A new and powerful family of parametric shapes extends the basic quadric surfaces and solids, yielding a variety of useful forms.

Superquadrics and Angle-Preserving Transformations

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Over the past 20 years, a great deal of interest has developed in the use of computer graphics and numerical methods for three-dimensional design. Significant progress in geometric modeling is being made, predominantly for objects best represented by lists of edges, faces, and vertices. One long-term goal of this work is a unified mathematical formalism, to form the basis of an interactive and intuitive design environment in which designers can simulate three-dimensional scenes with shading and texture, produce usable design images, verify numerical machining-control commands, and set up finite-element meshwork for structural and dynamic analysis.

A new collection of smooth parametric objects and a new set of three-dimensional parametric modifiers show potential for helping to achieve this goal. The superquadric primitives and angle-preserving transformations extend the traditional geometric primitives—quadric surfaces and parametric patches—used in existing design packages, producing a new spectrum of flexible forms. Their chief advantage is that they allow complex solids and surfaces to be constructed and altered easily from a few interactive parameters.

Mathematical preliminaries

Given two two-dimensional curves,

$$\underline{h}(\omega) = \begin{bmatrix} h_1(\omega) \\ h_2(\omega) \end{bmatrix}, \quad \omega_0 \le \omega \le \omega_1$$

and

$$\underline{m}(\eta) = \begin{bmatrix} m_1(\eta) \\ m_2(\eta) \end{bmatrix}, \quad \eta_0 \le \eta \le \eta_1,$$

the spherical product $\underline{x} = \underline{m} \otimes \underline{h}$ of the two curves is a surface defined by

$$\underline{x}(\eta, \omega) = \begin{bmatrix} m_1(\eta)\, h_1(\omega) \\ m_1(\eta)\, h_2(\omega) \\ m_2(\eta) \end{bmatrix}, \quad \begin{matrix} \omega_0 \le \omega \le \omega_1 \\ \eta_0 \le \eta \le \eta_1 \end{matrix}$$

Geometrically, $\underline{h}(\omega)$ is a horizontal curve vertically modulated by $\underline{m}(\eta)$; $m_1(\eta)$ changes the relative scale of $\underline{h}$, while $m_2(\eta)$ raises and lowers it. η is a north-south parameter, like latitude, whereas ω is an east-west parameter, like longitude (see Figure 1). Spherical product surfaces can be rescaled by a separate vector $\underline{a}$,

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \text{ by letting } \underline{\hat{m}} = \begin{bmatrix} m_1(\eta) \\ a_3\, m_2(\eta) \end{bmatrix}, \quad \underline{\hat{h}} = \begin{bmatrix} a_1 h_1(\omega) \\ a_2\, h_2(\omega) \end{bmatrix},$$

$$\text{yielding } \underline{\hat{x}} = \underline{\hat{m}} \otimes \underline{\hat{h}} = \begin{bmatrix} a_1\, m_1(\eta)\, h_1(\omega) \\ a_2\, m_1(\eta)\, h_2(\omega) \\ a_3\, m_2(\eta) \end{bmatrix}.$$

The spherical product derives its name from the surface of the unit sphere (Figure 2), which is produced when the half circle

$$\underline{m}(\eta) = \begin{bmatrix} \cos \eta \\ \sin \eta \end{bmatrix}, \quad -\pi/2 \le \eta \le \pi/2$$

is crossed with the full circle

$$\underline{h}(\omega) = \begin{bmatrix} \cos \omega \\ \sin \omega \end{bmatrix}, \quad -\pi \le \omega < \pi;$$

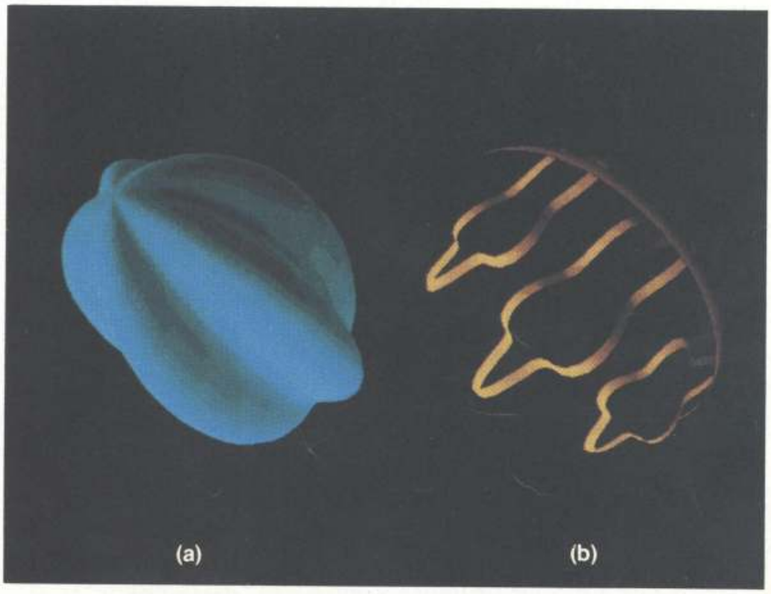

Figure 1. A spherical cross-product: (a) this closed surface was formed by crossing the vertical ellipse with a horizontal star shape (b).

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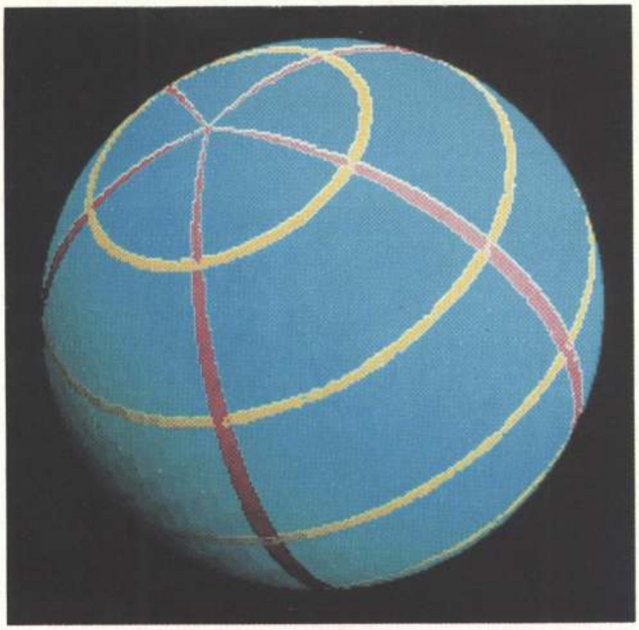


Figure 2. The sphere: η is the north-south parameter; ω is the east-west parameter.

$$\text{i.e., } x = \underline{m} \otimes \underline{h} = \begin{bmatrix} \cos\eta\cos\omega \\ \cos\eta\sin\omega \\ \sin\eta \end{bmatrix}, \begin{array}{l} -\pi/2 \le \eta \le \pi/2 \\ -\pi \le \omega < \pi \end{array}$$

The meridional half circle determines the radius of the longitudinal full circle, and determines the height above and below the X-Y plane.

The basic trigonometric forms of the standard quadric surfaces are representable in this product form:

Ellipsoids: $(x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 = 1$

$$\underline{x}(\eta, \omega) = \begin{bmatrix} \cos\eta \\ a_3 \sin\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \cos\omega \\ a_2 \sin\omega \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \cos\eta\cos\omega \\ a_2 \cos\eta\sin\omega \\ a_3 \sin\eta \end{bmatrix}, \begin{array}{l} -\pi/2 \le \eta \le \pi/2 \\ -\pi \quad \le \omega < \pi \end{array}$$

Hyperboloids of one sheet: $(x_1/a_1)^2 + (x_2/a_2)^2 - (x_3/a_3)^2 = 1$

$$\underline{x}(\eta, \omega) = \begin{bmatrix} \sec\eta \\ a_3 \tan\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \cos\omega \\ a_2 \sin\omega \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \sec\eta\cos\omega \\ a_2 \sec\eta\sin\omega \\ a_3 \tan\eta \end{bmatrix}, \begin{array}{l} -\pi/2 < \eta < \pi/2 \\ -\pi \quad \le \omega < \pi \end{array}$$

Hyperboloids of two sheets: $(x_1/a_1)^2 - (x_2/a_2)^2 - (x_3/a_3)^2 = 1$

$$\underline{x}(\eta, \omega) = \begin{bmatrix} \sec\eta \\ a_3 \tan\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \sec\omega \\ a_2 \tan\omega \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \sec\eta\sec\omega \\ a_2 \sec\eta\tan\omega \\ a_3 \tan\eta \end{bmatrix}, \begin{array}{l} -\pi/2 < \eta < \pi/2 \\ -\pi/2 < \omega < \pi/2 \text{ (sheet 1)} \\ \pi/2 < \omega < 3\pi/2 \text{ (sheet 2)} \end{array}$$

A torus may be thought of as an extended quadric surface: $(r-a)^2 = (x_3/a_3)^2 = 1$

where $r = \sqrt{(x_1/a_1)^2 + (x_2/a_2)^2}$;

$$\underline{x}(\eta, \omega) = \begin{bmatrix} a + \cos\omega \\ a_3 \sin\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \cos\omega \\ a_2 \sin\omega \end{bmatrix} = \begin{bmatrix} a_1(a+\cos\eta)\cos\omega \\ a_2(a+\cos\eta)\sin\omega \\ a_3 \sin\eta \end{bmatrix}$$

where $-\pi \le \eta < \pi$
and $-\pi \le \omega < \pi$.

The torus has the same trigonometric form as the ellipsoid, except that the modulating function is offset by a constant, and that η ranges over the full 360 degrees (see Figure 3).

Notation used in this article

$\alpha, \beta, \eta, \omega$ $\alpha_0, \alpha_1, \eta_1$	Surface parameters or angles are indicated by Greek letters. Particular values of the parameters are indicated by numerical subscripts.
$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$	Column vectors are indicated by a single underscore. Vector components are indicated by numerical subscripts.
$\underline{\underline{M}}$	Matrices are indicated by capital characters with two underscores. Components are indicated by subscript characters or subscript numerals. m_{j1} is the first column of $\underline{\underline{M}}$, m_{j2} is the second column of $\underline{\underline{M}}$, etc.
†	Transpose operator for matrices.
$\underline{a} \wedge \underline{b}$	Vector cross product of $\underline{a}$ and $\underline{b}$. $\underline{a} \wedge \underline{b} = \epsilon_{ijk}\, a_j b_k$.
$\underline{a} \cdot \underline{b}$	Scalar dot product of vector $\underline{a}$ and vector $\underline{b}$. $\underline{a} \cdot \underline{b} = a_i b_i$
i, j, k, i, j, k $a_i a_i = a_1^2 + a_2^2 + a_3^2$	Subscript characters are used in cartesian tensor notation.
$\underline{x}_\eta = \partial \underline{x} / \partial\eta$	Partial derivatives are indicated by parameter subscripts except for these special forms: $C_\eta = \cos\eta$, $C_\omega = \cos\omega$, $S_\eta = \sin\eta$, $S_\omega = \sin\omega$, $S_1 = \sin(\phi_1)$, $S_2 = \sin(\phi_2)$, etc.
$y_{j,i} = \partial y_j / \partial x_i$ $y_{j,\eta} = \partial y_j / \partial\eta$	Partial derivatives are also indicated by a comma in the subscript.
$\dfrac{d}{ds} =$	Derivative with respect to arclength parameter s.
$\dfrac{d}{dt} =$	Derivative with respect to curve parameter t.

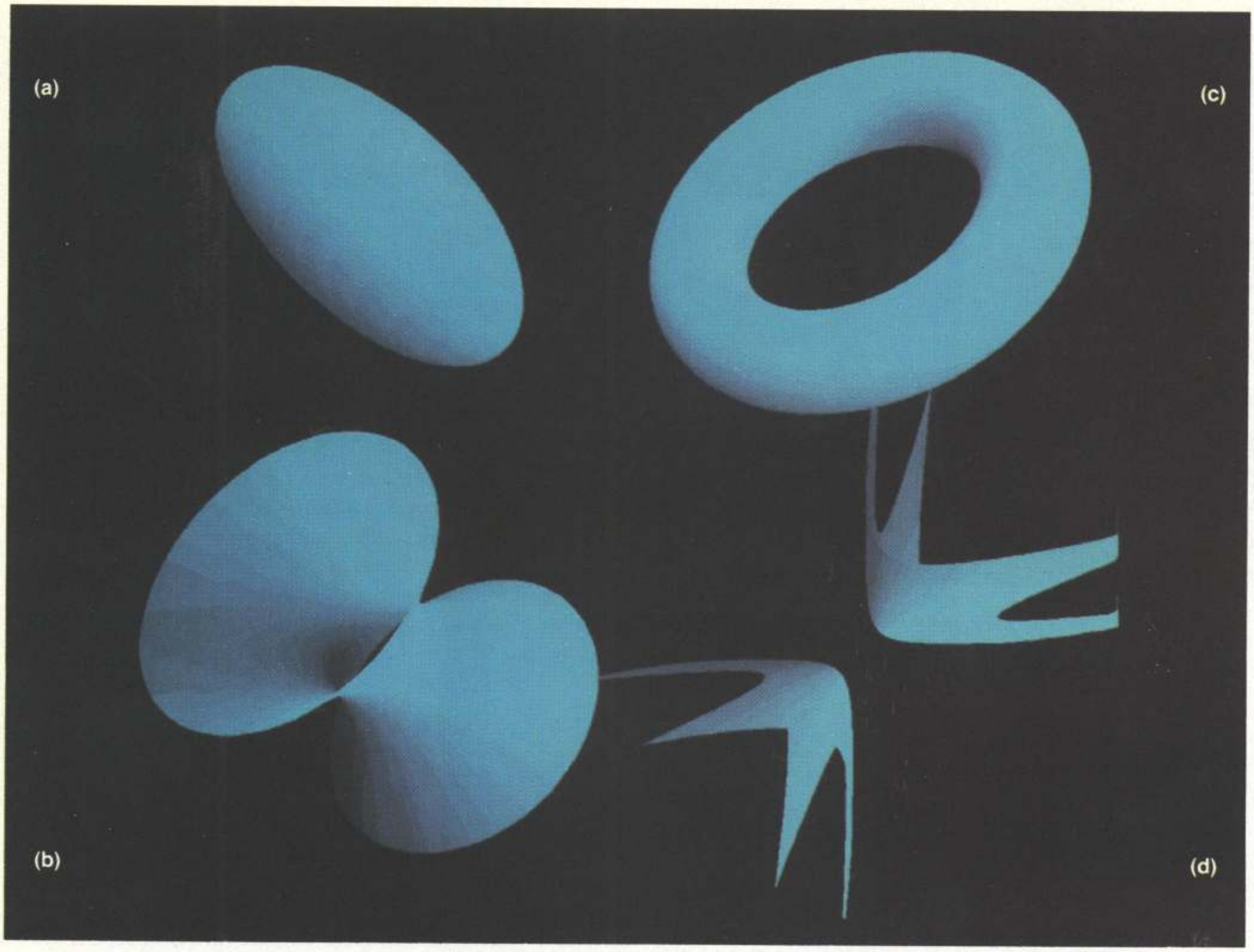


Figure 3. Quadrics: ellipsoid (a), hyperboloid of one sheet (b), toroid (c), and hyperboloid of two sheets (d).

Tangent vectors and normal vectors are important geometric entities associated with any three-dimensional surface:

$$\underline{x} = \underline{x}(\eta, \omega), \text{ where } (\eta, \omega) \in D;$$

$\underline{x}_\eta$ and $\underline{x}_\omega$ are tangent to the surface $\underline{x}$, and are used to determine the local length scale along the curve

$$\hat{\underline{x}} = \underline{x}(\eta(t), \omega(t)), \; t_0 \leq t \leq t_1$$

which is a curve lying in the surface $\underline{x}$.
Differential length is given by

$$ds = |\,\underline{x}_\eta \dot{\eta} + \underline{x}_\omega \dot{\omega}\,|\, dt, \text{ where } \dot{} = d/dt.$$

These differential lengths determine the size of the surface patches used in the image. If both $\underline{x}_\eta$ and $\underline{x}_\omega$ are finite and nonzero, the direction of the normal vector to the surface is given by

$$\underline{n}(\eta,\omega) = \underline{x}_\eta \wedge \underline{x}_\omega;$$

i.e.,

$$\underline{n} = \begin{bmatrix} x_{2,\eta}\, x_{3,\omega} - x_{3,\eta}\, x_{2,\omega} \\ -\, x_{1,\eta}\, x_{3,\omega} + x_{3,\eta}\, x_{1,\omega} \\ x_{1,\eta}\, x_{2,\omega} - x_{2,\eta}\, x_{1,\omega} \end{bmatrix}.$$

The unit normal $\underline{N}$ is obtained by dividing $\underline{n}$ by its magnitude:

$$\underline{N}(\eta, \omega) = \underline{n}(\eta, \omega) / |\, \underline{n}(\eta, \omega)\,|$$

If $\underline{x}_\eta$ or $\underline{x}_\omega$ are zero or infinite at a point $\underline{x}(\eta_0, \omega_0)$, a unique normal vector will exist only if the limit

$$\lim_{(\eta,\omega) \to (\eta_0, \omega_0)} \frac{\underline{x}_\eta \wedge \underline{x}_\omega}{|\underline{x}_\eta \wedge \underline{x}_\omega|}$$

exists. This limit does not exist at a cusp in the surface (see Figure 4). The primary application of the normal vector in raster graphics is the calculation of the diffuse reflection component of shading on an object, via Lambert's law:

$$I \propto f(\underline{N} \cdot \underline{L}) = f(\cos\theta)$$

where $\underline{L}$ is a unit vector along the light ray; i.e., the shading contribution from different light sources depends on the cosine of the angle between the light ray and the normal vector at that point on the surface. Another application of the tangent and normal vectors is the normal curvature κ of the surface, which is calculated from a curve, $\underline{x}(\eta(t), \omega(t))$, embedded in the surface $\underline{x}(\eta, \omega)$.

$$\kappa = \frac{D\dot{\eta}^2 + 2 D_1 \dot{\eta}\dot{\omega} + D_2 \dot{\omega}^2}{E\dot{\eta}^2 + 2F\dot{\eta}\dot{\omega} + G\dot{\omega}^2}$$

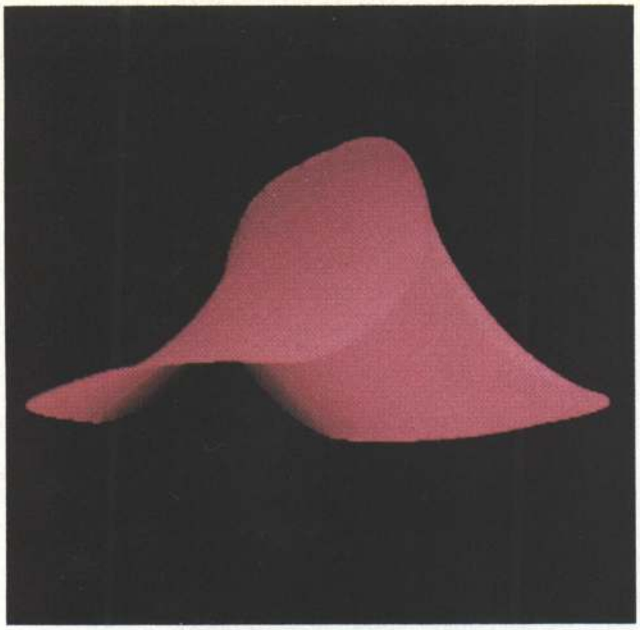


Figure 4. A shaded surface: more than one normal exists at the cusp.

where
$$D = \underline{N} \cdot \underline{x}_{\eta\eta}$$
$$D_1 = \underline{N} \cdot \underline{x}_{\eta\omega}$$
$$D_2 = \underline{N} \cdot \underline{x}_{\omega\omega}$$
and
$$E = \underline{x}_{\eta} \cdot \underline{x}_{\eta}$$
$$F = \underline{x}_{\eta} \cdot \underline{x}_{\omega}$$
$$G = \underline{x}_{\omega} \cdot \underline{x}_{\omega}$$

In regions of higher curvature, more points are needed for optimal sampling. In this way, the patches will smoothly tile the surface of the object and avoid sharp angular bends (see Figure 5).

Superquadrics in canonical positions

Superquadric solids are based on the four parametric forms of the quadric surfaces, in which each trigonometric function is raised to an exponent. In two dimensions the sine-cosine curve

$$\begin{aligned} x &= a \cos^{\epsilon} \theta \\ y &= b \sin^{\epsilon} \theta \end{aligned} \qquad \text{or } (x/a)^{2/\epsilon} + (y/b)^{2/\epsilon} = 1$$
$$-\pi \leq \theta < \pi$$

is called a superellipse (see Figure 6a), while the secant-tangent curve

$$\begin{aligned} x &= a \sec^{\epsilon} \theta \\ y &= b \tan^{\epsilon} \theta \end{aligned} \qquad \text{or } (x/a)^{2/\epsilon} - (y/b)^{2/\epsilon} = 1$$
$$-\pi/2 \leq \theta < \pi/2, \ \pi/2 \leq \theta \leq 3\pi/2$$

is called a superhyperbola (see Figure 6b).

The spherical product of pairs of these types of curves produces a uniform mathematical representation for the superquadrics. The two exponents are squareness parameters; they are used to pinch, round, and square off portions of the solid shapes, to soften the sharpness of square, and to produce edges and fillets of any arbitrary degree of roundness.

$\epsilon <$ 1: shape is somewhat square.
$\epsilon \sim$ 1: shape is round.
$\epsilon \sim$ 2: shape has a flat bevel.
$\epsilon >$ 2: shape is pinched.

Superquadrics are mathematical solids because (1) their surfaces divide three-dimensional space into three distinct regions: inside, outside, and surface boundary; and (2) a well-behaved inside-outside function determines the region in which an arbitrary point falls. For example, the unit sphere

$$\underline{x}(\eta, \omega) = \begin{bmatrix} \cos \eta \cos \omega \\ \cos \eta \sin \omega \\ \sin \eta \end{bmatrix} \begin{array}{l} -\pi/2 \leq \eta \leq \pi/2 \\ -\pi \leq \omega < \pi \end{array}$$

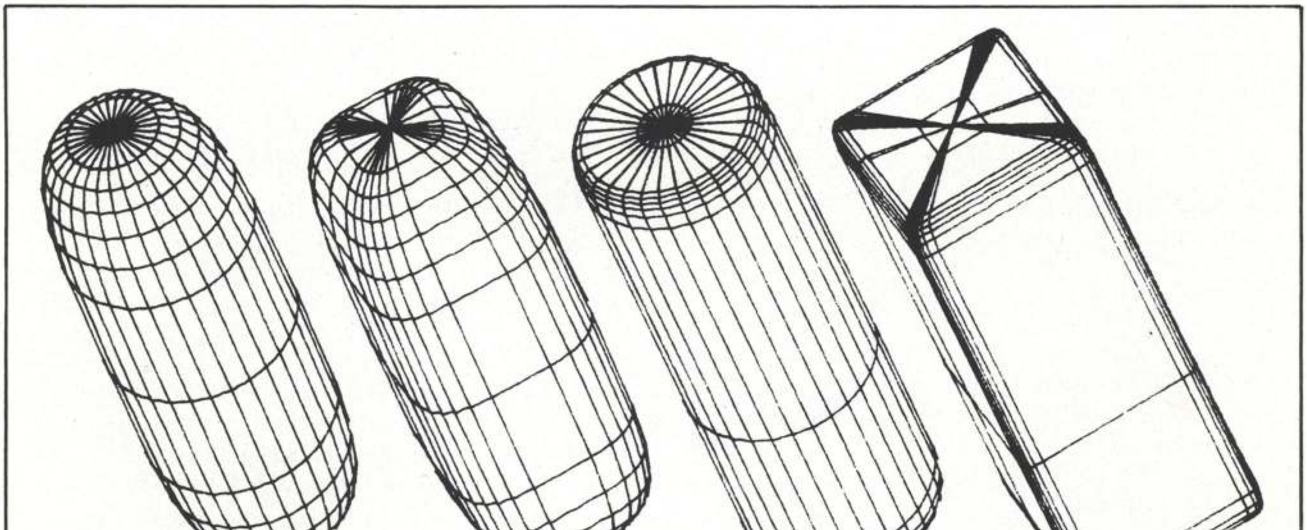
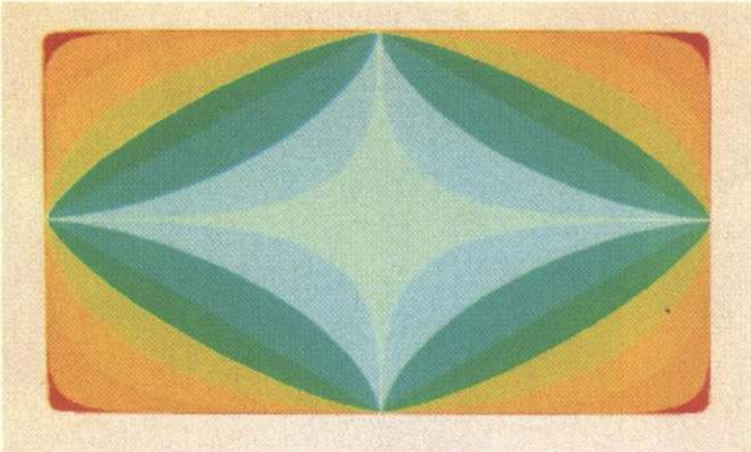
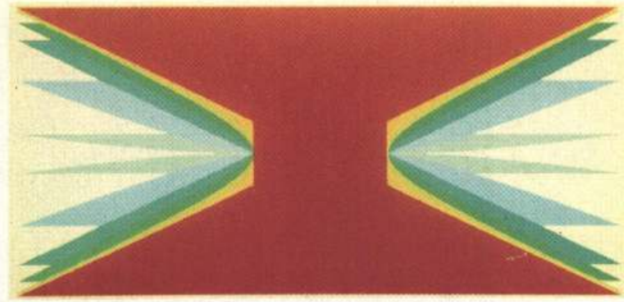


Figure 5. Curvature-dependent sampling.



(a)



(b)

Figure 6. Two-dimensional basis functions: superellipses (a) and superhyperbolas (b). The shape progression is square, round, bevel, and pinch as exponents increase from 0 to 3.

has the inside-outside function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

$$\text{If } \begin{cases} f(x_0, y_0, z_0) = 1, (x_0, y_0, z_0) \text{ is on the surface.} \\ f(x_0, y_0, z_0) > 1, (x_0, y_0, z_0) \text{ lies outside the sphere.} \\ f(x_0, y_0, z_0) < 1, (x_0, y_0, z_0) \text{ lies inside the sphere.} \end{cases}$$

Similar functions will be listed for all the superquadrics.

The bounding surface is not included in the solid description of an object because two objects could intersect solely at the boundary itself. In any solid modeling scheme, no operation on a set of solids should yield a nonsolid result, like a point, curve, or surface.

A surprising result is that the superquadric normal vectors lie on a dual superquadric form; that is, if the normal vectors are translated to originate from a single point, a second superquadric form is generated, whose own dual is the original. (The unit normal vectors are then easily obtained from these normal vectors.) The results are summarized below.

Superellipsoids (see Figure 7).

Position vector of surface:

$$\underline{x}(\eta, \omega) = \begin{bmatrix} C_\eta^{\epsilon_1} \\ a_3 S_\eta^{\epsilon_1} \end{bmatrix} \otimes \begin{bmatrix} a_1 C_\omega^{\epsilon_2} \\ a_2 S_\omega^{\epsilon_2} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 C_\eta^{\epsilon_1} C_\omega^{\epsilon_2} \\ a_2 C_\eta^{\epsilon_1} S_\omega^{\epsilon_2} \\ a_3 S_\eta^{\epsilon_1} \end{bmatrix}, \begin{matrix} -\pi/2 \leq \eta \leq \pi/2 \\ -\pi \leq \omega < \pi \end{matrix}$$

ϵ_1 is the squareness parameter in the north-south direction; ϵ_2 is the squareness parameter in the east-west direction. Cuboids are produced when both ϵ_1 and ϵ_2 are <1. Cylindroids are produced when $\epsilon_2 \sim 1$ and $\epsilon_1 < 1$. Pillow shapes are produced when $\epsilon_1 \sim 1$ and $\epsilon_2 < 1$. Pinched shapes are produced when either ϵ_1 or $\epsilon_2 > 2$. Flat-beveled shapes are produced when either ϵ_1 or $\epsilon_2 = 2$.

Normal vector:

$$\underline{n}(\eta, \omega) = \begin{bmatrix} 1/a_1 C_\eta^{2-\epsilon_1} C_\omega^{2-\epsilon_2} \\ 1/a_2 C_\eta^{2-\epsilon_1} S_\omega^{2-\epsilon_2} \\ 1/a_3 S_\eta^{2-\epsilon_1} \end{bmatrix}$$

Inside-outside function:

$$f(x,y,z) = \left((x/a_1)^{2/\epsilon_2} + (y/a_2)^{2/\epsilon_2}\right)^{\epsilon_2/\epsilon_1} + (z/a_3)^{2/\epsilon_1}$$

Superhyperboloids of one piece (see Figure 8).

Position vector of surface:

$$\underline{x}(\eta, \omega) = \begin{bmatrix} \sec^{\epsilon_1} \eta \\ a_3 \tan^{\epsilon_1} \eta \end{bmatrix} \otimes \begin{bmatrix} a_1 C_\omega^{\epsilon_2} \\ a_2 S_\omega^{\epsilon_2} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \sec^{\epsilon_1} \eta \, C_\omega^{\epsilon_2} \\ a_2 \sec^{\epsilon_1} \eta \, S_\omega^{\epsilon_2} \\ a_3 \tan^{\epsilon_1} \eta \end{bmatrix}, \begin{matrix} -\pi/2 < \eta < \pi/2 \\ -\pi \leq \omega < \pi \end{matrix}$$

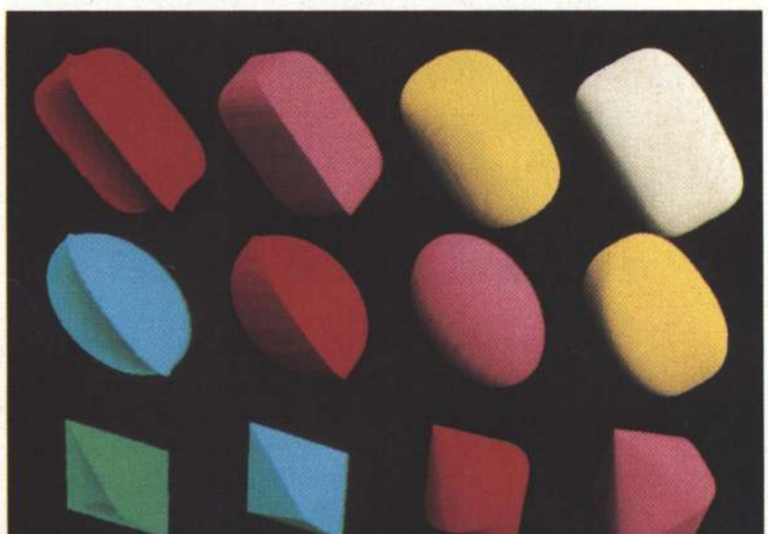


Figure 7. Superquadric ellipsoids. Exponents range from .3 to 3. The north-south exponent, ε_1, increases from top to bottom. The east-west exponent, ε_2, increases from right to left.

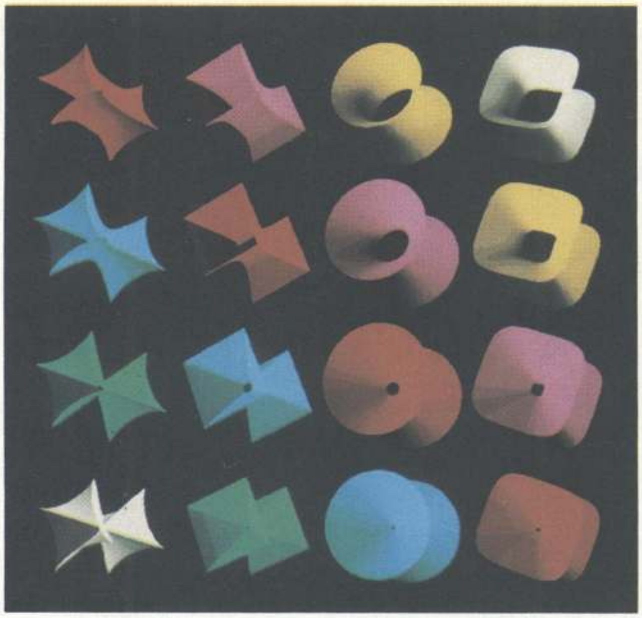


Figure 8. Superquadric hyperboloids of one sheet. Exponents increase as described in Figure 7.

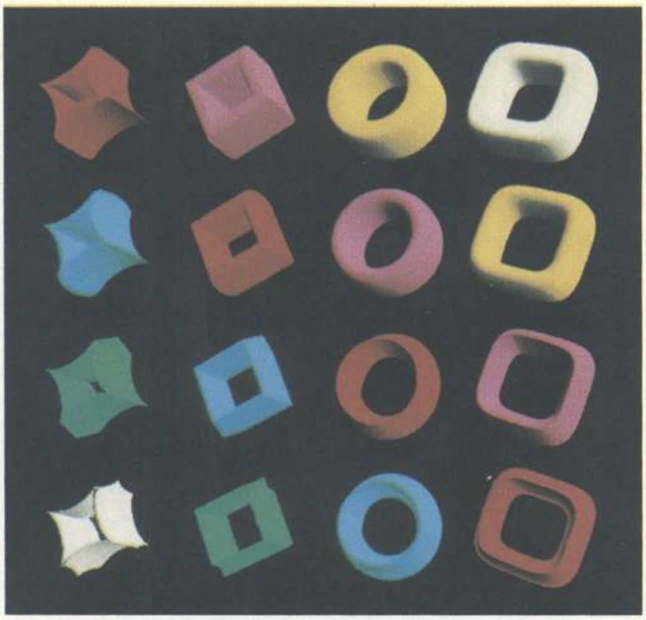


Figure 10. Superquadric toroids. Exponents increase as described in Figure 7.

Figure 9. Superquadric hyperboloids of two sheets. Exponents increase as described in Figure 7.

Normal vector:

$$\underline{n}(\eta, \omega) = \begin{bmatrix} 1/a_1 \sec^{2-\epsilon_1}\eta \cos^{2-\epsilon_2}\omega \\ 1/a_2 \sec^{2-\epsilon_1}\eta \sin^{2-\epsilon_2}\omega \\ 1/a_3 \tan^{2-\epsilon_1}\eta \end{bmatrix}$$

Inside-outside function:

$$f(x,y,z) = ((x/a_1)^{2/\epsilon_2} + (y/a_2)^{2/\epsilon_2})^{\epsilon_2/\epsilon_1} - (z/a_3)^{2/\epsilon_1}$$

Superhyperboloids of two pieces (see Figure 9).

Surface:

$$\underline{x}(\eta, \omega) = \begin{bmatrix} \sec^{\epsilon_1}\eta \\ a_3 \tan^{\epsilon_1}\eta \end{bmatrix} \otimes \begin{bmatrix} a_1 \sec^{\epsilon_2}\omega \\ a_2 \tan^{\epsilon_2}\omega \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \sec^{\epsilon_1}\eta \sec^{\epsilon_2}\omega \\ a_2 \sec^{\epsilon_1}\eta \tan^{\epsilon_2}\omega \\ a_3 \tan^{\epsilon_1}\eta \end{bmatrix}, \quad \begin{array}{l} -\pi/2 < \eta < \pi/2 \\ -\pi/2 < \omega < \pi/2 \text{ (piece 1)} \\ \pi/2 < \omega < 3\pi/2 \text{ (piece 2)} \end{array}$$

Normal vector:

$$\underline{n}(\eta, \omega) = \begin{bmatrix} 1/a_1 \sec^{2-\epsilon_1}\eta \sec^{2-\epsilon_2}\omega \\ 1/a_2 \sec^{2-\epsilon_1}\eta \tan^{2-\epsilon_2}\omega \\ 1/a_3 \tan^{2-\epsilon_1}\eta \end{bmatrix}$$

Inside-outside function:

$$f(x,y,z) = ((x/a_1)^{2/\epsilon_2} - (y/a_2)^{2/\epsilon_2})^{\epsilon_2/\epsilon_1} - (z/a_3)^{2/\epsilon_1}$$

Supertoroids (see Figure 10).

$$\underline{x}(\eta, \omega) = \begin{bmatrix} a_4 + C_\eta^{\epsilon_1} \\ a_3 S_\eta^{\epsilon_1} \end{bmatrix} \otimes \begin{bmatrix} a_1 C_\omega^{\epsilon_2} \\ a_2 S_\omega^{\epsilon_2} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 (a_4 + C_\eta^{\epsilon_1}) C_\omega^{\epsilon_2} \\ a_2 (a_4 + C_\eta^{\epsilon_1}) S_\omega^{\epsilon_2} \\ a_3 S_\eta^{\epsilon_1} \end{bmatrix}, \quad \begin{array}{l} -\pi \le \eta < \pi \\ -\pi \le \omega < \pi \end{array}$$

$$\underline{n}(\eta, \omega) = \begin{bmatrix} 1/a_1 C_\eta^{2-\epsilon_1} C_\omega^{2-\epsilon_2} \\ 1/a_2 C_\eta^{2-\epsilon_1} S_\omega^{2-\epsilon_2} \\ 1/a_3 S_\eta^{2-\epsilon_1} \end{bmatrix}$$

$$f(x,y,z) = (((x/a_1)^{2/\epsilon_2} + (y/a_2)^{2/\epsilon_2})^{\epsilon_2/2} - a_4)^{2/\epsilon_1} + (z/a_3)^{2/\epsilon_1}$$

where $a_4 = \hat{a}/\sqrt{(a_1^2 + a_2^2)}$, and $\hat{a}$ is the torus radius. Note that the dual of the normal vector is a degenerate torus, different from the original.

The inside of each solid is given by $f(x,y,z) < 1$.
The surface is indicated by $f(x,y,z) = 1$.
The outside is indicated by $f(x,y,z) > 1$.

The existence of the inside-outside functions means that superquadrics can be manipulated by means of solid boolean operations, such as union, intersection, and subtraction (see Figure 11).

Translation and rotation

The superquadric equations describe the solids in standard positions and orientations. It is usually necessary to translate and rotate the objects to the desired configurations. The rigid body transformations are invertible; thus, the original inside-outside function can be used after a function inversion. The manipulations on a solid operate on

a canonical surface, $\underline{x} = \underline{x}(\eta, \omega)$;
a normal vector, $\underline{n} = \underline{n}(\eta, \omega)$;
and an inside-outside function, $f = f(x,y,z)$.

The translated and rotated solid $\hat{S}$ is given by

$$\text{surface: } \underline{\hat{x}} = \underline{\underline{M}}\,\underline{x} + \underline{b},$$

where $\underline{\underline{M}}$ is a rotation matrix, and $\underline{b}$ is a displacement vector,

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

normal: $\underline{\hat{n}} = \underline{\underline{M}}\,\underline{n}$.

The new inside-outside function is calculated by inverting the transformation and substituting into the old inside-outside function; i.e.,

$$\hat{f}(\hat{x},\hat{y},\hat{z}) = f(x,y,z),$$

where

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\underline{M}}^{\dagger} \begin{bmatrix} \hat{x}-b_1 \\ \hat{y}-b_2 \\ \hat{z}-b_3 \end{bmatrix}$$

Note that the inverse of the rotation matrix, $\underline{\underline{M}}^{-1}$, is the same as its transpose, $\underline{\underline{M}}^{\dagger}$, because rotation matrices are orthogonal. For a general transformation $\mathfrak{I}$, the new surface is defined by

$$\underline{\hat{x}} = \mathfrak{I}(\underline{x})$$

and the new inside-outside function is defined by

$$\hat{f}(\underline{\hat{x}}) = f(\mathfrak{I}^{-1}(\underline{\hat{x}}))$$

over any domain for which the inverse transformation $\mathfrak{I}^{-1}$ exists.

Superquadrics are mathematically quite simple, involving a few sines, cosines, and exponents. The ellipsoids and toroids are bounded, simplifying the input schemes for solid modeling. The control parameters affect global properties of the shapes in a comprehensible manner, and normal vector information is available continuously over the surface, even across edges that are very square. The solids can be easily modified by bending and twisting, and have the potential to become widespread in three-dimensional geometric design.

Angle-preserving transformations

This section explains a family of invertible transforms developed to bend and twist mathematical objects in three dimensions. At each point in a transformed object, the bundle of embedded local vectors has been rotated away from its original orientation, without disturbing the angular relationship between the vectors in the bundle (see Figure 12). The method simplifies the calculations for normal and tangent vectors and can be used to preserve or modify volume, surface area, or arclength. These are shear-free deformations; they are coupled to a local ex-

Figure 11. Solid modeling with superquadrics.

Figure 12. Progressive deformation of a rectangular flat sheet. Vector bundles are differentially rotated to produce the new forms.

pansion coefficient to model the chief geometric characteristics of flexible physical objects.

At any point $\underline{x}$, the jacobian matrix $\underline{\underline{J}}(\underline{x})$ for the transform

$$\underline{y} = \underline{F}(\underline{x})$$

shall be a rotation matrix $\underline{\underline{M}}(\underline{x})$, multiplied by an expansion factor $p(\underline{x})$:

$$\underline{\underline{J}}(\underline{x}) = p(\underline{x})\,\underline{\underline{M}}(\underline{x})$$

In cartesian tensor notation, $J_{ij} = y_{i,j} = p(\underline{x})\, m_{ij}(\underline{x})$. Expansion is indicated when $p > 1$, compression when $p < 1$. Volume is conserved when $p = 1$. $\underline{x} = \underline{x}(\eta, \omega)$ is a surface to be transformed; the new tangent vectors $\underline{y}_\eta$ and $\underline{y}_\omega$ are calculated by the chain rule from the old tangent vectors $\underline{x}_\eta$ and $\underline{x}_\omega$:

$$[*] \quad \begin{aligned} \underline{y}_\eta &= \underline{\underline{J}}\,\underline{x}_\eta = p\,\underline{\underline{M}}\,\underline{x}_\eta \equiv \underline{\underline{M}}\,\underline{x}_\eta \\ \underline{y}_\omega &= \underline{\underline{J}}\,\underline{x}_\omega = p\,\underline{\underline{M}}\,\underline{x}_\omega \equiv \underline{\underline{M}}\,\underline{x}_\omega \end{aligned}$$

The direction of the normal vector on the surface $\underline{y}$, $\underline{n}^{(y)}$ is calculated via the same transformation M, from the old normal, $n^{(x)}$ on the surface $\underline{x}$.

$$\underline{n}^{(y)} = \underline{\underline{M}}\,\underline{n}^{(x)}.$$

This result is derived as follows:

$$\begin{aligned} \underline{n}^{(y)} &= \underline{y}_\eta \wedge \underline{y}_\omega \\ &= \underline{\underline{J}}\underline{x}_\eta \wedge \underline{\underline{J}}\underline{x}_\omega = p^2 \underline{\underline{M}}\underline{x}_\eta \wedge \underline{\underline{M}}\underline{x}_\omega \end{aligned}$$

or

$$n_i^{(y)} = p^2 \epsilon_{ijk}\, m_{j\hat{j}}\, x_{\hat{j},\eta}\, m_{k\hat{k}}\, x_{\hat{k},\omega}.$$

Since $[\epsilon_{ijk}]$ is an isotropic tensor,

$$m_{i\hat{i}}\,\epsilon_{\hat{i}\hat{j}\hat{k}} = \epsilon_{ijk}\, m_{j\hat{j}}\, m_{k\hat{k}},$$

so

$$\begin{aligned} n_i^{(y)} &= p^2 m_{i\hat{i}}\,\epsilon_{\hat{i}\hat{j}\hat{k}}\, x_{\hat{j},\eta}\, x_{\hat{k},\omega} \\ &= p^2 m_{i\hat{i}} (\underline{x}_\eta \wedge \underline{x}_\omega)_{\hat{i}} \\ &= p^2 m_{i\hat{i}}\, n_{\hat{i}}^{(x)}. \end{aligned}$$

□

Along any space curve

$$\underline{x} = \begin{bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \end{bmatrix}, \quad s_0 \le s \le s_1,$$

the transformed curve is given by

$$\underline{y}(s) = \int_{s_0}^{s} p(x(\hat{s}))\,\underline{\underline{M}}(\underline{x}(\hat{s}))\,\underline{x}'(\hat{s})\,d\hat{s}.$$

This is the main transforming equation.

Example 1. Inextensible curves in two dimensions: Starting with the curve

$$\underline{x} = \begin{bmatrix} s \\ 0 \end{bmatrix}, 0 \le s \le L,$$

i.e., starting with a segment joining (0,0) and $(L,0)$, let

$$\underline{\underline{M}}(s) = \begin{bmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{bmatrix}.$$

The derivative is given by

$$\underline{x}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so

$$\begin{aligned} \underline{y}(s) &= \int_0^s \begin{bmatrix} \cos\theta(\hat{s}) & -\sin\theta(\hat{s}) \\ \sin\theta(\hat{s}) & \cos\theta(\hat{s}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\hat{s} \\ &= \int_0^s \begin{bmatrix} \cos\theta(\hat{s}) \\ \sin\theta(\hat{s}) \end{bmatrix} d\hat{s}. \end{aligned}$$

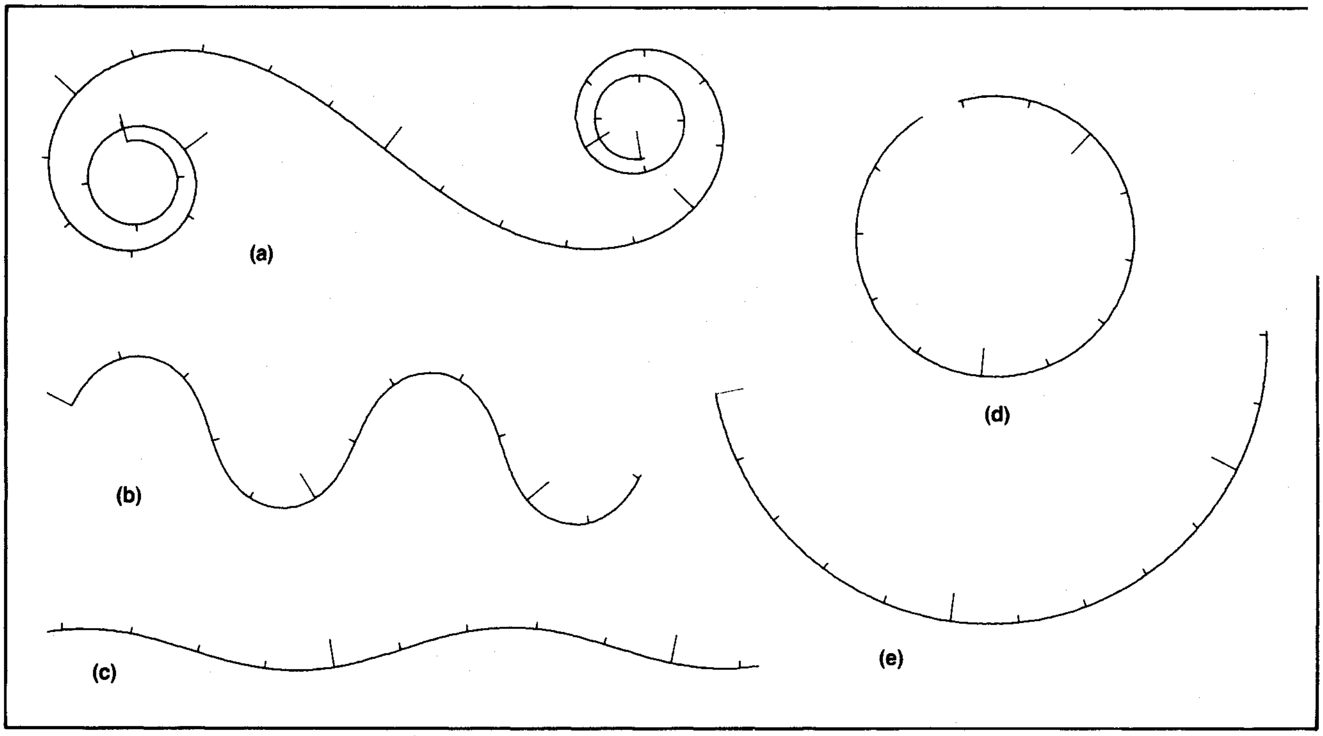


Figure 13. Deformations of a line segment: (a) a cornu spiral where $\theta = .06\,s^2 - .1$, (b) a sine-generated wave where $\theta = \cos s$, (c) a sine-generated wave where $\theta = .25 \cos s$, (d) a circular segment where $\theta = .5\,s$, and (e) a circular segment where $\theta = .25\,s$. Equally spaced rules on the curve indicate integer values of arclength s.

$$T = \text{tangent vector} = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}$$

$$N = \text{unit normal vector} = \underline{\underline{M}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{bmatrix}$$

Note that $T' = \theta' \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{bmatrix} = \theta' N$.

Therefore, θ' is the curvature, κ, of the curve $\underline{y}(s)$. θ' has the appropriate sign (positive or negative) that maintains $N(s)$ as a continuous field of vectors. If curvature were forced to be positive, there would be discontinuities in N, which could be inconvenient in some modeling situations (see Figure 13).

A circular deformation is produced when the curvature $\kappa = \theta' = \text{constant} = k$. For this situation, θ is a linear function of arclength, s:

$$\theta = ks + s_o.$$

The radius of curvature is $1/k$.

Inextensible coiled curves are produced by nonlinear curves, $\theta = \theta(s)$. $\theta(s)$ and the resulting transformation curves are shown in Figure 13. Curvature is high when $|\theta'|$ is large. θ' can be used for curvature-dependent algorithms to sample the curve $\underline{y}(s)$, as shown in Figure 5.

Example 2. Transformations of space curves:

$$x = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{y}(s) = \int_o^s \underline{\underline{M}}(\hat{s}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d\hat{s}$$

The tangent vector $\underline{T}$ is the first column of $\underline{\underline{M}}$, m_{j1}. The two other columns, m_{j2} and m_{j3}, are two mutually orthogonal vectors, $\underline{N}_1$ and $\underline{N}_2$, normal to the curve $\underline{y}(s)$. $\underline{N}_1$ and $\underline{N}_2$ are not the normal vector $\underline{N}$ and binormal vector $\underline{B}$ of the *curve,* although all four vectors lie in the curve's normal plane; $\underline{N}_1$ and $\underline{N}_2$ are the two principal normal vectors of the *transformation* $\underline{\underline{M}}$.

$\underline{T}$, $\underline{N}_1$ and $\underline{N}_2$ can be used as a local vector coordinate system (see Figure 14). The transformation coordinate system ($\underline{T}$, $\underline{N}_1$, $\underline{N}_2$) changes its orientation along the length of the curve trajectory. The angular velocity $\underline{\Omega}$ of the transformation gives three curvature parameters in terms of the local coordinate system, measuring bending in three mutually perpendicular planes. However, the torsion and curvature of the curve itself differ from the torsion and curvature of the transformation, as can be seen in the vertical line in Figure 14. The straight curve has zero curvature and torsion, although the coordinate system twists around it.

The differentiation rules are as follows:

$$\begin{aligned} \underline{T}' &= \Omega_3 \underline{N}_1 - \Omega_2 \underline{N}_2 \\ \underline{N}_1' &= -\Omega_3 \underline{T} + \Omega_1 \underline{N}_2 \\ \underline{N}_2' &= \Omega_2 \underline{T} - \Omega_1 \underline{N}_1 \end{aligned}$$

where $' = d/ds$. $\underline{\Omega}$ may be computed by noting that

$$\begin{aligned} \Omega_1 &= \underline{N}_2' \cdot \underline{N}_1' \\ \Omega_2 &= \underline{T} \cdot \underline{N}_2' \\ \Omega_3 &= \underline{N}_1 \cdot \underline{T}' \end{aligned}$$

The above differentiation rules for transformations replace the Frenet-Serret formulas, which relate the tangent, normal, and binormal vectors of curves to their derivatives.

Frenet-Serret formulas:

$$\begin{aligned} \underline{T}' &= \kappa \underline{N} \\ \underline{N}' &= \tau \underline{B} - \kappa \underline{T} \\ \underline{B}' &= -\tau \underline{N} \end{aligned}$$

κ is curvature of the curve.

τ is torsion of the curve.

$\kappa = \underline{N} \cdot \underline{T}'$.

$\tau = \underline{N} \cdot \underline{B}'$.

The equations coincide when $\Omega_2 = 0$.

Note that

$$\underline{N} = \frac{\Omega_3}{\sqrt{(\Omega_2^2 + \Omega_3^2)}} \underline{N}_1 - \frac{\Omega_2}{\sqrt{(\Omega_2^2 + \Omega_3^2)}} \underline{N}_2$$

and that $\kappa = \sqrt{(\Omega_2^2 + \Omega_3^2)}$.

Example 3. Transformations using the ZXZ euler rotation matrix:

for the ZXZ euler rotation matrix produced by

$\phi_1 = \phi_1(s)$: first rotation, around Z axis

$\phi_2 = \phi_2(s)$: second rotation, around X axis

$\phi_3 = \phi_3(s)$: third rotation, around Z axis

$$\underline{\underline{M}}(s) = \begin{bmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2 & -S_2 \\ 0 & S_2 & C_2 \end{bmatrix} \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\underline{T}(s), \underline{N}_1(s), \underline{N}_2(s)\right]$$

where

$$\underline{T}(s) = \begin{bmatrix} C_1C_3 - S_1C_2S_3 \\ C_1S_3 + S_1C_2C_3 \\ S_1 S_2 \end{bmatrix}$$

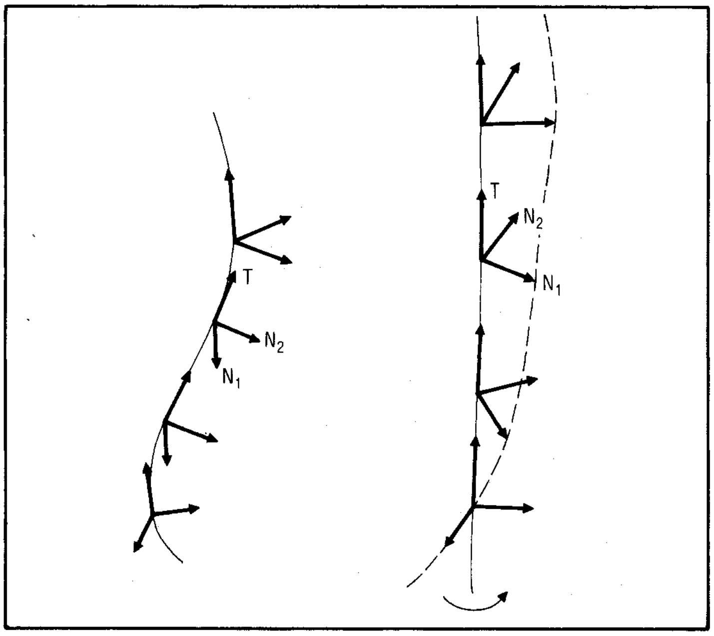


Figure 14. Orientation of the axes along the transformed curve.

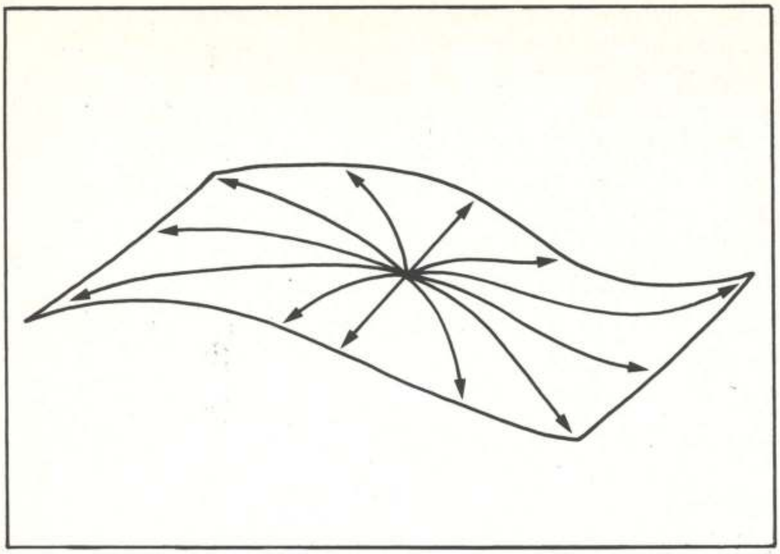


Figure 15. A set of transformation curves in a rectangular sheet.

$$\underline{N}_1(s) = \begin{bmatrix} S_1C_3 - C_1C_2S_3 \\ -S_1S_3 + C_1C_2C_3 \\ C_1S_2 \end{bmatrix}$$

$$\underline{N}_2(s) = \begin{bmatrix} S_2S_3 \\ -S_2C_3 \\ C_2 \end{bmatrix}$$

where

$$S_1 = \sin(\phi_1(s)) \quad S_2 = \sin(\phi_2(s)) \quad S_3 = \sin(\phi_3(s))$$
$$C_1 = \cos(\phi_1(s)) \quad C_2 = \cos(\phi_2(s)) \quad C_3 = \cos(\phi_3(s))$$

$$\underline{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} -S_3S_2\phi_1' - \phi_2'C_3 \\ C_3S_2\phi_1' - S_3\phi_2' \\ -C_2\phi_1' \quad -\phi_3' \end{bmatrix}$$

$$\underline{\underline{M}}' = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix} \underline{\underline{M}}$$

Transforms of 3-D surfaces and solids

A surface or solid is transformed much in the same manner as a space curve. First the object is broken down completely into a set of disjoint curves $\underline{x}(s)$ (emanating from a common origin), such that the union of the points in the curves is equal to the set of points in the object (see Figure 15). Each curve is transformed via the main transforming equation,

$$\hat{\underline{x}}(s) = \int J\, \underline{x}'(s)\, ds$$

to produce the transform of the object, $\hat{\underline{x}}(s)$.

The integrand, however, must be an exact differential, so that the results are independent of the integration path, and so that the surface tangent and normal vectors may be calculated using [*]. This consistency requirement is very restrictive and, generally, difficult to satisfy. For instance, the only nonrigid, solid-angle-preserving transformations operate on spheres.

To ease these restrictions, a small expansion and shear term is introduced. Volumetric and angular properties will be conserved within a tolerance λ. This is a realistic approximation because most material solids and surfaces stretch and shear to a small degree. To satisfy the consistency requirement for the solids and surfaces, the small, undetermined correction term, $\lambda\underline{\underline{Q}}$, is introduced to the jacobian matrix,

$$\underline{\underline{J}} = \underline{\underline{M}} + \lambda\, \underline{\underline{Q}}, \text{ where } \det(Q) = 1.$$

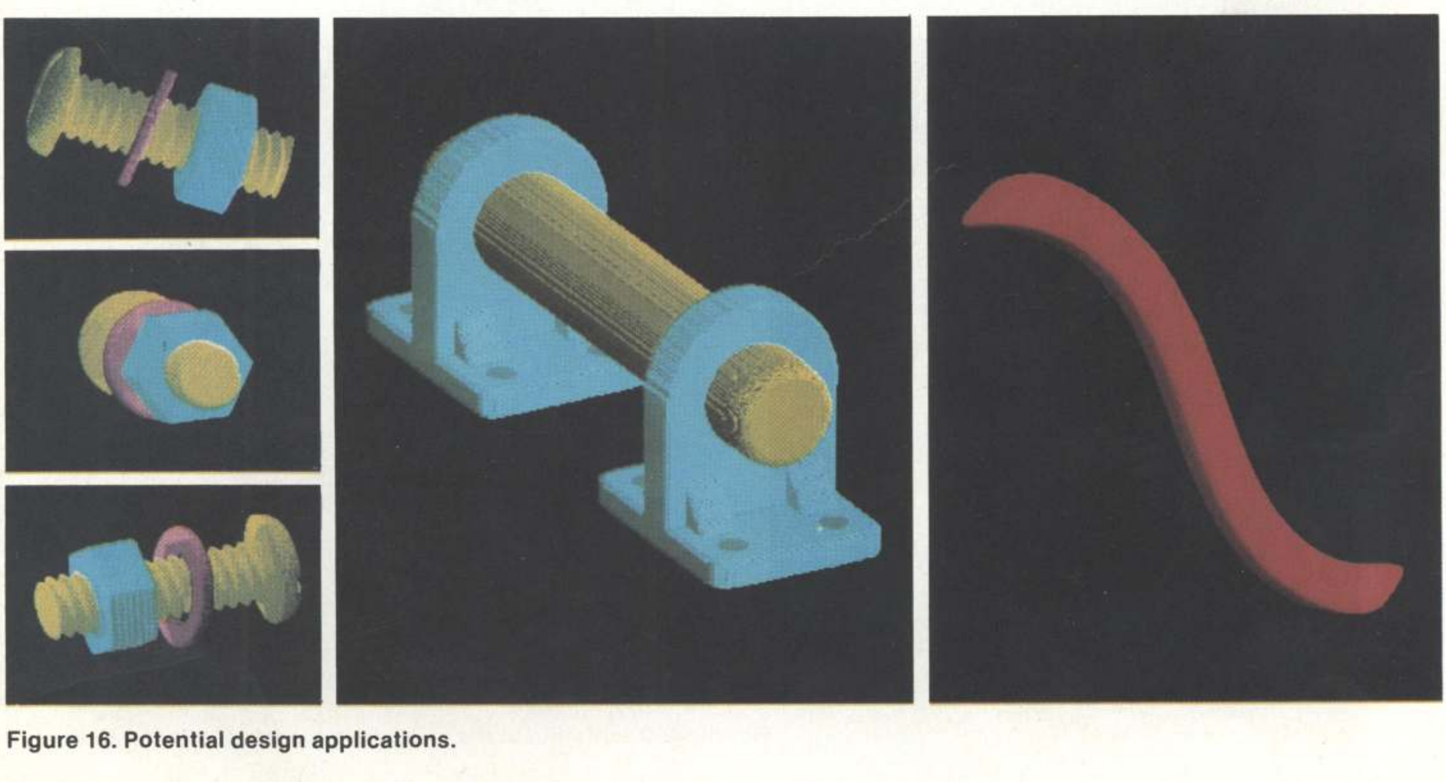


Figure 16. Potential design applications.

The transformation becomes

$$\hat{\underline{x}}(s) = \int (\underline{\underline{M}} + \lambda \underline{\underline{Q}})\, \underline{x}'\, ds$$

$$= \int (\underline{T} + \lambda \underline{Q}_1)\, dx + (\underline{N}_1 + \lambda \underline{Q}_2)\, dy + (\underline{N}_2 + \lambda \underline{Q}_3).$$

For the integrand to be an exact differential

$$\begin{aligned} (\underline{T} + \lambda \underline{Q}_1)_y &= (\underline{N}_1 + \lambda \underline{Q}_1)_x \\ (\underline{T} + \lambda \underline{Q}_1)_z &= (\underline{N}_2 + \lambda \underline{Q}_3)_x \\ (\underline{N}_1 + \lambda \underline{Q}_2)_z &= (\underline{N}_2 + \lambda \underline{Q}_3)_y \end{aligned}$$

using the differentiation rules for T, N_1, and N_2, we obtain

$$\begin{aligned} & -\Omega_2^{(y)}\, \underline{N}_2 + \Omega_3^{(y)}\, \underline{N}_1 + \lambda\, \underline{Q}_{1,y} \\ = & -\Omega_3^{(x)}\, \underline{T} + \Omega_1^{(x)}\, \underline{N}_2 + \lambda\, \underline{Q}_{2,x} \end{aligned}$$

and other similar relations involving Ω and λ. These relations are satisfied to order λ if all of the curvatures Ω_i are of order λ, and the derivatives of Q are of order 1.

$1 + \lambda$ is an upper limit on the expansion factor, because

$$|\underline{\underline{M}} + \lambda\, \underline{\underline{Q}}| \le |\underline{\underline{M}}| + \lambda |\underline{\underline{Q}}| = 1 + \lambda.$$

Using the ZXZ euler rotation matrix, all of the preceding relations will be satisfied to order λ if, for each of the angles ϕ_1, ϕ_2, and ϕ_3,

$$\left|\frac{\partial \phi}{\partial x}\right| + \left|\frac{\partial \phi}{\partial y}\right| + \left|\frac{\partial \phi}{\partial z}\right| < \lambda .$$

In other words, for small deformation rates, the volume or surface area of the object is preserved locally within a factor of $1 + \lambda$.

Superquadrics generalize the basic quadric surfaces and solids, producing a continuum of useful forms with rounded edges and filleted faces. Angle-preserving transformations operate on a predefined surface or space curve, bending and twisting the object into a new form. Together, the new primitives and operators have potential design applications wherever flexible operations are needed, or where volume, surface area, or arclength must be conserved (see Figure 16). They provide a powerful extension to the classical design shapes. ■

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