* and the *S-renewal equations*. The S-renewal function is defined to be the sum of the ordinary renewal functions of the component renewal processes. The S-renewal equation is derived by conditioning on the time of the first event which occurs in the superposition process. The solution of the S-renewal equations is derived in Section 2, and in Section 3 by letting $t \to \infty$, we derive the *Key Superposition Renewal Theorem*. Just like the ordinary Key Renewal Theorem, the Key Superposition Renewal Theorem is useful in studying the asymptotic behavior of the superposition process. In particular, we study its application to the calculation of the excess life, current life and the total life in the superposition process. We also apply the Key Superposition Renewal Theorem to the study of $(\sum_{i=1}^{p} GI_i)/M/1/1$ queueing systems. Numerical studies are carried out to compare various characteristics of the $(\sum_{i=1}^{p} GI_i)/M/1/1$ with that of the $M/M/1/1$ systems.

The S-renewal function, S-renewal equation and the Key superpositon theorem we derive in this paper is a continuous state space version of the Markov renewal function, Markov renewal equation and the limit theorems given in Chapter 10 of Cinlar (1975). In a Markov renewal process, the transition state space is a countable set. In the superposition of renewal processes case, the state space is the set of all possible ages of the processes at the time when an event occurs in the superposition process.

This paper is organized as follows : In Section 2 of this paper, we generalize the definitions of renewal functions and renewal integral equations associated with ordinary renewal processes to the S-renewal functions and S-renewal equations of p independent renewal processes. The solution of this S-renewal equation is presented. In Section 3, we study the asymptotic behavior of the solution of the S-renewal equations and apply it to various examples. This leads to the generalization of the Ordinary Key Renewal Theorem to the Key Superposition Renewal Theorem. Theorems stated in Sections 2 and 3 are proved in Section 4. The Central Limit Theorem for the superposition process will be proved in Section 5. In addition, the asymptotic results for the renewal function associated with ordinary renewal processes will be extended to the S-renewal function of the superposition process.

2 S-Renewal Functions and S-Renewal Equations

In this section, the definitions of the S-renewal function and the S-renewal equation for the superposition of renewal processes are given. The solution of the S-renewal equation is derived. Suppose that there are p independent ordinary renewal processes in operation simultaneously. Let F_i, $i = 1, 2, \ldots, p$, be the probability distribution functions for the successive interevent times of the ith process. Consider the sequence of events formed by superposing the individual processes. At the time when an event occurs in the superposition process, one of the processes. say process i, probabilistically starts over. In addition, the others have age $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_p$ respectively, where the D_j s are random variables. Suppose that at time 0. the processes have age $d_1, \ldots, d_p$ respectively. Let $\mathbf{d} = (d_1, d_2, \ldots, d_p)$ and $N(t, \mathbf{d})$ be the number of events that

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occur in the superposition of these p renewal processes during the time interval $(0, t]$. Then

$$N(t, \mathbf{d}) = \sum_{i=1}^{p} N_i^{d_i}(t). \tag{2.1}$$

$\{N_i^{d_i}(t); t \geq 0\}$ is an ordinary renewal process with initial age d_i, $1 \leq i \leq p$. It is also a delayed renewal process such that the successive occurrence times between events have a common distribution function F_i. The initial delay distribution function is $F_i^{d_i}$, some of the d_i s may be zero, and

$$F_i^{d_i}(t) = \frac{F_i(t + d_i) - F_i(d_i)}{1 - F_i(d_i)} \tag{2.2}$$

provided that $F_i(t) < 1$ for all $t > 0$. Here $F_i^{d_i}(t) = \mathcal{P}(X_i^{d_i} \leq t) = \mathcal{P}(X_i \leq t + d_i \mid X_i > d_i)$. $X_i^{d_i}$ is the initial time to the first renewal of the ith process, and X_i has a distribution function F_i. Before the *S-renewal function* for the superposition process $\{N(t, \mathbf{d}); t \geq 0\}$ is defined, let us define the concept of the convolution of any two increasing functions. Let A and B be nondecreasing functions, continuous from the right, with $A(0) = B(0) = 0$. Define the convolution of A and B, denoted $A * B$, by

$$A * B(t) = \int_0^t B(t - s) \, dA(s), \qquad t \geq 0. \tag{2.3}$$

The expected number of events that occur in the time interval $(0, t]$ is given by

$$\begin{aligned} H(t, \mathbf{d}) &= \mathcal{E}(N(t, \mathbf{d})) = \sum_{i=1}^{p} \mathcal{E}(N_i^{d_i}(t)) \\ &= \sum_{i=1}^{p} H_i^{d_i}(t) = \sum_{i=1}^{p} F_i^{d_i} * \sum_{j=0}^{\infty} (F_i)_j(t) = \sum_{i=1}^{p} F_i^{d_i} * [1 + H_i(t)]. \end{aligned} \tag{2.4}$$

where $(F_i)_0(t) = 1$ and $F_i^{d_i} * (F_i)_0(t) = F_i^{d_i}(t)$ for all t. $(F_i)_j(t) = (F_i)_{j-1} * F_i(t)$ is the j-fold convolution of the distribution function F_i. $H_i^{d_i}(t)$ is the expected number of renewals in the delayed renewal process $\{N_i^{d_i}(t); t \geq 0\}$ in the time interval $(0, t]$. $H_i(t) = \sum_{j=1}^{\infty} (F_i)_j(t)$ is the renewal function associated with the distribution F_i. Define $H(t, \mathbf{d})$ to be the *S-renewal function* of the superposition process $\{N(t, \mathbf{d}), t \geq 0\}$. $H(t, \mathbf{d})$ is finite for all finite t and d_i, $i = 1, \ldots, p$, because the $H_i(t)$s are.

Theorem 2.1

$$H(t, \mathbf{d}) = h(t, \mathbf{d}) + \mathcal{I}(H(t, \mathbf{d})) \tag{2.5}$$

where

$$\mathcal{I}(H(t,\mathbf{d})) \;\; = \;\; \sum_{i=1}^{p} \int_{0}^{t} H(t-s,\mathbf{u}(i,s,\mathbf{d})) \prod_{j=1,j\neq i}^{p} (1-F_{j}^{d_j}(s))\,dF_{i}^{d_i}(s). \qquad (2.6)$$

$$\mathbf{u}(i,s,\mathbf{d}) = (s+d_1, s+d_2, \ldots, s+d_{i-1}, 0, s+d_{i+1}, \ldots, s+d_p) \;\; and \;\; h(t,\mathbf{d}) = 1 - \prod_{i=1}^{p}(1-F_i^{d_i}(t)).$$

Proof: Conditioning on the time $T_1(\mathbf{d})$ at which the first event occurs in the process $N(t,\mathbf{d})$, and counting the expected number of events that occur thereafter,

$$\mathcal{E}[N(t,\mathbf{d}) \mid T_1(\mathbf{d}) = X_i^{d_i} = s] = \begin{cases} 0 & \text{if } s > t \\ 1 + H(t-s, \mathbf{u}(i,s,\mathbf{d})) & \text{if } s \leq t \end{cases} \qquad (2.7)$$

In words, no events occur in $(0,t]$ if the first event of the superposition process occurs after time t. On the other hand, if $T_1(\mathbf{d}) = X_i^{d_i} = s < t$, then one event occurs at time s, and on the average, $H(t-s, \mathbf{u}(i,s,\mathbf{d}))$ additional events will occur during the time interval $(s,t]$. At time s, the ith process probabilistically starts over, and the others now have ages $s+d_j$, $j = 1, 2, \ldots, i-1, i+1, \ldots, p$. An application of the law of total probability yields

$$H(t,\mathbf{d}) \;\; = \;\; \mathcal{P}(T_1(\mathbf{d}) \leq t) + \sum_{i=1}^{p} \int_{0}^{t} H(t-s, \mathbf{u}(i,s,\mathbf{d}))\,d\mathcal{P}(T_1(\mathbf{d}) = X_i^{d_i} \leq s). \qquad (2.8)$$

Now

$$\mathcal{P}(T_1(\mathbf{d}) \leq t) = 1 - \mathcal{P}(X_i^{d_i} > t; i=1,\ldots,p) = 1 - \prod_{i=1}^{p}(1-F_i^{d_i}(t)) = h(t,\mathbf{d}), \qquad (2.9)$$

and for $s \leq t$,

$$\mathcal{P}(T_1(\mathbf{d}) = X_i^{d_i} \leq s) \;\; = \;\; \mathcal{P}(X_i^{d_i} \leq X_j^{d_j}, X_i^{d_i} \leq s \;\; ; \;\; j=1,\ldots,p \;\; \& \;\; j \neq i)$$

$$= \;\; \int_{0}^{s} \prod_{j=1,j\neq i}^{p} (1-F_j^{d_j}(x))\,dF_i^{d_i}(x). \qquad (2.10)$$

It follows that

$$d\mathcal{P}(T_1(\mathbf{d}) = X_i^{d_i} \leq s) = \prod_{j=1,j\neq i}^{p} (1-F_j^{d_j}(s))\,dF_i^{d_i}(s). \qquad (2.11)$$

This completes the proof of the theorem. $\qquad\qquad \square$

Let $T_n(\mathbf{d})$ be the random time of the nth event in the superposition process $N(t,\mathbf{d})$. Let

$\delta_i(T_n(\mathbf{d}))$ be the age of the ith delayed renewal process $N_i^{d_i}(t)$ at time $T_n(\mathbf{d})$. Defining $H(t,\mathbf{d}) = 0$ if $t < 0$ and $\boldsymbol{\delta}(T_n(\mathbf{d})) = (\delta_1(T_n(\mathbf{d})), \ldots, \delta_p(T_n(\mathbf{d})))$, we have

$$\mathcal{I}(H(t,\mathbf{d})) = \mathcal{E}(N(t - T_1(\mathbf{d}), \boldsymbol{\delta}(T_1(\mathbf{d})))). \tag{2.12}$$

$\mathcal{I}(H(t,\mathbf{d}))$ is therefore the expected number of events that occur in the random interval $(0, t - T_1(\mathbf{d})]$ of the superposition process $N(t, \boldsymbol{\delta}(T_1(\mathbf{d})))$.

The integral equation derived in Theorem (2.1) is called an *S-renewal equation* for the superposition of p independent renewal processes. It is an extension of the renewal integral equation associated with an ordinary renewal equation to the superposition of p independent renewal processes. More generally, we consider integral equation of the form

$$A(t,\mathbf{d}) = a(t,\mathbf{d}) + \sum_{i=1}^{p} \int_0^t A(t - s, \mathbf{u}(i,s,\mathbf{d})) \prod_{j=1, j \neq i}^{p} (1 - F_j^{d_j}(s)) \, dF_i^{d_i}(s)$$

$$= a(t,\mathbf{d}) + \mathcal{E}(A(t - T_1(\mathbf{d}), \boldsymbol{\delta}(T_1(\mathbf{d})))) \tag{2.13}$$

where a is a known function of $p + 1$ variables and $a(t,\mathbf{d}) = 0$ if $t < 0$. The solution of this integral equation is given in the Theorem (2.2) below.

We first define some notation. Let $\mathcal{S} = \{1, 2, \ldots, p\}$ and $\mathcal{S}_j = \{1, 2, \ldots, j - 1, j + 1, \ldots, p\}$. We consider partitions of the set $\mathcal{S}_j$ into two disjoint sets π_{ij} and π_{ij}^c. π_{ij} is a subset of $\mathcal{S}_j$ with i elements. Let σ_{ij} be the set of all partitions of $\mathcal{S}_j$. Also, we can partition the set $\mathcal{S}$ into two disjoint set π_i and π_i^c. π_i is a subset of $\mathcal{S}$ with i elements. Let σ_i be the set of all partitions of $\{1, 2, \ldots, p\}$ into the two sets π_i and π_i^c.

Theorem 2.2 *Suppose a is a function of $p + 1$ variables and it satisfies the following properties:*

1. *$a(t,\mathbf{d}) = 0$ for $t < 0$,*

2. *For every finite $T > 0$, there exists a finite constant κ_T such that*

$$\sup_{0 \leq t \leq T} \mid a(t,\mathbf{d}) \mid \leq \kappa_T \text{ for all } d_1, \ldots, d_p \geq 0.$$

The function a is therefore a bounded function over finite intervals of the first variable, and it is uniformly bounded in the other variables.

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Then, there exists one and only one function A with the following properties.

1. For every finite $T > 0$, there exists a finite constant C_T such that

$$\sup_{0 \leq t \leq T} \mid A(t, \mathbf{d}) \mid \leq C_T \text{ for all } d_1, \ldots, d_p \geq 0.$$

2. A satisfies the S-renewal equation (2.13).

This function is

$$A(t, \mathbf{d}) = a(t, \mathbf{d}) + \sum_{n=1}^{\infty} \mathcal{E}(a(t - T_n(\mathbf{d}), \delta(T_n(\mathbf{d})))) \tag{2.14}$$

By conditioning on all the different ways of obtaining n events in the superposition process $N(t, \mathbf{d})$, we have

$$A(t, \mathbf{d}) = a(t, \mathbf{d}) + \sum_{i=0}^{p-1} \mathcal{U}_i(a(t, \mathbf{d})) \tag{2.15}$$

where

$$\mathcal{U}_i(a(t, \mathbf{d})) = \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f(\delta) \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau), \tag{2.16}$$

$$f(\delta) = a(t - \tau, \delta) \prod_{l \in \pi_{ij}^c} (1 - F_l^{d_l}(\tau)) \prod_{k \in \pi_{ij}} (1 - F_k(\tau - s_k)), \tag{2.17}$$

$$\delta = (\delta_1, \ldots, \delta_p) \text{ and } \delta_k = \begin{cases} \tau + d_k & \text{if } k \in \pi_{ij}^c \\ \tau - s_k & \text{if } k \in \pi_{ij} \\ 0 & \text{if } k = j \end{cases} \tag{2.18}$$

The proof of the theorem will be given in Section 4. Here, we offer some explanatory comments. In the solution of the S-renewal equation above, $i + 1$, $i = 0, 1, \ldots, p - 1$, is equal to the number of processes having had at least one event occur by time $T_n(\mathbf{d}) = \tau$. π_{ij} is the set of component processes having had at least one event occur by time $T_n(\mathbf{d}) = \tau$. π_{ij}^c is the set of component processes whose first event occurs after time $T_n(\mathbf{d})$. j is the component process giving the nth event in the superposition process $N(t, \mathbf{d})$. σ_{ij} is the set of all possible ways in which the above conditions hold. δ_k is the current age of the kth process, $k = 1, 2, \ldots, p$, at time $T_n(\mathbf{d}) = \tau$.

For $k \in \pi_{ij}$, $l \in \pi_{ij}^c$, and $s_k < \tau$, $[\prod_{l \in \pi_{ij}^c}(1 - F_l^{d_l}(\tau))][\prod_{k \in \pi_{ij}}(1 - F_k(\tau - s_k))dH_k^{d_k}(s_k)]dH_j^{d_j}(\tau)$ is the probability that the following events happen simultaneously: An event occurs in the superposition process at $(s_k - ds_k, s_k]$, this event comes from the kth component process. Also, no event occurs in the kth process in the time interval $(s_k, \tau]$. An event occurs in the superposition process at $(\tau - d\tau, \tau]$, this event comes from the jth component process. The first event of the lth process occurs at time $> \tau$.

When $p = 2$, the solution of the S-renewal equation is given by

$$
\begin{aligned}
A(t, d_1, d_2) \;=\;& a(t, d_1, d_2) + \int_0^t a(t - \tau, 0, \tau + d_2)(1 - F_2^{d_2}(\tau))\, dH_1^{d_1}(\tau) \\
& + \int_0^t a(t - \tau, \tau + d_1, 0)(1 - F_1^{d_1}(\tau))\, dH_2^{d_2}(\tau) \\
& + \int_0^t \int_0^\tau a(t - \tau, 0, \tau - s)(1 - F_2(\tau - s))\, dH_2^{d_2}(s)\, dH_1^{d_1}(\tau) \\
& + \int_0^t \int_0^\tau a(t - \tau, \tau - s, 0)(1 - F_1(\tau - s))\, dH_1^{d_1}(s)\, dH_2^{d_2}(\tau). \quad (2.19)
\end{aligned}
$$

In the special case that the function $a(t, \mathbf{d})$ has a product form, the solution of the S-renewal equation also has a product form. In addition, as $t \to \infty$ there is a Key Renewal Theorem for the process $N(t, \mathbf{d})$.

Corollary 2.3 *For $i = 1, 2, \ldots, p$, let F_i be the distribution function of a positive random variable with mean μ_i. Suppose that g_i, $i = 1, 2, \ldots, p$, is directly Riemann integrable and*

$$
a(t, \mathbf{d}) = \prod_{i=1}^{p} \frac{g_i(t + d_i)}{1 - F_i(d_i)}. \tag{2.20}
$$

Let A be the solution of the S-renewal equation (2.13). Then

$$
A(t, \mathbf{d}) = \prod_{i=1}^{p} \left(\frac{g_i(t + d_i)}{1 - F_i(d_i)} + g_i * H_i^{d_i}(t) \right). \tag{2.21}
$$

Furthermore, if F_i, $i = 1, 2, \ldots, p$, is not arithmetic, then

$$
\lim_{t \to \infty} A(t, \mathbf{d}) = \begin{cases} \prod_{i=1}^{p} \left(\dfrac{1}{\mu_i} \int_0^\infty g_i(x)\, dx \right) & \text{if } \mu_i < \infty,\ i = 1, \ldots, p \\[2mm] 0 & \text{otherwise} \end{cases} \tag{2.22}
$$

Proof: The proof is straightforward but tedious. Substitute Expression (2.20) into Equation (2.15) in Theorem (2.2) above and simplify. It is easily checked that

$$
\begin{aligned}
\mathcal{U}_i(a(t,\mathbf{d})) & \\
&= \sum_{j=1}^{p}\sum_{\sigma_{ij}}\left[\prod_{l\in\pi_{ij}^{c}}\frac{g_l(t+d_l)}{1-F_l(d_l)}\right]\int_0^t\prod_{k\in\pi_{ij}}\left[\int_0^{s_j}g_k(t-s_k)\,dH_k^{d_k}(s_k)\right]g_j(t-s_j)\,dH_j^{d_j}(s_j) \\
&= \sum_{\sigma_{i+1}}\left[\prod_{l\in\pi_{i+1}^{c}}\frac{g_l(t+d_l)}{1-F_l(d_l)}\right]\left[\prod_{k\in\pi_{i+1}}g_k*H_k^{d_k}(t)\right]. \qquad (2.23)
\end{aligned}
$$

Hence

$$
\begin{aligned}
A(t,\mathbf{d}) &= \prod_{i=1}^{p}\frac{g_i(t+d_i)}{1-F_i(d_i)}+\sum_{i=0}^{p-1}\sum_{\sigma_{i+1}}\left[\prod_{l\in\pi_{i+1}^{c}}\frac{g_l(t+d_l)}{1-F_l(d_l)}\right]\left[\prod_{k\in\pi_{i+1}}g_k*H_k^{d_k}(t)\right] \\
&= \prod_{i=1}^{p}\left(\frac{g_i(t+d_i)}{1-F_i(d_i)}+g_i*H_i^{d_i}(t)\right) \qquad (2.24)
\end{aligned}
$$

Finally, from the ordinary Key Renewal Theorem, if F_i, $i=1,2,\ldots,p$, is not arithmetic, then

$$
\lim_{t\to\infty}\left[\frac{g_i(t+d_i)}{1-F_i(d_i)}+g_i*H_i^{d_i}(t)\right]=\begin{cases}\dfrac{1}{\mu_i}\displaystyle\int_0^{\infty}g_i(x)\,dx & \text{if }\mu_i<\infty \\ 0 & \text{if }\mu_i=\infty\end{cases}. \qquad (2.25)
$$

This completes the proof of Equation (2.22) and Corollary (2.3). $\qquad\square$

Examples : (1) When $p=2$ and $a(t,d_1,d_2)=1$, we have after some algebra, $A(t,d_1,d_2)=1+H_1^{d_1}(t)+H_2^{d_2}(t)$. (2) When $a(t,\mathbf{d})=1$, it is easily checked that the solution of the S-renewal equation corresponding to a is given by $A(t,\mathbf{d})=1+\sum_{i=1}^{p}H_i^{d_i}(t)$. (3) When $a(t,\mathbf{d})=\prod_{i=1}^{p}(1-F_i^{d_i}(t))$, we can apply Corollary (2.3) with $g_i(t)=1-F_i(t)$ to find the solution of the S-renewal equation. It is easily checked that $A(t,\mathbf{d})=1$. (4) When $a(t,\mathbf{d})=h(t,\mathbf{d})=1-\prod_{i=1}^{p}(1-F_i^{d_i}(t))$, we can use the linearity property of integrals and the results of examples (2) and (3) above to obtain $A(t,\mathbf{d})=H(t,\mathbf{d})=\sum_{i=1}^{p}H_i^{d_i}(t)$.

Corollary (2.3) tells us that as $t\to\infty$, $A(t,\mathbf{d})$ is independent of the initial age d_i, $i=1,2,\ldots,p$, of the component processes. In other words, in the limit as $t\to\infty$, the effect of the initial age d_i of the process $N_i^{d_i}(t)$ disappears. In the next section, the above corollary is used to calculate the limiting remaining life of the superposition process $N(t,\mathbf{d})$.

3 Key Superposition Renewal Theorem and Applications

In this section, it will be shown that under some regularity conditions on $a(t, \mathbf{d})$, we can generalize the ordinary Key Renewal Theorem to the Key Superposition Renewal Theorem. We will see in the theorem below that $\lim_{t \to \infty} A(t, \mathbf{d})$ is independent of the initial ages d_i, $1 \le i \le p$, of the component processes. Before the theorem is given, we need the following two definitions.

Definition 3.1 *Let g be a function defined on $\mathcal{R}_+^m$. For every positive δ and $m, n_i = 1, 2, \ldots$, $i = 1, 2, \ldots, m$, let*

$$\underline{m}_{n_1, n_2, \ldots, n_m} = \min\{g(x_1, x_2, \ldots, x_m) : (n_i - 1)\delta \le x_i \le n_i\delta \, ; \, i = 1, 2, \ldots, m\},$$

$$\overline{m}_{n_1, n_2, \ldots, n_m} = \max\{g(x_1, x_2, \ldots, x_m) : (n_i - 1)\delta \le x_i \le n_i\delta \, ; \, i = 1, 2, \ldots, m\},$$

$$\underline{\sigma}(\delta) = \delta^m \sum_{i=1}^{m} \sum_{n_i=1}^{\infty} \underline{m}_{n_1, n_2, \ldots, n_m} \quad and \quad \overline{\sigma}(\delta) = \delta^m \sum_{i=1}^{m} \sum_{n_i=1}^{\infty} \overline{m}_{n_1, n_2, \ldots, n_m} \, .$$

Then g is said to be directly Riemann integrable if both series $\underline{\sigma}(\delta)$ and $\overline{\sigma}(\delta)$ converge absolutely for every positive δ, and the difference $\overline{\sigma}(\delta) - \underline{\sigma}(\delta)$ goes to 0 as $\delta \to 0$.

Definition 3.2 *Let a be a function defined on $\mathcal{R}_+^{p+1}$. For every positive h and $k = 1, 2, \ldots$, let*

$$\underline{c_k}(y_1, \ldots, y_p) = \begin{cases} \min a(x, y_1, \ldots, y_p) & if \, (k-1)h \le x < kh \\ 0 & otherwise \end{cases} , \quad (3.1)$$

$$\overline{c_k}(y_1, \ldots, y_p) = \begin{cases} \max a(x, y_1, \ldots, y_p) & if \, (k-1)h \le x < kh \\ 0 & otherwise \end{cases} . \quad (3.2)$$

For $i = 1, 2, \ldots, p$, let F_i be the distribution function of a positive random variable with finite mean μ_i. Define a $1 \times p$ random vector $\mathbf{D}$ such that

$$\mathbf{D} = (D_1, \ldots, D_{i-1}, 0, D_{i+1}, \ldots, D_p) \quad with \, probability \quad \frac{1}{\mu_i} \bigg/ \sum_{j=1}^{p} \frac{1}{\mu_j} .$$

$D_1, D_2, \ldots, D_p$ are independent random variables and D_i follows the limiting current life distribution of F_i, $i = 1, \ldots, p$. This means that for $z > 0$,

$$\mathcal{P}(D_i > z) = \frac{1}{\mu_i} \int_z^{\infty} (1 - F_i(y)) \, dy. \quad (3.3)$$

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Also, let

$$\underline{\sigma}(h) = h \sum_{k=1}^{\infty} \mathcal{E}(\underline{c_k}(\mathbf{D})) \quad and \quad \overline{\sigma}(h) = h \sum_{k=1}^{\infty} \mathcal{E}(\overline{c_k}(\mathbf{D})).$$

Then a is said to be directly Riemann integrable with respect to $(\mathbf{F}, \mathbf{D}) = ((F_1, D_1), \ldots, (F_p, D_p))$

if both series $\underline{\sigma}(h)$ *and* $\overline{\sigma}(h)$ *converge absolutely for every positive h, and the difference* $\overline{\sigma}(h) - \underline{\sigma}(h)$

goes to 0 as $h \rightarrow 0$.

Theorem 3.3 (Key Superposition Renewal Theorem for p independent processes)

For $i = 1, 2, \ldots, p$, *let* F_i *be the distribution function of a positive random variable with finite mean* μ_i. *Suppose that*

1. $a(t, \mathbf{d}) = 0$ *for t negative,*

2. *For every finite* $T > 0$, *there exists a finite constant* κ_T *such that*

$$\sup_{0 \le t \le T} | a(t, \mathbf{d}) | \le \kappa_T \text{ for all } d_1, \ldots, d_p \ge 0.$$

3. *A is the solution of the S-renewal equation (2.13).*

4. *The function*

$$b_j(t, d_1, \ldots, d_{j-1}, d_{j+1}, \ldots, d_p) = a(t, d_1, \ldots, d_{j-1}, 0, d_{j+1}, \ldots, d_p) \prod_{i=1, i \ne j}^{p} (1 - F_i(d_i))$$

defined on $\mathcal{R}_+^p$ *is directly Riemann integrable for all* $j = 1, 2, \ldots, p$.

5. *The function a is directly Riemann integrable with respect to* $(\mathbf{F}, \mathbf{D}) = ((F_1, D_1), \ldots, (F_p, D_p))$.

If F_i, $i = 1, \ldots, p$, *is not arithmetic and* $\mu = \left[\sum_{i=1}^{p} \frac{1}{\mu_i} \right]^{-1}$, *then*

$$\lim_{t \to \infty} A(t, \mathbf{d}) = \begin{cases} \sum_{i=1}^{p} \frac{1}{\mu_i} \int_0^{\infty} \mathcal{E}(a(D_1, \ldots, D_{i-1}, 0, D_{i+1}, \ldots, D_p)) \, dx & \text{if } \mu < \infty \\ 0 & \text{if } \mu = \infty \end{cases}$$

$$= \begin{cases} \frac{1}{\mu} \int_0^{\infty} \mathcal{E}(a(x, \mathbf{D})) \, dx & \text{if } \mu < \infty \\ 0 & \text{if } \mu = \infty \end{cases} \qquad (3.4)$$

Proof: The proof will be given in Section 4.

From Corollary (2.3) and the Key Superposition Renewal Theorem, we can conclude that as $t \to \infty$, $A(t, \mathbf{d})$ is independent of the initial ages d_i, $i = 1, 2, \ldots, p$, of the component processes. All we need to do is to average the initial age by the corresponding limiting current life distribution of the individual independent component processes.

Applications

Example 1 : Consider the $(\sum_{i=1}^{p} GI_i)/M/1/1$ queue, this is a queueing system with an arrival process which is the superposition of p independent renewal processes. There is no waiting room in this system. A customer that arrives and finds the server busy leaves the system and never returns. Assume that at time $t = 0$, the server is busy and the ages of the renewal processes are $d_1, d_2, \ldots, d_p$.

Part (a) : Let $A(t, \mathbf{d})$ be the probability that the server is busy at time t. By the memorylessness property of the exponential distribution and the independence of the service times, at time $t = 0$, the residual service time X of the customer being served at that time has the same exponential distribution as any service time. Conditioning on the time T_1 of the first arrival to the system, we have

$$
A(t, \mathbf{d}) = \begin{cases} 1 & \text{if } T_1 > t \text{ and } X > t \\ 0 & \text{if } T_1 > t \text{ and } X \le t \\ A(t - s, \mathbf{u}(i, s, \mathbf{d})) & \text{if } T_1 = s \le t \text{ and } I_1 = i \end{cases} \tag{3.5}
$$

where I_j is an indicator function such that $I_j = i$ if the jth arrival comes from the ith process. $A(t, \mathbf{d})$ satisfies the S-renewal equation with

$$
a(t, \mathbf{d}) = \mathcal{P}(T_1 > t, X > t) = e^{-at} \prod_{i=1}^{p} (1 - F_i^{d_i}(t)) \tag{3.6}
$$

where $1/a$ is the mean service time. By the Key Superposition Renewal Theorem, we have

λ	e_1	e_2	e_3	e_{10}
0.1	0.060	0.039	0.028	0.008
0.2	0.111	0.070	0.049	0.014
0.3	0.157	0.094	0.064	0.018
0.4	0.194	0.110	0.075	0.021
0.5	0.224	0.125	0.082	0.023
0.6	0.247	0.133	0.087	0.024
0.7	0.262	0.137	0.089	0.025
0.8	0.270	0.138	0.090	0.025
0.9	0.271	0.138	0.090	0.026

Table 3.1: Proportion of error when $a = 1$ in Example 1(a)

$$\lim_{t \to \infty} A(t, \mathbf{d})$$

$$= \sum_{i=1}^{p} \frac{1}{\mu_i} \left[\int_0^\infty e^{-ax}(1 - F_i(x)) \left(\prod_{j=1, j \neq i}^{p} \int_0^\infty \frac{(1 - F_j(x + y_j))}{\mu_j} \right) \left(\prod_{j=1, j \neq i}^{p} dy_j \right) dx \right]. \quad (3.7)$$

Expression (3.7) gives us the limiting probability that the server is busy at time t when t is large. This is independent of the initial conditions of the system. In the special case when

$$F_i(x) = G_p(x) = \begin{cases} 1 - \exp\left[-\left(\frac{\lambda}{1 - \lambda}\right)\left(\frac{x}{p} - 1\right)\right] & \text{if } \frac{x}{p} \geq 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.8)$$

for all $i \in \{1, 2, \ldots, p\}$, the total arrival rate to the system is $\lambda < 1$. Equation (3.7) simplifies to

$$\begin{aligned} L_p &= p\left(\frac{\lambda}{p}\right)^p \int_0^\infty e^{-at}(1 - G_p(x)) \left(\int_0^\infty 1 - G_p(x + y) \, dy \right)^{p-1} dx \\ &= \lambda \int_0^p e^{-ax}\left(1 - \frac{\lambda x}{p}\right)^{p-1} dx + \frac{\lambda(1 - \lambda)^p}{(1 - \lambda)a + \lambda} e^{-ap} \\ &= \sum_{i=0}^{p-1} (-1)^i \left(\frac{\lambda}{a}\right)^{i+1} \left[\prod_{j=1}^{i} \left(1 - \frac{j}{p}\right) \right] \left(1 - (1 - \lambda)^{p-1-i} e^{-ap}\right) + \frac{\lambda(1 - \lambda)^p}{(1 - \lambda)a + \lambda} e^{-ap}. \quad (3.9) \end{aligned}$$

As expected, Expression (3.9) converges to $L_a = \lambda/(\lambda + a)$ as $p \to \infty$. Let $\epsilon_p = (L_p - L_a)/L_a$. e_p is the proportion of error for the Poisson approximation as defined in Albin (1982). From

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Table (3.1) above, the percentage error could be as large as 13.8% when $p = 2$. This suggests that the Poisson approximation is not appropriate for p small. The Key Superposition Renewal Theorem is necessary to obtain the exact solution in the analysis of $(\sum_{i=1}^{p} GI_i)/M/1/1$ systems.

Part (b) : Let $B(t,\mathbf{d})$ be the probability that the first customer arriving after time t finds the server busy. Again, conditioning on the time T_1 of the first arrival to the system, we have

$$
B(t,\mathbf{d}) \;=\;
\begin{cases}
1 & \text{if } T_1 > t \text{ and } X > T_1 \\[4pt]
0 & \text{if } T_1 > t \text{ and } X \le T_1 \\[4pt]
B(t - s, \mathbf{u}(i,s,\mathbf{d})) & \text{if } T_1 = s \le t, \text{ and } I_1 = i
\end{cases}
\tag{3.10}
$$

$B(t,\mathbf{d})$ satisfies the S-renewal equation with

$$
a(t,\mathbf{d}) = \mathcal{P}(X > T_1 > t) = e^{-at}\left[\prod_{i=1}^{p}(1 - F_i^{d_i}(t))\right] - \int_t^\infty \left[\prod_{i=1}^{p}(1 - F_i^{d_i}(x))\right]ae^{-ax}\,dx.
\tag{3.11}
$$

By the Key Superposition Renewal Theorem, we have

$$
\begin{aligned}
\lim_{t \to \infty}& B(t,\mathbf{d}) \\
=\; & \sum_{i=1}^{p}\frac{1}{\mu_i}\left[\int_0^\infty e^{-ax}(1 - F_i(x))\left(\prod_{j=1,j\neq i}^{p}\int_0^\infty \frac{(1 - F_j(x + y_j))}{\mu_j}\right)\left(\prod_{j=1,j\neq i}^{p} dy_j\right)dx\right.\\
& \left. - \int_0^\infty\int_x^\infty ae^{-az}(1 - F_i(z))\left(\prod_{j=1,j\neq i}^{p}\int_0^\infty \frac{(1 - F_i(z + y_j))}{\mu_j}\right)\left(\prod_{j=1,j\neq i}^{p} dy_j\right)dz\,dx\right]
\end{aligned}
\tag{3.12}
$$

In the special case when $F_i(x) = G_p(x)$ for all $i \in \{1,2,\ldots,p\}$, equation (3.12) simplifies to

$$
m_p = \lambda \int_0^p e^{-ax}(1 - ax)\left(1 - \frac{\lambda x}{p}\right)^{p-1} dx + \left[\frac{(1 - ap)\lambda(1 - \lambda)^p}{(1 - \lambda)a + \lambda} - \frac{a\lambda(1 - \lambda)^{p+1}}{((1 - \lambda)a + \lambda)^2}\right]e^{-ap}.
\tag{3.13}
$$

As expected, m_p converges to $m_a = [\lambda/(\lambda + a)]^2$ as $p \to \infty$. Let $\epsilon_p = (m_p - m_a)/m_a$. From Table (3.2) below, the percentage error could be as large as 17.5% when $p = 2$.

Example 2 : Consider the superposition of two independent renewal processes. Define the process $X(t)$ such that $X(t) = i$, $i = 1,2$, if at time t, the last event came from the ith process. Let $A(t,d_1,d_2)$ be the probability that $X(t) = 1$ given that at time $t = 0$, the ages of the renewal processes are d_1 and d_2 respectively. Conditional on the time T_1 of the first arrival, $A(t,d_1,d_2)$ satisfies the S-renewal equation with

$$
a(t,d_1,d_2) = \mathcal{P}(T_1 > t)I(d_1 < d_2) = \left[\prod_{i=1}^{2}(1 - F_i^{d_i}(t))\right]I(d_1 < d_2)
\tag{3.14}
$$

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λ	ϵ_1	ϵ_2	ϵ_3	ϵ_{10}
0.1	-0.154	-0.213	-0.195	-0.068
0.2	-0.046	-0.136	-0.131	-0.044
0.3	0.057	-0.066	-0.075	-0.025
0.4	0.154	-0.004	-0.028	-0.011
0.5	0.242	0.049	0.011	0.000
0.6	0.318	0.093	0.042	0.009
0.7	0.382	0.128	0.067	0.015
0.8	0.430	0.155	0.086	0.020
0.9	0.461	0.175	0.100	0.024

Table 3.2: Proportion of error when $a = 1$ in Example 1(b)

where

$$I(d_1 < d_2) = \begin{cases} 1 & \text{if } d_1 < d_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

By the Key Superposition Renewal Theorem, we have

$$\lim_{t \to \infty} A(t, d_1, d_2) = \int_0^\infty \frac{(1 - F_1(x))}{\mu_1} \int_x^\infty \frac{(1 - F_2(y))}{\mu_2} \, dy \, dx = \mathcal{P}(R_2 > R_1) \tag{3.16}$$

where R_i, $i = 1, 2$, is the limiting current life distribution of the ith process.

Example 3 : Limiting Distributions of the Remaining Life and the Current Life

Let $\gamma_t(\mathbf{d})$ be the remaining life at time t of the process $N(t, \mathbf{d})$. For any fixed $z > 0$, set $A_z(t, \mathbf{d}) = \mathcal{P}(\gamma_t(\mathbf{d}) > z)$. Conditioning on the time T_1 of occurrence of the first event in the process $N(t, \mathbf{d})$, we can derive the S-renewal equation for $A_z(t, \mathbf{d})$. In this case, $a_z(t, \mathbf{d}) = \prod_{i=1}^p (1 - F_i^{d_i}(t + z))$. Let $g_i^z(t) = 1 - F_i(t + z)$. $a_z(t, \mathbf{d})$, therefore, has the product form specified in Corollary (2.3). It follows from that corollary

$$\lim_{t \to \infty} \mathcal{P}(\gamma_t(\mathbf{d}) > z) = \prod_{i=1}^p (\frac{1}{\mu_i} \int_z^\infty (1 - F_i(x)) \, dx), \tag{3.17}$$

which is the product of the individual limiting remaining life distributions of the component processes. This is as expected, because the p renewal processes are independent. The same

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argument in Karlin and Taylor (1975) page 193 can be used to show that the limiting distribution of the current life for the superposition of independent renewal processes is also given by Equation (3.17).

Example 4 : Limiting Distribution of the Total Life

Let $\beta_t(\mathbf{d})$ be the total life at time t of the process $N(t,\mathbf{d})$. For a fixed $z > 0$, set $A_z(t,\mathbf{d}) = \mathcal{P}(\beta_t(\mathbf{d}) > z)$. Again, we can condition on the time $T_1(\mathbf{d})$ of occurrence of the first event in the process $N(t,\mathbf{d})$ to derive an integral equation. $A_z(t,\mathbf{d})$ satisfies the S-renewal equation with

$$a_z(t,\mathbf{d}) = \prod_{i=1}^{p}(1 - F_i^{d_i}(\max(t,z))). \tag{3.18}$$

By Theorem (3.3), if $\mu_1,\ldots,\mu_p < \infty$ so that $\mu = \left[\sum_{i=1}^{p}\frac{1}{\mu_i}\right]^{-1} < \infty$, then

$$\lim_{t\to\infty}\mathcal{P}(\beta_t(\mathbf{d}) > z) \tag{3.19}$$

$$= \sum_{i=1}^{p}\frac{1}{\mu_i}\int_0^\infty(1 - F_i(\max(x,z)))\prod_{j=1,j\neq i}^{p}\left[\int_0^\infty \frac{(1 - F_j(\max(x,z) + y_j))}{\mu_j}dy_j\right]dx$$

$$= \sum_{i=1}^{p}\frac{1}{\mu_i}\left[\int_z^\infty x\,d\left[-(1 - F_i(x))\prod_{j=1,j\neq i}^{p}\left[\int_x^\infty \frac{(1 - F_j(y_j))}{\mu_j}dy_j\right]\right]dx\right] = \frac{1}{\mu}\int_z^\infty x\,dG(x)$$

where

$$G(x) = 1 - \sum_{i=1}^{p}\frac{\frac{1}{\mu_i}}{\frac{1}{\mu}}(1 - F_i(x))\prod_{j=1,j\neq i}^{p}\left[\int_x^\infty \frac{(1 - F_j(y))}{\mu_j}dy\right]. \tag{3.20}$$

Accordingly, we have established the limiting distribution of the total life is

$$\lim_{t\to\infty}\mathcal{P}(\beta_t(\mathbf{d}) \leq z) = \frac{1}{\mu}\int_0^z x\,dG(x) = B(z). \tag{3.21}$$

A similar argument from Karlin and Taylor (1975) page 195 can be used to show that the mean limiting total life is at least as big as the mean interval time in the superposition process $N(t,\mathbf{d})$. This fact is consistent with the analogous result for ordinary renewal processes.

4 Proof of Theorems

Proof of Theorem 2.2: We first verify that A specified by equation (2.15) fulfills the requisite boundedness properties. Using standard renewal theory, H_i is nondecreasing and finite. It

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follows that

$$H_i^{d_i}(t) = F_i^{d_i} * [1 + H_i(t)] = F_i^{d_i}(t) + \int_0^t F_i^{d_i}(t-s)\, dH_i(s) \le 1 + H_i(t) < \infty \quad \text{for all } t. \quad (4.1)$$

Hence, for every T, we have

$$
\begin{aligned}
\sup_{0 \le t \le T} |A(t, \mathbf{d})| &\le \kappa_T \left[1 + \sum_{i=0}^{p-1} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^T \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau) \right] \\
&= \kappa_T \left[1 + \sum_{i=0}^{p-1} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^T \left[\prod_{k \in \pi_{ij}} H_k^{d_k}(\tau) \right] dH_j^{d_j}(\tau) \right] \\
&= \kappa_T \left[1 + \sum_{i=0}^{p-1} \sum_{\sigma_{i+1}} \int_0^T d\left[\prod_{k \in \pi_{i+1}} H_k^{d_k}(\tau) \right] \right] \\
&= \kappa_T \prod_{i=1}^{p} (1 + H_i^{d_i}(T)) = C_T < \infty. \quad (4.2)
\end{aligned}
$$

establishing that the Expression (2.15) satisfies property (1) of the theorem. Next we want to show that Expression (2.15) solves the S-renewal equation. Conditional on

1. the time $T_1(\mathbf{d})$ of the first event that occurs in the process $N(t, \mathbf{d})$, and

2. $\delta_i(T_1(\mathbf{d}))$, $i = 1, \ldots, p$, the age of the ith component process $N_i^{d_i}(t)$ at time $T_1(\mathbf{d})$,

we have

$$T_n(\mathbf{d}) = T_1(\mathbf{d}) + T_{n-1}(\boldsymbol{\delta}(T_1(\mathbf{d}))) \quad (4.3)$$

and

$$\boldsymbol{\delta}(T_n(\mathbf{d})) = (\delta_1(T_{n-1}(\boldsymbol{\delta}(T_1(\mathbf{d})))), \ldots, \delta_p(T_{n-1}(\boldsymbol{\delta}(T_1(\mathbf{d}))))) = \boldsymbol{\delta}(T_{n-1}(\boldsymbol{\delta}(T_1(\mathbf{d})))). \quad (4.4)$$

In words, knowing $T_1(\mathbf{d})$ and $\delta_i(T_1(\mathbf{d}))$, $i = 1, \ldots, p$, then the times of n events in the process $N(t, \mathbf{d})$ are equal to $T_1(\mathbf{d})$ plus the times of $n - 1$ events in the process $N(t, \boldsymbol{\delta}(T_1(\mathbf{d})))$. Also, the age of the ith component process at the time of n events, $T_n(\mathbf{d})$, is the same as the age of the delayed process $\{N_i^{\delta_i(T_1(\mathbf{d}))}(t); t \ge 0\}$ at the times of $n - 1$ events. It follows that for all $n > 1$,

$$
\begin{aligned}
&\mathcal{E}(A(t - T_n(\mathbf{d}), \boldsymbol{\delta}(T_n(\mathbf{d})))) \\
&= \mathcal{E}[\mathcal{E}(A(t - T_n(\mathbf{d}), \boldsymbol{\delta}(T_n(\mathbf{d}))) \mid T_1(\mathbf{d}), \boldsymbol{\delta}(T_1(\mathbf{d})))] \\
&= \mathcal{E}[\mathcal{E}(A(t - T_1(\mathbf{d}) - T_{n-1}(\boldsymbol{\delta}(T_1(\mathbf{d}))), \boldsymbol{\delta}(T_{n-1}(\boldsymbol{\delta}(T_1(\mathbf{d}))))) \mid T_1(\mathbf{d}), \boldsymbol{\delta}(T_1(\mathbf{d})))]. \quad (4.5)
\end{aligned}
$$

We can now solve the S-renewal equation by successive approximations, and we use equality (4.5) to simplify each step in the approximation to obtain

$$A(t, \mathbf{d})$$

$$= a(t, \mathbf{d}) + \mathcal{E}(A(t - T_1(\mathbf{d}), \delta(T_1(\mathbf{d}))))$$

$$= a(t, \mathbf{d}) + \mathcal{E}(a(t - T_1(\mathbf{d}), \delta(T_1(\mathbf{d}))))$$

$$\quad + \mathcal{E}[\mathcal{E}(A(t - T_1(\mathbf{d}) - T_1(\delta(T_1(\mathbf{d}))), \delta(T_1(\delta(T_1(\mathbf{d}))))) \mid T_1(\mathbf{d}), \delta(T_1(\mathbf{d})))]$$

$$= a(t, \mathbf{d}) + \mathcal{E}(a(t - T_1(\mathbf{d}), \delta(T_1(\mathbf{d})))) + \mathcal{E}(A(t - T_2(\mathbf{d}), \delta(T_2(\mathbf{d}))))$$

$$= \ldots$$

$$= a(t, \mathbf{d}) + \sum_{j=1}^{n-1} \mathcal{E}(a(t - T_j(\mathbf{d}), \delta(T_j(\mathbf{d})))) + \mathcal{E}(A(t - T_n(\mathbf{d}), \delta(T_n(\mathbf{d})))). \qquad (4.6)$$

Next we want to show that

$$\lim_{n \to \infty} | \mathcal{E}(A(t - T_n(\mathbf{d}), \delta(T_n(\mathbf{d})))) | = 0 \qquad (4.7)$$

for every fixed t. Let $I_n(N(t, \mathbf{d}))$ be the indicator function such that $I_n(N(t, \mathbf{d}))$ is equal to j if the nth event of the superposition process comes from the jth component process. Observe that by conditioning on

1. $i + 1, 0 \leq i \leq \min\{n - 1, p - 1\} = \tilde{p}$, the number of processes having had at least one event occur by time $T_n(\mathbf{d})$,

2. $I_n(N(t, \mathbf{d})) = j, j = 1, 2, \ldots, p$,

3. For each $i = 0, \ldots, \tilde{p}$, and $j = 1, 2, \ldots, p$,

   (a) $\pi_{ij} \subseteq \{1, \ldots, j - 1, j + 1, \ldots, p\}$ and $| \pi_{ij} | = i$,

   (b) $N_k^{d_k}(T_n(\mathbf{d})) = \begin{cases} m_k + 1 & \text{if } k \in \pi_{ij} \cup \{j\} \\ 0 & \text{otherwise} \end{cases}$,

   (c) $\sum_{k \in \pi_{ij} \cup \{j\}} m_k = n - i - 1$, and $m_k \geq 0; k = 1, \ldots, p$,

   (d) $T_n(\mathbf{d}) = \tau$, and $\delta_k(T_n(\mathbf{d})) = \delta_k = \begin{cases} \tau + d_k & \text{if } k \in \pi_{ij}^c \\ \tau - s_k & \text{if } k \in \pi_{ij} \\ 0 & \text{if } k = j \end{cases}$,

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we have

$$| \mathcal{E}(A(t - T_n(\mathbf{d}), \delta(T_n(\mathbf{d})))) |$$

$$= \; \Big| \sum_{i=0}^{\tilde{p}} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f_A(\delta) \Big[\sum_{\theta_{ij}^n} \Big(\prod_{k \in \pi_{ij}} dF_k^{d_k} * (F_k)_{m_k}(s_k) \Big) dF_j^{d_j} * (F_j)_{m_j}(\tau) \Big] \Big|$$

$$\leq \; C_t \sum_{i=0}^{\tilde{p}} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \Big[\sum_{\theta_{ij}^n} \Big(\prod_{k \in \pi_{ij}} F_k^{d_k} * (F_k)_{m_k}(\tau) \Big) dF_j^{d_j} * (F_j)_{m_j}(\tau) \Big]$$

$$\leq \; C_t \sum_{i=0}^{\tilde{p}} \sum_{\sigma_{i+1}} \Big[\sum_{\theta_{i+1}^n} \Big(\prod_{k \in \pi_{i+1}} F_k^{d_k} * (F_k)_{m_k}(t) \Big) \Big], \tag{4.8}$$

where

$$\theta_{ij}^n \; = \; \{ (m_1, \ldots, m_p) : m_j + \sum_{k \in \pi_{ij}} m_k = n - i - 1, \quad m_k \geq 0, \quad k = 1, \ldots, p \},$$

$$\theta_{i+1}^n \; = \; \{ (m_1, \ldots, m_p) : \sum_{k \in \pi_{i+1}} m_k = n - i - 1, \quad m_k \geq 0, \quad k = 1, \ldots, p \},$$

and

$$f_A(\delta) \; = \; A(t - \tau, \delta) \prod_{l \in \pi_{ij}^c} (1 - F_l^{d_l}(\tau)) \prod_{k \in \pi_{ij}} (1 - F_k(\tau - s_k)).$$

For $i = 0, 1, \ldots, \tilde{p}$,

$$\sum_{n=i+1}^{\infty} \sum_{\theta_{i+1}^n} \Big(\prod_{k \in \pi_{i+1}} F_k^{d_k} * (F_k)_{m_k}(t) \Big) = \prod_{k \in \pi_{i+1}} H_k^{d_k}(t) < \infty. \tag{4.9}$$

This means that $\sum_{\theta_{i+1}^n} (\prod_{k \in \pi_{i+1}} F_k^{d_k} * (F_k)_{m_k}(t))$ is the nth term of a finite series and hence converges to 0 as $n \to \infty$. Expression (4.8) is now bounded by the sum of a finite number of terms, each of which converges to 0 as $n \to \infty$. This completes the proof of Expression (4.7). Letting $n \to \infty$ in Equation (4.6), we have Result (2.14) and

$$\sum_{n=1}^{\infty} \mathcal{E}(a(t - T_n(\mathbf{d}), \delta(T_n(\mathbf{d})))) \tag{4.10}$$

$$= \; \sum_{i=0}^{p-1} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f(\delta) \Big[\sum_{n=i+1}^{\infty} \sum_{\theta_{ij}^n} \Big(\prod_{k \in \pi_{ij}} dF_k^{d_k} * (F_k)_{m_k}(s_k) \Big) dF_j^{d_j} * (F_j)_{m_j}(\tau) \Big].$$

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Finally, observe that

$$\sum_{n=i+1}^{\infty} \sum_{\theta_{ij}^n} \left[\prod_{k \in \pi_{ij}} dF_k^{d_k} * (F_k)_{m_k}(s_k) \right] dF_j^{d_j} * (F_j)_{m_j}(\tau) = \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau). \quad (4.11)$$

This completes the proof of the theorem. □

Proof of Theorem 3.1 : Key Superposition Renewal Theorem The proof is a direct extension of the proof of the ordinary Key Renewal Theorem given in Feller (1971) or Cinlar (1975). It is carried out in two steps.

Step 1: Let

$$a(x,\mathbf{d}) = \begin{cases} c(\mathbf{d}) & \text{if } 0 \le L \le x \le U < \infty \\ 0 & \text{otherwise} \end{cases}, \quad (4.12)$$

where $c(\mathbf{d})$ is a function independent of x. Since the function a is independent of the first variable, it follows that $\kappa_T = \kappa$ which is independent of T. Let $I_A(t)$ be the indicator function defined by $I_A(t) = 1$ if $t \in A$ and 0 otherwise. Observe that

1. $\lim_{t \to \infty} a(t,\mathbf{d}) = \lim_{t \to \infty} c(\mathbf{d}) I_{\{L \le t \le U\}}(t) = 0.$

2. We know that $F_l^{d_l}$ is a nondecreasing function such that $F_l^{d_l}(\infty) = \lim_{\tau \to \infty} F_l^{d_l}(\tau) = 1$, i.e., given any $\epsilon > 0$, there exists $\tau_o(\epsilon)$ such that for $\tau > \tau_o(\epsilon)$ and $\forall l \in \{1,2,\ldots,p\}$, we have

$$1 - F_l^{d_l}(\tau) < \frac{\epsilon \mu}{(U-L)\kappa \mid \sigma \mid} = u < 1 \quad (4.13)$$

where $\sigma = \max_{1 \le i,j \le p} \mid \sigma_{ij} \mid$. Also, by Proportion (3.5.1) part (iii) in Ross (1983), given any $\epsilon > 0$, there exists $t_o(\epsilon)$ such that for all $t > t_o(\epsilon)$ and $\forall k \in \{1,2,\ldots,p\}$,

$$\mid H_k^{d_k}(t-L) - H_k^{d_k}(t-U) \mid < \frac{(\epsilon+1)(U-L)}{\mu_k} \quad (4.14)$$

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3. For $i = 0, 1, \ldots, p - 2$ and $t > \max(\tau_o(\epsilon) + U, t_o(\epsilon))$,

$$\sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f(\delta) \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau)$$

$$\leq \kappa u \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_{t-U}^{t-L} \left[\prod_{k \in \pi_{ij}} \int_0^\tau (1 - F_k(\tau - s_k)) dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau)$$

$$= \kappa u \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_{t-U}^{t-L} \left[\prod_{k \in \pi_{ij}} F_k^{d_k}(\tau) \right] dH_j^{d_j}(\tau) \leq \kappa u \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_{t-U}^{t-L} dH_j^{d_j}(\tau)$$

$$= \kappa u \sum_{j=1}^{p} |\sigma_{ij}| [H_j^{d_j}(t - L) - H_j^{d_j}(t - U)]$$

$$\leq \kappa u |\sigma| \sum_{j=1}^{p} \frac{(\epsilon + 1)(U - L)}{\mu_j} = \epsilon(\epsilon + 1) \tag{4.15}$$

Since ϵ can be made arbitrarily small provided that t is large enough, it follows that when $i = 0, 1, \ldots, p - 2$,

$$\lim_{t \to \infty} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f(\delta) \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau) = 0. \tag{4.16}$$

4. Next observe that by Proposition (3.5.1) part (iii) in Ross (1983), we have

$$\frac{1}{\mu} \int_0^\infty \mathcal{E}(a(x, \mathbf{D})) \, dx$$

$$= \sum_{j=1}^{p} \left(\frac{U - L}{\mu_j} \right) \mathcal{E}(c(D_1, \ldots, D_{j-1}, 0, D_{j+1}, \ldots, D_p))$$

$$= \lim_{t \to \infty} \sum_{j=1}^{p} \int_{t-U}^{t-L} \left[\underbrace{\int_0^\infty \cdots \int_0^\infty}_{p-1 \text{ times}} c(\mathbf{y_j}) \prod_{k=1, k \neq j}^{p} \left(\frac{(1 - F_k(y_k))}{\mu_k} \, dy_k \right) \right] dH_j^{d_j}(\tau) \tag{4.17}$$

where $\mathbf{y_j} = (y_1, \ldots, y_{j-1}, 0, y_{j+1}, \ldots, y_p)$.

5. Let $\mathbf{s} = (s_1, \ldots, s_p)$ and $\mathbf{v}(j, \tau, \mathbf{s}) = (\tau - s_1, \ldots, \tau - s_{j-1}, 0, \tau - s_{j+1}, \ldots, \tau - s_p)$. For $i = p - 1$,

$$\lim_{t \to \infty} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f(\delta) \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau) \tag{4.18}$$

$$= \lim_{t \to \infty} \sum_{j=1}^{p} \int_{t-U}^{t-L} \left[\underbrace{\int_0^\tau \cdots \int_0^\tau}_{p-1 \text{ times}} c(\mathbf{v}(j, \tau, \mathbf{s})) \prod_{k=1, k \neq j}^{p} \left((1 - F_k(\tau - s_k)) dH_k^{d_k}(s_k) \right) \right] dH_j^{d_j}(\tau)$$

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It is easily deduced from condition (4) of this Theorem and by the extension of the proof for Ordinary Key Renewal Theorem to higher dimensions that

$$\lim_{\tau \to \infty} \underbrace{\int_0^\tau \cdots \int_0^\tau}_{p-1 \text{ times}} c(\mathbf{v}(j,\tau,\mathbf{s})) \prod_{k=1,k\neq j}^{p} \left((1 - F_k(\tau - s_k)) \, dH_k^{d_k}(s_k) \right)$$

$$= \underbrace{\int_0^\infty \cdots \int_0^\infty}_{p-1 \text{ times}} c(\mathbf{y_j}) \prod_{k=1,k\neq j}^{p} \left(\frac{(1 - F_k(y_k))}{\mu_k} \, dy_k \right) \tag{4.19}$$

Since $t - L \geq \tau \geq t - U$, hence $\tau \to \infty$ as $t \to \infty$ and

$$\lim_{t \to \infty} \sum_{j=1}^{p} \sum_{\sigma_{ij}} \int_0^t \underbrace{\int_0^\tau \cdots \int_0^\tau}_{i \text{ times}} f(\delta) \left[\prod_{k \in \pi_{ij}} dH_k^{d_k}(s_k) \right] dH_j^{d_j}(\tau) = \frac{1}{\mu} \int_0^\infty \mathcal{E}(a(x,\mathbf{D})) \, dx.$$

Step 2: Let $a(x, y_1, \ldots, y_p)$ be a function that satisfies the conditions stated in the theorem. For fixed $h > 0$ and $k = 1, 2, \ldots$, let

$$\underline{a}(x, y_1, \ldots, y_p) = \sum_{k=1}^{\infty} \underline{c_k}(y_1, \ldots, y_p) I_{[(k-1)h,kh)}(x) \tag{4.20}$$

and

$$\overline{a}(x, y_1, \ldots, y_p) = \sum_{k=1}^{\infty} \overline{c_k}(y_1, \ldots, y_p) I_{[(k-1)h,kh)}(x) \tag{4.21}$$

where $\underline{c_k}(y_1, \ldots, y_p)$ and $\overline{c_k}(y_1, \ldots, y_p)$ are defined in Equations (3.1) and (3.2). Then

$$\underline{a}(y_1, \ldots, y_p) \leq a(y_1, \ldots, y_p) \leq \overline{a}(y_1, \ldots, y_p).$$

By step 1 and condition (4) of the theorem, it is easily shown that if $\underline{A}(t, \mathbf{d})$ and $\overline{A}(t, \mathbf{d})$ are the solutions of the S-renewal equations corresponding to $\underline{a}$ and $\overline{a}$, then

$$\lim_{t \to \infty} \underline{A}(t, \mathbf{d}) = \frac{h}{\mu} \sum_{k=1}^{\infty} \mathcal{E}(\underline{c_k}(\mathbf{D})) = \frac{1}{\mu} \int_0^\infty \mathcal{E}(\underline{a}(x, \mathbf{D})) \, dx \tag{4.22}$$

and

$$\lim_{t \to \infty} \overline{A}(t, \mathbf{d}) = \frac{h}{\mu} \sum_{k=1}^{\infty} \mathcal{E}(\overline{c_k}(\mathbf{D})) = \frac{1}{\mu} \int_0^\infty \mathcal{E}(\overline{a}(x, \mathbf{D})) \, dx. \tag{4.23}$$

But $\underline{A}(t, \mathbf{d}) \leq A(t, \mathbf{d}) \leq \overline{A}(t, \mathbf{d})$ and hence all limit values of $A(t, \mathbf{d})$ lie between $\dfrac{h}{\mu} \sum_{k=1}^{\infty} \mathcal{E}(\underline{c_k}(\mathbf{D}))$ and $\dfrac{h}{\mu} \sum_{k=1}^{\infty} \mathcal{E}(\overline{c_k}(\mathbf{D}))$. By condition (5) of the theorem, it follows that

$$\lim_{t \to \infty} A(t, \mathbf{d}) = \frac{1}{\mu} \int_0^\infty \mathcal{E}(a(x, \mathbf{D})) \, dx. \tag{4.24}$$

$\square$

5 Asymptotic Results of N(t, d) and H(t, d)

In this section, we extend some of the asymptotic results for ordinary delayed renewal process to the superposition of p renewal processes. Let $N(t; p) = N(t, 0, \ldots, 0)$ be the counting process associated with the superposition of p independent renewal processes $N_i(t)$, $i = 1, 2, \ldots, p$, each of these processes restarted at time $t = 0$. The next theorem tells us that $N(t; p)$ is asymptotically normally distributed as $t \to \infty$.

Theorem 5.1 Central Limit Theorem for Superposition of p Renewal Processes

For $i = 1, \ldots, p$, let μ_i and σ_i^2, assumed finite, represent the mean and variance of an interarrival time of the ith renewal process $N_i(t)$. Then

$$\lim_{t \to \infty} \mathcal{P} \left\{ \frac{N(t; p) - t \sum_{i=1}^{p} \frac{1}{\mu_i}}{\sqrt{t \sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3}}} < y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, dx. \tag{5.1}$$

Proof: The proof is straightforward. By the Central Limit Theorem for ordinary renewal processes [Karlin and Taylor (1975)], we have for $i = 1, 2, \ldots, p$,

$$N_i(t) \sim N\left(\frac{t}{\mu_i}, \frac{t\sigma_i^2}{\mu_i^3}\right) \text{ as } t \to \infty. \tag{5.2}$$

Since all the component processes are independent,

$$N(t; p) = \sum_{i=1}^{p} N_i(t) \sim N\left(t \sum_{i=1}^{p} \frac{1}{\mu_i}, t \sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3}\right). \tag{5.3}$$

$\square$

Let T_n be the time of occurrence of the nth event in the superposition process $N(t; p)$. The theorem above can now be used to show that T_n is also asymptotically normal as $n \to \infty$.

Theorem 5.2 *Let T_n be the time of occurrence of the nth event in the superposition process $N(t; p)$. The asymptotic Result (5.1) can be inverted to show that as $n \to \infty$,*

$$T_n \sim N\left(n \Big/ \sum_{i=1}^{p} \frac{1}{\mu_i}, n \sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3} \Big/ (\sum_{i=1}^{p} \frac{1}{\mu_i})^3\right). \tag{5.4}$$

Proof: First, observe the equivalence of the sets of events

$$\{T_n > t\} \quad \text{if and only if} \quad \Big\{N(t;p) = \sum_{i=1}^{p} N_i(t) < n\Big\}.$$

Write

$$t = \frac{n}{\sum\limits_{i=1}^{p} \frac{1}{\mu_i}} + y_n \sqrt{n \sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3} \Big/ \Big(\sum_{i=1}^{p} \frac{1}{\mu_i}\Big)^3} \tag{5.5}$$

so that

$$n - t \sum_{i=1}^{p} \frac{1}{\mu_i} = -y_n \Big(\sum_{i=1}^{p} \frac{1}{\mu_i}\Big) \sqrt{n \sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3} \Big/ \Big(\sum_{i=1}^{p} \frac{1}{\mu_i}\Big)^3}. \tag{5.6}$$

Then

$$\mathcal{P}(T_n > t) = \mathcal{P}(N(t;p) < n) = \mathcal{P}\Bigg(\frac{N(t;p) - t \sum\limits_{i=1}^{p} \frac{1}{\mu_i}}{\sqrt{t \sum\limits_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3}}} < \frac{n - t \sum\limits_{i=1}^{p} \frac{1}{\mu_i}}{\sqrt{t \sum\limits_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3}}}\Bigg). \tag{5.7}$$

By Expressions (5.5) and (5.6) above,

$$\Big(n - t \sum_{i=1}^{p} \frac{1}{\mu_i}\Big) \Big/ \sqrt{t \sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3}} = -y_n \Big\{1 + y_n \sqrt{\sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3} \Big/ n \sum_{i=1}^{p} \frac{1}{\mu_i}}\Big\}^{-1/2}. \tag{5.8}$$

Now fix $y_n = y$ and let $n \to \infty$. Then

$$\lim_{t \to \infty} -y_n \Big\{1 + y_n \sqrt{\sum_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3} \Big/ n \sum_{i=1}^{p} \frac{1}{\mu_i}}\Big\}^{-1/2} = -y. \tag{5.9}$$

Furthermore, by Expression (5.5), $t \to \infty$ as $n \to \infty$. Hence by Theorem (5.1) above,

$$\lim_{n \to \infty} \mathcal{P}\Bigg(\frac{T_n - n \Big/ \sum\limits_{i=1}^{p} \frac{1}{\mu_i}}{\sqrt{n \sum\limits_{i=1}^{p} \frac{\sigma_i^2}{\mu_i^3} \Big/ \Big(\sum\limits_{i=1}^{p} \frac{1}{\mu_i}\Big)^3}} > y\Bigg) = \lim_{n \to \infty} \mathcal{P}(T_n > t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-y} e^{-x^2/2} \, dx. \tag{5.10}$$

$\square$

The following theorem is an extension of Proposition (3.5.1) in Ross (1983). The proof of the theorem is straightforward.

Theorem 5.3 *Let μ be the mean interarrival time for the superposition process $N(t, \mathbf{d})$.*

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1. With probability 1, $\lim\limits_{t\to\infty} \dfrac{N(t,\mathbf{d})}{t} = \dfrac{1}{\mu}$.

2. $\lim\limits_{t\to\infty} \dfrac{H(t,\mathbf{d})}{t} = \dfrac{1}{\mu}$.

3. If F_i, $i = 1, 2, \ldots, p$, are not arithmetic, then

$$\lim_{t\to\infty}\left(H(t+h,\mathbf{d}) - H(t,\mathbf{d})\right) = \frac{h}{\mu}.$$

4. For $i = 1, 2, \ldots, p$, let F_i be nonarithmetic distributions with finite variance σ_i^2. Then we have

$$\lim_{t\to\infty}\left(H(t,\mathbf{d}) - \frac{t}{\mu}\right) = \sum_{i=1}^{p} \frac{\sigma_i^2 - \mu_i^2}{2\mu_i^2}.$$

This is an asymptotic expansion of $H(t,\mathbf{d})$.

Proof: Use Proposition (3.5.1) in Ross (1983). $\qquad\square$

Remarks: Theorem (5.3) also holds when the component processes are dependent.

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