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SUPERPROCESSES AND PARTIAL DIFFERENTIAL EQUATIONS[1]

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The subject of this article is a class of measure-valued Markov processes. A typical example is *super-Brownian motion*. The Laplacian Δ plays a fundamental role in the theory of Brownian motion. For super-Brownian motion, an analogous role is played by the operator $\Delta u - \psi(u)$, where a nonlinear function ψ describes the branching mechanism. The class of admissible functions ψ includes the family $\psi(u) = u^\alpha$, $1 < \alpha \le 2$.

Super-Brownian motion belongs to the class of continuous state branching processes investigated in 1968 in a pioneering work of Watanabe. Path properties of super-Brownian motion are well known due to the work of Dawson, Perkins, Le Gall and others. Partial differential equations involving the operator $\Delta u - \psi(u)$ have been studied independently by several analysts, including Loewner and Nirenberg, Friedman, Brezis, Véron, Baras and Pierre. Connections between the probabilistic and analytic theories have been established recently by the author.

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Introduction.

1. A prototype of Brownian motion is a chaotic movement of one particle. Super-Brownian motion describes a cloud arising as the limit of a system of independent Brownian particles which die at random times, leaving a random number of offspring. The limit is taken as the mass of each particle tends to 0 but the expected total mass at each time t does not change.

Denote by ξ_t the position of the Brownian particle at time t. To every open set D in the state space E there corresponds a random point ξ_τ, where $\tau = \inf\{t: \xi_t \notin D\}$ is the first exit time from D. More generally, to every open set Q in $S = [0, \infty) \times E$ there corresponds a random point $(\tau, \xi_\tau) \in S$, where $\tau = \inf\{t: (t, \xi_t) \notin Q\}$. These random points play a key role in the theory of Brownian motion. In particular, they are crucial for the probabilistic approach to elliptic and parabolic differential equations involving the Laplacian.

For super-Brownian motion, an analogous role belongs to random measures X_τ on S describing the mass distribution of the cloud at the first exit time from Q (more precisely, we freeze, before passing to the limit, each particle at its first exit time from Q). The mass distribution X_t at time t is a particular case corresponding to $Q = [0, t) \times E$. In contrast to ξ_τ, which can be defined through ξ_t, it is impossible in general to define X_τ in terms of X_t. For this reason, we consider the family $\{X_\tau\}$ rather than $\{X_t\}$ as our principal subject.

The stochastic process X_t is a Markov process with transition probabilities $P_{r,\nu}$. The family X_τ also has a Markov property. However, to state this property, it is necessary to extend the set $\{P_{r,\nu}\}$ to a set $\{P_\mu\}$ indexed by measures on S: while $P_{r,\nu}$ describes the evolution which starts from the mass distribution ν at time r, the probability measure P_μ corresponds to the initial cloud spread in time and space (in the discrete approximation, particles—"immigrants"—appear at various times and various points of E).

We introduce the family $X = (X_\tau, P_\mu)$ by describing the joint probability distribution of $X_{\tau_1}, \dots, X_{\tau_n}$ relative to P_μ for every finite collection $\tau_1, \dots, \tau_n$ and every μ. The construction is applicable to an arbitrary right Markov process ξ and to first exit times from finely open sets in the global state space S. Besides ξ, a superprocess X is characterized by an additive functional K of ξ which describes the branching intensity and by a function ψ which determines the branching mechanism.

2. This article consists of three parts and an Appendix. References outside each part include the part number. For instance, we write (II.3.1) for formula (3.1) in Part II (Theorem A.1.2 means Theorem 1.2 in the Appendix). Within Part II, we write (3.1), not (II.3.1).

Part I is devoted to the general theory of superprocesses. We construct them in two different ways. The first is based on an explicit formula for the finite-dimensional distributions and it involves solving certain integral equations (cf. [96], [21], [23] and [35]). The second construction uses a passage to the limit from branching particle systems (cf. [24] and [26]). The proofs are moved to the Appendix. We also discuss the Markov property, regularity properties and the concept of a part of a superprocess.

In Part II, we concentrate on the case when ξ is a diffusion determined by a second-order elliptic differential operator L. We call the corresponding X a superdiffusion. Parabolic and elliptic partial differential equations involving the operator $Lv - \psi(v)$ can be investigated in terms of the corresponding superdiffusion.

In the elliptic case, L and ψ are independent of time, and both ξ and X are time-homogeneous Markov processes. We put $P_\nu = P_{0,\nu}$ and $\bar{X}_\tau(B) = X_\tau(\mathbb{R}_+ \times B)$ for every Borel set B in E. For a broad class of domains D and functions $f \geq 0$, the solution of the first boundary value problem,

$$Lv = \psi(v) \quad \text{in } D, \tag{1}$$

$$v = f \quad \text{on } \partial D, \tag{2}$$

can be obtained by the formula

$$v(x) = -\log P_{\delta_x} \exp\left\{-\int f(x)\bar{X}_\tau(dx)\right\}, \tag{3}$$

where δ_x is Dirac's measure at x.

An analogous problem for the parabolic equation

$$\dot{v} + Lv = \psi(v) \quad \text{in } Q \tag{4}$$

can be solved with X_τ substituted for $\bar{X}_\tau$. The solution of a more general nonhomogeneous parabolic equation can be obtained by using the part of X in Q (see Theorem II.3.1).

Most results in Part II are stated for $\psi(z) = z^\alpha$, $1 < \alpha \leq 2$. We only sketch the proofs and refer the reader to [25] and [28] for details. In contrast to the linear equations, the problem (1), (2) can be solved for functions f with values in $[0, +\infty]$. For instance,

$$(5) \qquad v(x) = -\log P_{\delta_x}\{\bar{X}_\tau = 0\}$$

is the solution corresponding to $f = +\infty$. The maximal solution of (1) in an arbitrary domain D can be obtained by the formula

$$(6) \qquad v(x) = -\log P_{\delta_x}\{R \subset D\},$$

where R is the range of X, that is, the minimal closed set in E which supports all measures X_t (it also supports, P_μ-a.s., $\bar{X}_\tau$ for all τ and all μ). We call a set B R-polar if $P_{\delta_x}\{R \cap B = \emptyset\} = 1$ for all $x \notin B$. Clearly, a closed set $\Gamma \subset E$ is R-polar if and only if (1) has no solutions in $D = E \setminus \Gamma$ except 0. We prove that an analytic set B is R-polar if and only if the condition

$$\int_B g(x,y)\nu(dy) \text{ belongs to } L^\alpha(E)$$

implies that $\nu(B) = 0$. Here $g(x,y)$ is Green's function of the Brownian motion with a positive constant killing rate (see Theorem II.12.3). (This is equivalent to the following statement: B is R-polar if and only if $B_{2,\alpha'}(B) = 0$, where $\alpha' = \alpha/(\alpha - 1)$ and $B_{2,\alpha'}$ is a Bessel capacity; see [70].) We also give a test of R-polarity in terms of the Hausdorff measure (Theorem II.12.5).

All these results follow easily from their parabolic counterpart. The maximal solution of (4) in an arbitrary open set $Q \subset S$ is given by the formula

$$(7) \qquad v(r,x) = -\log P_{r,\delta_x}\{G \subset Q\},$$

where G is the graph of X, that is, the minimal closed set in S which supports all measures X_t (it supports, P_μ-a.s., every measure X_τ). The class of analytic G-polar sets A can be characterized by the property

$$\int_A p(r,x;s,y)\nu(ds,dy) \text{ belongs to } L^\alpha(S)$$

only if $\nu(A) = 0$ [$p(r,x;s,y)$ is the transition density of the Brownian motion with a positive constant killing rate]. The characterization of this class in terms of the restricted Hausdorff measure was obtained recently by Sheu [84].

Section II.5 can be considered as a step toward the Martin boundary theory for equation (4). All solutions v of (4) are described in terms of the behavior of "the cloud" near ∂Q. More precisely, we approximate Q by an increasing sequence of regular bounded domains Q_n, and we put

$$(8) \qquad v(r,x) = -\log P_{r,\delta_x} e^{-Z},$$

where Z depends on the limit behavior of X_{τ_n} (τ_n is the first exit time from Q_n). Formulae (3), (5), (6) and (7) are special cases of (8).

A new concept of G-regularity is introduced in Section II.6. It is motivated by the probabilistic definition of regular points of Q for a diffusion ξ. Recall that a point (r^0, x^0) of ∂Q is called regular if a particle which starts at time r^0 from point x^0 has probability 0 of staying in Q during any time interval (r^0, t).

Analogously, we say that $(r^0, x^0) \in \partial Q$ is G-regular if $P_{r^0, \delta_{x^0}}\{G(r^0, t] \subset Q\} = 0$ for every $t > r$, where $G(r^0, t]$ is the intersection of G with $(r^0, t] \times E$. We prove that (r^0, x^0) is G-regular if and only if the maximal solution v of (4) tends to ∞ at (r^0, x^0).

Part III contains a survey of the literature and general comments. An early look through this part can be useful for understanding our motivation and the relation of the present paper to the previous work.

3. Notation. For every measurable space $(E, \mathscr{B})$, we denote by $\mathscr{M}(E)$ the set of all finite measures on $\mathscr{B}$. We consider $\mathscr{M}(E)$ as a measurable space with the measurable structure generated by the functions $f_B(\mu) = \mu(B)$ with $B \in \mathscr{B}$. The expression $\langle f, \mu \rangle$ stands for the integral of f with respect to μ, and $|\mu|$ means $\langle 1, \mu \rangle$.

We write $f \in \mathscr{B}$ if f is a $\mathscr{B}$-measurable function. Writing $f \in p\mathscr{B}$ $(b\mathscr{B})$ means that, in addition, f is positive (bounded). We put

$$bp\mathscr{B} = (b\mathscr{B}) \cap (p\mathscr{B}), \qquad b_c\mathscr{B} = \{f: f \in \mathscr{B}, -c \le f \le c\},$$

$$b_c p\mathscr{B} = (b_c\mathscr{B}) \cap (p\mathscr{B}).$$

If E is a topological space, then $\mathscr{B}_E$ stands for the Borel σ-algebra in E.

We set $\mathbb{R}_+ = [0, \infty)$, $S = \mathbb{R}_+ \times E$. We say that a function f on S is supported by an interval $[0, a)$ if it vanishes on $[a, \infty) \times E$, and we put $f \in \mathscr{H}$ if $f \in bp\mathscr{B}_S$ is supported by some interval $[0, a)$, $a > 0$. If $\{\mathscr{F}_\Delta\}$ is a family of σ-algebras indexed by subsets Δ of $\mathbb{R}_+$, then $\mathscr{F}_{\le t} = \mathscr{F}[0, t]$ and $\mathscr{F}_{\ge t} = \mathscr{F}[t, \infty)$.

Following Dellacherie and Meyer ([14], III, 16), we say that a metrizable topological space is *Luzin* if it is homeomorphic to a Borel subset of a compact metrizable space. A measurable space $(E, \mathscr{B})$ is called *Luzin* if it is isomorphic to $(L, \mathscr{B}_L)$, where L is a Luzin metrizable space. A kernel from a measurable space $(E, \mathscr{B})$ to a measurable space $(\tilde{E}, \tilde{\mathscr{B}})$ is a function $k(x, B)$ such that $k(x, \cdot)$ is a measure on $\tilde{\mathscr{B}}$ and $k(\cdot, B)$ is a $\mathscr{B}$-measurable function on E.

Part I. Superprocesses.

1. Definition and fundamental properties. A superprocess X is determined by three parameters: a Markov process ξ, which describes the spatial motion; an additive functional K of ξ, which defines intensity of branching; and a function ψ, which determines the branching mechanism.

1.1. *Markov processes.* A *Markov process* $\xi = (\xi_t, \Pi_{r,x})$ on the time interval $\mathbb{R}_+ = [0, \infty)$ with a random birth time α is a combination of the following elements:

(a) a measurable space $(\Omega^0, \mathscr{F}^0)$ (the sample space);
(b) a measurable function α: $\Omega^0 \to \mathbb{R}_+$ (the birth time);
(c) a measurable space $(E, \mathscr{B})$ (the state space at time t);

(d) for every pair $\omega \in \Omega^0$, $t \in [\alpha(\omega), +\infty)$, a point $\xi_t(\omega)$ in E (the state at time $t \geq \alpha$);
(e) for every $r \in \mathbb{R}_+$, $x \in E$, a probability measure $\Pi_{r,x}$ on $(\Omega^0, \mathscr{F}^0)$ (transition probabilities).

We consider a one-point extension $E \cup \{\partial\}$ of E and we set $\xi_t(\omega) = \partial$ for $t < \alpha(\omega)$. Denote by $\mathscr{F}^0(\Delta)$ the minimal σ-algebra in Ω^0 which contains all sets $\{\omega: \xi_t(\omega) \in B\}$ for $t \in \Delta$, $B \in \mathscr{B}$, and all sets $\{\omega: \xi_t(\omega) = \partial\}$ for $t \in \Delta$. We assume that $\mathscr{F}^0 \supset \mathscr{F}^0(\mathbb{R}_+)$ and that the following hold:

1.1.A. For every $Z \in \mathscr{F}^0$, the function $f(r, x) = \Pi_{r,x} Z$ is measurable with respect to $\mathscr{B}_{\mathbb{R}_+} \times \mathscr{B}$.
1.1.B. For every (r, x), $\Pi_{r,x}\{\alpha = r, \xi_\alpha = x\} = 1$.
1.1.C. (Markov property.) For every $r < t \in \mathbb{R}_+$, $F \in p\mathscr{F}^0_{\leq t}$, $Z \in p\mathscr{F}^0_{\geq t}$,

$$\Pi_{r,x}(FZ) = \Pi_{r,x}(F\Pi_{t,\xi_t} Z).$$

The transition function of ξ is defined by the formula

$$p(r, x; t, B) = \Pi_{r,x}\{\xi_t \in B\}, \quad \text{for } r < t \in \mathbb{R}_+,\ x \in E,\ B \in \mathscr{B}. \tag{1.1}$$

It follows from 1.1.C that

$$\begin{aligned} &\Pi_{r,x}\{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\} \\ &\quad = \int_{B_1} \cdots \int_{B_n} p(r, x; t_1, dy_1) \cdots p(t_{n-1}, y_{n-1}; t_n, dy_n), \end{aligned} \tag{1.2}$$

for all $n \geq 2$, $r < t_1 < \cdots < t_n \in \mathbb{R}_+$, $B_1, \dots, B_n \in \mathscr{B}$. Formulae (1.1) and (1.2) can be used for constructing a Markov process starting from a Markov transition function p.

We put $S = \mathbb{R}_+ \times E$, $\mathscr{B}_S = \mathscr{B}_{\mathbb{R}_+} \times \mathscr{B}$, and we call $(S, \mathscr{B}_S)$ the *global state space*. [Sometimes it is useful to consider a variable state space $(E_t, \mathscr{B}_t)$; then the global state space is a measurable subset of $\mathbb{R}_+ \times E$.]

Condition 1.1.A makes it possible to introduce the measures

$$\Pi_\mu = \int_S \Pi_{r,x} \mu(dr, dx), \quad \text{for } \mu \in \mathscr{M}(S), \tag{1.3}$$

$$\Pi_{r,\nu} = \int_E \Pi_{r,x} \nu(dx), \qquad \text{for } \nu \in \mathscr{M}(E). \tag{1.4}$$

1.2. *Right processes.* We say that a process ξ is *right* if the following hold:

1.2.A. The global state space S is a metrizable Luzin space.
1.2.B. All paths of ξ are right-continuous.
1.2.C. For every $r < u \in \mathbb{R}_+$, every $\nu \in \mathscr{M}(E)$ and every $Z \in p\mathscr{F}^0_{\geq u}$, $\Pi_{t,\xi_t} Z$ is right-continuous on $[r, u)$ $\Pi_{r,\nu}$-a.s.

An important class of right processes is diffusions, which we consider in Part II.

1.3. *Exit times.* To every set $Q \subset S$ there corresponds the *first exit time*

$$\tau = \inf\{t: t \geq \alpha, (t, \xi_t) \notin Q\}. \tag{1.5}$$

If ξ is right, then there exists an increasing family of σ-algebras $\mathscr{A}_t \supset \mathscr{F}^0_{\leq t}$ such that the following hold:

1.3.A. The first exit time τ from any set $Q \in \mathscr{B}_S$ is a stopping time relative to the filtration $\mathscr{A}_t$.

1.3.B. The Markov property 1.1.C holds for every $F \in p\mathscr{A}_t$ and every $Z \in p\mathscr{F}^0_{\geq t}$.

1.3.C. (Strong Markov property.) For every stopping time τ and for all $\mu \in \mathscr{M}(S)$, $Z \in p\mathscr{F}^0_{\geq t}$,

$$\Pi_\mu\{Z|\mathscr{A}_\tau\} = \Pi_{\tau, \xi_\tau} Z \quad \Pi_\mu\text{-a.s. on } \{\tau < t\}. \tag{1.6}$$

[The family $\mathscr{A}_t$ can be constructed as follows. Consider the intersection $\mathscr{F}^0_{t+}$ of $\mathscr{F}^0_{\leq u}$ over all $u > t$ and the completion $(\mathscr{F}^0)^\mu$ of $\mathscr{F}^0$ relative to μ, and denote by $\mathscr{A}^\mu_t$ the minimal σ-algebra which contains $\mathscr{F}^0_{t+}$ and all sets $A \in (\mathscr{F}^0)^\mu$ such that $P_\mu(A) = 0$. $\mathscr{A}_t$ is the intersection of $\mathscr{A}^\mu_t$ over all $\mu \in \mathscr{M}(S)$.]

Let τ be the first exit time from Q. Put $(r, x) \in Q^0$ if $\Pi_{r,x}\{\tau > r\} = 1$. A set $Q \in \mathscr{B}_S$ is called *finely open* if $Q^0 = Q$. We denote by $\mathscr{T}$ the set of all exit times from finely open sets $Q \in \mathscr{B}_S$. (Restriction to finely open sets is justified by the fact that the first exit time from an arbitrary set $Q \in \mathscr{B}_S$ coincides, a.s., with the first exit time from Q^0.) Suppose that τ and $\tilde{\tau}$ are the first exit times from finely open sets Q and $\tilde{Q}$. Clearly, $\tau \leq \tilde{\tau}$ if and only if $Q \subset \tilde{Q}$.

1.4. *Additive functionals.* An *additive functional* K of a Markov process ξ is a measure $K(\omega, dt)$ on $[\alpha(\omega), \infty)$ depending on $\omega \in \Omega^0$ in such a way that $K(\cdot, (r, s))$ is measurable relative to the universal completion of $\mathscr{F}^0(r, s)$. In this paper, we consider only additive functionals of the form

$$K(B) = \int_B k(s, \xi_s)\, ds, \qquad B \in \mathscr{B}_{\mathbb{R}_+}, \tag{1.7}$$

where $k \in p\mathscr{B}_S$ is the subject of the following condition:

1.4.A. For every $a \in \mathbb{R}_+$, there exists a constant c_a such that $k(s, x) \leq c_a$ for all $r \in [0, a)$, $x \in E$.

[We set $k(s, \partial) = 0$.] Superprocesses corresponding to a more general class of additive functionals are constructed in [24] (see Section III.2.2).]

1.5. *Parameter ψ.* The *branching parameter* ψ is a transformation from $bp\mathscr{B}_S$ to $\mathscr{B}_S$. We assume that the values of $\psi(z)$ at time s depend only on the values of z at the same time, that is, $\psi(z)(s, x) = \psi^s(z^s)(x)$, where $z^s(x) = z(s, x)$ and ψ^s is an operator from $bp\mathscr{B}$ to $\mathscr{B}$. In the most important case of *local branching*, the value of $\psi(z)$ at (s, x) depends only on $z(s, x)$ and therefore $\psi(z)(s, x) = \psi[s, x; z(s, x)]$, where $\psi(s, x; t)$ is a function from $S \times \mathbb{R}_+$ to $\mathbb{R}$.

1.6. *Definition of a superprocess.* Fix a measurable space $(\Omega, \mathscr{F})$. A random measure on $\mathscr{M}(S)$ is a measurable mapping from Ω to $\mathscr{M}(S)$. Suppose that to every $\tau \in \mathscr{T}$ there corresponds a random measure X_τ on S, and to every $\mu \in \mathscr{M}(S)$ there corresponds a probability measure P_μ on $(\Omega, \mathscr{F})$. We say that (X_τ, P_μ) is a *superprocess with parameters* (ξ, K, ψ) if the following hold:

1.6.A. For every $f \in \mathscr{H}$, $\tau \in \mathscr{T}$, $\mu \in \mathscr{M}(S)$,

$$P_\mu \exp\langle -f, X_\tau\rangle = e^{-\langle v, \mu\rangle}, \tag{1.8}$$

where

$$v(r,x) + \Pi_{r,x}\int_r^\tau \psi(v)(s,\xi_s)\,dK_s = \Pi_{r,x} f(\tau, \xi_\tau). \tag{1.9}$$

1.6.B. For $n \geq 2$, the joint probability distribution of $X_{\tau_1}, \dots, X_{\tau_n}$ is described as follows. Let

$$I = \{1, \dots, n\}, \qquad \tau_I = \min\{\tau_1, \dots, \tau_n\}, \qquad \lambda = \min\{i : \tau_i = \tau_I\}. \tag{1.10}$$

For every $f_i \in \mathscr{H}$, $i \in I$, put

$$\langle f_I, X_I\rangle = \sum \langle f_i, X_{\tau_i}\rangle. \tag{1.11}$$

Then

$$P_\mu \exp[-\langle f_I, X_I\rangle] = \exp\langle -v_I, \mu\rangle, \tag{1.12}$$

where the functions v_I are determined recursively by the integral equations

$$v_I(r,x) + \Pi_{r,x}\int_\alpha^{\tau_I} \psi(v_I)(s,\xi_s)\,dK_s = \Pi_{r,x} G_I, \tag{1.13}$$

with

$$G_I = [f_\lambda + v_{I-\lambda}](\tau_\lambda, \xi_{\tau_\lambda}). \tag{1.14}$$

[Note that 1.6.B with $n = 1$ and $v_\varnothing = 0$ coincides with 1.6.A.]

We say that two superprocesses $X = (X_\tau, P_\mu)$ and $\tilde{X} = (\tilde{X}_\tau, \tilde{P}_\mu)$ are *equivalent* if, for every $\mu \in \mathscr{M}(S)$, the finite-dimensional distributions of X_τ relative to P_μ coincide with the finite-dimensional distributions of $\tilde{X}_\tau$ relative to $\tilde{P}_\mu$.

1.7. *Existence.* The existence of a superprocess is proved for a local branching defined by the function

$$\begin{aligned}\psi(s,x;t) &= a(s,x)t + b(s,x)t^2 \\ &\quad + \int_{[0,1)} (e^{-ut} - 1 + ut)\, n(s,x;du) \\ &\quad + \int_{[1,\infty)} (e^{-ut} - 1)\, \tilde{n}(s,x;du),\end{aligned} \tag{1.15}$$

where $a \in \mathscr{B}_S$, $b \in p\mathscr{B}_S$, n is a kernel from S to $[0, 1)$ and $\tilde{n}$ is a kernel from S to $[1, \infty)$ such that the following holds:

1.7.A. For every finite interval Δ,

$$a, \quad b, \quad \int_{[0,1)} u^2 n(s, x; du) \quad \text{and} \quad \int_{(1,\infty)} u\tilde{n}(s, x; du) \tag{1.16}$$

are bounded on $S_\Delta = \Delta \times E$.

A more general (nonlocal) branching is described by the formula

$$\begin{aligned} \psi^s(z)(x) &= a(s,x)z(x) + b(s,x)z(x)^2 \\ &\quad + \int_0^1 \left[e^{-uz(x)} - 1 + uz(x)\right] n(s,x;du) \\ &\quad - \int_E \gamma(s,x;dy)z(y) + \int_{\mathscr{M}} (e^{-\langle z,\eta\rangle} - 1) m(s,x;d\eta). \end{aligned} \tag{1.17}$$

Here $\mathscr{M} = \mathscr{M}(E)$; a, b and n are as in (1.15); γ is a kernel from S to E; m is a kernel from S to $\mathscr{M}$ such that the following holds:

1.7.B. For every finite Δ,

$$\gamma(s, x; E) \quad \text{and} \quad \int_{\mathscr{M}} |\eta| m(s, x; d\eta) \tag{1.18}$$

are bounded on S_Δ.

Formula (1.15) is a particular case of (1.17) with $\gamma = 0$ and $m(s, x; d\eta)$ concentrated on the set $\eta = u\delta_x$, where $u \geq 1$. (The measure m is the image of $\tilde{n}$ under the mapping $u \to u\delta_x$ and it satisfies 1.7.B if and only if $\tilde{n}$ satisfies 1.7.A.)

In addition to 1.7.A and 1.7.B, we need to impose one of the following assumptions:

1.7.C. For every finite Δ, $\int 1_{|\eta| < \beta} |\eta| m(s, x; d\eta) \to 0$ uniformly on S_Δ as $\beta \to 0$.
1.7.C′. For every finite Δ, $\int_0^\beta u^2 n(s, x; du) \to 0$ uniformly on S_Δ as $\beta \to 0$.

THEOREM 1.1. *The following conditions are sufficient for existence and uniqueness (up to equivalence) of a superprocess with parameters* (ξ, K, ψ):

(a) ξ *is a right Markov process.*
(b) K *satisfies condition* 1.4.A.
(c) ψ *is given by formula* (1.17) *with* a, b, γ, m *and* n *subject to conditions* 1.7.A *and* 1.7.B *and one of conditions* 1.7.C *or* 1.7.C′.

Condition (c) *holds if* ψ *is given by* (1.15) *with* a, b, n *and* $\tilde{n}$ *subject to condition* 1.7.A.

Theorem 1.1 follows immediately from Theorems A.2.3 and A.3.2.

REMARK. A superprocess X can be constructed for an arbitrary Markov process ξ in a Luzin state space (E, B). However, in general, the first exit time τ from a set $Q \in \mathscr{B}_S$ is not necessarily a stopping time, and the index set $\mathscr{T}$ must be restricted accordingly.

1.8. We will concentrate on superprocesses with local branching. Formula (1.15) can be rewritten in the form

$$(1.19)\quad \psi(s,x;t) = a(s,x)t + b(s,x)t^2 + \int_0^\infty (e^{-ut} - 1 + ut)n(s,x;du)$$

with modified a and n. Note the following:

1.8.A. For every finite Δ, $\int_0^\infty (u \wedge u^2)n(s,x;du)$ is bounded on S_Δ.

This condition replaces 1.7.A for n and $\tilde{n}$.
Note that, for every $f \in bp\mathscr{B}_S$,

$$(1.20)\qquad P_\mu\langle f, X_\tau\rangle = \Pi_\mu\left\{ f(\tau,\xi_\tau)\exp\left[-\int_\alpha^\tau a(s,\xi_s)\,dK_s\right]\right\}.$$

Indeed, the left-hand side is equal to the derivative of $P_\mu \exp\langle -\lambda f, X_\tau\rangle$ with respect to λ at $\lambda = 0$, and (1.20) follows from (1.8), (1.9) and Lemma A.1.6 if $f \in \mathscr{H}$. A monotone passage to the limit leads to the general formula.

If $a \geq 0$ in (1.19) (we call this case *subcritical*), then ψ is monotone increasing in t. On the other hand, it follows from (1.8) that v is a monotone increasing functional of f. By the monotone convergence theorem, (1.8) and (1.9) can be extended to all functions $f \in p\mathscr{B}_S$ and the same is true for (1.12) and (1.13).

Suppose that $a = 0$. Then, by (1.20),

$$(1.20a)\qquad P_\mu\langle f, X_\tau\rangle = \Pi_\mu f(\tau,\xi_\tau),$$

for all μ, τ and f. We call this case *critical*. In particular, by setting $a = b = 0$, $n(x,du) = \tilde{c}u^{-1-\alpha}\,du$ with $\tilde{c} = c\alpha(\alpha-1)/\Gamma(2-\alpha)$, we get

$$(1.21)\qquad \psi = cz^\alpha, \qquad 1 < \alpha < 2.$$

1.9. *Properties of superprocesses.* These properties are stated in Theorems 1.2–1.4. The first of them was proved in [30], Section 1.5.

THEOREM 1.2. *Suppose that ψ is given by formula* (1.19). *We have the following*:

1.9.A. *For every $B \in \mathscr{B}_S$, $X_\tau \in \mathscr{M}(B)$ [i.e., $X_\tau(B^c) = 0$] P_μ-a.s. if $(\tau,\xi_\tau) \in B$ Π_μ-a.s.*

If τ is the first exit time from Q, then the following hold:

1.9.B. *$X_\tau \in \mathscr{M}(Q^c)$ P_μ-a.s. for every $\mu \in \mathscr{M}(S)$.*
1.9.C. *$P_\mu\{X_\tau = \mu\} = 1$ for all $\mu \in \mathscr{M}(Q^c)$.*

THEOREM 1.3 (Markov property). *For all* $\tau \in \mathscr{T}$, $\mu \in \mathscr{M}(S)$ *and* $Z \in p\mathscr{F}_{\geq \tau}$,

$$P_\mu\{Z | \mathscr{F}_{\leq \tau}\} = P_{X_\tau} Z \quad P_\mu\text{-a.s.} \tag{1.22}$$

Here $\mathscr{F}_{\leq \tau}$ *is the* σ*-algebra in* Ω *generated by* $X_{\tau'}$, $\tau' \leq \tau$, *and* $\mathscr{F}_{\geq \tau}$ *is the* σ*-algebra generated by* $X_{\tau'}$, $\tau' \geq \tau$.

PROOF. By the multiplicative systems theorem, it is sufficient to prove that

$$P_\mu YZ = P_\mu(Y P_{X_\tau} Z) \tag{1.23}$$

for $Y = \exp\{-\langle f_I, X_I \rangle - \langle f_0, X_\tau \rangle\}$ and $Z = \exp\{-\langle h, X_\tau \rangle - \langle f_J, X_J \rangle\}$, where $\tau = \tau_0$, $\tau_i \leq \tau_0 \leq \tau_j$ and $i \leq 0 \leq j$ for $i \in I$ and $j \in J$, and $\langle f_I, X_I \rangle$ is given by (1.11). By 1.9.C, $P_\eta \exp[\langle -h, X_\tau \rangle F] = \exp\langle -h, \eta \rangle P_\eta F$, for every $\eta \in \mathscr{M}(Q^c)$ and every F. By 1.9.B, we can apply this to $\eta = X_\tau$ and therefore we can assume that $h = 0$ (otherwise, we replace f_0 by $f_0 + h$).

For every $\tau \in \mathscr{T}$, we set

$$(\Psi_\tau v)^r(x) = v^r(x) + \Pi_{r,x} \int_r^\tau \psi^s(v^s)(\xi_s)\, dK_s. \tag{1.24}$$

Equation (1.13) can be written in a shorter form:

$$(\Psi_\tau v_I)^r(x) = \Pi_{r,x} G_I. \tag{1.25}$$

For every set K, we put

$$\begin{gathered} v_K(r,x) = -\log P_{r,\delta_x} \exp\langle -f_K, X_K \rangle, \\ \tau(K) = \inf\{\tau_k : k \in K\}, \qquad \lambda(K) = \min\{k : \tau_k = \tau(K)\}. \end{gathered} \tag{1.26}$$

By (1.12), (1.13) and (1.25), v_K satisfies the equation

$$\Psi_{\tau(K)} v_K(r,x) = \Pi_{r,x}\left[f_{\lambda(K)} + v_{K - \lambda(K)}\left(\tau_{\lambda(K)}, \xi_{\tau_{\lambda(K)}}\right)\right] \tag{1.27}$$

and, for every μ,

$$P_\mu \exp\langle -f_K, X_K \rangle = \exp\langle -v_K, \mu \rangle. \tag{1.28}$$

We prove (1.23) by induction on $|I|$. First, suppose that $I = \varnothing$ and take $K = \{0\} \cup J$. Clearly, $\lambda(K) = 0$ and $\tau(K) = \tau$. Note that

$$P_\mu Y P_{X_\tau} Z = P_\mu \exp\langle -\tilde{f}, X_\tau \rangle = \exp\langle -\tilde{v}, \mu \rangle, \tag{1.29}$$

where $\tilde{f} = f_0 + v_J$ and

$$\tilde{v}(r,x) = -\log P_{r,\delta_x}\langle -\tilde{f}, X_\tau \rangle$$

satisfies the equation $\Psi_\tau \tilde{v} = \Pi_{r,x} \tilde{f}(\tau, \xi_\tau)$. By (1.27), v_K satisfies the same equation and therefore $v_K = \tilde{v}$.

Now let $I \neq \varnothing$ and let τ_i be the first exit time from $Q_i \subset Q$. Then $\sigma = \tau(I)$ is the first exit time from the intersection U of all Q_i, $i \in I$. By applying the already proved particular case of (1.23) to σ, we get

$$P_\mu YZ = P_\mu P_{X_\sigma} YZ, \qquad P_\mu Y P_{X_\tau} Z = P_\mu X_\sigma\left[Y P_{X_\tau} Z \right]. \tag{1.30}$$

Consider disjoint Borel sets $\Gamma_i \subset Q_i^c$ such that U^c is the union of Γ_i, and denote by N_i the restriction of X_σ to Γ_i. By (1.12), 1.9.B and 1.9.C,

$$P_{X_\sigma} YZ = \prod_{i \in I} \{\exp\langle -f_i, N_i\rangle P_{N_i} \exp[-\langle f_{I-i}, X_{I-i}\rangle - \langle f_0, X_\tau\rangle] Z\},$$

$$P_{X_\sigma}[YP_{X_\tau}Z] = \prod_{i \in I} \{\exp\langle -f_i, N_i\rangle P_{N_i} \exp[-\langle f_{I-i}, X_{I-i}\rangle - \langle f_0, X_\tau\rangle] P_{X_\tau}Z\},$$

and (1.23) follows from (1.30) and the induction. □

THEOREM 1.4. *Let $\tau_1, \dots, \tau_n, \dots \in \mathscr{T}$ and let $\tau_n \uparrow \tau$ or $\tau_n \downarrow \tau$. If a sequence $f_n \in \mathscr{B}_S$ is uniformly bounded and if $f_n(\tau_n, \xi_{\tau_n}) \to f(\tau, \xi_\tau)$ Π_μ-a.s., then $\langle f_n, X_{\tau_n}\rangle \to \langle f, X_\tau\rangle$ P_μ-a.s.*

This follows from [30]: In the case $\tau_n \uparrow \tau$, we can apply Theorem 4.1; the case $\tau_n \downarrow \tau$ can be covered by a slight modification of the proof of Theorem 3.1 or Theorem 4.1.

1.10. *Markov process $(X_t, P_{r,\nu})$.* To every superprocess X, there corresponds a Markov process $(X_t, P_{r,\nu})$ in the state space $\mathscr{M}(E)$ defined by

$$X_t(B) = X_{\tau_t}[\iota_t(B)], \quad \text{for } B \in \mathscr{B},$$
$$(1.31) \qquad P_{r,\nu} = P_{\iota_t(\nu)}, \quad \text{for } r \in \mathbb{R}_+, \nu \in \mathscr{M}(E),$$

where τ_t is the first exit time from $S_{<t}$ and $\iota_t(x) = (t, x)$ is an imbedding of E into S. By 1.9.A, $X_t = X_{\tau_t}$ $P_{r,\nu}$-a.s. for all $r < t$ and $\nu \in \mathscr{M}(E)$ because $\Pi_{r,x}\{\tau_t = t\} = 1$ for all $r < t$ and $x \in E$. The Markov property of $(X_t, P_{r,\nu})$ follows from Theorem 1.2.

THEOREM 1.5. *There exists a right version of the process $(X_t, P_{r,\nu})$. It can be chosen in such a way that the following hold:*

1.10.A. *The function $P_{t,X_t}Z$ is, P_μ-a.s., right-continuous in t on $[r, u)$ for every $Z \in p\mathscr{F}_{\geq u}$ and every $\mu \in \mathscr{M}(S_{<r})$.*
1.10.B. *$\langle f^t, X_t\rangle$ is, P_μ-a.s., right-continuous on $[r, \infty)$ for all $\mu \in \mathscr{M}(S_{<r})$ and all $f \in b\mathscr{B}_S$ such that $f^t(\xi_t)$ is, Π_μ-a.s., right-continuous on $[r, \infty)$.*

This follows from Theorems 2.1 and 3.1 in [30].

Since $\iota_t(E)$ coincides with $S_t = \{t\} \times E$, we can interpret X_t as the restriction of the measure X_{τ_t} to S_t. With this interpretation, we have the following lemma.

LEMMA 1.1. *For all measurable subsets B of S_t and all $\tau \in \mathscr{T}$,*

$$(1.32) \qquad X_\tau(B) \leq X_t(B) \quad a.s.$$

PROOF. Put $U = X_\tau(B)$ and $V = X_t(B) = X_{\tau_t}(B)$. We claim that for every bounded Borel function f on $\mathbb{R}^2$ and for every $\nu \in \mathscr{M}(S)$,

$$P_\nu f(U,V) = P_{\hat{\nu}} f(U,V),$$

where $\hat{\nu}$ is the restriction of ν to $S_{\le t}$. By the multiplicative system theorem, it is sufficient to prove this for $f(u,v) = e^{-ru-sv}$ with $r, s \ge 0$, in which case the left-hand side is equal to $P_{\hat{\nu}} f(U,V) P_{\nu-\hat{\nu}} f(U,V)$ and $P_{\nu-\hat{\nu}}\{U = V = 0\} = 1$ by 1.9.A. By taking $f(u,v) = 1_{u>v}$, we get $P_\nu\{U > V\} = P_{\hat{\nu}}\{U > V\}$.

Let $\sigma = \tau \wedge \tau_t$. By Theorem 1.3,

$$P_\mu\{U > V\} = P_\mu P_{X_\sigma}\{U > V\} = P_\mu P_{\hat{X}_\sigma}\{U > V\}. \tag{1.33}$$

P_μ-a.s., X_σ is concentrated on $(Q \cap S_{<t})^c$; therefore $\hat{X}_\sigma$ is concentrated on $(Q \cap S_t) \cup (Q^c \cap S_{<t})$ and $P_{\hat{X}_\sigma}\{U > V\} = 0$ by 1.9.A. Clearly, (1.33) implies (1.32). □

1.11. *Parts of a superprocess.* We define two versions $\tilde{X}$ and $\hat{X}$ for a part of X in a finely open set $Q \in \mathscr{B}_S$. Let τ_t be the first exit time from $Q_{<t} = Q \cap S_{<t}$. We denote by $\tilde{X}_t$ the restriction of X_{τ_t} to the t-section Q_t of Q. The family $\tilde{X} = (\tilde{X}_t, P_{r,\nu})$ is a Markov process in $\mathscr{M}(Q_t)$. The process $(X_t, P_{r,\nu})$ of Section 1.10 can be considered as the part of X in $Q = S$. Theorems 2.1 and 3.1 in [30] imply the following generalization of Theorem 1.5.

THEOREM 1.6. *There exists a right version of* $\tilde{X}$ *which satisfies the following conditions*:

1.11.A. *The function* $P_{t,\tilde{X}_t} Z$ *is*, P_μ-*a.s., right-continuous in* t *on* $[r,u)$ *for every* $Z \in p\mathscr{F}^{\tilde{X}}_{\le u}$ *and every* $\mu \in \mathscr{M}(Q_{<r})$.

1.11.B $\langle f^t, \tilde{X}_t \rangle$ *is*, P_μ-*a.s., right-continuous on* $[r,\infty)$ *for all* $\mu \in \mathscr{M}(Q_{<r})$ *and all* $f \in b\mathscr{B}_Q$ *such that* $f^t(\xi_t)$ *is*, Π_μ-*a.s., right-continuous on* $[r,\tau)$.

REMARK. Analogous results have been proved in [30] for more general increasing right continuous families τ_t.

The second version of the part of X is defined in the next theorem.

THEOREM 1.7. *There is an* $\mathscr{M}(S)$-*valued process* $\hat{X}_t$ *such that, for every* $\mu \in \mathscr{M}(S)$,

$$P_\mu\{\hat{X}_t = \tilde{X}_t\} = 1 \tag{1.34}$$

and $\langle f^t, \hat{X}_t \rangle$ *is*, P_μ-*a.s., right-continuous on* $\mathbb{R}_+$ *if* f *is a bounded Borel function on* $\mathbb{R}_+ \times S$ *and if* $f^t(t \wedge \tau, \xi_{t\wedge\tau})$ *is*, Π_μ-*a.s., right-continuous on* $\mathbb{R}_+$. (*Here* τ *is the first exit time from* Q.)

(Cf. Theorem 5.2 in [30].)

Clearly, any two versions of $\hat{X}_t$ are P_μ-indistinguishable for each $\mu \in \mathscr{M}(S)$, and every version of $\tilde{X}_t$ is P_μ-indistinguishable from $\hat{X}_t$ on $[r, \infty)$ if $\mu \in \mathscr{M}(Q_{<r})$. Moreover, for arbitrary $\rho \in p\mathscr{B}_S$ and $\mu \in \mathscr{M}(S)$, $\langle \rho^t, \hat{X}_t \rangle$ is P_μ-indistinguishable from a $\mathscr{B} \times \mathscr{F}$-measurable function and therefore, for every measure γ on $\mathbb{R}_+$, the integral

$$\int_{\mathbb{R}_+} \langle \rho^t, \hat{X}_t \rangle \gamma(dt)$$

is defined up to P_μ-equivalence. Besides,

$$(1.35)\quad \int_r^\infty \langle \rho^t, \hat{X}_t \rangle \gamma(dt) = \int_r^\infty \langle \tilde{\rho}^t, \tilde{X}_t \rangle \gamma(dt) \quad P_\mu\text{-a.s. for all } \mu \in \mathscr{M}(Q_{<r}),$$

where $\tilde{\rho}^t(x) = \rho^t(t, x)$.

THEOREM 1.8. *Suppose that $\psi(z_1) \le \psi(z_2)$ if $z_1 \le z_2$. For all $\mu \in \mathscr{M}(S)$ and all $f, \rho \in p\mathscr{B}_S$,*

$$(1.36)\quad P_\mu \exp\left\{\int_{\mathbb{R}_+} \langle -\rho^t, \hat{X}_t \rangle \gamma(dt) - \langle f, X_\tau \rangle \right\} = \exp\langle -v, \mu \rangle,$$

where

$$(1.37)\quad \begin{aligned} v(r, x) &+ \Pi_{r,x} \int_\alpha^\tau \psi(v)(s, \xi_s)\, dK_s \\ &= \Pi_{r,x}\left[\int_\alpha^\tau \tilde{\rho}^t(\xi_t) \gamma(dt) + f(\tau, \xi_\tau)\right]. \end{aligned}$$

PROOF. We can pass to the limit in (1.36) and (1.37) along any monotone increasing sequences γ_n and ρ_n. Therefore it is sufficient to prove the theorem for γ concentrated on a finite interval $[a, b]$ and for bounded ρ.

STEP 1. Let γ be concentrated on a finite set $\{t_1 < \cdots < t_n\}$. Then, by (1.34),

$$\int_{\mathbb{R}} \langle \rho^t, \hat{X}_t \rangle \gamma(dt) = \sum_1^n \langle f_i, X_{\tau_i} \rangle \quad \text{a.s.},$$

where $f_i = 1_Q \gamma\{t_i\} \rho^{t_i}$, $\tau_i = \tau_{t_i}$. The equations (1.36) and (1.37) follow from (1.12) and (1.13).

STEP 2. Let ρ be continuous on $[a, b] \times (Q \cup \partial Q)$. Take a partition $\Lambda = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and put

$$Y = \int_a^b \langle \rho^t, \hat{X}_t \rangle \gamma(dt) + \langle f, X_\tau \rangle, \qquad Y_\Lambda = \sum_1^n \langle \rho^{t_i}, \hat{X}_{t_i} \rangle \gamma(t_{i-1}, t_i] + \langle f, X_\tau \rangle.$$

By Step 1,

$$(1.38)\quad P_\mu \exp(-Y_\Lambda) = \exp\langle -v_\Lambda, \mu \rangle,$$

where

$$v_\Lambda(r,x) = -\log P_{r,\delta_x} \exp(-Y_\Lambda)$$

satisfies

$$(1.39)\qquad \begin{aligned} v_\Lambda(r,x) &+ \Pi_{r,x} \int_\alpha^\tau \psi(v_\Lambda)(s,\xi_s)\, dK_s \\ &= \Pi_{r,x}\left[\int_\alpha^\tau \tilde{\rho}^{\beta(t)}(\xi_{\beta(t)})\gamma(dt) + f(\tau,\xi_\tau)\right], \\ \beta(t_i) &= t_i, \quad \text{for } t_{i-1} \le t < t_i. \end{aligned}$$

Suppose $\Lambda_1 \subset \cdots \subset \Lambda_n \subset \cdots$ and the union of Λ_n is everywhere dense in $[a,b]$. Then $Y_{\Lambda_n} \to Y$, $v_{\Lambda_n} \to v$, and we get (1.36) and (1.37) by passing to the limit in (1.38) and (1.39).

STEP 3. It is easy to see that, if (1.36) and (1.37) hold for ρ_n and if $\rho_n \to \rho$ boundedly, then they hold for ρ. It follows from Step 2 that they hold for all bounded ρ. □

1.12. *Historical processes.* The historical process $\hat{\xi}$ for ξ is a process whose state space at time t is the path of ξ before t. The corresponding superprocess $\hat{X}$ (called a *historical superprocess*) will be used in the next section for proving the S property of X.

We start from a Markov process ξ in a metric space E. We assume that Ω^0 consists of all right-continuous paths and that $\xi_t(w) = w_t \in E$ for $t \ge \alpha$, $w \in \Omega^0$. Let $w_{\le t}$ stand for the restriction of w to $[0,t]$. By setting

$$\hat{\xi}_t(w) = w_{\le t},$$

we define a stochastic process with the same Ω^0 and α and with a variable state space $E_{\le t}$. The transition probabilities for $\hat{\xi}$ are defined by the formula

$$(1.40)\qquad \int F(y_{\le t})\hat{\Pi}_{r,x_{\le r}}(dy_{\le t}) = \int F(x_{\le r}, y_{(r,t]})\Pi_{r,x_r}(dy_{(r,t]})$$

(cf. (1.31) in [26]).

The global state space $\hat{S}$ of $\hat{\xi}$ can be identified with the union of $E_{\le t}$ over all $t \in \mathbb{R}_+$. With every set $Q \subset S$ we associate a set $\hat{Q} \subset \hat{S}$ defined by the condition $w_{\le t} \in \hat{Q}$ if, for every $s \in [0,t]$, $(s, w_s) \in Q$ or $w_s = \partial$. If Q is a finely open set for ξ, then $\hat{Q}$ is a finely open set for $\hat{\xi}$. Moreover, if τ is the first exit time of ξ from Q, then $\{t < \tau\} = \{\hat{\xi}_t \in \hat{Q}\}$ and therefore the first exit time of $\hat{\xi}$ from $\hat{Q}$ is also τ.

LEMMA 1.2. *Let C be a finely open set for $\hat{\xi}$ which contains with every path $w_{\le t}$ its restriction to any interval $[0,s]$ with $s < t$. To every partition $\Lambda = \{0 = t_0 < \cdots < t_n\}$ there corresponds a finely open set*

$$C_\Lambda = \bigcup_{i=1}^n \{w_{\le s}: t_{i-1} \le s < t_i,\, w_{\le t_{i-1}} \in C\}$$

which contains C. Let τ and τ_Λ be the first exit times of $\hat{\xi}$ from C and C_Λ. If $\Lambda_1 \subset \cdots \subset \Lambda_n \subset \cdots$ and if the union $\tilde{\Lambda}$ of Λ_n is everywhere dense in $\mathbb{R}_+$, then $\tau_{\Lambda_n} \downarrow \tau$.

PROOF. $C_\Lambda \supset C$ because $\{w_{\le s} \in C\} \subset \{w_{\le t_{i-1}} \in C\} \subset \{w_{\le s} \in C_\Lambda\}$ for $t_{i-1} \le s < t_i$. Hence $\tau_{\Lambda_n} \downarrow \sigma \ge \tau$. On the other hand, for every $s \in \Lambda_n$, $\{w_{\le s} \in C_{\Lambda_n}\} \subset \{w_{\le s} \in C\}$ and therefore, for $s \in \tilde{\Lambda}$, $\{s < \sigma\} \subset \{s < \tau_{\Lambda_n}\} \subset \{w_{\le s} \in C_{\Lambda_n}\} \subset \{w_{\le s} \in C\} \subset \{s < \tau\}$.

1.13. *S property.* This property is stated in the following theorem.

THEOREM 1.9. *Suppose that $f \in bp\mathscr{B}_S$ and that $f(t, \xi_t)$ is, a.s., right-continuous in t. Then, for every $\tau \in \mathscr{T}$,*

$$\{\langle f, X_t\rangle = 0 \text{ for all } t\} \subset \{\langle f, X_\tau\rangle = 0\} \quad a.s. \tag{1.41}$$

PROOF. Let $\hat{\xi}$ be the historical process for ξ. Formula $j(w_{\le t}) = (t, w_t)$ maps the global state space $\hat{S}$ of $\hat{\xi}$ onto the global space S of ξ and it induces mappings from $bp\mathscr{B}_S$ to $bp\mathscr{B}_{\hat{S}}$ and from $\mathscr{M}(\hat{S})$ to $\mathscr{M}(S)$ such that $\langle f, \hat{\mu}\rangle = \langle \hat{f}, \mu\rangle$ and $\hat{f}(t, \hat{\xi}_t) = f(t, \xi_t)$. The superprocess $\hat{X}$ with parameters $(\hat{\xi}, K, \psi)$ is connected with X by the formula $j\hat{X}_t = X_t$ and therefore

$$\Omega_f = \{\langle f, X_t\rangle = 0 \text{ for all } t\} = \left\{\langle \hat{f}, \hat{X}_t\rangle = 0 \text{ for all } t\right\}.$$

Let Q be the finely open set associated with τ. By Theorem 1.4 and Lemma 1.2 with $C = \hat{Q}$, (1.41) will be proved if we show that

$$\left\langle \hat{f}, \hat{X}_{\tau_\Lambda}\right\rangle = 0 \quad \text{a.s. on } \Omega_f, \tag{1.42}$$

for every $\Lambda = \{0 = t_0 < \cdots < t_n\}$. By 1.9.A, $\hat{X}_{\tau_\Lambda}$ is concentrated, a.s., on $\hat{S}_{t_0} \cup \cdots \cup \hat{S}_{t_n}$ and (1.42) follows from Lemma 1.1. □

Only small adjustments are needed to prove the following, more general, result:

THEOREM 1.10. *Let $\tilde{X}$ be the part of a superprocess X in a finely open set Q. Suppose that $f \in bp\mathscr{B}_{\bar{Q}}$ and that $f(t, \xi_t)$ is, a.s., right-continuous in t on $[0, \tau]$, where τ is the first exit time from Q. Then, for every finely open set $U \subset Q$,*

$$\left\{\langle f, \tilde{X}_t\rangle = 0 \text{ for all } t\right\} \subset \{\langle f, X_\sigma\rangle = 0\} \quad a.s., \tag{1.43}$$

where σ is the first exit time from U.

2. Branching particle systems.

2.1. *Branching.* The first stochastic model of branching appeared in 1874 in the problem of the family name extinction posed by Galton and solved by Watson [96]. The branching mechanism is determined here by the probabilities q_n, $n = 0, 1, 2, \ldots$, that a father has n sons.

The next step—branching with several types of particles—was made in 1947 in [54] (an output of a seminar of Kolmogorov held at Moscow University). Here the branching mechanism is defined by the probabilities $q(i; k_1, \dots, k_a)$ for a particle of type i to generate k_1 particles of type $1, \dots, k_a$ particles of type a.

In general, the type of a particle can be described by a point of an arbitrary measurable space $(E, \mathscr{B})$. A particle located at $x \in E$ generates a random number n of particles, with a probability distribution $q_n(x; dy_1, \dots, dy_n)$. An offspring configuration $y_1, \dots, y_n$ can be interpreted as a measure $\nu = \delta_{y_1} + \cdots + \delta_{y_n}$, where δ_y is the Dirac measure at y.

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, and let $\mathscr{M} = \mathscr{M}(E, \mathbb{Z}_+)$ be the space of all $\mathbb{Z}_+$-valued finite measures on E. A probability measure q on $\mathscr{M}$ is defined by a sequence of symmetric measures $q_n(dy_1, \dots, dy_n)$ on $(E^n, \mathscr{B}^n)$ such that $\Sigma q_n(E^n) = 1$. In particular, to every finite measure μ on $(E, \mathscr{B})$ there corresponds the Poisson measure on $\mathscr{M}$ with intensity μ given by

$$q_n(dy_1, \dots, dy_n) = \lambda^n (n!)^{-1} e^{-\lambda} m(dy_1) \cdots m(dy_n), \tag{2.1}$$

with $\lambda = \mu(E)$, $m = \mu/\lambda$.

Denote by $b_1 p\mathscr{B}$ the set of all functions $z \in p\mathscr{B}$ such that $z \le 1$, and put $z^\nu = z(y_1) \cdots z(y_n)$ for $\nu = \delta_{y_1} + \cdots + \delta_{y_n}$. A probability measure q on $\mathscr{M}$ is uniquely determined by the generating function

$$\begin{aligned} \varphi(z) &= \int_{\mathscr{M}} q(d\nu) z^\nu \\ &= \sum_n \int q_n(dy_1, \dots, dy_n) z(y_1) \cdots z(y_n), \qquad z \in b_1 p\mathscr{B}, \end{aligned} \tag{2.2}$$

or by the Laplace transform

$$L(f) = \varphi(e^{-f})(x) = \int q(d\nu) e^{-\langle f, \nu \rangle}, \qquad f \in bp\mathscr{B}. \tag{2.3}$$

For the Poisson measure, we have

$$\begin{aligned} \varphi(z) &= \exp\langle z - 1, \mu \rangle, \\ L(f) &= \exp\langle e^{-f} - 1, \mu \rangle. \end{aligned} \tag{2.4}$$

The general branching mechanism can be described by a stochastic kernel $q(s, x; d\nu)$ from $S = \mathbb{R}_+ \times E$ to $\mathscr{M}$. It defines the probability distribution of offspring given that the act of procreation takes place at time s at point x. We associate with q a transformation in the space $b_1 p\mathscr{B}_S$, given by the formula

$$\begin{aligned} \varphi(z)(s, x) &= \int q(s, x; d\nu) z^\nu \\ &= \sum_n \int q_n(s, x; dy_1, \dots, dy_n) z(s, y_1) \cdots z(s, y_n). \end{aligned} \tag{2.5}$$

In the case of a local branching, the offspring is born at x and $\varphi(z)(s, x) =$

$\varphi[s, x; z(s, x)]$, where

$$\varphi(s, x; z) = \sum p_n(s, x) z^n \tag{2.6}$$

is the generating function for the number of children.

2.2. *Branching particle systems.* A branching particle system is a system of particles moving in the space E, dying and producing at their death times random offspring. The particles alive at time t are indistinguishable. The only interaction between the particles is that the birth time of offspring coincides with death time of their parent. The motion of each particle is described by a Markov process ξ, the branching mechanism by a generating function (2.5) and their death time by a continuous additive functional K of ξ: The probability of surviving during time interval (r, t) and of dying between t and $t + dt$ is equal to $H(r, t)\, dK_t$, where

$$H(r, t) = e^{-K(r,t)}. \tag{2.7}$$

Consider an $\mathscr{M}$-valued stochastic process X_t, where $X_t(B)$ is the number of particles at time t in a set B. Denote by $P_{r,\nu}$ the probability law for X given that the process starts at time r from $\nu \in \mathscr{M}$. We have two fundamental equations: If $f \in p\mathscr{B}$ and $Z_f = \exp\langle -f, X_t\rangle$, then

$$P_{r,\nu_1+\nu_2} Z_f = P_{r,\nu_1} Z_f P_{r,\nu_2} Z_f, \tag{2.8}$$

for all $\nu_1, \nu_2 \in \mathscr{M}$, and

$$\begin{aligned} P_{r,\delta_x} Z_f = \Pi_{r,x}\Big[& H(r,t)\exp(-f(\xi_t)) \\ & + \int_r^t H(r,s)\, dK_s \int_{\mathscr{M}} q(s, \xi_s; d\nu) P_{s,\nu} Z_f \Big]. \end{aligned} \tag{2.9}$$

Equation (2.8) follows from the independence of the evolution of any two parts of the population living at time r. In (2.9), we deal with a process started at time r by one particle located at x: The first term corresponds to the case when this particle is still alive at time t; the second term corresponds to its death at time $s \in (r, t)$.

Put

$$h(r, x) = -\log P_{r,\delta_x} \exp\langle -f, X_t\rangle. \tag{2.10}$$

Clearly, (2.8) and (2.9) imply that

$$P_{r,\nu} \exp\langle -f, X_t\rangle = \exp\left\{-\int h(r,x)\nu(dx)\right\} \tag{2.11}$$

and

$$e^{-h(r,x)} = \Pi_{r,x}\left[H(r,t)\exp[-f(\xi_t)] + \int_r^t H(r,s)\, dK_s\, \varphi(e^{-h})(s, \xi_s)\right]. \tag{2.12}$$

2.3. *Enriched model.* $X = (X_t, P_{r,\nu})$ is a Markov process in $\mathscr{M}$ in the sense of Section 1.1. We enrich the model by expanding the family of random measures X_t, $t \in \mathbb{R}_+$, to a family X_τ, $\tau \in \mathscr{T}$, and the class of probability measures $P_{r,\nu}$, $r \in \mathbb{R}_+$, $\nu \in \mathscr{M}(E, \mathbb{Z}_+)$, to a class P_η, $\eta \in \mathscr{M}(S, \mathbb{Z}_+)$.

Every $\eta \in \mathscr{M}(S, \mathbb{Z}_+)$ has the form $\eta = \Sigma \delta_{r_i, x_i}$. The corresponding P_η is the probability law of a particle system initiated by a finite number of "immigrants": (r_i, x_i) is the time and the place of entry for the immigrant i.

For every particle a, we trace backward its historical path (s, w_s), which consists of the path of a and all its ancestors starting from an immigrant i. Consider all historical paths which exit from Q and identify those that coincide till the first exit time τ from Q. The ends of these paths form a configuration (t_j, y_j), $j = 1, 2, \dots,$ in S. We put

$$X_\tau = \sum \delta_{t_j, y_j}.$$

Note that $X_\alpha = \eta$ P_η-a.s. (α is the first exit time from $Q = \varnothing$).

The same arguments as in Sections 2.2 lead to the equations

$$P_\eta \exp\langle -f, X_\tau \rangle = \exp\langle -h, \eta \rangle, \tag{2.13}$$

$$e^{-h(r,x)} = \Pi_{r,x}\Big[H(\alpha, \tau) \exp[-f(\xi_\tau)] + \int_\alpha^\tau H(r,s)\, dK_s\, \varphi(e^{-h})(s, \xi_s) \Big]. \tag{2.14}$$

Using the notation of Section 1.6, we can describe the joint probability distribution of $X_{\tau_1}, \dots, X_{\tau_n}$ by

$$P_\mu \exp[-\langle f_I, X_I \rangle] = \exp\langle -h_I, \mu \rangle, \tag{2.15}$$

where

$$h_I^r(x) = -\log P_{r,\delta_x} \exp[-\langle f_I, X_I \rangle] \tag{2.15a}$$

is determined recursively by the equations

$$\begin{aligned} \exp[-h_I(r,x)] &= \Pi_{r,x}\Big\{ H(\alpha, \tau_I) F \\ &\quad + \int_\alpha^{\tau_I} H(r,s)\, dK_s \varphi[\exp(-h_I)](s, \xi_s) \Big\}, \\ F &= \exp\{-(f_\lambda + h_{I-\lambda})(\tau_I, \xi_{\tau_I})\}. \end{aligned} \tag{2.16}$$

(By definition, $h_\varnothing = 0$.)

2.4. *Poisson initial data.* To every measure q on $\mathscr{M}(S, \mathbb{Z}_+)$ there corresponds a measure

$$P_q = \int q(d\eta) P_\eta.$$

If q_μ is the Poisson measure with intensity μ, then

$$P_{q_\mu} \exp[-\langle f_I, X_I \rangle] = \exp\langle -u_I, \mu \rangle, \tag{2.17}$$

where $u_I = 1 - \exp(-h_I)$ satisfies

$$u_I^r(x) + \Pi_{r,x} \int_\alpha^{\tau_I} \tilde{\varphi}(u_I)(s, \xi_s)\, dK_s = \Pi_{r,x} G_I. \tag{2.18}$$

Here

$$\tilde{\varphi}(z) = \varphi(1 - z) - 1 + z,$$
$$G_I = [1 + (u_{I-\lambda} - 1)\exp(-f_\lambda)](\tau_\lambda, \xi_{\tau_\lambda})$$

(we set $u_I = 0$ for $I = \varnothing$).

Formulae (2.17) and (2.18) follow from (2.15) and (2.16) and Lemma A.1.5 (cf. proof of formulae (1.9) and (1.11) in [24]).

3. Superprocesses as the limit of branching particle systems.

3.1. *Heuristic passage to the limit.* If the mass of each particle is β, then the mass distribution βX_τ belongs to $\mathscr{M}(S, \beta\mathbb{Z}_+)$. To get a superprocess with parameters (ξ, K, ψ), we replace the initial intensity μ by μ/β, the functional K by $\sigma_\beta K$ for some constant $\sigma_\beta > 0$, the generating function φ by φ_β and we pass to the limit as $\beta \to 0$. It follows from (2.17) and (2.18) that

$$P_{q_{\mu/\beta}}[-\beta\langle f_I, X_I \rangle] = \exp\langle -v_I^\beta, \mu \rangle, \tag{3.1}$$

$$v_I^\beta(r, x) + \Pi_{r,x} \int_\alpha^\tau \psi_\beta(v_I^\beta)(s, \xi_s)\, dK_s = \Pi_{r,x} G_I^\beta, \tag{3.2}$$

where $v_I^\beta = u_I/\beta$,

$$G_I^\beta = \left[\frac{1}{\beta}(1 - \exp(-\beta f_\lambda)) + \exp(-\beta f_\lambda) v_{I-\lambda}^\beta\right](\tau_\lambda, \xi_{\tau_\lambda}) \tag{3.3}$$

and

$$\begin{aligned} \psi_\beta(z)(s, x) &= \beta^{-1}\sigma_\beta \tilde{\varphi}_\beta(\beta z)(s, x) \\ &= \frac{\sigma_\beta}{\beta}\left[\varphi_\beta(1 - \beta z) - 1 + \beta z\right], \quad \text{for } 0 \le \beta z \le 1. \end{aligned} \tag{3.4}$$

Suppose that $\psi_\beta(z) \to \psi(z)$ as $\beta \to 0$. Then by passing to the limit in (3.1), (3.2) and (3.3), we get equations (1.12), (1.13) and (1.14).

3.2. *Rigorous results.* Some restrictions are needed to justify the passage to the limit in Section 3.1. It is convenient to state them in terms of the operators $\psi_\beta^s\colon \mathscr{B} \to \mathscr{B}$ connected with the functions (3.4) by

$$\psi_\beta(z)(s, x) = \psi_\beta^s(z^s)(x), \quad \text{for } z^s(x) = z(s, x) \tag{3.5}$$

(cf. Section 1.5). We introduce into $b\mathscr{B}$ the uniform norm $\|z\| = \sup_x |z(x)|$.

THEOREM 3.1. *Suppose that* ξ *is right and* K *satisfies condition* 1.4.A. *The passage to the limit described in Section* 3.1 *is legitimate if the operators* ψ_β^s *given by* (3.4) *and* (3.5) *and the mappings* ψ^s *from* $bp\mathscr{B}$ *to* $b\mathscr{B}$ *satisfy, on every finite interval* Δ, *the following assumptions*:

3.2.A. *There is a function* $A(c)$ *such that, for all* $z, \tilde{z} \in b_c p\mathscr{B}$ *and all* s,

$$\|\psi^s(z) - \psi^s(\tilde{z})\| \le A(c)\|z - \tilde{z}\|.$$

3.2.B. *There is a function* $\alpha(\beta, c)$ *such that, for every* $c \ge 0$, $\alpha(\beta, c) \to 0$ *as* $\beta \downarrow 0$ *and*

$$\|\psi_\beta(z) - \psi(z)\| \le \alpha(\beta, c),$$

for all $z \in b_c p\mathscr{B}$.

3.2.C. *There is a constant* C *such that*

$$\psi_\beta(z) + C(\|z\| + 1) \ge 0,$$

for all $\beta > 0$ *and* z *subject to the condition* $0 \le \beta z \le 1$.

More precisely, if $f_i \in \mathscr{H}$ *for all* $i \in I$, *then the solutions* v_f^β *of* (3.2) *and* (3.3) *converge, as* $\beta \to 0$, *to the unique solutions of* (1.13) *and* (1.14), *and the Laplace transforms* (3.1) *converge to the Laplace transforms* (1.12).

Theorem 3.1 is proved in Section A.3.1.

3.3. *Interpretation of* ψ *in terms of branching particle systems*. Possible values for the branching parameter ψ of a superprocess are described in Section 1.5. To understand the heuristic meaning of the coefficients which appear in formulas (1.15), (1.17) and (1.19), we investigate how various values of ψ can be obtained by the limiting procedure in Theorem 3.1.

We say that constants σ_β and generating functions φ_β form an *approximating family for* ψ if ψ_β defined by (3.4) and ψ satisfy conditions 3.2.A–3.2.C. It is easy to check that, if (σ^i, φ^i) is an approximating family for ψ^i, $i = 1, \dots, k$, and if $c^1, \dots, c^k$ are positive constants, then

$$\sigma = \sum c^i \sigma^i, \qquad \varphi = \frac{1}{\sigma} \sum c^i \sigma^i \varphi^i$$

is an approximating family for $\psi = \Sigma c^i \psi^i$.

This remark allows us to construct an approximating family for any ψ of the form (1.17) with time-independent coefficients subject to conditions 1.7.A–1.7.C, starting from a few examples (Examples 3.3.A–3.3.E). (This will be used in the proof of Theorem A.3.2.)

EXAMPLE 3.3.A. The functions

$$(3.6) \quad \sigma = 1, \qquad \varphi(z)(x) = \int_E \exp[z(y) - 1]\gamma(x, dy) + 1 - \gamma(x, E)$$

(independent of β) form an approximating family for

$$\psi(z)(x) = z(x) - \int \gamma(x, dy) z(y), \tag{3.7}$$

assuming that $\gamma(x, E) \le 1$ for all x.

This is clear from the formula

$$\psi_\beta(z)(x) = z(x) - \frac{1}{\beta} \int_E [1 - e^{-\beta z(y)}] \gamma(x, dy).$$

EXAMPLE 3.3.B. For

$$\psi(z)(x) = a(x) z(x), \tag{3.8}$$

with $|a| \le 1$, an approximating family is given by

$$\sigma_\beta = \frac{1}{\beta}, \qquad \varphi_\beta(z) = z + \beta a(1 - z) + \beta(1 - z)^2. \tag{3.9}$$

Indeed, φ_β with $0 < \beta \le \frac{1}{3}$ is a generating function and $\psi_\beta(z) = \psi(z) + \beta z^2$.

Note that, in Examples 3.3.A and 3.3.B, ψ is a linear operator and therefore (1.9) has a solution of the form

$$v(r, x) = \int v(r, x; dt, dy) f(t, y),$$

where $v(r, x; B)$ is the solution corresponding to $f = 1_B$. Since the Laplace functional determines a random measure up to equivalence, we conclude from (1.8) that, P_μ-a.s.,

$$X_\tau(B) = \int \mu(dr, dx) v(r, x; B), \quad \text{for all } B.$$

EXAMPLE 3.3.C. The functions

$$\sigma_\beta = \beta, \qquad \varphi_\beta(z)(x) = \int (\exp(\langle z - 1, \eta \rangle / \beta) - 1) m(x, d\eta) + 1 \tag{3.10}$$

provide an approximating family for

$$\psi(z) = \int (\exp(-\langle z, \eta \rangle) - 1) m(x, d\eta), \tag{3.11}$$

assuming that $m(x, d\eta)$ is a kernel from E to $\mathscr{M}$ such that $m(x, \mathscr{M}) \le 1$ and m satisfies 1.7.B.

Indeed, $\psi_\beta(z) = \beta z + \psi(z)$ and, by 1.7.B, ψ satisfies 3.2.A.

EXAMPLE 3.3.D. Suppose that ψ is given by (3.11), and let $\int |\eta| m(x, d\eta) \le 1$ and $\int 1_{|\eta| < \beta} |\eta| m(x, d\eta) \to 0$ uniformly as $\beta \to 0$. Then the formula

$$\sigma_\beta(x) = \beta^{1/2},$$

$$\varphi_\beta(z) = 1 + \beta^{1/2} \int_{\mathscr{M}_\beta} (\exp(\langle z - 1, \eta \rangle / \beta) - 1) m(x, d\eta), \tag{3.12}$$

with $\mathscr{M}_\beta = \{|\eta|^2 > \beta\}$, gives an approximating family for ψ.

Indeed, (3.12) is a generating function since $m(x,\mathscr{M}_\beta) \le \beta^{-1/2}$ by Chebyshev's inequality. Clearly,

$$\psi_\beta(z) = \beta^{1/2}z + \int_{\mathscr{M}_\beta}(\exp(-\langle z,\eta\rangle) - 1)m(x,d\eta)$$

satisfies 3.2.B and 3.2.C, and ψ satisfies 3.2.A.

The offspring distributions corresponding to φ_β in Examples 3.3.A, 3.3.C and 3.3.D are the mixtures of an atom concentrated at 0 and Poisson measures with various intensities η. In particular, in the case of Example 3.3.D, the intensities η are weighted according to the measure $\beta^{1/2}m(x,\beta\, d\eta)$ restricted to $\beta^{-1}\mathscr{M}_\beta$.

EXAMPLE 3.3.E. Put

$$\psi(z)(x) = b(x)z(x)^2 + \int_0^1 [e^{-uz(x)} - 1 + uz(x)]n(x,du). \tag{3.13}$$

If $0 \le b(x)$ and $2b(x) + \int_0^1 u^2 n(x,du) \le 1$, then an approximating family for ψ can be obtained by the formula

$$\sigma_\beta = \frac{1}{\beta}, \qquad \varphi_\beta(z) = z + \beta^2\psi\left(\frac{1-z}{\beta}\right). \tag{3.14}$$

Indeed, $\psi_\beta = \psi$ for all β, which implies 3.2.B. Conditions 3.2.A and 3.2.C follow from the inequality

$$0 \le e^{-\alpha} - 1 + \alpha \le \tfrac{1}{2}\alpha^2, \quad \text{for all } \alpha \ge 0. \tag{3.15}$$

It remains to show that φ_β is a generating function. We have

$$\varphi_\beta(z)(x) = \sum_0^\infty p_k^\beta(x)z(x)^k,$$

where

$$p_0^\beta(x) = \beta^2\psi\left(\frac{1}{\beta}\right)(x), \qquad p_1^\beta(x) = 1 - b - \beta\int_0^1(1 - e^{-u/\beta})un(x,du),$$

$$p_k^\beta(x) = \frac{\beta^2}{k!}\int_0^1\left(\frac{u}{\beta}\right)^k e^{-u/\beta}n(x,du), \quad \text{for } k \ge 2.$$

Clearly, $\varphi_\beta(1) = 1$ and $p_k \ge 0$ for all k.

Appendix to Part I. Construction of superprocesses. Section 1 of the Appendix contains miscellaneous results which are used in the rest of the Appendix and in the main part of the article. The existence of a superprocess with parameters (ξ, K, ψ) is proved in Sections 2 and 3 by two different methods. In Section 2, we solve the integral equations (I.1.9) and (I.1.13), and we prove that the right-hand sides in (I.1.12) are the Laplace transforms of a compatible family of probability measures. In Section 3, we get a superprocess by passing to the limit from branching processes. By combining the results of both sections, we arrive at Theorem 1.1.

1. Miscellanea.

1.1. *Gronwall's lemma.*

LEMMA 1.1. *Suppose that Borel functions $h_n \geq 0$ satisfy, for all $r \in [0, a)$, the following conditions*:

$$h_0(r) \leq N, \tag{1.1}$$

$$h_n(r) \leq p + q\int_r^a h_{n-1}(s)\,ds, \quad \text{for } n = 1, 2, \dots, \tag{1.2}$$

where p, q and N are positive constants. Then

$$h_n(r) \leq pe^{q(a-r)} + \frac{Nq^n(a-r)^n}{n!}. \tag{1.3}$$

In particular, if $h_0 = 0$ or if h_n does not depend on n, then

$$h_n(r) \leq pe^{q(a-r)}. \tag{1.4}$$

PROOF. By induction in n, we get

$$h_n(r) \leq p\sum_{k=0}^{n-1} \frac{q^k(a-r)^k}{k!} + \frac{Nq^n(a-r)^n}{n!}.$$

Clearly, this implies (1.3) and (1.4). □

1.2.

THEOREM 1.1. *Let $(E, \mathscr{B})$ be a measurable space. Suppose that operators Ψ_r^s, $r < s \in [0, a)$ in the space $b\mathscr{B}$ and a measurable function $g^r(x)$ on $[0, a) \times E$ have the following properties*:

1.2.A. *There is a function $A(c)$ such that*

$$\|\Psi_r^s(z) - \Psi_i^s(\tilde{z})\| \leq A(c)\|z - \tilde{z}\|,$$

for all r, s and all $z, \tilde{z} \in b_c\mathscr{B}$.

1.2.B. *There is a constant C such that*

$$\Psi_r^s(z) + C(\|z\| + 1) \geq 0,$$

for all $z \in b\mathscr{B}$ and all r, s, x.

1.2.C. *There is a constant K such that $\|g^r\| \leq K$ for all r.*

Then the equation

$$v^r + \int_r^a \Psi_r^s(v^s)\,ds = g^r, \qquad r \in [0, a), \tag{1.5}$$

has a unique solution. More precisely, if

(1.6)
$$v_0^r = 0,$$
$$v_n^r = g^r - \int_r^a \Psi_r^s(v_{n-1}^s)\, ds, \quad \textit{for } n \geq 1,$$

then, for all $r \in [0, a)$ *and all* n,

(1.7)
$$\|v_n^r\| \leq L$$

and

(1.8)
$$\|v_n^r - v^r\| \leq \varepsilon_n,$$

where L *and* ε_n *depend only on the function* A *and the constants* C, K *and* a, *and* $\varepsilon_n \to 0$ *as* $n \to \infty$.

PROOF. By (1.6), 1.2.B and 1.2.C,

$$\|v_n^r\| \leq K + \int_r^a C(\|v_{n-1}^s\| + 1) ds,$$

and therefore $h_n(r) = \|v_n^r\|$ satisfy (1.1), with $N = 0$, and (1.2), with $p = K + Ca$ and $q = C$. By Lemma 1.1, (1.7) holds with $L = pe^{-qa}$. By (1.6) and 1.2.A,

$$\|v_{n+1}^r - v_n^r\| \leq \tilde{q} \int_r^a \|v_n^s - v_{n-1}^s\|\, ds,$$

where $\tilde{q} = A(L)$ and, by (1.3), $\|v_{n+1}^r - v_n^r\| \leq L\tilde{q}^n a^n / n!$, for all $r \in [0, a)$. Clearly, this implies (1.8). The uniqueness follows easily from Lemma 1.1. □

1.3.

THEOREM 1.2. *Let* 1.2.A *hold for* Ψ_r^s, $r < s \in [0, a)$. *Suppose that the operators* $\Psi_r^s(\beta)$, $r < s \in [0, a)$, $\beta > 0$, *and measurable functions* $g^r(x)$ *and* $g^r(\beta, x)$ *satisfy the following conditions*:

1.3.A. *There is a function* $\alpha(\beta, c)$ *such that* $\alpha(\beta, c) \to 0$ *as* $\beta \downarrow 0$ *and*

$$\|\Psi_r^s(\beta, z) - \Psi_r^s(z)\| \leq \alpha(\beta, c),$$

for all r, s *and* β *and all* $z \in b_c\mathscr{B}$.

1.3.B. *There is* C *such that*

$$\Psi_r^s(\beta, z) + C(\|z\| + 1) \geq 0,$$

for all $z \in b\mathscr{B}$ *and all* r, s, x *and* β.

1.3.C. *There is* K *such that* $|g^r(\beta, x)| \leq K$, *for all* r, x *and* β.

1.3.D. *There exists a function* $\delta(\beta)$ *such that* $\delta(\beta) \to 0$ *as* $\beta \to 0$, *and* $\|g^r(\beta) - g^r\| \leq \delta(\beta)$ *for all* r *and* β.

If

(1.9)
$$v_\beta^r + \int_r^a \Psi_r^s(\beta, v_\beta^s)\, ds = g_\beta^r, \quad \textit{for all } r, \beta,$$

then v_β^r converges, as $\beta \to 0$, to the unique solution of equation (1.5). *More precisely, for all $r \in [0, a)$ and all β,*

$$\|v_\beta^r\| \le L \tag{1.10}$$

and

$$\|v_\beta^r - v^r\| \le \varepsilon(\beta), \tag{1.11}$$

where L and $\varepsilon(\beta)$ depend only on the functions A, α and δ and the constants C, K and a, and $\varepsilon(\beta) \to 0$ as $\beta \to 0$.

PROOF. By 1.3.B and (1.9),

$$\|v_\beta^r\| \le K + \int_r^a C\big(\|v_\beta^s\| + 1\big)\, ds$$

and, by Lemma 1.1, (1.10) holds with $L = [K + Ca]e^{Ca}$. By 1.2.A, 1.3.A and (1.10),

$$\big\|\Psi_r^s(\beta, v_\beta^s) - \Psi^s(\beta', v_{\beta'}^s)\big\| \le A(L)\big\|v_\beta^s - v_{\beta'}^s\big\| + \tilde{p},$$

where $\tilde{p} = \alpha(\beta, \lambda) + \alpha(\beta', L)$. By (1.9) and 1.3.D, $h(r) = \|v_\beta^r - v_{\beta'}^r\|$ satisfies (1.2), with $q = A(L)$ and $p = a\tilde{p} + \delta(\beta) + \delta(\beta')$, and (1.11) follows from (1.4).

1.4. We introduce into the space $bp\mathscr{B}$ the topology of bounded convergence. Every probability P on $\mathscr{M} = \mathscr{M}(E)$ is uniquely determined by its Laplace functional,

$$L_P(f) = \int_{\mathscr{M}} \exp(-\langle f, \nu\rangle) P(d\nu), \qquad f \in bp\mathscr{B}. \tag{1.12}$$

In Lemmas 1.2–1.4 we assume that $(E, \mathscr{B})$ is a Luzin measurable space.

LEMMA 1.2. *Let P_n be probability measures on $\mathscr{M}$. If L_{P_n} converges to a continuous functional L, then L is also a Laplace functional of a probability P on $\mathscr{M}$.*

A proof can be found in, for instance, [23] (see Lemma 2.1 there).

LEMMA 1.3. *Consider $E_n \in \mathscr{B}$ such that $E_n \uparrow E$, and put $\mathscr{M}_n = \mathscr{M}(E_n)$. Suppose that P_n is a probability measure on $\mathscr{M}_n$ and that the Laplace functionals L_n of P_n satisfy the following condition, for all n and all $f \in bp\mathscr{B}$:*

$$L_n(f_n) = L_{n+1}(f_{n+1}), \tag{1.13}$$

where $f_n = 1_{E_n} f$. Then there exists a unique probability measure P on $\mathscr{M}$ such that

$$L_P(f) = L_n(f), \tag{1.14}$$

for all n and all f which vanish outside some E_n.

PROOF. By restricting every measure $\mu \in \mathscr{M}_{n+1}$ to E_n, we define a measurable mapping p_n from $\mathscr{M}_{n+1}$ to $\mathscr{M}_n$. By (1.13), P_n coincides with the image of P_{n+1} under p_n. The probability P is the inverse image of the laws P_n (see [14], Chapter III, Theorem 53). □

1.5. A functional L on $bp\mathscr{B}$ is called *positive definite* if

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j L(f_i + f_j) \geq 0,$$

for every $n = 1, 2, \dots,$ all $f_1, \dots, f_n \in bp\mathscr{B}$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

LEMMA 1.4. *A functional L on $bp\mathscr{B}$ is the Laplace functional of a probability measure P if and only if it is continuous, positive definite, $Lf \geq 0$ for all f and $L(0) = 1$.*

A proof can be found in the Appendix to [35].

1.6.

REMARK. We need a multivariate form of Lemmas 1.2–1.4. Consider a finite number of measurable spaces $(E_i, \mathscr{B}_i)$, $i = 1, \dots, k$, and put $\mathscr{M}_i = \mathscr{M}(E_i)$ and $G_i = \mathscr{B}p\mathscr{B}_i$. To every probability measure P on the product space $\mathfrak{M} = \mathscr{M}_1 \times \cdots \times \mathscr{M}_k$ there corresponds the Laplace functional

$$L_P(f_1, \dots, f_k) = \int_{\mathfrak{M}} \exp \sum \langle -f_i, \nu_i \rangle P(d\nu_1, \dots, d\nu_k), \tag{1.15}$$

$$f_1 \in G_1, \dots, f_k \in G_k.$$

To apply our lemmas to such functionals, it is sufficient to note that the measurable space $\mathfrak{M}$ is isomorphic to $\mathscr{M}(E)$, where E is the union of E_i: To every $\nu \in \mathscr{M}(E)$ there corresponds an element $(\nu_1, \dots, \nu_k)$ of $\mathfrak{M}$, where ν_i is the restriction of ν to E_i. Analogously, $G_1 \times \cdots \times G_k$ can be identified with $bp\mathscr{B}$. After this identification, (1.15) takes the form (1.12).

1.7. Let $\xi = (\xi_t, \Pi_{r,x})$ be a right Markov process, and let τ be a stopping time relative to the filtration $\mathscr{A}_t$ (we use the notation of Section I.1.3). Put $C \in \mathscr{F}^0_{\geq\tau}$ if $A \in \mathscr{F}^0$ and if, for each r, $\{C, \tau > r\} \in \mathscr{F}^0_{>r}$.

LEMMA 1.5. *Suppose that K is an additive functional of ξ given by* (I.1.7), *with $k \in bp\mathscr{B}_S$. Let $w, \rho \in b\mathscr{B}_S$, $F \in b\mathscr{F}^0_{\geq\tau}$ and let w and $\Pi_{r,x}|F|$ be supported by $[0, a)$. Put*

$$H(r,t) = \exp\left\{-\int_r^t \rho(s, \xi_s)\, dK_s\right\}.$$

The equation

$$g(r,x) = \Pi_{r,x}\left[H(\alpha, \tau)F + \int_\alpha^\tau H(\alpha, s) w(s, \xi_s)\, dK_s\right] \tag{1.16}$$

implies

(1.17) $$g(r,x)+\Pi_{r,x}\int_\alpha^\tau(\rho g)(s,\xi_s)\,dK_s=\Pi_{r,x}\left[F+\int_\alpha^\tau w(s,\xi_s)\,dK_s\right].$$

If g is bounded and supported by $[0,a)$, *then* (1.17) *implies* (1.16).

PROOF. Put

$$d\hat{K}_t=\rho(t,\xi_t)\,dK_t,\qquad Y_s=1_{s<\tau}H(s,\tau)F,$$

$$Z_s=1_{s<\tau}\int_s^\tau H(s,u)w(u,\xi_u)\,dK_u.$$

Since $\{s<\tau\}\in\mathscr{A}_s$ and $Y_s,Z_s\in\mathscr{F}^0_{\geq s}$, we have, by I.1.1.C,

$$\Pi_{r,x}1_{s<\tau}(k\rho)(s,\xi_s)(Y_s+Z_s)=\Pi_{r,x}1_{s<\tau}(k\rho)(s,\xi_s)\Pi_{s,\xi_s}(Y_s+Z_s)$$

and, therefore,

(1.18) $$\begin{aligned}\Pi_{r,x}\int_\alpha^\tau g^s(\xi_s)\,d\hat{K}_s&=\Pi_{r,x}\int_\alpha^\tau\Pi_{s,\xi_s}(Y_s+Z_s)\,d\hat{K}_s\\&=\Pi_{r,x}\int_\alpha^\tau(Y_s+Z_s)\,d\hat{K}_s.\end{aligned}$$

Note that

$$\int_r^t H(s,t)\,d\hat{K}_s=1-H(r,t).$$

Therefore,

(1.19) $$\Pi_{r,x}\int_\alpha^\tau Y_s\,d\hat{K}_s=\Pi_{r,x}F[1-H(\alpha,\tau)]$$

and, by Fubini's theorem,

(1.20) $$\Pi_{r,x}\int_\alpha^\tau Z_s\,dK_s=\Pi_{r,x}\int_\alpha^\tau w^u(\xi_u)[1-H(\alpha,u)]\,dK_u.$$

Clearly, (1.17) follows from (1.18)–(1.20) and (1.16).

Suppose that $\tilde{g}$ is bounded, supported by $[0,a)$ and satisfies (1.17), and let g be given by (1.16). Then $h(r)=\|g^r-\tilde{g}^r\|$ is bounded and satisfies the condition

$$h(r)\leq q\int_r^a h(s)\,ds,\quad\text{for } r\in[0,a),$$

where q is an upper bound for $\|\rho^s k^s\|$ on $[0,a)$. By Lemma 1.1, $h=0$. □

1.8.

LEMMA 1.6. *Suppose K is as in Lemma* 1.5, $f\in bp\mathscr{B}_S$ *is supported by an interval* $[0,b]$ *and operators* ψ^s, $s\in[0,b]$, *in* $bp\mathscr{B}$ *satisfy the following condi-*

tions:

1.8.A. $\psi^s(0) = 0$.

1.8.B. $\|\psi^s(z) - \psi^s(\tilde{z})\| \le c\|z - \tilde{z}\|$, *for all* s, z *and* $\tilde{z}$.

1.8.C. $\psi^s(z) = a^s z + R^s(z)$, *where* a^s *are uniformly bounded*, R^s *are monotone increasing* [*i.e.*, $R^s(z_1) \ge R(z_2)$ *if* $z_1 \ge z_2 \ge 0$] *and* $R^s(\lambda)/\lambda \to 0$ *as* $\lambda \downarrow 0$.

If $v_\lambda^r \in bp\mathscr{B}$ *satisfy*

$$v_\lambda^r = 0, \text{ for } r \ge b,$$

$$v_\lambda^r(x) + \Pi_{r,x}\int_\alpha^\tau \psi^s(v_\lambda^s)(\xi_s)\, dK_s = \lambda\Pi_{r,x} f(\tau, x_\tau), \text{ for } r < b, \tag{1.21}$$

then

$$\lim_{\lambda\downarrow 0} \frac{v_\lambda^r(x)}{\lambda} = \Pi_{r,x} H(\alpha, \tau) f(\tau, \xi_\tau), \tag{1.22}$$

where

$$H(r, s) = \exp\left\{-\int_r^s a^t(\xi_t)\, dK_t\right\}.$$

Proof. Without any loss of generality, we can assume that $\tau \le b$. It follows from Lemma 1.1 that v_λ^r is determined uniquely by (1.21). By applying Lemma 1.5 to $\rho = a$, $g = v$ and $w = av - \psi(v)$, we get that (1.21) is satisfied if

$$v_\lambda^r(x) + \Pi_{r,x}\int_\alpha^\tau H(\alpha, s) R^s(v_\lambda^s)(\xi_s)\, dK_s = \lambda\Pi_{r,x} H(\alpha, \tau) f(\tau, \xi_\tau). \tag{1.23}$$

Hence (1.21) implies (1.23). It follows from (1.23) that $\|v_\lambda^r\|/\lambda \le c$, for all r and λ. By 1.8.C, $R^s(v_\lambda^s)/\lambda \le R^s(c\lambda)/\lambda \to 0$ as $\lambda \to 0$, and (1.22) follows from (1.23) by the dominated convergence theorem. □

2. Direct construction.

2.1. A real-valued function v on an Abelian semigroup G is called *negative definite* if

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j v(g_i + g_j) \le 0, \tag{2.1}$$

for every $n \ge 2$, all $g_1, \dots, g_n \in G$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\Sigma\lambda_i = 0$. It is known (see [3], page 74) that the following holds:

2.1.A. If v is negative definite, then $L(f) = e^{-v(f)}$ is positive definite.

We deal with the case when $G = bp\mathscr{B}$. We also consider $G = G_1 \times \cdots \times G_m$, where $G_i = bp\mathscr{B}_i$, but this more general case can be reduced to the previous one by identifying G with the space of bounded positive measurable functions on the union E of E_i (cf. Section 1.6.). Denote by $\mathfrak{N}(G)$ the class of all negative definite functions v such that $v(g) \geq 0$ for all g and $v(0) = 0$. Clearly, it contains all positive additive functions and it is closed under addition and pointwise convergence. We also use the following property:

2.1.B. If $v_u \in \mathfrak{N}(G)$ for every u in a measure space $(U, \mathscr{U}, \eta)$ and if $v_u(g)$ is η-integrable in u for each g, then

$$v(g) = \int v_u(g)\eta(du)$$

also belongs to $\mathfrak{N}(G)$.

2.2. We denote by $\mathfrak{N}_E(G)$ the set of all mappings v from G to $bp\mathscr{B}$ such that $v(g)(x)$ belongs to $\mathfrak{N}(G)$, for every $x \in E$.

THEOREM 2.1. *Suppose that ξ is a right Markov process in $(E, \mathscr{B})$, $k \in bp\mathscr{B}_S$ and ψ^s, $s \in \mathbb{R}_+$, are operators from $bp\mathscr{B}$ to $\mathscr{B}$ such that the following hold*:

2.2.A. *If $v \in \mathfrak{N}_E(G)$, then $-\psi^s[v(g)](x)$ is negative definite in g, for every $x \in E$.*

2.2.B. *For every s, $\psi^s(0) = 0$ and ψ^s is continuous relative to bounded convergence.*

Suppose that, in addition, the following conditions hold on every finite interval Δ:

2.2.C. $\psi^s(z) \leq \lambda z$, *for all $s \in \Delta$ and all $z \in bp\mathscr{B}$.*

2.2.D. $\|\psi^s(z) - \psi^s(\tilde{z})\| \leq A(c)\|z - \tilde{z}\|$, *for all $s \in \Delta$ and all $z, \tilde{z} \in b_c p\mathscr{B}$.*

2.2.E. $\psi^s(z) + C(\|z\| + 1) \geq 0$, *for all $s \in \Delta$ and all $z \in bp\mathscr{B}$.*

Then there exists a unique (up to equivalence) superprocess with parameters (ξ, K, ψ).

PROOF. We continue ψ^s to $b\mathscr{B}$, preserving conditions 2.2.B, 2.2.D and 2.2.E by setting $\psi^s(z) = \psi^s(|z|)$. We fix a and restrict ourselves to functions f_i supported by the interval $[0, a)$.

Let λ be a positive constant. By Lemma 1.5 (applied to $\rho = \lambda$ and $g = v_I$), equation (I.1.13) follows from

$$v_I^r(x) = \Pi_{r,x} H_\lambda(\alpha, \tau_I) G_I + \Pi_{r,x} \int_\alpha^{\tau_I} H_\lambda(\alpha, s) \Phi^s(v_I^s)(\xi_s)\, dK_s, \tag{2.2}$$

where $H_\lambda(r, s) = e^{-\lambda K(r,s)}$ and $\Phi^s(z) = \lambda z - \psi^s(z)$ if we show that G_I is bounded and v_I^r is a bounded solution of (2.2).

Define a sequence $v_{I,n}$ by the recursive formula

$$v_{I,0} = 0,$$

$$v_{I,n}^r(x) = \Pi_{r,x} H_\lambda(\alpha, \tau_I) G_I + \Pi_{r,x} \int_\alpha^{\tau_I} H_\lambda(\alpha, s) \Phi^s(v_{I,n-1}^s)(\xi_s)\, dK_s. \tag{2.3}$$

We shall prove that there exist constants K_m and L_m and functions $\varepsilon_n(m) \to 0$ as $n \to \infty$ such that, if $|I| = m$ and $\|f_i\| \le c$, for $i = 1, \dots, m$, then

$$|G_I| \le K_m, \tag{2.4}$$

$$\|v_I^r\| \le L_m, \quad \text{for all } r \in [0, a) \tag{2.5}$$

and

$$\|v_{I,n}^r - v_I^r\| \le \varepsilon_n(m), \quad \text{for all } r \in [0, a) \text{ and all } n. \tag{2.6}$$

Clearly, $v_{I,n}$ are supported by $[0, a)$ and they satisfy (1.6) with

$$\Psi_r^s(z) = -\Pi_{r,x} 1_{s<\tau_I} H_\lambda(a, s)\Phi^s(z)(\xi_s)k(s, \xi_s),$$

$$g^r(x) = \Pi_{r,x} H_\lambda(a, \tau_I) G_I.$$

Conditions 1.2.A and 1.2.B follow from 2.2.D and 2.2.E.

Note that (2.4) and 1.2.C hold for $m = 1$. Therefore (2.5) and (2.6) for $m = 1$ follow from (1.7) and (1.8). Suppose that (2.4)–(2.6) are true for $m - 1$. Then (2.4) holds for m, with $K_m = c + L_{m-1}$. Hence 1.2.C holds with $K = K_m$ and, by (1.7) and (1.8), conditions (2.5) and (2.6) are satisfied for m. Clearly, (2.2) follows from (2.3)–(2.6).

Let us show that G_I and v_I are positive if $f_i \ge 0$, for all $i \in I$. Take λ to satisfy 2.2.C on $\Delta = [0, a)$. If $G_{I-i} \ge 0$, for all $i \in I$, then $G_I \ge 0$ and $v_{I,n} \ge 0$ for all n, by (2.3) and 2.2.C. It follows from (2.6) that $v_I \ge 0$ on $[0, a)$.

Put $\mathscr{B}_a = \mathscr{B}_{[0,a)} \times \mathscr{B}$. Let $I = \{1, \dots, m\}$. We consider G_I as a function of $(f_1, \dots, f_m) \in (bp\mathscr{B}_a)^m$ with values in $b\mathscr{F}^0$. Analogously, we interpret v_I as a function from $(bp\mathscr{B}_a)^m$ to $b\mathscr{B}_a$. We prove, by induction on m, that these functions are negative definite. If our statement is true for $m - 1$ and if $|I| = m$, then, obviously, G_I is negative definite. If $v_{I,n-1}$ is negative definite, then, by (2.3), 2.2.A and 2.1.B, $v_{I,n}$ is negative definite. By (2.6), v_I has the same property.

In the same way, by using 2.2.B, we prove that v_i are continuous relative to the bounded convergence in $(bp\mathscr{B}_a)^m$ and vanish for $f_1 = \cdots = f_m = 0$.

Hence,

$$L_I(f_1, \dots, f_m) = \exp\langle -v_I, \mu\rangle \tag{2.7}$$

satisfies all conditions of Lemma 1.4 and, by this Lemma, there exists a unique probability measure on $\mathscr{M}(S_\Delta)^m$ with Laplace transform given by (2.7). By Lemma 1.3 and Section 1.6, there exists a unique probability measure P_μ on $\mathscr{M}(S)^m$ such that (I.1.12) and (I.1.13) hold for all $f_i \in \mathscr{H}$. It is easy to see that $L_I(f_1, \dots, f_m) = L_J(f_1, \dots, f_{m-1})$ if $J = \{1, \dots, m-1\}$ and $f_m = 0$. Therefore the existence of the stochastic process (X_τ, P_μ) subject to conditions I.1.6.A and I.1.6.B follows from Kolmogorov's theorem.

2.3.

THEOREM 2.2. *The conclusion of Theorem* 2.1 *is true if there exist operators* ψ_β^s, $\beta > 0$, *which satisfy conditions* 2.2.A *through* 2.2.D *and such that, for* *every finite interval* Δ *the following hold*:

2.3.A. $\|\psi_\beta^s(z) - \psi^s(z)\| \le \alpha(\beta, c)$, *for all* $s \in \Delta$, $\beta > 0$, $0 \le z \le c$ *with* $\alpha(\beta, c) \to 0$ *as* $\beta \to 0$.

2.3.B. $\psi_\beta^s(z) + C(\|z\| + 1) \ge 0$, *for all* $s \in \Delta$, $\beta > 0$, $z \in B_+(E)$.

PROOF. By Theorem 2.1, there exist stochastic processes (X_τ^β, P_μ) which satisfy I.1.6.A and I.1.6.B with ψ replaced by ψ_β. By Theorem 1.2, the corresponding solutions of (I.1.9) and (I.1.13) converge to the solutions v_I corresponding to ψ. It follows from Lemma 1.2 that $\exp\langle -v_I, \mu\rangle$ are the Laplace transforms of a compatible family of finite-dimensional distributions of a stochastic process (X_τ, P_μ). □

2.4.

THEOREM 2.3. *Suppose that* ξ *is right*, K *satisfies conditions* I.1.4.A *and* ψ *is given by formula* (I.1.17) *with* a, b, γ, m *and* n *subject to conditions* I.1.7.A, I.1.7.B *and* I.1.7C′. *Then there exists a unique* (*up to equivalence*) *superprocess with parameters* (ξ, K, ψ).

PROOF. To simplify notation, we consider only the case when a, b, γ, m and n do not depend on time. For $z \in bp\mathscr{B}$,

$$0 \le \int_{\mathscr{M}} (1 - \exp(-\langle z, \eta\rangle)) m(x, d\eta) \le (\|z\| + 1) \int 1 \wedge |\eta| m(x, d\eta) \tag{2.8}$$

and

$$0 \le \int_0^1 [\exp(-uz(x)) - 1 + uz(x)] n(x, du) \le \tfrac{1}{2} z(x)^2 \int_0^1 u^2 n(x, du), \tag{2.9}$$

and, therefore, $\psi(z) \in b\mathscr{B}$. Put

$$\tilde{\psi}(z)(x) = a(x) z(x) - \int_E \gamma(x, dy) z(y) + \int_{\mathscr{M}} (\exp(-\langle z, \eta\rangle) - 1) m(x, d\eta)$$

and note that

$$\psi(z)(x) = \tilde{\psi}(z)(x) + \int_{[0,1]} F_z(x, u) N(x, du),$$

where

$$F_z(x, u) = \begin{cases} \left[e^{-uz(x)} - 1 + uz(x)\right] u^{-2}, & \text{for } u > 0, \\ \tfrac{1}{2} z(x)^2, & \text{for } u = 0, \end{cases}$$

and

$$N(x, du) = u^2 n(x, du) \quad \text{on } (0, 1], \qquad N(x, \{0\}) = 2b(x).$$

Denote by N_β the measure concentrated on $[\beta, 1]$ such that $N_\beta = N$ on $(\beta, 1]$

and $N(x, \{\beta\}) = 2b(x)$. Consider

$$\psi_\beta(z)(x) = \tilde{\psi}(z)(x) + \int_{[0,1]} F_z(x,u) N_\beta(x, du)$$
$$= \tilde{\psi}(z)(x) + \int_{[0,1]} [e^{-uz(x)} - 1] n_\beta(x, du) + a_\beta(x) z(x),$$

where

$$n_\beta(x, du) = u^{-2} N_\beta(x, du), \qquad a_\beta(x) = \int_{[0,1]} u n_\beta(x, du).$$

Suppose that $v \in \mathfrak{N}_E(G)$. It follows from 2.1.A that $\Phi(v) = 1 - \exp(-\langle v, \eta \rangle)$ belongs to $\mathfrak{N}(G)$ for every finite measure η. By 2.1.B, ψ_β satisfies 2.2.A. Since $a_\beta(x)$ is bounded, ψ_β satisfies 2.2.B also. It is easy to check conditions 2.2.C, 2.2.D and 2.3.B by using (2.8) and (2.9). Condition 2.3.A holds because

$$|\psi_\beta(z) - \psi(z)| \le 2\beta(x) |F_z(x, \beta) - F_z(x, 0)| + \tfrac{1}{2} \|z\|^2 \int_0^\beta u^2 n(x, du) + m^\beta(x, E).$$

Therefore Theorem 2.3 follows from Theorem 2.2. □

3. Passage to the limit.

3.1.

PROOF OF THEOREM I.3.1. Conditions I.3.2.A–I.3.2.C hold for all $z \in b\mathscr{B}$ and all $\beta > 0$ if we set

$$\psi^s(z) = \psi^s(|z|),$$
$$\psi^s_\beta(z) = \begin{cases} \psi^s_\beta(|z|), & \text{if } \|\beta z\| \le 1, \\ 0, & \text{if } \|\beta z\| > 1. \end{cases}$$

We fix a and assume that all functions f_i are supported by $[0, a)$. Clearly, all functions $v_I(\beta)$ have the same property, and (I.3.2), restricted to $[0, a)$, has the form (1.5) with

$$\Psi^s_r(\beta, z) = \Pi_{r,x} 1_{s < \tau_I} \psi^s_\beta(z)(\xi_s) k(s, \xi_s), \qquad g^r_\beta(x) = \Pi_{r,x} G^\beta_I. \tag{3.1}$$

If, in addition, v_I is supported by $[0, a)$, then (I.1.13), restricted to $[0, a)$, has the form (1.5) with

$$\Psi^s_r(z) = \Pi_{r,x} 1_{s < \tau_I} \psi^s(z)(\xi_s) k(s, \xi_s), \qquad g^r(x) = \Pi_{r,x} G_I. \tag{3.2}$$

The conditions I.3.2.A–I.3.2.C of our theorem imply 1.2.A and 1.3.A.

We prove, by induction on $m = |I|$, that if $\|f_i\| \le c$ for all $i \in I$, then

$$\|G^\beta_I\| \le K_m, \quad \text{for all } \beta > 0, \tag{3.3}$$

$$\|v^r_I(\beta)\| \le L_m, \quad \text{for all } \beta > 0 \text{ and all } r \in [0, a), \tag{3.4}$$

where K_m and L_m depend only on c and a. Moreover, the unique solution v_I of (I.1.13) satisfies the condition

$$(3.5) \qquad \|v_I^r(\beta) - v_I^r\| < \varepsilon_n(\beta), \quad \text{for all } \beta > 0 \text{ and all } r \in [0, a),$$

where $\varepsilon_n(\beta) \to 0$ as $\beta \to 0$ and they depend only on c, a, α and C.

If (3.3) and (3.4) hold for $m - 1$, then (3.3) holds for m, with $K_m = c + L_{m-1}$. Hence 1.3.C holds with $K = K_m$ and, by (1.10), (3.4) is true with some L_m. If (3.5) is satisfied for $m - 1$, then 1.3.D is satisfied and (1.11) implies (3.5).

Let $I = \{1, \dots, m\}$. For any fixed β, r and x, $v_I^r(\beta, x)$ is a continuous functional of $(f_1, \dots, f_m) \in (bp\mathscr{B}_a)^m$ (relative to bounded convergence). By (3.4) and (3.5), the same is true for $v_I^r(x)$ and for the functional (2.7). The concluding part of the proof is almost identical to the corresponding part of proof of Theorem 2.1. The only difference is that now we use Lemma 1.2 instead of Lemma 1.4. □

3.2.

THEOREM 3.2. *Suppose that ξ and K satisfy the conditions of Theorem* I.1.1 *and that ψ is given by* (I.1.17), *with a, b, γ, m and n subject to conditions* I.1.7.A–I.1.7.C. *Then there exists a unique (up to equivalence) superprocess with parameters (ξ, K, ψ).*

PROOF. If the coefficients are independent of the time parameter s, then Theorem 3.2 follows from Theorem I.3.1 and Section I.3.3. The arguments in Section I.3.3 are applicable also to the case when conditions I.1.7.A–I.1.7.C are satisfied on a fixed interval Δ and the coefficients $a, b, \dots$ vanish for $s \notin \Delta$. In the general case, we consider functions a_i, b_i, γ_i, m_i and n_i which coincide with a, b, γ, m and n on the interval $\Delta_i = [0, i)$ and vanish outside Δ_i, and we obtain Theorem 3.2 by applying Theorem 2.2. □

Part II. Superdiffusion and partial differential equations.

1. Diffusion and PDEs.

1.1. *Diffusion.* We start from a differential operator

$$L = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

in $S = \mathbb{R}_+ \times \mathbb{R}^d$ with coefficients which depend on $(r, x) \in S$ and satisfy the following conditions:

1.1.A. (Uniform ellipticity.) There exists a constant $\gamma > 0$ such that

$$\sum_{i,j} a_{ij} u_i u_j \ge \gamma \sum_i u_i^2, \quad \text{for all } (r, x) \in S,\ u_1, \dots, u_d \in \mathbb{R}.$$

1.1.B. a_{ij} and b_i are bounded, continuous in (r, x) and they satisfy Hölder's conditions: There exist constants $A > 0$ and $0 < \alpha \le 1$ such that

$$|a_{ij}(r,x) - a_{ij}(s,y)| \le A\left[|r-s|^{\alpha/2} + |x-y|^{\alpha}\right],$$
$$|b_i(r,x) - b_i(r,y)| \le A|x-y|^{\alpha},$$

for all r, x and y.

Under these conditions, there exists a function $p(r, x; t, y)$ such that, for every bounded continuous function f,

$$u(r,x) = \int_E p(r,x;t,y) f(y)\, dy$$

satisfies the equation

$$\dot{u} + Lu = 0 \quad \text{in } S_{<t} = [0,t) \times \mathbb{R}^d$$

and the condition

$$u(r,x) \to f(x) \quad \text{as } r \uparrow t.$$

(p is called a *fundamental solution* of the equation $\dot{u} + Lu = 0$.) (The existence of a fundamental solution is proved, for instance, in [37], Chapter 1.)

The *diffusion with generator* L is a Markov process in $E = \mathbb{R}^d$ with continuous paths and with the transition function $p(r, x; t, dy) = p(r, x; t, y)\, dy$. For the existence of such process, see, for example, [87] or [17]. Clearly, all diffusions satisfy conditions I.1.2.A–I.1.2.C.

1.2. *Maximum principle.* Let Q be an open set in S. Let $\mathbb{C}(Q)$ denote the set of all continuous functions in Q. Put $y \in \mathbb{C}^1(Q)$ if u and $\partial u/\partial x_i$, $i = 1, \dots, d$, belong to $\mathbb{C}(Q)$, and put $u \in \mathbb{C}^2(Q)$ if u, $\dot{u}$ and $\partial u/\partial x_i$, $i = 1, \dots, d$, and $\partial^2 u/\partial x_i\, \partial x_j$, $i, j = 1, \dots, d$, belong to $\mathbb{C}(Q)$. Let $\mathbb{C}^2 = \mathbb{C}^2(S)$ and let $\mathbb{C}_0^2$ stand for the class of all $y \in \mathbb{C}^2$ which vanish outside a compact set.

LEMMA 1.1 (Maximum principle). *Suppose that Q is a bounded open set and that $u \in \mathbb{C}^2(Q)$ satisfies the conditions*

$$\dot{u} + Lu \ge 0 \quad \textit{in } Q,$$
$$\limsup u(r,x) \le 0 \quad \textit{as } (r,x) \to (t,a) \in \partial Q.$$

Then $u \le 0$ in Q.

[See, e.g., [37], Chapter 2.]

1.3. *Perron solution.* We consider classical solutions of the equation

$$\dot{u} + Lu = 0 \quad \text{in } Q, \tag{1.1}$$

that is, we assume that $u \in \mathbb{C}^2(Q)$ and that (1.1) holds at every point of Q. The proof of the following result is based on the maximum principle.

THEOREM 1.1. *Let Q be bounded and let f be a bounded continuous function on ∂Q. Then there exists a unique function u in Q such that the following hold*:

(a) $u \le v$ *if*

$$v \in \mathbb{C}^2(Q), \qquad \dot{v} + Lv \le 0 \quad \text{in } Q, \qquad \liminf_{z \to c} v(z) \ge f(c), \quad \text{for all } c \in \partial Q; \tag{1.2}$$

(b) $u \ge v$ *if*

$$v \in \mathbb{C}^2(Q), \qquad \dot{v} + Lv \ge 0 \quad \text{in } Q, \qquad \limsup_{z \to c} v(z) \le f(c), \quad \text{for all } c \in \partial Q. \tag{1.3}$$

The function u is a solution of (1.1).

We call u the *Perron solution*. [The pioneering work of Perron appeared in 1923. The general result (in the elliptic setting) is due to Wiener and Brelot. For the parabolic setting (and $L = \Delta$) see [15].]

THEOREM 1.2. *Let $(\xi_t, \Pi_{r,x})$ be a diffusion with generator L. The Perron solution of* (1.1) *is given by the formula*

$$u(r,x) = \Pi_{r,x} f(\tau, \xi_\tau), \tag{1.4}$$

where $\tau = \inf\{t\colon \tau \ge \alpha, (t, \xi_t) \notin Q\}$ is the first exit time from Q.

PROOF. (i) Suppose that $h \in \mathbb{C}_0^2$, $\mathscr{F}[r,t]$ is the σ-algebra generated by ξ_s, $s \in [r,t]$, and

$$M_t = h(t, \xi_t) - \int_r^t (\dot{h} + Lh)(s, \xi_s)\, ds, \qquad t \ge r. \tag{1.5}$$

Then $(M_t, \mathscr{F}[r,t], \Pi_{r,x})$ is a martingale. (This follows immediately from Itô's formula if we describe the diffusion ξ by a stochastic differential equation.) Since τ is bounded, by the optional sampling theorem,

$$\begin{aligned} h(r,x) &= \Pi_{r,x} M_0 = \Pi_{r,x} M_\tau \\ &= \Pi_{r,x} h(\tau, \xi_\tau) - \Pi_{r,x} \int_r^\tau (\dot{h} + Lh)(s, \xi_s)\, ds. \end{aligned} \tag{1.6}$$

(ii) Consider a sequence of open sets $Q_n \uparrow Q$ such that $\overline{Q}_n \subset Q_{n+1}$. Let v satisfy (1.2). Consider a function $h_n \in \mathbb{C}_0^2$ which coincides with v on $\overline{Q}_n$. Let τ_n be the first exit time from Q_n. By (1.6),

$$h_n(r,x) = \Pi_{r,x} h_n(\tau_n, \xi_{\tau_n}) - \Pi_{r,x} \int_r^{\tau_n} (\dot{h}_n + Lh_n)(s, \xi_s)\, ds$$

and, therefore,

$$v(r,x) = \Pi_{r,x} v(\tau_n, \xi_{\tau_n}) - \Pi_{r,x} \int_r^{\tau_n} (\dot{v} + Lv)(s, \xi_s)\, ds \ge \Pi_{r,x} v(\tau_n, \xi_{\tau_n}), \tag{1.7}$$

for $(r, x) \in Q_n$. Clearly $(\tau_n, \xi_{\tau_n}) \to (\tau, \xi_\tau)$ and, by Fatou's lemma,

$$v(r,x) \geq \Pi_{r,x} \liminf v(\tau_n, \xi_{\tau_n}) \geq \Pi_{r,x} f(\tau, \xi_\tau) = u(r,x).$$

Analogously, $v \leq u$ if v satisfies (1.3). □

1.4. *The improved maximum principle.* We say that a subset T of ∂Q is *total* if

$$\Pi_{r,x}\{(\tau, \xi_\tau) \in T\} = 1, \quad \text{for all } (r,x) \in Q. \tag{1.8}$$

THEOREM 1.3. *Let Q be bounded and let G be a total subset of ∂Q. If $u \in \mathbb{C}^2(Q)$ is bounded above and satisfies the conditions*

$$\dot{u} + Lu \geq 0 \quad \textit{in } Q, \tag{1.9}$$

$$\limsup u(r,x) \leq 0 \quad \textit{as } (r,x) \to (t,a) \in T, \tag{1.10}$$

then $u \leq 0$ in Q.

PROOF. Set $f = \varphi \vee 0$, where $\varphi(c) = \limsup u(r,x)$ as $(r,x) \to c$. Clearly, u satisfies (1.3) and, by Theorem 1.2, $u(r,x) \leq \Pi_{r,x} f(\tau, \xi_\tau)$. The right-hand side is equal to 0 because $(\tau, \xi_\tau) \in T$ $\Pi_{r,x}$-a.s., and $f = 0$ on T.

1.5. *The mean value property.*

THEOREM 1.4. *Suppose that Q is a bounded domain and T is a total subset of ∂Q. If u is a bounded continuous function on $Q \cup T$ which satisfies* (1.1), *then*

$$u(r,x) = \Pi_{r,x} u(\tau, \xi_\tau), \quad \textit{for all } (r,x) \in Q. \tag{1.11}$$

PROOF. Let Q_n and τ_n be the same as in the proof of Theorem 1.2. Then $u(r,x) = \Pi_{r,x} u(\tau_n, \xi_{\tau_n})$ [cf. (1.7)]. Since $(\tau_n, \xi_{\tau_n}) \to (\tau, \xi_\tau) \in T$ $\Pi_{r,x}$-a.s., we get (1.11) by the dominated convergence theorem.

COROLLARY. *The set of all solutions of* (1.1) *is closed under bounded convergence.*

Indeed, if (1.11) holds for u_n and if $u_n \to u$ boundedly, then (1.11) holds for u, which implies (1.1) by Theorem 1.2.

1.6. *Regular points.* Put $\tau_+ = \inf\{t: t > \alpha, (t, \xi_t) \notin Q\}$. We say that a point $c = (t,a)$ of ∂Q is *regular* and we write $c \in \partial_r Q$ if

$$\Pi_{t,a}\{\tau_+ = t\} = 1. \tag{1.12}$$

This definition is equivalent to the following: A point $(t,a) \in \partial Q$ is regular if, for every $t' > t$,

$$\Pi_{r,x}\{\tau > t'\} \to 0 \quad \text{as } (r,x) \to (t,a). \tag{1.13}$$

We refer to [17], Chapter 13, for the proof, which is based on the following properties of ξ:

1. (Strong Feller property.) The function

$$u(r,x) = \Pi_{r,x} f(\xi_t) = \int p(r,x;t,y) f(y)\, dy, \qquad r < t,\ x \in E,$$

is continuous for every bounded Borel f.
2. For every $\beta > 0$,

$$\sup_{r,x} \Pi_{r,x}\Big\{ \sup_{r \le s \le r+h} |\xi_s - \xi_r| \ge \beta \Big\} \to 0 \quad \text{as } h \to 0.$$

The role of regularity is revealed by the following theorem.

THEOREM 1.5. *If (t,a) is a regular point of ∂Q and if f is a bounded function on ∂Q which is continuous at (t,a), then*

$$u(r,x) = \Pi_{r,x} f(\tau, \xi_\tau) \to f(t,x) \quad \textit{as } (r,x) \to (t,a).$$

To prove that $c = (t,a) \in \partial Q$ is regular, it is sufficient to construct a *barrier* at c, that is, a continuous function u on $\bar{Q}$ such that

$$\text{(1.14)} \qquad \dot{u} + Lu \le 0 \ \text{ in } Q, \qquad u(c) = 0, \qquad u(z) > 0 \ \text{ for } z \ne c.$$

Indeed, for every $t' > t$, the infimum β of u on the set $Q \cap S_{\ge t'}$ is strictly positive and, by Chebyshev's inequality,

$$\Pi_{r,x}\{\tau > t'\} \le \Pi_{r,x}\{u(\tau, x_\tau) \ge \beta\} \le \Pi_{r,x} u(\tau, \xi_\tau)/\beta.$$

The arguments in the proof of Theorem 1.2 show that $\Pi_{r,x} u(\tau, \xi_\tau) \le u(r,x)$ and (1.14) implies (1.13).

By using a barrier of the form

$$\text{(1.15)} \qquad u(r,x) = \varepsilon^{-2p} - |(r,x) - (r',x')|^{-2p},$$

we prove the *regularity test*: A point $c = (t,a) \in \partial Q$ is regular if there exists $c' = (r', x')$, with $x' \ne a$, such that

$$|(r,x) - (r',x')| > |(t,a) - (r',x')|,$$

for all $(r,x) \in \bar{Q}$ sufficiently close to c and different from c.

[If $\varepsilon = |c - c'|$ and p is sufficiently large, then the function (1.15) is a barrier at c in the intersection of Q with the $\varepsilon/2$-neighborhood of c.]

By applying this test to a simple rectangle $(r_1, r_2) \times (a_1, b_1) \times \cdots \times (a_d, b_d)$, we get that every point $(t,a) \in \partial Q$ with $r_1 < t < r_2$ is regular. [Clearly, $(t,a) \in \partial Q$ is regular if $t = r_2$, and it is irregular if $t = r_1$, $a \in (a_1, b_1) \times \cdots \times (a_d, b_d)$.]

1.7. *Regular domains.* We say that an open set Q is *regular* if the set $\partial_r Q$ of all regular points is total in ∂Q. All simple rectangles are regular domains. An example of an irregular open set is given by $Q = \{(r,x): r \ne 1\}$. (Indeed, every point of $\partial Q = \{1\} \times E$ is irregular.)

LEMMA 1.2. *If U is a regular open set, then $U \cap Q$ is regular for every open set Q such that $\bar{U} \cap \partial Q \subset \partial_r Q$.*

PROOF. Note that $\partial(U \cap Q) = N_1 \cup N_2$, where $N_1 = \bar{U} \cap \partial Q$ and $N_2 = \partial U \cap Q$. Clearly, N_1 and $N_2 \cap \partial_r U$ are contained in $\partial_r(U \cap Q)$. Their union is total in $\partial_r(U \cap Q)$ because $\partial_r U$ is total in ∂U. □

Theorems 1.2, 1.3 and 1.5 imply the following theorem.

THEOREM 1.6. *If Q is a bounded regular open set, then the first boundary value problem,*

$$(1.16)\qquad \begin{aligned} \dot{u} + Lu &= 0 \quad \text{in } Q,\\ u &= f \quad \text{on } \partial_r Q, \end{aligned}$$

has a unique bounded solution for every bounded continuous function f on $\partial_r Q$. Moreover,

$$u(r,x) = \Pi_{r,x} f(\tau, \xi_\tau).$$

We introduce a special class of regular open sets and we prove that every open set can be approximated by these sets. We start from compact sets of the form $[a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$, which we call *cells*. A finite union of cells is called a *simple compact set*, and the totality of all its interior points is called a *simple open set*.

The boundary ∂C of a cell $C = [a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$ consists of $2(d+1)$ d-dimensional faces. We distinguish two horizontal faces: the top $\{b_0\} \times [a_1, b_1] \times \cdots \times [a_d, b_d]$ and the bottom $\{a_0\} \times [a_1, b_1] \times \cdots \times [a_d, b_d]$. We call the rest vertical faces.

LEMMA 1.3. *If F is a $(d-1)$-dimensional face of a cell C, then*

$$\varphi(r,x) = \Pi_{r,x}\{(t, \xi_t) \in F \text{ for some } t > r\} = 0, \quad \text{for all } (r,x) \in S.$$

PROOF. If the face F is horizontal, then $F \subset \{a\} \times H$ for some $a \in \mathbb{R}_+$ and a $(d-1)$-dimensional affine subspace H of $E = \mathbb{R}^d$. Therefore, $\varphi(r,x) \le \Pi_{r,x}\{\xi_a \in H\} = 0$. If F is vertical, then $F \subset \mathbb{R}_+ \times H$, where H is a $(d-2)$-dimensional affine subspace of E. Therefore $\varphi(r,x) \le \Pi_{r,x}\{\xi_t \in H$ for some $t > r\} = 0$. □

THEOREM 1.7. *Every simple open set is regular. For an arbitrary open set Q, there exists a sequence of simple open sets $Q_n \uparrow Q$ such that $\bar{Q}_n \subset Q_{n+1}$.*

PROOF. Let Q be the set of all interior points of a simple compact set $K = \bigcup_1^n K_i$. We can assume that the intersection of every two cells $K_i \cap K_j$ is either empty or it is a common face of both cells. Note that $\partial Q = \bigcup_1^N F_k$, where $F_1, \dots, F_N$ are d-dimensional cells which enter the boundary of exactly one of the K_i. We denote by F_k^0 the set of all points of F_k which do not belong to any

$(d-1)$-dimensional face of K_i. By Lemma 1.3, to prove that Q is regular, it is sufficient to show that, for every k, either $\Pi_{r,x}\{(\tau,\xi_\tau)\in F_k^0\}=0$ for all $(r,x)\in Q$ or $F_k^0\subset\partial_r Q$. Clearly, the first case takes place if F_k is the bottom of K_i. If F_k is the top of K_i, then $F_k^0\subset\partial_r Q$. If F_k is vertical, then $F_k^0\subset\partial_r Q$ by the regularity test.

To construct Q_n, consider a partition of $S=\mathbb{R}_+^{d+1}$ into cells with vertices in the lattice $2^{-n}\mathbb{Z}_+^{d+1}$ and take the union K^N of all cells whose ε_n-neighborhoods are contained in Q. The set Q_n of all interior points of K^n satisfies the requirements of Theorem 1.7 if $\varepsilon_n=(d+1)^{1/2}2^{-n}$. □

1.8. The *parabolic potential* is defined by

$$F(r,x)=\Pi_{r,x}\int_r^\infty\rho(s,\xi_s)\,ds=\int_r^\infty\int_E p(r,x;s,y)\rho(s,y)\,ds\,dy, \tag{1.17}$$

where ρ is a Borel function in S.

THEOREM 1.8. *If ρ is bounded and vanishes outside a compact set, then $F\in\mathbb{C}^1$. Suppose, in addition, ρ is continuous in (r,x) and locally Hölder continuous in x uniformly in r [i.e., for every (r^0,x^0), there exists a neighborhood U and constants $A>0$ and $0<\alpha\le 1$ such that $|\rho(r,x)-\rho(r,y)|\le A|x-y|^\alpha$, for all $(r,x),(r,y)\in U$]. Then $F\in\mathbb{C}^2$ and*

$$\dot F+LF=-\rho \quad \text{in } S. \tag{1.18}$$

A proof for the case $L=\Delta$ is contained in [15], Section I, XVII.6. For the general case see [37], Theorem 1.9.

2. Parts of a diffusion.

2.1. *Definition*. Denote by Q_t the t-section of an open set $Q\subset S$ and by $\mathscr{B}_t$ the Borel σ-algebra in Q_t. Put

$$p_Q(r,x;t,B)=\Pi_{r,x}\{\xi_t\in B,\tau>t\},\quad \text{for } r<t,\ x\in Q_r,\ B\in\mathscr{B}_t, \tag{2.1}$$

where τ is the first exit time from Q. The Markov property of ξ implies that for all $(r,x)\in Q$, $r<t_1<\cdots<t_n$, $B_1\in\mathscr{B}_{t_1},\dots,B_n\in\mathscr{B}_{t_n}$,

$$\begin{aligned}&\Pi_{r,x}\{\xi_{t_1}\in B_1,\dots,\xi_{t_n}\in B_n,\tau>t_n\}\\ &\quad=\int_{B_1}\cdots\int_{B_n}p_Q(r,x;t_1,dy_1)\cdots p_Q(t_{n-1},y_{n-1};t_n,dy_n).\end{aligned} \tag{2.2}$$

By applying this formula to $n=2$, $t_1=s$, $t_2=t$, $B_1=Q_s$ and $B_2=B$, we get

$$p_Q(r,x;t,B)=\int_{Q_t}p_Q(r,x;s,dy)p_Q(s,y;t,B), \tag{2.3}$$

for $r<s<t$ and $B\in\mathscr{B}_t$. Hence p_Q is a Markov transition function in Q_t, and the restriction $\tilde\xi_t$ of ξ_t to the interval $[\alpha,\tau)$ is a Markov process with the transition function p_Q. We call $\tilde\xi$ the *part of the diffusion* ξ *in* Q.

By the strong Markov property of ξ,

$$\Pi_{r,x}\{\tau \le t,\, \xi_t \in B\} = \Pi_{r,x}\{\tau \le t,\, \Pi_{\tau,\xi_\tau}\{\xi_t \in B\}\}$$

and, therefore,

$$p_Q(r,x;t,B) = p(r,x;t,B) - \Pi_{r,x} 1_{\tau \le t} p(\tau,\xi_\tau;t,B). \tag{2.4}$$

Since $p(r,x;t,dy) = p(r,x;t,y)\,dy$, we conclude that

$$p_Q(r,x;t,B) = \int_B p_Q(r,x;t,y)\,dy, \tag{2.5}$$

where

$$p_Q(r,x;t,y) = p(r,x;t,y) - \Pi_{r,x} 1_{\tau \le t} p(\tau,\xi_\tau;t,y). \tag{2.6}$$

2.2. The *parabolic potential in Q* is given by the formula

$$F_Q(r,x) = \Pi_{r,x}\int_r^\tau \rho(s,\xi_s)\,ds = \int_Q p_Q(r,x;s,y)\rho(s,y)\,ds\,dy, \tag{2.7}$$

where ρ is a Borel function in Q. [We set $p_Q(r,x;t,y) = 0$ for $t < r$.]

THEOREM 2.1. *If Q and ρ are bounded, then $F_Q \in \mathbb{C}^1(Q)$ and, for every $(t,a) \in \partial_r Q$,*

$$F_Q(r,x) \to 0 \quad as\ (r,x) \to (t,a),\ (r,x) \in Q. \tag{2.8}$$

Suppose, in addition, that ρ is continuous in (r,x) and locally Hölder continuous in x uniformly in r. Then $F_Q \in \mathbb{C}^2(Q)$ and

$$\dot{F}_Q + LF_Q = -\rho \quad in\ Q. \tag{2.9}$$

PROOF. It follows from (2.4) that $F_Q(r,x) = F(r,x) - \Pi_{r,x}F(\tau,\xi_\tau)$, where F is given by (1.17), and our statement follows from Theorems 1.2 and 1.7. □

3. Superdiffusion and PDEs.

3.1. *Probabilistic solution of a nonlinear PDE.* A superprocess X with parameters (ξ, K, ψ) is called a *superdiffusion* if ξ is a diffusion. Recall that ψ is a mapping from $bp\mathscr{B}_S$ to $\mathscr{B}_S$ (see Section I.1.5). We assume that $dK_t = dt$ and that the following hold:

3.1.A. $\psi(z) \in \mathbb{C}^1$ for every $z \in \mathbb{C}^1$.
3.1.B. $\psi(z_1)(s,x) \le \psi(z_2)(s,x)$ if $z_1(s,x) \le z_2(s,x)$.

In particular, 3.1.A holds if $\psi(z)(s,x) = \psi[s,x;z(s,x)]$, with a continuously differentiable function $\psi(s,x;t)$. 3.1.B holds if $\psi(s,x;t)$ is given by (I.1.19), with $a \ge 0$ (the subcritical case).

THEOREM 3.1. *Let τ be the first exit time from Q and let $\tilde{X}$ be the part of X in Q. If $\rho \ge 0$ is bounded and belongs to $\mathbb{C}^1(Q)$ and if $f \ge 0$ is a bounded*

Borel function on ∂Q, then

$$v(r,x) = -\log P_{r,\delta_x} \exp\left\{ \int_{\mathbb{R}_+} \langle -\rho^t, \tilde{X}_t \rangle \, dt - \langle f, X_\tau \rangle \right\} \tag{3.1}$$

belongs to $\mathbb{C}^2(Q)$ and is a solution of

$$\dot{v} + Lv = \psi(v) - \rho \quad \textit{in } Q. \tag{3.2}$$

In particular,

$$v(r,x) = -\log P_{r,\delta_x} \exp\langle -f, X_\tau \rangle \tag{3.3}$$

is a solution of

$$\dot{v} + Lv = \psi(v) \quad \textit{in } Q. \tag{3.4}$$

If v is defined by (3.1) *and if f is continuous at $c \in \partial_r Q$, then*

$$v(z) \to f(c) \quad \textit{as } z \to c. \tag{3.5}$$

PROOF. The proof is based on Theorem I.1.8. Formula (3.1) is a particular case of (1.36), with $\mu = \delta_{r,x}$. Formulae (3.2) and (3.5) follow from (I.1.37) and Theorems 1.2, 1.5 and 2.1. For details we refer the reader to [28], Section 1.3, Proof of Theorem 1.1. (Theorem 1.1 in [28] is stated for a special class of functions ψ but the proof does not need any change.) □

3.2. *Comparison principle.*

THEOREM 3.2. *Let ψ satisfy* 3.1.B, *let Q be bounded and let T be total in ∂Q. Suppose*

$$u, v \in \mathbb{C}^2(Q), \qquad \dot{u} + Lu - \psi(u) \le \dot{v} + Lv - \psi(v) \quad \textit{in } Q \tag{3.6}$$

and

$$u - v \textit{ is bounded below}, \qquad \liminf(u - v) \ge 0 \quad \textit{on } T. \tag{3.7}$$

Then $u \ge v$ in Q.

PROOF. Let $w = v - u$. If the theorem is false, then $\tilde{Q} = Q \cap \{w > 0\} \ne \varnothing$. By 3.1.B, $\dot{w} + Lw \ge \psi(v) - \psi(u) \ge 0$ in $\tilde{Q}$. Note that $\tilde{T} = \partial\tilde{Q} \cap \{Q \cup T\}$ is a total subset of $\partial\tilde{Q}$. If $c \in \tilde{T}$, then either $c \in Q$ and then $w(c) = 0$ or $c \in T$ and then $\limsup w \le 0$ at c. This contradicts Theorem 1.3. □

3.3. *Upper bounds.* It follows from Theorem 3.2 that if

$$\dot{u}^0 + Lu^0 - \psi(u^0) \le 0 \quad \text{in } Q, \tag{3.8}$$

then u^0 is an upper bound for every solution v of (3.2) for which

$$v - u^0 \text{ is bounded above} \quad \text{and} \quad \limsup(v - u^0) \le 0 \text{ on } T. \tag{3.9}$$

We say that u^0 is an *absolute barrier* in Q if u^0 satisfies (3.8) and

$$\lim u^0 = +\infty \quad \text{on } T, \tag{3.10}$$

for a total subset T of ∂Q. Clearly, u^0 is an upper bound for all bounded solutions of (3.2). [We remind the reader that u^0 is positive, as are all functions in the domain of ψ.]

From this point on, we assume that

$$\psi(z) = z^\alpha, \qquad 1 < \alpha \le 2. \tag{3.11}$$

The following results are proved (in a slightly different but equivalent form) in the Appendix to [28].

3.3.A. There exist constants $a_1 > 0$ and $a_2 \ge 0$ (which depend only on α and bounds for the coefficients a_{ij}, and b_i of the operator L) such that

$$u^0(r,x) = \left[(a_1 R + a_2)R^3(R + r^0 - r)^{-1}(R^2 - |x - x^0|^2)^{-2}\right]^{1/(\alpha-1)} \tag{3.12}$$

is an absolute barrier in the cylinder

$$U_R(r^0, x^0) = \{(r,x): 0 < r - r^0 < R, |x - x^0| < R\}. \tag{3.13}$$

3.3.B. If v is a bounded solution of (3.2) and if

$$\{(r,x): 0 \le r - r^0 < R, |x - x^0| < R\} \subset Q,$$

then

$$v(r^0, x^0) \le \left[a_1 R^{-1} + a_2 R^{-2}\right]^{1/(\alpha-1)}. \tag{3.14}$$

Indeed, by the comparison principle, (3.12) is an upper bound for v in $U_R(r^0, x^0)$ and v is continuous at (r^0, x^0).

It follows from 3.3.B that the class of all bounded solutions of (3.2) is locally uniformly bounded. Clearly, 3.3.B also implies the following:

3.3.C. There exist no solutions of (3.2) in S except $v = 0$.

By using Theorem 3.2 and 3.3.B, we deduce an upper bound near the boundary:

3.3.D. Suppose that B is a relatively open subset of ∂Q. For every $b \in B$ and every $A > 0$ there exist $R > 0$ and $N < \infty$ such that

$$v(z) \le N \quad \text{for all } z \in Q \cap U_R(b) \tag{3.15}$$

if

$$\dot v + Lv = v^\alpha \text{ in } Q \quad \text{and} \quad \limsup_{z \to b} v(z) \le A \text{ for all } b \in B. \tag{3.16}$$

Proof can be found in [28], page 961.

3.4. *The first boundary value problem.* We are prepared now to discuss the problem

$$\begin{aligned} \dot v + Lv &= v^\alpha \quad \text{in } Q, \\ v &= f \quad \text{on } \partial_r Q \end{aligned} \tag{3.17}$$

in a bounded regular open set Q. We only sketch the proofs and refer the reader to [28], Section 1, for details.

It follows from Theorems 3.1 and 3.2 that the following hold:

3.4.A. If $f \geq 0$ is a bounded continuous function on $\partial_r Q$, then the unique bounded solution of (3.17) is given by (3.3).

3.4.B. (The mean value property.) If $T \subset \partial_r Q$ is a total subset of ∂Q and if a bounded continuous function v in $Q \cup T$ satisfies (3.4), then

$$v(r,x) = -\log P_{r,\delta_x} \exp\langle -v, X_\tau\rangle \quad \text{in } Q.$$

The mean value property and upper bounds (3.14)–(3.16) and Theorem 3.1 yield the following:

3.4.C. Let $v_n \to v$ pointwise in an arbitrary open set Q. If $\dot{v}_n + Lv_n = v_n^\alpha$ in Q, then $v \in \mathbb{C}^2(Q)$ and $\dot{v} + Lv = v^\alpha$ in Q. Suppose, in addition, that $B \subset \partial_r Q$ is relatively open in ∂Q and f is a bounded continuous function on B. If $v_n = f$ on B, then $v = f$ on B. [Here, as in (3.17), writing "$v = f$ on B" means "for every $(t,a) \in B$, $v(r,x) \to f(t,a)$ as $(r,x) \to (t,a)$, $(r,x) \in Q$."]

In contrast to the linear case, the first boundary value problem (3.17) can be solved for functions f with infinite values.

3.4.D. Let f be a continuous function from $\partial_r Q$ to $[0,\infty]$. Then (3.3) is the minimal solution of (3.17).

To prove 3.4.D, we apply 3.4.A to bounded functions $f_k = f \wedge k$ and pass to the limit using 3.4.C and 3.4.B.

Another passage to the limit is needed to prove the following:

3.4.E. For every closed $\Gamma \subset \partial Q$,

$$v(r,x) = -\log P_{r,\delta_x}\{X_\tau(\Gamma) = 0\} \tag{3.18}$$

satisfies (3.4) and $v = \infty$ on the set of all interior (relative to ∂Q) points of Γ. If, in addition, $A = \partial Q \setminus \Gamma \subset \partial_r Q$, then $v = 0$ on A.

For every $\mu \in \mathscr{M}(S)$,

$$P_\mu\{X_\tau(\Gamma) = 0\} = \exp(-\langle v, \mu\rangle). \tag{3.19}$$

4. Graph of a superdiffusion.

4.1. *Random closed sets.* Let $\omega \to F(\omega)$ be a map from Ω to the space of all closed sets in S. We say that F is a random closed set relative to a σ-algebra $\mathscr{F}$ in Ω if

$$\{\omega: F(\omega) \cap K \neq \varnothing\} \in \mathscr{F}, \quad \text{for all compact sets } K. \tag{4.1}$$

This is equivalent to the condition

$$\{\omega: F(\omega) \cap U \neq \varnothing\} \in \mathscr{F}, \quad \text{for all open sets } U. \tag{4.2}$$

The following general result is based on the Choquet theory of capacity. It is proved, for instance, in [69], Chapter 2.

THEOREM 4.1. *Let $\mathscr{F}^*$ stand for the universal completion of $\mathscr{F}$. If F is a random closed set relative to $\mathscr{F}$, then*

$$\{\omega\colon F(\omega) \cap A \neq \varnothing\} \in \mathscr{F}^*, \quad \text{for all analytic sets } A, \tag{4.3}$$

and, if P is a probability on $(\Omega, \mathscr{F})$, then

$$P\{F \cap A \neq \varnothing\} = \sup P\{F \cap K \neq \varnothing\} = \inf P\{F \cap U \neq \varnothing\}, \tag{4.4}$$

where K runs over all compact subsets of A, and U runs over all open sets U which contain A.

4.2. *Graph of X.* Let X_t be a right version of X. (As in Section I.1.10, we interpret X_t as a measure on $S_t = \{t\} \times E$.) The *graph* of X is the minimal closed set $G \subset S$ such that, for every $t \in \mathbb{R}_+$, X_t is concentrated on G. Clearly, G is defined uniquely up to equivalence relative to the family of measures $\{P_\mu, \mu \in \mathscr{M}(S)\}$. The *graph* G_Q *of X in an open set Q* is defined analogously through the part $\tilde{X}$ of X in Q. Clearly, $G_Q \subset \overline{Q}$.

For every open set U, there exists a bounded continuous function $f \geq 0$ such that $U = \{f > 0\}$, and Theorem I.1.6 implies the following:

4.2.A. For every open set U and an arbitrary subset Λ everywhere dense in $\mathbb{R}$,

$$\begin{aligned}\{G_Q \cap U \neq \varnothing\} &= \left\{\tilde{X}_t(U) > 0 \text{ for some } t \in \mathbb{R}_+\right\}\\ &= \left\{\tilde{X}_t(U) > 0 \text{ for some } t \in \Lambda\right\} \quad \text{a.s.}\end{aligned}$$

It follows from 4.2.A that G_Q is a random closed set relative to the σ-algebra $\hat{\mathscr{F}}_X$ generated by X_ε, $\varepsilon \in \mathscr{F}$. We have the following:

4.2.B. G_Q coincides with the support of the measure $Y(dt, dx) = \gamma(dt)\tilde{X}_t(dx)$, where γ is an arbitrary Radon measure on $\mathbb{R}$ such that $\operatorname{supp}\gamma = \mathbb{R}_+$.
4.2.C. If σ is the first exit time from an open set $U \subset Q$, then the restriction of X_σ to Q is, a.s., concentrated on G_Q.
4.2.D. $G_Q = \varnothing$ P_μ-a.s. if $\mu(Q) = 0$.

We get 4.2.B from 4.2.A. To prove 4.2.C, note that, by Theorem I.1.10, for every open set V,

$$\begin{aligned}\{G_Q \cap V = \varnothing\} &= \left\{\tilde{X}_t(V) = 0 \text{ for all } t\right\} \subset \{X_\sigma(V) = 0\}\\ &= \{\operatorname{supp} X_\sigma \cap V = \varnothing\} \quad \text{a.s.}\end{aligned}$$

If $\mu(Q) = 0$, then, by I.1.9.C, $X_{\tau \wedge t}(Q) = \mu(Q) = 0$ P_μ-a.s., which implies 4.2.D.

4.3.

LEMMA 4.1. *Let σ be the first exit time from an open set $U \subset Q$. Then*

$$\{X_\sigma(Q) = 0\} \subset \left\{G_Q \subset \overline{U}\right\} \quad a.s. \tag{4.5}$$

PROOF. By the Markov property (Theorem I.1.3),

$$(4.6)\qquad \begin{aligned} &P_\mu\{X_{\sigma\wedge t}(Q_{<t}) > 0,\ X_\sigma(Q) = 0\} \\ &\quad = P_\mu\{X_{\sigma\wedge t}(Q_{<t}) > 0,\ P_{X_{\sigma\wedge t}}[X_\sigma(Q) = 0]\}. \end{aligned}$$

Since $X_{\sigma\wedge t}$ is concentrated, a.s., on $U^c_{<t}$ and since $U^c_{<t} \cap Q_{<t} \subset U^c \cap Q$, we have $X_{\sigma\wedge t}(Q \cap U^c) > 0$ a.s. if $X_{\sigma\wedge t}(Q_{<t}) > 0$. Suppose $\nu(Q \cap U^c) > 0$ for some ν, and let $\hat{\nu}$ be the restriction of ν to $Q \cap U^c$. Then, by I.1.9.C, $P_{\hat{\nu}}$-a.s., $X_\sigma(Q) = \hat{\nu}(Q) > 0$ and, therefore, $P_\nu\{X_\sigma(Q) = 0\} \le P_{\hat{\nu}}\{X_\sigma(Q) = 0\} = 0$. Therefore (4.6) implies $\{X_\sigma(Q) = 0\} \subset \{X_{\sigma\wedge t}(Q_{<t}) = 0\}$ a.s.

By applying the Markov property of X once more, we get

$$(4.7)\qquad \begin{aligned} P_\mu\{X_\sigma(Q) = 0,\ \tilde{X}_t(\bar{U}^c) > 0\} &\le P_\mu\{X_{\sigma\wedge t}(Q_{<t}) = 0,\ \tilde{X}_t(\bar{U}^c) > 0\} \\ &= P_\mu\{X_{\sigma\wedge t}(Q_{<t}) = 0,\ F(X_{\sigma\wedge t})\} \quad \text{a.s.}, \end{aligned}$$

where $\tilde{X}$ is the part of X in Q and $F(\nu) = P_\nu\{\tilde{X}_t(\bar{U}^c) > 0\}$. If $\nu(Q_{<t}) = 0$, then $P_\nu\{X_{\sigma\wedge t} = \nu\} = 1$ and $F(\nu) = 0$ if $\nu \in \mathscr{M}(\bar{U})$. Since $X_{\sigma\wedge t}$ is concentrated on $\bar{U}_{<t} \subset \bar{U}$, (4.7) implies that, for every t, $\{X_\sigma(Q) = 0\} \subset \{\tilde{X}_t \in \mathscr{M}(\bar{U})\}$, and (4.5) follows from 4.2.A.

4.4. *Graph and PDEs.* Put $\mu \in \mathscr{M}_c(Q)$ if the support of μ is compact and is contained in Q. Let $\mathscr{M}_c = \mathscr{M}_c(S)$.

THEOREM 4.2. *Suppose that $Q \subset \tilde{Q}$ are two open sets and Γ is a closed set such that $\tilde{Q} \cap \partial Q \subset \Gamma \subset Q^c$ and $A = \partial Q \cap \Gamma^c \subset \partial_r Q$. Then*

$$(4.8)\qquad v(r,x) = -\log P_{r,\delta_x}\{G_{\tilde{Q}} \cap \Gamma = \varnothing\}$$

is the maximal positive solution of the problem

$$(4.9)\qquad \begin{aligned} \dot{v} + Lv &= v^\alpha \quad \text{in } Q, \\ v &= 0 \quad \text{on } A. \end{aligned}$$

For every $\mu \in \mathscr{M}_c$,

$$(4.10)\qquad G \text{ is compact } P_\mu\text{-a.s.},$$

and, for every $\mu \in \mathscr{M}_c(Q)$,

$$(4.11)\qquad P_\mu\{G_{\tilde{Q}} \cap \Gamma = \varnothing\} = \exp(-\langle v, \mu\rangle).$$

PROOF. By Theorem 1.7, there exists a sequence of regular open sets $U_n \uparrow \Gamma^c$ such that $\bar{U}_n \subset U_{n+1}$. By Lemma 1.2, the sets $Q_n = U_n \cap Q$ are regular. Clearly, $Q_n \uparrow Q$, $\bar{Q}_{n-1} \cap Q \subset Q_n$ and $\partial Q \cap \partial Q_n \uparrow A$.

The rest of the proof can be found in [28], Section 2.3. We sketch the main steps here.

Put $B_n = Q \cap \partial Q_n$, $\Gamma_n = \bar{B}_n$, $A_n = \partial Q_n \setminus \Gamma_n$ and $Z_n = X_{\tau_n}(\Gamma_n)$, where τ_n is the first exit time from Q_n. By 4.2.C and Lemma 4.1, $\{G_{\tilde{Q}} \subset \bar{Q}_{n-1}\} \subset$

$\{Z_n = 0\} \subset \{G_{\tilde{Q}} \subset \overline{Q}_n\}$ P_μ-a.s. if $\mu \in \mathscr{M}(Q_n)$. Therefore, for every $\mu \in \mathscr{M}_c(Q)$,

$$\bigcup\{Z_n = 0\} = \bigcup\left\{G_{\tilde{Q}} \subset \overline{Q}_n\right\} = C \quad P_\mu\text{-a.s.,} \tag{4.12}$$

where $C = \{G_{\tilde{Q}}$ is compact and $G_{\tilde{Q}} \cap \Gamma = \varnothing\}$.

By 3.4.E, $P_\mu\{Z_n = 0\} = \exp\langle -v_n, \mu\rangle$, where $v_n(r,x) = -\log P_{r,\delta_x}\{Z_n = 0\}$ satisfies the following conditions:

$$\dot{v}_n + Lv_n = v_n^\alpha \quad \text{in } Q_n, \qquad v_n = 0 \quad \text{on } A_n.$$

Therefore, by (4.12),

$$\lim\langle v_n, \mu\rangle = -\log P_\mu(C) \tag{4.13}$$

and $v(r,x) = \lim v_n(r,x) = -\log P_{r,\delta_x}(C)$. By 3.4.C, v satisfies (4.9) and, by the dominated convergence theorem, (4.13) implies that $P_\mu(C) = \exp(-\langle v, \mu\rangle)$. By taking $\tilde{Q} = S$ and $\Gamma = \varnothing$ and by applying 3.3.C, we get (4.10), which implies (4.11). The maximality of v follows easily from the comparison principle.

4.5.

Corollary. *For every open set* $Q \subset \tilde{Q}$,

$$v(r,x) = -\log P_{r,\delta_x}\{G_{\tilde{Q}} \subset Q\} \tag{4.14}$$

is the maximal solution of

$$\dot{v} + Lv = v^\alpha \quad \text{in } Q. \tag{4.15}$$

In particular, this is true for

$$v(r,x) = -\log P_{r,\delta_x}\{G_Q \subset Q\} = -\log P_{r,\delta_x}\{G \subset Q\}. \tag{4.16}$$

To prove this statement, take $\Gamma = Q^c$ in (4.8) and then put $\tilde{Q} = Q$ and $\tilde{Q} = S$ in (4.14).

4.6. *CB property*. We say that the *B property* holds for $Z \in \hat{\mathscr{F}}_X$ if

$$P_{\mu_1+\mu_2}Z = P_{\mu_1}ZP_{\mu_2}Z, \quad \text{for all } \mu_1, \mu_2 \in \mathscr{M}(S). \tag{4.17}$$

By the definition, the B property holds for a set C if it holds for $Z = 1_C$. We say that the *CB property* holds for $Z \in \hat{\mathscr{F}}_X$, $\mu \in \mathscr{M}$, if

$$\log P_\mu Z = \int \log P_{r,\delta_x}Z\mu(dr, dx). \tag{4.18}$$

In contrast to the B property, the CB property can be destroyed by bounded passage to the limit. Note that the CB property is equivalent to

$$P_\mu Z = \exp(-\langle v, \mu\rangle), \tag{4.19}$$

where

$$v(r,x) = -\log P_{r,\delta_x}Z. \tag{4.20}$$

According to (I.1.8), this equation is satisfied for an arbitrary $\mu \in \mathscr{M}(S)$ if $Z = \exp\langle -f, X_\tau \rangle$. By (3.19), the CB property holds for $\{X_\tau = 0\}$ and every $\mu \in \mathscr{M}(S)$.

It follows from (4.11) (applied to $Q = \Gamma^c$ and $\tilde{Q} = S$) that the CB property for $\{G \cap \Gamma = \varnothing\}$ and μ holds if Γ is closed and if $\operatorname{supp} \mu$ is compact and disjoint from Γ.

In general, this is not true. For instance, the CB property for $C = \{G \cap \Gamma = \varnothing\}$ and μ does not hold if $\Gamma = S_{\le t}$, $\mu(\Gamma) = 0$ but $\Gamma \cap \operatorname{supp} \mu \neq \varnothing$. However we have the following:

THEOREM 4.3. *Let A be an analytic set and let $B = \operatorname{supp} \mu$ be disjoint from A. Then the CB property holds for $\{G \cap A = \varnothing\}$ and μ.*

PROOF. Arguments in the proof of Theorem 4.2 show that

$$\{X_{\tau_n} = 0\} \uparrow \{G \subset Q\} \quad P_\mu\text{-a.s.}, \tag{4.21}$$

if τ_n is the first exit time from $Q_n = \{z: \operatorname{dist}(z, Q^c) > 1/n\}$ and if $\operatorname{dist}(Q^c, \operatorname{supp} \mu) > 0$ [cf. (4.12)].

First, suppose that A is compact. By applying (4.21) to $Q = A^c$, we get

$$\{X_{\tau_n} = 0\} \uparrow \{G \cap A = \varnothing\} \quad P_\mu\text{-a.s.}$$

Choose n_0 such that $\operatorname{supp} \mu \subset Q_{n_0}$ and put $\sigma = \tau_{n_0}$. Since X_σ is supported by a compact set $\{z: \operatorname{dist}(z, A) = 1/n_0\}$ which is disjoint from A, we have $P_{X_\sigma}\{G \cap A = \varnothing\} = \exp\langle -u, X_\sigma \rangle$ for some u and, by the Markov property, $P_\mu\{G \cap A = \varnothing\} = P_\mu P_{X_\sigma}\{G \cap A = \varnothing\} = \exp\langle -v, \mu \rangle$ for some v. This equation is true for every μ with support disjoint from A. By applying it to $\mu = \delta_{r,x}$, $(r, x) \notin A$, we prove the statement of our theorem. The case of an arbitrary analytic A can be reduced to the case of a compact A as in [28] (see part (ii) in the proof of Lemma 2.3 there). □

5. Probabilistic representation for an arbitrary solution.

5.1. Suppose that Q is an arbitrary open set in S and let ψ satisfy 3.1.A and 3.1.B. In this section we establish a 1–1 correspondence between the solutions of

$$\dot{v} + Lv = \psi(v) \quad \text{in } Q \tag{5.1}$$

and a class of random variables Z determined by the behavior of the superdiffusion near the boundary of Q.

Fix a sequence of regular bounded open sets $Q_n \uparrow Q$ such that $\overline{Q}_n \subset Q_{n+1}$. Let τ_n be the first exit time from Q_n. It follows from the Markov property of X that $\{X_{\tau_n}, P_\mu\}$ is a Markov chain. We claim that, if v is a solution of (5.1), then $F_n = \exp\langle -v, X_{\tau_n} \rangle$ is a martingale relative to $(\mathscr{F}_{\le \tau_n}, P_\mu)$ for every $\mu \in \mathscr{M}(S)$. Indeed, by the mean value property,

$$P_{r, \delta_x} F_n = e^{-v(r,x)} \quad \text{in } Q_n. \tag{5.2}$$

By I.1.9.C, this is true also in Q_n^c. Since the CB property holds for F_n, we have $P_\mu F_n = \exp(-\langle v, \mu\rangle)$ and, therefore,

$$P_\mu\{F_n | \mathscr{F}_{\le \tau_m}\} = P_{X_{\tau_m}} F_n = F_m \quad P_\mu\text{-a.s.,} \tag{5.3}$$

for all $\mu \in \mathscr{M}(S)$ and all $m < n$.

The bounded martingale F_n converges a.s. and therefore there exists, P_μ-a.s., the limit

$$Z = \lim_{n\to\infty} \langle v, X_{\tau_n}\rangle \tag{5.4}$$

in the topology of $[0, \infty]$. Clearly, Z is measurable with respect to the intersection of all σ-algebras $\mathscr{F}_{\ge \tau_n}$. We call such a Z a tail functional of $\{X_{\tau_n}\}$.

THEOREM 5.1. *Formulae* (5.4) *and*

$$v(r, x) = -\log P_{r, \delta_x} e^{-Z} \tag{5.5}$$

establish a 1–1 *correspondence between the solutions v of equation* (5.1) *and the class of positive functions Z with the following properties*:

5.1.A. *Z is a tail functional of $\{X_{\tau_n}\}$.*
5.1.B. *The CB property holds for e^{-Z} and $\mu \in \mathscr{M}_c(Q)$.*
5.1.C. $P_{r,\delta_x}\{Z < \infty\} > 0$, *for all $(r, x) \in Q$.*

(*We do not distinguish functions Z which are equivalent with respect to the family $\{P_\mu, \mu \in \mathscr{M}\}$.*)

PROOF. Formula (5.4) determines a class of equivalent functions Z subject to condition 5.1.A. Moreover,

$$P_\mu e^{-Z} = \lim P_\mu F_n \tag{5.6}$$

and 5.1.B follows from the CB property for F_n. Formulae (5.2) and (5.6) imply (5.5). Since $v < \infty$, (5.5) implies 5.1.C.

Now suppose that Z satisfies 5.1.A–5.1.C and let v be given by (5.5). By 5.1.C, $v < \infty$ in Q. By 5.1.A and the Markov property of $\{X_{\tau_n}, P_\mu\}$,

$$P_\mu\{e^{-Z} | \mathscr{F}_{\le \tau_n}\} = P_{X_{\tau_n}} e^{-Z} \quad P_\mu\text{-a.s.,} \tag{5.7}$$

for all $\mu \in \mathscr{M}$ and all n. If $\mu \in \mathscr{M}_c(Q)$, then $X_{\tau_c} \in \mathscr{M}_c(Q)$ P_μ-a.s., by I.1.9.A, and

$$P_{X_{\tau_n}} e^{-Z} = \exp\langle -v, X_{\tau_n}\rangle, \tag{5.8}$$

by 5.1.B. We get from (5.7) and (5.8) that

$$e^{-Z} = \lim_{n\to\infty} \exp\langle -v, X_{\tau_n}\rangle \quad P_\mu\text{-a.s.,}$$

which implies (5.4). By (5.5), (5.7) and (5.8),

$$e^{-v(r,x)} = P_{r,\delta_x} e^{-Z} = P_{r,\delta_x} \exp\langle -v, X_{\tau_n}\rangle.$$

By Theorem 3.1, v satisfies (5.1). □

5.2. We say that Z is a *shift-invariant functional* of $\{X_{\tau_n}\}$ if there exists a measurable function G from $\mathscr{M}^\infty$ to $\mathbb{R}$ such that

$$Z = G(X_{\tau_k}, X_{\tau_{k+1}}, \dots, X_{\tau_{k+n}}, \dots) \quad P_\mu\text{-a.s.,} \tag{5.9}$$

for all $\mu \in \mathscr{M}(S)$ and all k. Clearly, this condition holds for Z determined by (5.4) if we put

$$G(\mu_1, \mu_2, \dots, \mu_n, \dots) = \limsup \langle v, \mu_n \rangle.$$

Therefore Theorem 5.1 holds with condition 5.1.A replaced by the following condition:

5.2.A. Z is a shift-invariant functional of $\{X_{\tau_n}\}$.

Our arguments also show that condition 5.1.B can be replaced by the following:

5.2.B. The CB property holds for e^{-Z} and for all $\mu \in \mathscr{M}(S)$.

6. G-regularity.

6.1. Denote by $G(\Delta)$ the intersection of the graph G with $S_\Delta = \Delta \times E$. Consider an open set $Q \subset S$ and put

$$T_{r+} = \sup\{t \colon t > r, G(r, t] \subset Q\} \tag{6.1}$$

(if the set in brackets is empty, then we set $T_{r+} = r$). We say that a point $(r^0, x^0) \in \partial Q$ is *G-regular* if

$$P_{r^0, \delta_{x^0}}\{T_{r^0+} > r^0\} = 0. \tag{6.2}$$

[By the Blumenthal 0–1 law, the probability (6.2) is always equal to 0 or 1.] Clearly, (6.2) is equivalent to the condition

$$P_{r^0, \delta_{x^0}}\{G(r^0, t] \subset Q\} = 0, \quad \text{for every } t > r^0. \tag{6.3}$$

LEMMA 6.1. *A point $(r^0, x^0) \in \partial Q$ is G-regular if and only if*

$$P_{r^0, \delta_{x^0}}\{G(r^0, \infty) \subset Q\} = 0. \tag{6.4}$$

PROOF. Clearly, (6.3) implies (6.4). By the Markov property of X,

$$P_{r^0, \delta_{x^0}}\{G(r^0, \infty) \subset Q\} = P_{r^0, \delta_{x^0}}\{\Omega^0, Y\}, \tag{6.5}$$

where $\Omega^0 = \{G(r^0, t] \subset Q\}$ and $Y = P_{t, X_t}\{G \subset Q\}\}$. By Theorem 4.2, for every $\omega \in \Omega^0$, the CB property holds for $\{G \subset Q\}$ and X_t. Hence $Y = \exp\langle -v, X_t\rangle$, where v is given by (4.16). By the Corollary in Section 4.5, v belongs to $\mathbb{C}^2(Q)$. If (6.4) holds, then, by (6.5), $\exp\langle -v, X_t\rangle = 0$ $P_{r^0, \delta_{x^0}}$-a.s. on the set Ω^0. On the other hand, v is bounded on a compact set $G(r^0, t] \cap S_t$ and, therefore, $\langle v, X_t\rangle < \infty$ $P_{r^0, \delta_{x^0}}$-a.s. on Ω^0, which implies (6.3). □

LEMMA 6.2. *Let* $B_R(x^0) = \{x: |x - x^0| < R\}$ *and let* $Q = (r^0, \infty) \times B_R(x^0)$. *Then* (r^0, x^0) *is a G-irregular point of Q. In other words,*

$$P_{r^0, \delta_{x^0}}\{\hat{T} > r^0\} = 1, \tag{6.6}$$

where

$$\hat{T} = \inf\left\{t: t > r^0, \operatorname{supp} X_t \cap B_R(x^0)^c \neq \varnothing\right\}. \tag{6.7}$$

PROOF. Let $\tilde{Q} = (t, \infty) \times B_R(x^0)$ with $t < r^0$. Note that $\{G \subset \tilde{Q}\} \subset \{G(r^0, \infty) \subset Q\}$. Therefore

$$P_{r^0, \delta_{x^0}}\{G(r^0, \infty) \subset Q\} \geq P_{r^0, \delta_{x^0}}\{G \subset \tilde{Q}\} = \exp\left[-v(r^0, x^0)\right],$$

where v is the maximal solution of $\dot{v} + Lv = v^{\alpha}$ in $\tilde{Q}$. By Lemma 6.1, (r^0, x^0) is G-irregular. □

LEMMA 6.3. *The G-regularity of a point* (r^0, x^0) *depends only on the shape of the open set Q in a neighborhood of* (r^0, x^0).

PROOF. Denote by $\tilde{Q}$ the intersection of Q with the set $U_R(r^0, x^0)$ given by (3.13). Let $T = T_{r^0+}$ and $\tilde{T} = \tilde{T}_{r^0+}$ be defined by (6.1) and an analogous formula for $\tilde{Q}$. Clearly, $\tilde{T} = \min\{T, \hat{T}, r^0 + R\}$, where $\hat{T}$ is determined by (6.7). By (6.6), $P_{r^0, \delta_{x^0}}\{\tilde{T} > r^0\} = P_{r^0, \delta_{x^0}}\{T > r^0\}$. □

LEMMA 6.4. *For every* (r^0, x^0),

$$G = (r^0, x^0) \cup G(r^0, \infty) \quad P_{r^0, \delta_{x^0}}\text{-}a.s. \tag{6.8}$$

PROOF. By 4.2.B, G is, $P_{r^0, \delta_{x^0}}$-a.s., the closure of the set $(r^0, x^0) \cup G(r^0, \infty)$. By (6.6), the intersection of $G(r^0, \infty)$ with S_{r^0} is contained in $B_R(x^0)$ for every $R > 0$. Hence it consists of a single point (r^0, x^0).

6.2. *Test of G-regularity.*

THEOREM 6.1. *A point* $(r^0, x^0) \in \partial Q$ *is G-regular if and only if the maximal solution v of* (4.15) *tends to* ∞ *at* (r^0, x^0).

PROOF. (i) Let σ_R be the first exit time from $U_R(r^0, x^0)$ given by (3.13). Put $\Omega^0 = \{\operatorname{supp} X_{\sigma_B} \subset Q\}$ and $Y = P_{X_{\sigma_B}}\{G \subset Q\}$. By 4.2.C and Lemma 6.4, $\operatorname{supp} X_{\sigma_B}$ is contained, $P_{r^0, \delta_{x^0}}$-a.s., in $G(r^0, \infty)$ and, by the Markov property, Theorem 4.2 and the Corollary in Section 4.5,

$$\begin{aligned} P_{r^0, \delta_{x^0}}\{G(r^0, \infty) \subset Q\} &\leq P_{r^0, \delta_{x^0}}\{\Omega^0, Y\} \\ &= P_{r^0, \delta_{x^0}}\left\{\Omega^0, \exp\left\langle -v, X_{\sigma_R}\right\rangle\right\} \\ &\leq P_{r^0, \delta_{x^0}} \exp\langle -\tilde{v}, X_{\sigma_R}\rangle, \end{aligned} \tag{6.9}$$

where $\tilde{v} = v$ in Q, $\tilde{v} = \infty$ on Q^c. Let N_R be the infimum of $\tilde{v}$ on $\overline{U}_R(r^0, x^0)$. If $v \to \infty$ at (r^0, x^0), then $N_R \to \infty$ as $R \downarrow 0$. By Lemma 5.1, a.s., $\langle 1, X_{\sigma_R} \rangle \to 1$ as $R \downarrow 0$. By (6.9),

$$P_{r^0, \delta_{x^0}}\{G(r^0, \infty) \subset Q\} \leq P_{r^0, \delta_{x^0}} \exp\langle -N_R, X_{\sigma_R} \rangle \to 0$$

and (r^0, x^0) is G-regular by Lemma 6.1.

(ii) Note that $\{G[s, \infty) \subset Q\} = \{G \subset Q \cup S_{<s}\} \supset \{G \subset Q\}$ and, therefore, $v_s(r, x) = -\log P_{r, \delta_x}\{G[s, \infty) \subset Q\} \leq -\log P_{r, \delta_x}\{G \subset Q\} = v(r, x)$. By the Corollary in Section 4.5, v_s is continuous in $Q \cup S_{<s}$ and therefore, for every $s > r^0$,

$$\liminf v(r, x) \geq \liminf v_s(r, x) = v_s(r^0, x^0) \tag{6.10}$$

as $(r, x) \to (r^0, x^0)$. Clearly,

$$v_s(r^0, x^0) \uparrow -\log P_{r^0, \delta_{x^0}}\{G(r^0, \infty) \subset Q\}. \tag{6.11}$$

If (r^0, x^0) is G-regular, then $v(r, x) \to \infty$ by (6.10), (6.11) and Lemma 6.1. □

6.3. *Examples.*

Example 6.1. Clearly, the boundary of $S_{<t}$ is G-regular. By Theorem 6.1, the maximal solution of (4.15) tends to ∞ at $\partial S_{<t}$. This follows also from an explicit formula

$$v(r, x) = [(\alpha - 1)(t - r)]^{-1/(\alpha - 1)},$$

which gives a solution of (4.15) such that $v \to \infty$ as $r \uparrow t$.

Example 6.2. For $S_{>t}$ the maximal solution

$$v(r, x) = -\log P_{r, \delta_x}\{G \subset Q\} = 0.$$

Clearly, ∂Q is G-irregular.

6.4. Consider a family of domains

$$Q_c = \{(t, x) : |x|^2 < c\rho(t), t > 0\}, \qquad c > 0, \tag{6.12}$$

where ρ is an increasing function such that $\rho(0) = 0$. It follows from a result in [13] (stated as Theorem 9.3.2.3 in [9]) that, if $L = \frac{1}{2}\Delta$ and if

$$\rho(t) = \left[\frac{2}{\alpha - 1} t |\log t|\right]^{1/2}, \tag{6.13}$$

then $z^0 = (0, 0)$ is G-irregular for Q_c if $c > 1$ and it is regular if $c < 1$. For $\alpha = 2$, this result was obtained earlier in [88].

An interesting problem is to get criteria for G-regularity and G-irregularity analytically by applying Theorem 6.1 (in the same spirit as was done for the heat equation in [77]). Let Q_c be given by (6.12) and put $Q = Q_1$. To prove that z^0 is G-irregular for Q_c, $c > 1$, it is sufficient to construct, for some $a > 0$, a function $u \geq 0$ in $U = Q \cap S_{<a}$ such that

$$\begin{aligned} \dot{u} + Lu &\leq u^{\alpha} && \text{in } U, \\ u &= \infty && \text{on } \partial Q \cap S_{(0,a)}, \\ \liminf u &< \infty && \text{at } z^0. \end{aligned}$$

Indeed, let $\tilde{Q} = Q_c \cup S_{>a}$. Note that $\partial U = A \cup B$, where $A \subset S_a$ and $B = \partial Q \cap S_{<a}$. Moreover, the set $T = \partial U \setminus \{z^0\}$ is total in ∂U. The function $v(r,x) = -\log P_{r,\delta_x}\{G \subset \tilde{Q}\}$ vanishes on A and therefore $u - v$ satisfies condition (3.7) (with Q replaced by U). Clearly, (3.6) also holds in U. By the comparison principle, $v \leq u$ in U. Hence v does not tend to ∞ at z^0 and, by Theorem 6.1, z^0 is a G-irregular point of $\tilde{Q}$. Obviously, this implies that z^0 is G-irregular for Q_c. A similar test can be given for G-regularity of z^0 for Q_c, $c < 1$.

6.5. If $c \in \partial_r Q$, then, for every $N \in \mathbb{R}_+$, there exists, by Theorem 3.1, a solution of (4.15) with the boundary value N at c. We conclude from Theorem 6.1 that c is G-regular for Q. The converse is false. If $d < \kappa_\alpha = 2\alpha/(\alpha - 1)$, then

$$v(r,x) = \left[(\alpha - 1)^{-1}(\kappa_\alpha - d)\right]^{1/(\alpha-1)}|x|^{-2/(\alpha-1)}$$

is a solution of $\dot{v} + \frac{1}{2}\Delta v - v^{\alpha} = 0$ in $Q = \{(r,x) \quad x \neq 0\}$. Hence every $c \in \mathbb{R}_+ \times \{0\} \subset \partial Q$ is G-regular. However, it is irregular for $d \geq 2$. Hence it is G-regular but irregular for $2 \leq d < \kappa_\alpha = 2 + 2/(\alpha - 1)$. (The existence of G-regular points which are not regular follows also from comparing the results of [13] and [77].)

7. *G*-polar sets.

7.1. Denote by $\mathscr{M}_F$ the class of all measures $\mu \in \mathscr{M}$ with their support disjoint from F.

DEFINITION. An analytic set A is called *G-polar* if

(7.1) $$P_\mu\{G \cap A \cap F = \varnothing\} = 1,$$

for every closed set F and every $\mu \in \mathscr{M}_F$.

Note that all analytic subsets of a G-polar set are G-polar.

[This is not true with a definition which seems more natural at first glance. Note that the condition

(7.2) $$P_\mu\{G \cap A = \varnothing\} = 1$$

can hold only for $\mu \in \mathscr{M}_A$ because $P_\mu\{\operatorname{supp}\mu \subset G\} = 1$. However, the class $\mathfrak{N}$ of analytic sets A which satisfy (7.2) for all $\mu \in \mathscr{M}_A$ does not fit the criteria to be called the class of G-polar sets: It is possible that $\tilde{A} \subset A$ and $A \in \mathfrak{N}$ but $\tilde{A} \notin \mathfrak{N}$ (take, e.g., $A = S_{\leq 1}$, $\tilde{A} = S_1$).]

It follows from Theorem 4.1 that A is G-polar if and only if all compact sets $K \subset A$ are G-polar.

7.2.

LEMMA 7.1. *An analytic set A is G-polar if and only if*

$$P_{r,\delta_x}\{G(r,\infty) \cap A = \varnothing\} = 1, \quad \textit{for all } (r,x). \tag{7.3}$$

REMARK. By Lemma 6.1, this implies that if a closed set Γ is G-polar, then $\Gamma = \partial(\Gamma^c)$ and all points of $\partial(\Gamma^c)$ are G-irregular.

PROOF OF LEMMA 7.1. If (7.1) holds, then, by taking $F = S_{\geq s}$, $\mu = \delta_{r,x}$, we get

$$P_{r,\delta_x}\{G[s,\infty) \cap A\} = 1 \tag{7.4}$$

for every $s > r$, which implies (7.3).

On the other hand, (7.3) implies that, for every F, $P_{r,\delta_x}\{G(r,\infty) \cap F \cap A = \varnothing\} = 1$. If $(r,x) \notin F$, then, by Lemma 6.4, $G \cap F \cap A = G(r,\infty) \cap F \cap A$ P_{r,δ_x}-a.s., and therefore (7.1) holds for $\mu = \delta_{r,x}$. By Theorem 4.2, (7.1) holds for every $\mu \in \mathscr{M}_F$.

7.3.

THEOREM 7.1. *An analytic set A is G-polar if and only if the following two conditions hold*:

7.3.A. *A contains no set $S_{<t}$.*
7.3.B. $P_{r,\delta_x}\{G \cap A = \varnothing\} = 1$, *for all $(r,x) \notin A$.*

PROOF. Clearly, 7.3.A and 7.3.B are necessary for G-polarity. Suppose that they are satisfied. By 7.3.A, for every r, there exist $r^0 < r$ and x^0 such that $(r^0, x^0) \notin A$. If $s > r$, then

$$0 = P_{r^0,\delta_{x^0}}\{G \cap A \neq \varnothing\} \geq P_{r^0,\delta_{x^0}}\{X_s(A_s) > 0\}, \tag{7.5}$$

where A_s is the s-section of A. By (7.5) and (I.1.20),

$$0 = \Pi_{r^0,x^0}\{\xi_s \in A_s\} = \int_{A_s} p(r^0, x^0; s, y)\, ds,$$

where p is the transition density of ξ. Since p is strictly positive, the Lebesgue measure of A_s is equal to 0. Therefore, for all x,

$$P_{r,\delta_x} X_s(A_s) = \int_{A_s} p(r,x;s,y)\, ds = 0. \tag{7.6}$$

If $t > s > r$, then, by (7.6) and by the Markov property,

$$\begin{aligned} P_{r,\delta_x}\{G[t,\infty) \cap A = \varnothing\} &= P_{r,\delta_x}\{X_s(A_s) = 0, G[t,\infty) \cap A = \varnothing\} \\ &= P_{r,\delta_x}\big\{X_s(A_s) = 0, P_{s,X_s}\{G[t,\infty) \cap A = \varnothing\}\big\}. \end{aligned} \tag{7.7}$$

By the CB property, for every ν,

$$\log P_{s,\nu}\{G \cap A \cap S_{\geq t} = \varnothing\} = \int \nu(dx) \log P_{s,\delta_x}\{G \cap A \cap S_{\geq t} = \varnothing\}.$$

If $\nu(A_s) = 0$, then the right-hand side vanishes by 7.3.B and $\log P_{s,\nu}\{G[t,\infty) \cap A = \varnothing\} = 0$. By (7.7), $P_{r,\delta_x}\{G[t,\infty) \cap A = \varnothing\} = 1$. Since this is true for all $t > r$, we get (7.3), and A is G-polar.

7.4.

THEOREM 7.2. *Suppose that a closed set* Γ *satisfies* 7.3.A *and let* $Q = \Gamma^c$. *The following five conditions are equivalent*:

7.4.A. Γ *is G-polar.*
7.4.B. $P_\mu\{G \subset Q\} = 1$, *for all* $\mu \in \mathscr{M}_\Gamma$.
7.4.C. $P_{r,\delta_x}\{G \subset Q\} = 1$ *for all* $(r, x) \in Q$.
7.4.D. *The maximal solution of* (4.15) *is equal to* 0.
7.4.E. *Equation* (4.15) *has no solutions except* 0.

PROOF. By Theorem 7.1, 7.4.A is equivalent to 7.4.C. By Theorem 4.2, the CB property holds for $\{G \subset Q\}$ and $\mu \in \mathscr{M}_\Gamma$ and therefore 7.4.B is equivalent to 7.4.C. By Section 4.5, the maximal solution of (4.15) is given by (4.16), which means that 7.4.C is equivalent to 7.4.D. The equivalence of 7.4.D and 7.4.E is obvious.

8. Range and R-polar sets.

8.1. Now we interpret X_t and all its parts $\tilde{X}_t$ as measures in E by using a natural correspondence between E and $\{t\} \times E$. The *range* R *of* X is the minimal closed subset of E which supports all measures X_t. The *range* R_D *of* X *in an open set* $D \subset E$ is the minimal closed set R_D which supports all measures $\tilde{X}_t$ (here $\tilde{X}$ is the part of X in $Q = \mathbb{R}_+ \times D$). If G is compact (which is true P_μ-a.s. for all $\mu \in \mathscr{M}_c$), then R is the projection of G on E and $\{R \cap B = \varnothing\} = \{G \cap (\mathbb{R}_+ \times B) = \varnothing\}$. This makes it possible to get information on R from the results on G obtained in the previous sections.

In this section we assume that the diffusion ξ is homogeneous (i.e., the coefficients of the operator L do not depend on time). As before, we assume that $K(dt) = dt$ and ψ is given by (3.11). The corresponding Markov process $(X_t, P_{r,\nu})$ has a stationary transition function. We set $P_\nu = P_{0,\nu}$ for every $\nu \in \mathscr{M}(E)$.

8.2. Put $a \in \partial_r D$ if $(0, a)$ is a regular point of $\mathbb{R}_+ \times D$. [Note that $\partial_r(\mathbb{R}_+ \times D) \subset \mathbb{R}_+ \times \partial D$.] The following result is a trivial implication of Theorem 4.2.

THEOREM 8.1. *Suppose that* $D \subset \tilde{D}$ *are two open sets and* B *is a closed set in* E *such that* $\tilde{D} \cap \partial D \subset B \subset D^c$ *and* $A = \partial D \cap B^c \subset \partial_r D$. *Then*

$$v(x) = -\log P_{\delta_x}\{R_{\tilde{D}} \cap B = \varnothing\} \tag{8.1}$$

is the maximal positive solution of the problem

$$Lv = v^\alpha \quad \text{in } D, \tag{8.2}$$

$$v = 0 \quad \text{on } A. \tag{8.3}$$

For every $\nu \in \mathscr{M}_c(E)$,

$$R \text{ is compact } P_\nu\text{-a.s.} \tag{8.4}$$

COROLLARY. *For every* D,

$$v(x) = -\log P_{\delta_x}\{R \subset D\} = -\log P_{\delta_x}\{R_D \subset D\} \tag{8.5}$$

is the maximal solution of (8.2).

8.3. Theorem 4.3 implies the following theorem:

THEOREM 8.2 (CB property). *If A is analytic and if $\nu \in \mathscr{M}_A$, then*

$$\log P_\nu\{R \cap A = \varnothing\} = \int \nu(dx) \log P_{\delta_x}\{R \cap A = \varnothing\}.$$

8.4. For every $s > 0$, we denote by R^s the minimal closed set in E which supports all X_t with $t \geq s$. An analytic set A is called *R-polar* if

$$P_\nu\{R^s \cap A = \varnothing\} = 1, \tag{8.6}$$

for all $s > 0$ and all $\nu \in \mathscr{M}(E)$.

THEOREM 8.3. *Each of the following conditions is necessary and sufficient for R-polarity of an analytic set A*:

8.4.A. $P_{\delta_x}\{R \cap A = \varnothing\} = 1$ *for all* $x \notin A$.
8.4.B. $\mathbb{R}_+ \times A$ *is G-polar*.

COROLLARY. *A closed set $\Gamma \neq E$ is R-polar if and only if* 0 *is the only solution of the equation $Lv = v^\alpha$ in $D = \Gamma^c$.*

8.5.

EXAMPLE. The function

$$v(r, x) = c|x|^{-2/(\alpha-1)} \tag{8.7}$$

satisfies the equation

$$\tfrac{1}{2}\Delta v = v^\alpha \quad \text{in } \mathbb{R}^d \setminus \{0\} \tag{8.8}$$

if

$$d < \kappa_\alpha = \frac{2\alpha}{\alpha - 1}, \qquad c = \left[(\alpha-1)^{-1}(\kappa_\alpha - d)\right]^{1/(\alpha-1)}. \tag{8.9}$$

Hence a single point is not R-polar if $d < \kappa_\alpha$. (If $d \geq \kappa_\alpha$, then a singleton is R-polar by the test in Theorem 12.3.)

9. Additive functionals of a diffusion.

9.1. In the next section we investigate additive functionals of superdiffusions. As a heuristic introduction to Section 10, we state here, without proofs, analogous results for additive functionals of diffusions. They are well known (in a slightly different form) to specialists and they can be proved, for instance, by arguments similar to those in Section 1.10.

Let ξ be a diffusion in a domain $Q \subset S$ with generator L. This is a Markov process on a random time interval $[\alpha, \zeta)$. (It can be constructed as the part in Q of a diffusion in S with ζ equal to the first exit time from Q.) An additive functional A of ξ is a random measure on $\mathbb{R}_+$ concentrated on $[\alpha, \zeta)$ such that $A(r,t)$ is measurable relative to the universal completion of $\mathscr{F}^0(r,t)$ (cf. Section I.1.4).

We introduce three levels of additive functionals.

9.2. *Ground level.* To every Borel function $f \geq 0$, there corresponds an additive functional

$$A(\Delta) = \int_\Delta f(s, \xi_s)\, ds.$$

Let $p(r,x;s,y)$ be the transition density of ξ. Then, for every $\mu \in \mathscr{M}(Q)$ and every Borel $h \geq 0$,

$$(9.1) \quad \Pi_\mu \int_{\mathbb{R}} h(s,\xi_s) A(ds) = \int_{Q\times Q} \mu(dr,dx) p(r,x;s,y) h(s,y) \eta(ds,dy),$$

where $\eta(ds,dy) = f(s,y)\, ds\, dy$. We call η the *characteristic measure of* A.

9.3. *First level.* Consider $k(r,x;s,y) = k(s-r, y-x)$, where

$$(9.2) \qquad k(r,x) = \begin{cases} \frac{1}{2}(2\pi r)^{-d/2} \exp\{-r/2 - |x|^2/2r\}, & \text{for } r > 0,\\ k(r,x) = 0, & \text{for } r \leq 0. \end{cases}$$

[Note that $\int k(r,x)\, dr\, dx = 1$ and that $2k(r,x;s,y)$ is the transition density of the Brownian motion with the killing rate $\frac{1}{2}$.] Put $\eta \in N_\infty$ if

$$(9.3) \qquad F(r,x) = \int k(r,x;s,y)\eta(ds,dy)$$

is bounded. Let $k_\varepsilon(r,x) = \varepsilon^{-d-1} k(r/\varepsilon, x/\varepsilon)$. Consider a *mollifier* $\rho_n = k_{\varepsilon_n}$, where $\varepsilon_n \downarrow 0$. Let

$$A_n(\Delta) = \int_\Delta f_n(s,\xi_s)\, ds,$$

where

$$(9.4) \qquad f_n(r,x) = \int \rho_n(r-s; x-y)\eta(ds,dy).$$

FACT. Suppose that $\mu(dr, dx) = q(r, x)\, dr\, dx$ with a bounded q. Then, for every $\eta \in N_\infty$ and every finite interval Δ, $A_n(\Delta)$ converges in $L^2(\Pi_\mu)$ to $A(\Delta)$, where A is an additive functional with the characteristic measure η [i.e., (9.1) holds for all $h \geq 0$].

9.4. *Second level.* Put $\eta \in \overline{N}_\infty$ if there exist $\eta_n \in N_\infty$ such that $\eta_n \uparrow \eta$. Clearly, $A_{\eta_n} \uparrow A$, where A is an additive functional of ξ with the characteristic measure η.

FACT. A σ-finite measure η belongs to $\overline{N}_\infty$ if and only if η does not charge any polar set. [We say that an analytic set A is polar if

$$\int_A k(r, x; s, y)\nu(ds, dy) \quad \text{belongs to } L^\infty(Q)$$

only if $\nu(A) = 0$.]

REMARK. All additive functionals with σ-finite characteristic measures can be obtained as functionals of the second level.

10. Additive functionals of a superdiffusion.

10.1. The superdiffusion X with parameters (L, α) in a domain $Q \subset S$ can be constructed as the part in Q of the superprocess with parameters (ξ, K, ψ), where ξ is the diffusion with generator L, $K(dt) = dt$ and $\psi(v) = v^\alpha$. We investigate a special class of additive functionals of X in three stages similar to the stages in Section 9. At one point we use analytical results proved for the domains of the form $Q = \Delta \times D$, where Δ is a finite interval and D is a bounded domain in E with a smooth boundary. This class is sufficient for our purposes.

10.2. *Ground level.* To every positive Borel function f on Q, there corresponds an additive functional

$$I(\Delta) = \int_\Delta \langle f, X_s \rangle\, ds.$$

Note that $I(\mathbb{R}_+) = \langle f, Y \rangle$, where $Y(ds, dx) = ds\, X_s(dx)$ and, for every $\mu \in \mathscr{M}(Q)$,

$$(10.1) \qquad P_\mu I(\mathbb{R}_+) = \int_{Q \times Q} \mu(dr, dx) p(r, x; s, y) \eta(ds, dy),$$

where $\eta(ds, dy) = f(s, y)\, ds\, dy$. By Theorem I.1.8,

$$(10.2) \qquad P_\mu \exp\{-I(\mathbb{R}_+)\} = \exp(-\langle u, \mu \rangle),$$

where

$$(10.3) \quad u(r, x) + \int_Q p(r, x; s, y) u(s, y)^\alpha\, ds = \int_Q p(r, x; s, y) \eta(ds, dy).$$

If f is bounded and belongs to $C^1(Q)$, then, by Theorem 3.1,

$$(10.4)\qquad \begin{aligned} \dot{u} + Lu - u^\alpha &= -f \quad \text{in } Q,\\ u &= 0 \qquad \text{on } \partial_r Q. \end{aligned}$$

10.3. *First level.* Put $\eta \in N_\alpha$ if the function F defined by (9.3) belongs to $L^\alpha(S)$. Consider a function ρ of class C^∞ with a compact support such that $\int \rho \, dr\, dx = 1$ and put $\rho^\varepsilon(r, x) = \varepsilon^{-d-1}\rho(r/\varepsilon, x/\varepsilon)$. Consider a mollifier $\rho_n = \rho^{\varepsilon_n}$, where $\varepsilon_n \downarrow 0$. Define functions f_n by formula (9.4). By the Corollary in Section 4.4 and by the comparison principle, there exist unique u_{mn} such that

$$\begin{aligned} \dot{u}_{mn} + Lu_{mn} - u_{mn}^\alpha &= \tfrac{1}{2}(f_m + f_n),\\ u_{mn} &= 0 \quad \text{on } \partial_r Q. \end{aligned}$$

It follows from results in [2] that, as $m, n \to \infty$, the u_{mn} converge in $L^\alpha(Q)$ to a solution of (10.4).

Put $\mu \in \mathfrak{M}_\alpha$ if $\mu(dr, dx) = \rho(r, x)\, dr\, dx$ with $\rho \in L^{\alpha'}(Q)$ $(1/\alpha + 1/\alpha' = 1)$. If $\mu \in \mathfrak{M}_\alpha$, then $\langle u_{mn}, \mu\rangle \to \langle u, \mu\rangle$ and therefore

$$\begin{aligned} &P_\mu\{\exp\langle -\tfrac{1}{2}f_n, Y\rangle - \exp\langle -\tfrac{1}{2}f_m, Y\rangle\}^2\\ &\qquad = \exp\langle -u_{nn}, \mu\rangle + \exp\langle -u_{mm}, \mu\rangle - 2\exp\langle -u_{mn}, \mu\rangle \to 0. \end{aligned}$$

Hence there exists I_η such that $\langle f_n, Y\rangle \to I_\eta$ in P_μ-probability (I_η can be chosen independently of μ; see [27]). It is easy to see that I_η satisfies (10.1) and (10.2). Moreover, if η is concentrated on a compact set K, then

$$(10.5)\qquad \{G \cap K = \varnothing\} \subset \{I_\eta = 0\} \quad P_\mu\text{-a.s.}$$

(see [27], Theorem 1.1).

10.4. *Second level.* The class $\overline{N}_\alpha$ is defined analogously to $\overline{N}_\infty$. To every $\eta \in \overline{N}_\alpha$, there corresponds an additive functional I_η subject to (10.1), (10.2) and (10.5).

We shall use the following result (see [2] or [27]).

THEOREM 10.1. *Denote by $\mathscr{S}_\alpha$ the class of analytic sets A such that*

$$\int_A k(r, x; s, y)\nu(ds, dy) \text{ belongs to } L^\alpha(S)$$

only if $\nu(A) = 0$. A measure η belongs to $\overline{N}_\alpha$ if and only if it does not charge any set $A \in \mathscr{S}_\alpha$.

11. *G*-polarity test.

11.1.

THEOREM 11.1. *The class of G-polar sets coincides with $\mathscr{S}_\alpha$.*

OUTLINE OF THE PROOF (cf. [28]). (i) Clearly, $A \in \mathscr{S}_\alpha$ if and only if $K \in \mathscr{S}_\alpha$ for all compact sets $K \subset A$. Therefore it is sufficient to prove the theorem for compact sets.

(ii) Let K be G-polar. Suppose

$$\int_K k(r,x;s,y)\nu(ds,dy) \text{ belongs to } L^\alpha(S)$$

and $\nu(K^c) = 0$. Then $\nu \in N_\alpha$ and, by Theorem 10.1, there exists $I_\nu \geq 0$ such that, for every $\mu \in \mathfrak{M}_\alpha$,

$$P_\mu I_\nu = \int_{Q\times Q} \mu(dr,dx)p(r,x;s,y)\nu(ds,dy) \tag{11.1}$$

and, by (10.5), $\{G \cap K = \varnothing\} \subset \{I_\nu = 0\}$ P_μ-a.s. Since K is G-polar, $P_\mu\{G \cap K = \varnothing\} = 1$ for all $\mu \in \mathscr{M}_K$. Hence $P_\mu\{I_\nu = 0\} = 1$ and, since $p(r,x;s,y) > 0$ for $r < s$, we conclude from (11.1) that $\nu = 0$.

(iii) To finish the proof, we need the following result on removable singularities for the equation $\dot{v} + Lv = v^\alpha$ (see [2]).

Let K be a compact subset of a simple rectangle Q. If $K \in \mathscr{S}_\alpha$ and if

$$\dot{v} + Lv = v^\alpha \quad \text{in } Q \setminus K, \tag{11.2}$$

then $v \in L^\alpha_{\text{loc}}(Q)$ and

$$\dot{v} + Lv = v^\alpha \quad \text{in } C_c^\infty(Q) \tag{11.3}$$

[i.e., $\int(-\dot{\varphi} + L\varphi)(v)\,dr\,dx = \int \varphi v^\alpha\,dr\,dx$, for every infinitely differentiable function φ whose support is compact and is contained in Q].

Moreover, we need the following extension of the maximum principle proved in [2]: If

$$\dot{v} + Lv \geq 0 \quad \text{in } C_c^\infty(Q), \tag{11.4}$$

$$\limsup v \leq 0 \quad \text{on } \partial_r Q, \tag{11.5}$$

then $v \leq 0$ a.e. in Q.

Note that

$$v(r,x) = -\log P_{r,\delta_x}\{G_Q \cap K = \varnothing\} \tag{11.6}$$

is the maximal solution of (11.2) subject to the condition

$$v = 0 \quad \text{on } \partial_r Q. \tag{11.7}$$

[This follows form Theorem 4.2 applied to $Q_1 = Q \setminus K \subset Q_2 = Q$ and $\Gamma = K \cup (\partial Q \setminus \partial_r Q)$.] Since (11.2) implies (11.3), the function (11.6) satisfies (11.4). By (11.7), it satisfies (11.5). Therefore $v = 0$ in Q and $P_{r,\delta_x}\{G_Q \cap K = \varnothing\} = 1$ for $(r,x) \in Q \setminus K$. Let $(r,x) \in Q \setminus K$. By (4.16),

$$\begin{aligned} P_{r,\delta_x}\{G \cap K = \varnothing\} &\geq P_{r,\delta_x}\{G \subset Q \setminus K\} = P_{r,\delta_x}\{G_{Q\setminus K} \subset Q \setminus K\} \\ &\geq P_{r,\delta_x}\{G_Q \subset Q \setminus K\} = P_{r,\delta_x}\{G_Q \cap K = \varnothing\}. \end{aligned}$$

Hence, $P_{r,\delta_x}\{G \cap K = \varnothing\} = 1$ for all $(r,x) \notin K$ and K is G-polar by Theorem 7.1. □

11.2.

THEOREM 11.2. *A singleton $\{c\}$ is G-polar if and only if $\alpha \geq (d+2)/d$.*

PROOF. To simplify notation, take $c = (0,0)$. Then

$$\int_{\{c\}} k(r,x;s,y)\nu(ds,dy) = ak_{-r}(x).$$

Clearly, $k_{-r}(x)$ belongs to $L^\alpha(S)$ if and only if

$$\int_S 1_{s>0} s^{-\alpha d/2} \exp\{-\alpha s/2\} \exp\{-d|x|^2/2s\}\, ds\, dx = \int_0^\infty e^{-\alpha s/2} s^{-(\alpha-1)d/2} < \infty,$$

which is equivalent to the condition $(\alpha - 1)d/2 < 1$.

12. Tests of R-polarity and H-polarity.

12.1. The results of this section follow from Theorem 11.1 and from analytical lemmas in [2].

We recall that, according to Theorem 8.3, an analytic set $A \subset E$ is R-polar if $\mathbb{R}_+ \times A$ is G-polar. We say that A is H-polar if $\{t\} \times A$ is G-polar for every $t \in \mathbb{R}$. Clearly, all R-polar sets are H-polar.

THEOREM 12.1. *An analytic set A is R-polar if $\Delta \times A$ is G-polar for an interval Δ of positive measure.*

Recently Sheu [84] proved that Theorem 12.1 remains true with Δ replaced by any Borel set Λ of positive Lebesgue measure.

THEOREM 12.2. *An analytic set A is H-polar if $\{t\} \times A$ is G-polar for some t.*

(In the case $\alpha = 2$, this theorem can be deduced also from the results in [34].)

PROBLEM. Suppose A is H-polar and Λ is a set of Lebesgue measure 0. Is $\Lambda \times A$ G-polar?

12.2. The Bessel kernel is defined by the formula

$$g_\beta(x,y) = g_\beta(y-x),$$

where

$$g_\beta(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\lambda x}(1+|\lambda|^2)^{-\beta/2}\, d\lambda = N_{d,\beta}|x|^\nu K_\nu(|x|).$$

Here $\nu = \frac{1}{2}(\beta - d)$, $N_{d,\beta}$ is a constant and K_ν is the Bessel function of the third kind of order ν. In particular,

(12.1) $$g_2(x) = \int_0^\infty k(r,x)\,dr,$$

where k is given by (9.2).

THEOREM 12.3. *Put $\nu \in \mathscr{E}_\alpha^\beta$ if*

$$\int_A g_\beta(x,y)\nu(dy) \text{ belongs to } L^\alpha(E)$$

implies $\nu(A) = 0$. An analytic set A is R-polar if and only if $A \in \mathscr{E}_\alpha^2$, and it is H-polar if and only if $A \in \mathscr{E}_\alpha^{2/\alpha}$.

12.3. An analytic set A is polar in the classical sense if

(12.2) $$\int_A g_2(x,v)\nu(dy) \text{ belongs to } L^\infty(E)$$

implies $\nu(A) = 0$.

THEOREM 12.4. *H-polarity is identical to the classical polarity in the case $\alpha = 2$.*

PROOF. Since the convolution of g_1 with itself is equal to g_2, the condition

$$\int_A g_1(x,y)\nu(dy) \text{ belongs to } L^2(E)$$

is equivalent to the condition

$$\int_{A\times A}\nu(dx)g_2(x,y)\nu(dy) < \infty.$$

It is well known that the latter condition is equivalent to (12.2).

A similar test can be proved for R-polarity. Let us write $k_d(r,x)$ and $g_{2,d}(r,x)$ for the functions (9.2) and (12.1), indicating explicitly their dependence on d. Put $g_{2,d}(x,y) = g_{2,d}(y-x)$.

THEOREM 12.4′. *If $\alpha = 2$ and $d \geq 4$, then A is R-polar if and only if*

(12.3) $$\int_{A\times A}\nu(dx)g_{2,d-2}(x,y)\nu(dy) < \infty \quad \textit{implies} \quad \nu(A) = 0.$$

PROOF. Note that $2\pi s k_d(s,x) = k_{d-2}(s,x)$ and

$$\int_{\mathbb{R}^d} k_d(r,x-z)k_d(r',z)\,dz = \tfrac{1}{2}k_d(r+r',x).$$

Therefore

$$\int_{\mathbb{R}^d} dy\left[\int_A g_{2,d}(x,y)\nu(dy)\right]^2 = (4\pi)^{-1}\int_{A\times A}\nu(dx)g_{2,d-2}(x,y)\nu(dy)$$

and (12.3) follows from Theorems 11.1 and 12.1.

12.4. The Hausdorff measure with an index γ is defined by the formula

$$H_\gamma(A) = \lim_{\varepsilon \downarrow 0} \inf \sum_i r_i^\gamma,$$

with the infimum taken over all countable coverings of A by open balls $U(x_i, r_i)$ of center x_i and radius $r_i < \varepsilon$. The Hausdorff dimension H-dim A is the supremum of γ such that $H_\gamma(A) > 0$. The Hausdorff codimension cd(A) of a set $A \subset \mathbb{R}^d$ is equal to $d - H$-dim A.

THEOREM 12.5. *Let $\kappa_\alpha = 2\alpha/(\alpha - 1)$ and $\gamma = d - \kappa_\alpha$. If $\gamma < 0$, then A is R-polar if and only if A is empty. If $\gamma > 0$, then*

$$\{\mathrm{cd}(A) > \kappa_\alpha\} \Rightarrow \{H_\gamma(A) < \infty\} \Rightarrow \{A \textit{ is } R\textit{-polar}\}$$

$$\Rightarrow \{H_{\gamma'}(A) = 0 \textit{ for all } \gamma' > \gamma\} \Rightarrow \{\mathrm{cd}(A) \geq \kappa_\alpha\}.$$

THEOREM 12.6. *Criteria for H-polarity can be obtained by replacing κ_α in Theorem* 12.5 *with $\lambda_\alpha = 2/(\alpha - 1) = \kappa_\alpha - 2$ (or d by $d + 2$).*

REMARK. The case $\gamma = 0$ can be treated by using the logarithmic Hausdorff measure and the corresponding generalization of the Hausdorff dimension (see [28]).

Part III. Historical notes and comments.

1. Homogeneous and inhomogeneous Markov processes.

1.1. In this paper, inhomogeneous processes are considered as the principal subject, and homogeneous processes as an important particular case. It is true that every inhomogeneous process can be made homogeneous by including time in the description of a state. However, there are processes (e.g., historical processes introduced in Section I.1.12) for which the natural state space changes in time and their reduction to homogeneous processes obscures the picture of the path evolution. Another advantage of the nonhomogeneous setting is the possibility of applying useful transformations which destroy the homogeneity (like monotone time change or approximation by piecewise constant processes). On the other hand, analytic formulae in the homogeneous case are usually simpler. The situation can be compared with the relationship between theories of elliptic and parabolic differential equations.

Most literature is focused on the homogeneous case. The inhomogeneous approach is presented, for instance, in [53], [9] and [58], and in monographs [16] and [87].

1.2. We build the global state space S of pairs (t, x), where x is a point of the state space E_t at time t. Another possibility is to start from S and to consider E_t as disjoint parts of S.

We consider processes on the time interval $\mathbb{R}_+$. In other articles (see, e.g., [18], [23], [24] and [28]) this interval is $\mathbb{R}$. The difference is inessential unless we are interested in the behavior of the path as $t \to -\infty$.

1.3. In contrast to some previous publications, we assume that the life interval contains the birth time α. In combination with condition I.1.2.B, this excludes the possibility of branching points at which the process jumps with positive probability to a distinct state.

1.4. The class of right processes considered in the present paper is close to the class of right processes introduced by Meyer and studied by Getoor, Sharpe and others (in the time homogeneous case, see, e.g., [83]). Both classes coincide if we exclude processes with branching points.

In [22], [26] and [29] we deal with a smaller class of regular processes. Denote by K_r the class of all probability measures of $\mathscr{F}^0_{>r}$ such that

$$\Pi\{Z|\mathscr{F}^0(a, s]\} = \Pi_{s, X_s} Z \quad \Pi = \text{a.s.},$$

for all $s > r$ and $Z \in p\mathscr{F}^0_{\geq s}$. We say that ξ is *regular* if it satisfies conditions 1.2.A and 1.2.B and the following stronger version of 1.2.C: For every $r \in \mathbb{R}_+$, every $\Pi \in K_r$ and every $Z \in p\mathscr{F}^0_{\geq u}$, $\Pi_{t, \xi_t} Z$ is right-continuous on $[r, u)$ Π-a.s. Every right process can be made regular by expanding the state spaces $(E_t, \mathscr{B}_t)$. On the other hand, if ξ is a regular process and if $\tilde{E}_t \in \mathscr{B}_t$ satisfy the condition

$$\Pi_{r, x}\{\xi_t \in \tilde{E}_t \text{ for all } t > r\} = 1, \quad \text{for all } r \in \mathbb{R}_+, x \in E_r,$$

then the restriction of ξ to $\tilde{E}_t$ is right but not necessarily regular. More detail on the relationship between these two classes can be found in [30].

The regularity properties of superprocesses stated in Theorems I.1.4–I.1.6 are proved in [30] by using the results of [29]. The proofs are based on ideas developed in [32], [35] and [36]. In particular, proofs of I.1.10.B and I.1.11.B (as well as Theorem 1.4) use a relationship between stopping times for X and ξ established in [35].

2. Superprocesses and branching measure-valued processes.

2.1. Let $(E, \mathscr{B})$ be a measurable Luzin space and let $\mathscr{M}(E)$ be the space of finite measures on E with the natural measurable structure. We say that a Markov process $X = (X_t, P_{r,\mu})$ in $\mathscr{M}(E)$ is a *branching process* if

$$(2.1) \qquad \log P_{r,\mu} \exp\langle -f, X_t\rangle = \int_E \log P_{r,\delta_x} \exp\langle -f, X_t\rangle \mu(dx),$$

for all $\mu \in \mathscr{M}(E)$, $r < t \in \mathbb{R}_+$, $f \in p\mathscr{B}$. This implies that

$$(2.2) \qquad P_{r,\mu+\nu} \exp\langle -f, X_t\rangle = P_{r,\mu} \exp\langle -f, X_t\rangle P_{r,\nu} \exp\langle -f, X_t\rangle,$$

for all $\mu, \nu \in \mathscr{M}(E)$. Heuristically, the branching property means the independence of evolutions for any parts of the population alive at time r. Condition

(2.1) is stronger than (2.2), but it follows from (2.2) if it is possible to introduce a topology into $\mathscr{M}(E)$ such that the following hold:

1. The set of measures supported by finite sets is everywhere dense.
2. $P_{r,\mu} \exp\langle -f, X_t\rangle$ is continuous in μ for a sufficiently large class of f. (Cf. the discussion of CB property in Section II.4.6.)

Clearly (I.1.8) implies (2.1) and, therefore, a superprocess with parameters (ξ, K, ψ) is a branching measure-valued process.

2.2. The problem of description of all branching measure-valued processes has attracted the attention of investigators since the late 1950s. Jiřina [47, 48] suggested the name continuous state branching processes (CSB processes). (We use this term only in the time-homogeneous setting.) Examples of CSB processes were studied by Lamperti [60] and Motoo [71]. A systematic theory of CSB processes with a compact metrizable space E was developed by Watanabe [94].

Watanabe proved that, for every Feller semigroup $\mathbb{P}_t$ and every ψ of the form (I.1.15) (with constant $a, b, n, \tilde{n}$) there exists a CSB process which satisfies the following conditions:

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -v, \mu\rangle, \tag{2.3}$$

$$v(x) + \int_0^t \mathbb{P}_{t-s}\psi(v)(x)\,ds = \mathbb{P}_t f(x), \quad \text{for } s \in [0,t], \tag{2.4}$$

which are a particular case of equations (I.1.8) and (I.1.9).

Silverstein [85] constructed a CSB process with parameters (ξ, K, ψ) for a process ξ with mass creation and annihilation and ψ of the form (I.1.17) under a number of technical conditions on the combination of ξ, K, ψ. An attempt to replace them by more natural restrictions is made in [24] and in the present paper. In particular, it is sufficient that a continuous functional K satisfies the following conditions:

2.2.A. $\Pi_{r,\mu} K(r,t) < \infty$ for all $r < t$ and all μ.
2.2.B. $\Pi_{r,x} \exp qK(r,t) < \infty$ for all $r < t$, $q > 0$ and all x.

(Cf. 1.2.C in [24].) In the present paper, to simplify proofs, we impose even stronger restrictions on K. A construction of superprocesses for a much wider class of K and ψ is given in a forthcoming book by the author (a part of a series of monographs published by the Centre de Rechearches Mathématiques, Université de Montréal).

2.3. The converse problem—characterization of the most general branching measure-valued processes—was studied in [95] (see also [80]) for the case of a finite space E. Branching measure-valued processes in an arbitrary state space E are investigated recently in [31].

2.4. The following result is proved in [65]: If ψ is a continuous real-valued function on $\mathbb{R}_+$ such that $\psi(0) = 0$ and if

$$\psi(z) = \lim_{\beta \downarrow 0} \left[a_\beta + b_\beta z + c_\beta \varphi_\beta (1 - \beta z)\right],$$

where a_β, b_β, $c_\beta \geq 0$ are constants and φ_β is a generating function, then

$$\psi(z) = az + bz^2 + \int_0^1 (e^{-uz} - 1 + uz) n(du) + \int_1^\infty (e^{-uz} - 1)\tilde{n}(du),$$

where $b \geq 0$, $\int u^2 n(du) < \infty$ and $\tilde{n}[1, \infty) < \infty$.

Clearly, this implies the following: If a superprocess with parameters (ξ, K, ψ) is obtained by the passage to the limit described in Theorem I.3.1 from a branching particle system with a local branching [i.e., if φ is given by (I.2.6)], then ψ belongs to the class given by (I.1.15).

3. Branching particle systems and enhanced models of superprocesses.

3.1. We refer to [43] for the early history of the theory of branching processes. Branching particle systems corresponding to a diffusion ξ and an additive functional K were studied in [86]. Special classes of such systems were investigated earlier in [82]. A general theory of branching particle systems was developed in [44].

3.2. Passage to the limit from branching particle systems to measure-valued processes can be found in [94] and [7]. More general theorems are proved in [33], Chapter 9, and in [80]. All these authors considered only the case when ξ is a process with a stationary Feller transition function. This restriction was dropped in [21] and [23]. Another construction of a superprocess over a non-Feller (but time homogeneous) process ξ was given in [35] and [36].

3.3. The complete description of a branching particle system is given by the random tree composed of the paths of all particles. The historical superprocess (see Section I.1.12) provides a continuous counterpart for this tree: $\hat{X}_\tau$ can be obtained from branching particle systems by the limit procedure of Section I.3.1—as the limit of discrete measures $\Sigma\delta_{w_{\leq\tau}(i)}$, where $w_{\leq\tau}(i)$, $i = 1, 2, \dots, n$, are "the historical paths" of all particles which exit from Q (each path is traced from the exit time back to the arrival of the forefather-immigrant).

Historical paths have been considered already by Kallenberg [49] in his "backward tree formula." Historical superprocesses were introduced and studied in [12], [24] and [26]. (The terminology in [12] differs from our terminology: "historical processes" are called "path processes" and the name "historical processes" is reserved for what we call "historical superprocesses.")

3.4. Let $\tau \in \mathscr{T}$. Denote by $E_{\le\tau}$ the set of paths obtained by restricting each path w to the interval $[0, \tau(w)]$. The measure $\hat{X}_\tau$ on $E_{\le\tau}$ also can be defined by

$$\hat{X}_\tau(B) = \lim \sum \hat{X}_{t_i}\{\xi_{\le t_i} \in B, t_{i-1} \le \tau < t_i\}, \tag{3.1}$$

where the limit (in probability) is taken over a sequence of partitions of $\mathbb{R}_+$ with the mesh tending to 0. We return to measures X_τ by setting

$$X_\tau(A) = \hat{X}_\tau\{(\tau, \xi_\tau) \in A\}. \tag{3.2}$$

The Markov property (I.1.22) was deduced in [26] from (3.1), (3.2) and the properties of the historical superprocess $\hat{X}$.

In [26] we consider a larger than $\hat{X}_\tau$, $\tau \in \mathscr{T}$, family of random measures

$$\hat{X}^\eta = \int \langle d\eta, \hat{X}_t \rangle$$

indexed by measure-valued additive functionals of ξ. [$\hat{X}_\tau$ corresponds to the measure η on $\mathbb{R}_+ \times E$ given by

$$\eta(\Delta, B) = 1_\Delta(\tau) 1_B(\hat{\xi}_\tau).]$$

3.5. The probabilistic structure of measures $\hat{X}_t$ has been studied in [12]. The results can be extended to measures $\hat{X}_\tau$. Since $\hat{X}_\tau$ is an infinitely divisible random element of $\mathscr{M} = \mathscr{M}(E_{\le\tau})$, we have

$$\log \hat{P}_\mu \exp\left(-\langle \hat{f}, \hat{X}_\tau \rangle\right) = -\int \hat{f}(w_{\le\tau}) \gamma(\mu, w_{\le\tau})$$
$$+ \int_{\mathscr{M}} \left(1 - \exp\left(-\langle \hat{f}, \nu \rangle\right)\right) R_\tau(\mu, d\nu),$$

where $\gamma(\mu, \cdot\,)$ and $R_\tau(\mu, \cdot\,)$ are positive measures (see, e.g., [50]). If we assume that $\hat{P}_\mu\{\hat{X}_\tau = 0\} > 0$, then $\gamma = 0$. By the CB property (Section II.4.6),

$$R_\tau(\mu, C) = \int \mu(dx_{\le r}) R_\tau(x_{\le r}, C).$$

The following property can be proved by using these formulae. Let σ be the first exit time from an open set $U \subset Q$ such that $\operatorname{dist}(U, Q^c) \lessdot 0$ and let $j_\sigma : E_{\le\tau} \to E_{\le\sigma}$ associate with every element of $E_{\le\tau}$ its restriction to $[0, \sigma)$. Then the image $\hat{X}_\tau^\sigma$ of $\hat{X}_\tau$ under ψ_σ is concentrated on a finite subset of $E_{\le\sigma}$ (cf. Proposition 3.3 in [12]). This fact can be used for constructing a superdiffusion as a limit of imbedded branching particle systems (cf. [64]).

3.6. A new type of path-valued processes was introduced by Le Gall [63], who used them to study superdiffusions with the quadratic branching parameter $\psi(t) = bt^2$. For Le Gall's process $\bar{\xi}_t$, the state space at time t is the space of paths over a random time interval $[0, \zeta_t]$, where ζ_t is not monotone (it is the reflecting Brownian motion in $\mathbb{R}_+$). The superdiffusion X (and the historical superdiffusion $\hat{X}$) can be constructed by using $\bar{\xi}$ and the local times of ζ at all levels. It is proved that a set C is R-polar for X if and only if it is not hit by the random set which consists of all terminal points of $\bar{\xi}_t$ [i.e., the points $\bar{\xi}_t(\zeta_t)$]. The G-polar sets can be characterized in a similar way. The process $\bar{\xi}_t$ is a symmetric Markov process (even if we start from inhomogeneous ξ). By applying the potential theory of such processes, Le Gall was able to get (for the case $\alpha = 2$) the necessary conditions for R-polarity given by Theorem II.12.3 and the necessary conditions for G-polarity provided in Theorem II.11.1.

A related trajectorial construction of superdiffusions by modeling the quadratic branching with the help of the Poisson point process of Brownian excursions was developed in Le Gall's earlier papers [61, 62]. Among applications of this construction, [62] contains a new proof of the properties of the support process obtained first by Perkins [74] (cf. Section 5 below).

4. Differential equations involving the operator $Lv - v^\alpha$.

4.1. We use [41] as the standard reference book on elliptic partial differential equations and [37] and [59] for references on parabolic equations. Analysts usually write the equation (II.1.1) with the reversed time direction. From the probabilistic point of view, the form (II.1.1) is preferable since it represents the backward Kolmogorov equation. The term v^α is (II.3.17) is often replaced by $|v|^{\alpha-1}v$. In our setting, this makes no difference since we are interested only in positive solutions.

The relationship between the equation $\dot{v} + Lv = \psi(v)$ and the Laplace transform of the superdiffusion X_t is well known [94]. A connection between the nonhomogeneous equation $\dot{v} + Lv = \psi(v) - \rho$ and $\int_0^\infty \langle \rho^t, X_t \rangle \, dt$ was established first in [45].

4.2. The elliptic equation

$$\Delta v = v^\alpha, \tag{4.1}$$

with $\alpha = (d+2)/(d-2)$ in a d-dimensional domain D, was investigated by Loewner and Nirenberg [68] in 1974. They established the following:

(a) If D is bounded and ∂D is smooth, then (4.1) has a unique positive solution which tends to $+\infty$ at the boundary.
(b) For an arbitrary D, there exists the maximal positive solution v_D.
(c) Suppose that ∂D is compact and $\tilde{D} = D \setminus K$, where $K \subset D$ is compact. If H-dim $K < d/2 - 1$, then $v_{\tilde{D}}$ is bounded near K; if K is a smooth hypersurface with dimension $> d/2 - 1$, then $v_{\tilde{D}} \to +\infty$ as x tends to K.

[Note that the critical codimension $d - (d/2 - 1) = d/2 + 1$ coincides with κ_α in Theorem II.12.5.]

4.3. Isolated singularities of (4.1) with $\alpha > 1$ have been studied in [6], [67] and [91]. (Recall that a probabilistic interpretation is known only for $1 < \alpha \le 2$.) It was established in [6] and [67] that the singularity is removable for $d \ge \kappa_\alpha$. The most complete results for $d < \kappa_\alpha$ have been obtained in [91]. In particular, it is proved that, if $3 \le d < \kappa_\alpha$ and $0 \in D$, then every positive solution v in $D \setminus \{0\}$ has either the form

$$q_{\alpha,d}|x|^{-2/(\alpha-1)}[1+\varepsilon(x)] \tag{4.2}$$

or the form

$$c|x|^{2-d}[1+\varepsilon(x)], \tag{4.3}$$

where $\varepsilon(x) \to 0$ as $x \to 0$, $q_{\alpha,d} = [2(\alpha-1)^{-1}(\kappa_\alpha - d)]^{1/(\alpha-1)}$ and $c \ge 0$ is a constant.

The maximal solution (4.2) can be obtained by the probabilistic formula

$$v(x) = -\log P_{\delta_x}\{R \subset D \setminus \{0\}\}. \tag{4.4}$$

In particular,

$$-\log P_{\delta_x}\{R \subset \mathbb{R}^d \setminus \{0\}\} = q_{\alpha,d}|x|^{-2/(\alpha-1)} \tag{4.5}$$

(cf. Section II.8.5). To get a probabilistic interpretation of solution (4.3), note that, for $d < \kappa_\alpha$, the function $k(r,x)$ in (II.9.2) belongs to $L^\alpha(S)$ and therefore the measure $\eta(ds,dy) = ds\,\delta_0(dy)$ is an element of class N_α (see Section II.10.3). The desired expression is

$$v(x) = -\log P_{\delta_x}\exp\{-cI_\eta(\mathbb{R}_+)\} \tag{4.6}$$

[cf. (II.10.2)].

Isolated singularities for the equation $\Delta v = \psi(v)$ with a continuous increasing function ψ have been studied in [89], [92] and [79]. [Note that all functions of the form (I.1.19) with $a \ge 0$ satisfy this condition.] Sufficient conditions of removability in terms of behavior of $\psi(t)$ as $t \to \infty$ were given in [90] and [1].

4.4. A singular solution of

$$\dot v + \Delta v - v^\alpha = 0 \quad \text{in } [0,t) \times D, \tag{4.7}$$

subject to the boundary condition

$$v \to \delta_y \quad \text{as } r \uparrow t, \tag{4.8}$$

where δ_y is the Dirac delta function at point y, appeared first in [4]. The authors proved that such a solution exists if and only if $\alpha < (d+2)/d$ (cf. Theorem II.11.2). We know that

$$v(r,x) = -\log P_{\delta_{r,x}}\exp(-I_\eta(\mathbb{R}_+)), \tag{4.9}$$

where $\eta = \delta_{t,y}$ is the Dirac measure at point (t,y).

By replacing I_η with $kI(\eta)$ in (4.9) and by letting k tend to ∞, we arrive at the *very singular solution*

$$v(r,x) = -\log P_{\delta_{r,x}}\{I_\eta(\mathbb{R}_+) = 0\} \tag{4.10}$$

of (4.7), constructed first in [5] and investigated in detail in [51]. Apparently, it coincides with the maximal solution

$$u(r,x) = -\log P_{\delta_{r,x}}\{(t,y) \notin G\} \tag{4.11}$$

in the domain $Q = S \setminus \{(t,y)\}$.

4.5. Removable singularities for elliptic equations involving a more general operator $Lv - v^\alpha$ were studied in [1]. Most results of [1] follow from the subsequent paper [2] devoted to the parabolic case. [1] was used in [25] and [2] in [28]. In the present paper, all results on the elliptic equations are deduced from the results for the parabolic case.

4.6. According to II.3.4.D, the first boundary value problem (II.3.17) has a solution if Q is bounded and regular and $f\colon \partial_r Q \to [0,\infty]$ is continuous. The uniqueness follows from the comparison principle if f is finite. Analogous results can be established for the elliptic case [see (1)–(3) in the Introduction]. It was proved independently in [93] and [55] that, if D is a bounded domain of class C^2 in $\mathbb{R}^d$, then there exists only one solution of the equation $Lv = v^\alpha$ which tends to ∞ at the boundary. However, the uniqueness theorem fails if $f = \infty$ at one point of ∂D even if D is a ball [56].

4.7. The elliptic version of the problem discussed in Section II.5 is to describe all solutions of the equation

$$Lv = v^\alpha \quad \text{in } D. \tag{4.12}$$

As in the parabolic case, the problem can be restated in probabilistic terms. It is natural to investigate, first, the case when $L = \Delta$ and D is a bounded domain with a smooth boundary. Let $g(x,y)$ be Green's function and $k(x,y)$ be the Poisson kernel for Δ in D. Equation (4.12) holds if there exists a harmonic function h such that

$$v(x) + \int_D g(x,y)v(y)^\alpha\, dy = h(x). \tag{4.13}$$

Clearly, h is the harmonic majorant of v, and v is the maximal solution of (4.12) dominated by h. Therefore (4.13) establishes a 1–1 correspondence between the class V_α of all solutions dominated by harmonic functions and a class H_α of positive harmonic functions. Let $m(dx) = \rho(x)\,dx$, where $\rho(x)$ is the distance from x to ∂D. We conjecture that H_α contains all positive harmonic functions which belong to $L^\alpha(m)$. It follows from [42] that this is true at least for functions $h(x) = k(x,y)$. Denote by M_α the class of measures μ for which

$$h(x) = \int_{\partial D} k(x,y)\mu(dy)$$

is an element of H_α. It is clear that, if $\mu_n \uparrow \mu$ and $\mu_n \in M_\alpha$, then $\mu \in M_\alpha$ assuming that the corresponding h is finite. Denote by K_α the class of Borel subsets Γ of ∂D such that

$$\int_\Gamma k(x,y)\nu(dy) \text{ belongs to } L^\alpha(m)$$

only if $\nu(\Gamma) = 0$. We conjecture that $\mu \in M_\alpha$ if and only if μ does not charge any set of class K_α. Motivated by the results in Sections II.11 and II.12, we also conjecture that K_α coincides with the class of Borel sets $\Gamma \subset \partial D$ such that $P_{\delta_x}\{R_D \cap \Gamma = \varnothing\} = 1$, for all $x \in D$. (Partial results in this direction have been proved by Le Gall and Sheu.)

4.8. The equations involving the operator $Lu + u^\alpha$ rather than $Lv - v^\alpha$ have also been studied extensively in the literature. Fujita [39] discovered that the equation

$$\dot{u} = \Delta u + u^\alpha \tag{4.14}$$

(after time reversal it takes the form $\dot{v} + \Delta v + v^\alpha = 0$) has no global solutions in $\mathbb{R}_+ \times \mathbb{R}^d$ if $d < 2/(\alpha - 1)$ and it has such solutions if $d > 2/(\alpha - 1)$. Fujita's critical value $2/(\alpha - 1)$ coincides with λ_α in Theorem II.12.6 but we have no explanation of this identity. The "blow-up" phenomenon for the equations involving $Lu + u^\alpha$ was studied in terms of branching particle systems in [72]. We refer to [40] and [38] for more recent developments.

5. Path properties of the super-Brownian motion.

5.1. The super-Brownian motion with quadratic branching parameter $\psi(z) = z^2$ has attracted the most attention and is investigated in great detail. The process has continuous paths. More precisely, $\langle f, X_t \rangle$ is continuous a.s. for any bounded Borel function f [78]. (Recall that, in general, $\langle f^t, X_t \rangle$ is right-continuous a.s. for a bounded continuous f.) The proof in [78] is based on nonstandard analysis. A standard proof for a broader class of processes is given in [76].

5.2. The *support process* $K_t = \operatorname{supp} X_t$ is studied in [11] and [75]. It is proved that K_t is right-continuous with left limits (in the topology induced by the Hausdorff metric in the space of compact sets) and that, for almost all ω, the following hold:

1. $K_t \subset K_{t-}$ for all $t > 0$;
2. $K_{t-} \setminus K_t$ is empty or a singleton for all $t > 0$;
3. $K_t = K_{t-}$ for each fixed $t > 0$.

It is easy to deduce from these results that the graph G of X is the union of all sets $\{t\} \times K_{t-}$, $t > 0$, and $\{0\} \times K_0$.

5.3. *Relations between X_t and the Hausdorff measures.* These were investigated in [10], [73]–[75], [11] and [12].

If $d = 1$, then $X_t(dx) = \rho(t, x)\,dx$ with a continuous ρ [78]. This result was established independently in [57]. (An earlier result in the same direction was obtained in [80].)

According to [10], for any d, the measure X_t is concentrated, a.s., on a random Borel set of Hausdorff dimension not larger than 2. Perkins [74] has proved that, for $d > 2$, $X_t(dx) = \rho_t(x)\eta(dx)$, where η is the Hausdorff measure corresponding to the function $\varphi(r) = r^2 \log\log(1/r)$. Moreover, a.s., $0 < c_d \le \rho_t \le C_d < \infty$ on K_t, for all $t > 0$. These results have been refined in [12]. It was shown that, for every fixed t, $\rho_t = \text{const.}$ a.s., and therefore K_t is a set-valued Markov process.

5.4. The fact that the range R of the super-Brownian motion is a.s. compact was established first in [46]. The necessary conditions for R- and H-polarity (which coincide with the conditions in Theorems II.12.4 and II.12.4′) were established first in [75] independently of any results obtained by analysts. However, the sufficiency part of these theorems (even for $\alpha = 2$) is not yet proved this way. Theorem II.12.5 and II.12.6 for $\alpha = 2$ were proved in [11]. The proofs are also purely probabilistic. (Note that our method is not applicable to more general random sets R_k—the sets of k-multiple points of X—studied in [11] and [75].)

5.5. In [75] and [12] the super-Brownian motion as well as the superprocesses corresponding to symmetric stable processes ξ were investigated. In this situation, the topological support K_t is, a.s., either the empty set or the entire state space, and it must be replaced by a random Borel set Λ_t supporting X_t.

6. Invariant measures.

6.1. The asymptotic behavior of branching particle systems and superprocesses in the critical case and the related problem of the classification of the equilibrium distributions have been studied by many authors (see, e.g., [8], [23], [49], [66] and [12]). A general entrance boundary theory for superprocesses was developed in [23]. We sketch here the results on equilibria for time-homogeneous diffusions which follow from this theory.

6.2. Let $p_t(x, dy)$ be a stationary Markov transition function in E. A measure m is called *p-invariant* if

$$\int m(dx)p_t(x, B) = m(B) \quad \text{for all } t, B.$$

We say that m is a *p-equilibrium* measure if, in addition, $m(E) = 1$. For the Brownian motion in $\mathbb{R}^d$, all invariant measures are given by the formula $m(dx) = \text{const.}\, dx$. (There exist no equilibrium measures.)

In general, the p-equilibria form a convex cone which is generated by its extremal elements.

6.3. Let ξ be a diffusion with a stationary transition function p and

$$\mathbb{P}_t f(x) = \int p_t(x, dy) f(y).$$

A superdiffusion X with parameters (ξ, ψ) can be defined in the space $\mathscr{M}_\rho = \{\nu\colon \langle \rho, \nu \rangle < \infty\}$ if $\rho \geq 0$ and $\mathbb{P}_t \rho / \rho$ is bounded for each t. Its transition function $\mathscr{P}$ is stationary. Set

$$V_t f(x) = -\log P_{\delta_x} \exp\langle -f, X_t \rangle,$$

$$W_f(x) = \int_0^\infty \psi(V_t f)\, dt.$$

To every p-invariant measure m there corresponds a unique $\mathscr{P}$-equilibrium measure M_m such that

$$\int_{\mathscr{M}_\rho} M_m(d\nu) \exp(-\langle f, \nu \rangle) = \exp(-\langle f - Wf, m \rangle), \qquad f \in L^1_+(m).$$

If $\langle Wf, m \rangle = \langle f, m \rangle < \infty$ for some $f > 0$, then M_m is concentrated at 0. We call it the *trivial* p-equilibrium.

The opposite extremal case is

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \langle W(\lambda f), m \rangle = 0,$$

for some $f > 0$. Then we say that m is *dissipative*. The map $m \to M_m$ is a 1–1 map from the set of all dissipative p-invariant measures m onto the set of all nontrivial *extremal* $\mathscr{P}$-equilibrium measures M such that

$$\int_{\mathscr{M}_\rho} M(d\nu) \langle \rho, \nu \rangle < \infty, \tag{6.1}$$

and the inverse mapping is given by the formula

$$\int_{\mathscr{M}_\rho} M(d\nu) \nu = m.$$

6.4. Suppose $\psi(z) = z^\alpha$. Since $V_t f \leq \mathbb{P}_t(f)$, m is dissipative if

$$\int_0^\infty \left\langle (\mathbb{P}_t f)^\alpha, m \right\rangle dt < \infty, \tag{6.2}$$

for some $f > 0$.

Let ξ be the Brownian motion in $\mathbb{R}^d$ and let $m(dx) = \text{const.}\, dx$. Take $f = p_1$, where

$$p_t(x) = (2\pi t)^{-d/2} e^{-x^2/2t}.$$

Then $\mathbb{P}_t f = p_{t+1}$ and $\langle (T_t f)^\alpha, m \rangle = \text{const.}\ (t+1)^{-d(\alpha-1)/2}$. Hence (6.2) holds if $d > 2/(\alpha - 1)$. Dawson proved that, if $\alpha = 2$ and $d = 1$ or 2, then M_m is trivial. Therefore in these cases there exist no nontrivial $\mathscr{P}$-equilibria subject to condition (6.1). [Recently, Bramson, Cox and Greven have shown that the last statement remains true even without condition (6.1).]

7. Other results.

7.1. A martingale approach was applied in [80] to superprocesses with quadratic branching and in [36] and [32] to superprocesses with more general branching described by the formula (I.1.15).

7.2. Superprocesses with quadratic branching possess moments of all orders. The diagrams for calculating these moments were suggested in [20]. By using these diagrams, a representation of all square-integrable functionals of X was obtained which is similar to the representation of functionals of Gaussian processes by the multiple Wiener–Itô integrals. As an application, the existence of local times and self-intersection local times for X has been investigated. (The first results on the local time for the super-Brownian motion were obtained in [45].)

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