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# Superselection Sectors with Braid Group Statistics and Exchange Algebras

## I. General Theory

K. Fredenhagen, K. H. Rehren*, and B. Schroer

Institut für Theorie der Elementarteilchen, Freie Universität Berlin, WE 4, Arnimallee 14, D-1000 Berlin 33

**Abstract**. The theory of superselection sectors is generalized to situations in which normal statistics has to be replaced by braid group statistics. The essential role of the positive Markov trace of algebraic quantum field theory for this analysis is explained, and the relation to exchange algebras is established.

### 1. Introduction

Superselection sectors in 4 dimensional quantum field theories are classified by the equivalence classes of irreducible representations of some compact group, the group of internal symmetries. All models seem to have this property, and recently, Doplicher and Roberts succeeded in deriving the existence of such a group from first principles [1]. Their treatment is based on the theory of superselection sectors [2] which has been developed in the framework of algebraic quantum field theory [3]. The basic result of the theory of superselection sectors is the intrinsic definition of statistics. There is, associated with each sector, an – up to equivalence – unique representation of the permutation group which describes the statistics of multiparticle states. In principle, the theory can be applied also to models in lower dimensional space time, however, there statistics has to be described, in general, by a representation of the braid group.[1]

In view of the recent progress in the analysis of representations of the braid group [4] it seems to be worthwhile to analyze those representations which occur in quantum field theory more closely. On the other hand, nowadays a lot of models are known, especially conformally covariant field theories in two dimensions, which exhibit a rich structure of superselection sectors which does not seem to fit the representation theory of some group. Actually, representations of the braid group

* Adress from October 1st, 1988: Inst. for Theor. Physics, Rijksuniversiteit, Princetonplein 5, P.B. 80.006, NL-3508 TA Utrecht, The Netherlands

[1] This fact seems to be well known to the experts. We thank D. Buchholz, S. Doplicher, J. Fröhlich, J. Roberts, and R. Tscheuschner for helpful discussions on this point

have been found in these models [5] which look very similar to the intrinsically defined representations arising from the algebraic framework.

It is the aim of this paper to utilize the results of the abstract analysis of first principles for the investigation of concrete models. After a brief description of the algebraic framework we introduce the statistics operators and show that they generate a representation of the braid group. We then consider the trace functional on the algebra of statistics operators which had been used in the normal statistics case for the analysis of possible representations of the permutation group. This functional is shown to define a Markov trace on the braid algebra, i.e. its value is determined up to some rescaling by the link which is obtained by closing the braid. The functional can be used to measure relative dimensions and is relevant for the metric in the space of scattering states.

We then describe how the $R$ matrices in exchange algebras [5] are connected with the statistics operators. We show that the exchange algebra is equivalent to a reduced version of the field bundle of DHR II [2]. This implies unitarity of the $R$-matrices and provides us with a Markov trace on the $R$-matrices. Finally we outline how fusion rules define a sequence of subalgebras of the algebras of observables and compare the structure with V. Jones' treatment of towers of algebras. Applications to low dimensional ($D=2,3$) quantum field theories, in particular to the ($d=1$) exchange algebras of light-cone fields, which are the building blocks of conformal $\mathrm{QFT}_2$, will be dealt with in a second part. For an outlook we refer to the concluding remarks of this paper.

Our work might be compared with the recent work of Buchholz et al. [6]. These authors analyze the superselection structure of the $U(1)$ current algebra on a 1-dimensional light cone. As a matter of fact all sectors of this model are abelian (see footnote 4) so only one-dimensional representations of the braid group occur. In contrast to this the main emphasis of our work lies on the analysis of nonabelian sectors.

Another recent approach to the understanding of superselection sectors in 2 and 3 dimensional theories is due to Fröhlich [7, 25]. Working in a Wightman framework, Fröhlich formulates commutation relations for charged fields which involve a matrix representation of the braid group and which are consistent with locality of observables (leaving aside the positivity issue). This looks similar but is not identical to the exchange algebras introduced in [5].

## 2. Statistics

Let us briefly describe the algebraic framework of quantum field theory.[2] There is a family of v. Neumann algebras $\mathscr{A}(O)$ in a Hilbert space $\mathscr{H}$ indexed by the closed double cones in Minkowski space such that the following properties hold:

(i) $$\mathscr{A}(O_1)\subset\mathscr{A}(O_2) \quad \text{if } O_1\subset O_2 \quad \text{(isotony)} .$$

$\mathscr{A}=\bigcup\mathscr{A}(O)$ is called the algebra of observables.

(ii) $$\mathscr{A}(O_1)\subset\mathscr{A}(O_2)' \quad \text{if } O_1\subset O_2' \quad \text{(locality)} .$$

[2] For a more detailed description see [2]

Here $\mathscr{A}(O_2)'$ is the commutant of $\mathscr{A}(O_2)$, i.e. the set of all bounded operators in $\mathscr{H}$ which commute with $\mathscr{A}(O_2)$; $O_2'$ denotes the spacelike complement of $O_2$.

(ii) is strengthened by the requirement that $\mathscr{A}(O)$ is the *maximal* algebra satisfying (ii).

(iii) $$\mathscr{A}(O')' = \mathscr{A}(O) \quad \text{(Haag – duality)} \ , \tag{2.2}$$

where $\mathscr{A}(O')$ is the algebra generated by the algebras $\mathscr{A}(O_1)$ with double cones $O_1 \subset O'$.

(iv) There is a representation $x \to \alpha_x$ of the translation group by automorphisms of $\mathscr{A}$ such that

$$\alpha_x(\mathscr{A}(O)) = \mathscr{A}(O+x) \ . \tag{2.3}$$

(v) Moreover, there is a strongly continuous unitary representation $x \to U(x)$ of the translation group in $\mathscr{H}$ implementing $\alpha_x$, i.e.

$$\alpha_x(A) = U(x) A U(-x) \tag{2.4}$$

such that the generators of $U$ (i.e. the energy-momentum operators) have spectrum in the closed forward light cone.

(vi) There is a vector $\Omega \in \mathscr{H}$ (representing the vacuum) such that

$$U(x)\Omega = \Omega \ . \tag{2.5}$$

$\Omega$ is unique up to a phase.

The Haag duality property (iii) requires some explanation. It should not be confused with other concepts of duality abundant in the literature. It may happen that the originally chosen net of observable algebras (e.g. the net generated by the energy-momentum tensor in conformal $\mathrm{QFT}_2$) does not have this property of duality. But Bisognano and Wichmann [8] have shown that for a net generated by Wightman fields one always can pass to the bidual net $\mathscr{B}(O) := \mathscr{A}(O')'$ which then has the desired property. The issue of maximalization of observable algebras in order to achieve Haag duality is in a subtle way related to the possible occurence of spontaneous symmetry breakdown [9] and (in two-dimensional situations) to Kramers-Wannier-Kadanoff duality. For the general analysis and classification of superselection sectors, it is only important that a Haag dual observable algebra exists; for discussing concrete models however [done in Part II] one is obliged to say something more specific.

The theory of locally generated superselection sectors to which we restrict ourselves in this paper analyzes representations $\pi$ of $A$ which have positive energy in the sense of (v) and are unitarily equivalent to the (identical) vacuum representation on the spacelike complement $O'$ of any double cone $O$, i.e. to each double cone $O$ there is a unitary $V: \mathscr{H}_\pi \to \mathscr{H}$ with

$$V\pi(A) = AV \ , \quad A \in \mathscr{A}(O') \ . \tag{2.6}$$

This unitary equivalence can be used to define a representation $\varrho$ in the vacuum Hilbert space $\mathscr{H}$ which is equivalent to $\pi$,

$$\varrho(A) = V\pi(A)V^{-1} \ , \quad A \in \mathscr{A} \ . \tag{2.7}$$

Due to (2.6), $\varrho$ acts trivially on $\mathscr{A}(O')$ and because of duality (2.2) it is actually an endomorphism of $\mathscr{A}$. $\varrho$ is called a "localized morphism." The fact that endomorphisms can be composed leads to a composition rule of sectors. A composed sector may be reducible, but all subrepresentations still have property (2.6)[3], and are therefore also equivalent to some localized morphism. Morphisms localized in spacelike separated regions commute (DHRI [2]).

Charged fields which interpolate between different superselection sectors can be defined in the so-called field bundle $\mathscr{F}$ introduced in DHR II [2]. State vectors are pairs $\{\varrho, \phi\}$, where $\varrho$ is a localized morphism and $\phi \in \mathscr{H}$. Fields are pairs $\{\varrho, A\}$ with $A \in \mathscr{A}$ and act on vectors according to

$$\{\varrho, A\}\{\varrho', \phi\} = \{\varrho'\varrho, \varrho'(A)\phi\} \ . \tag{2.8}$$

Observables are fields with $\varrho = \iota$ (the identity morphism). There is a large redundancy in this formalism which can be described by the action of intertwiners: if $T \in \mathscr{A}$ satisfies

$$\varrho'(A)T = T\varrho(A) \ , \quad A \in \mathscr{A} \ , \tag{2.9}$$

we call $T$ an intertwiner from $\varrho$ to $\varrho'$ and use the notation

$$T = (\varrho'|T|\varrho) \ . \tag{2.10}$$

Intertwiners act on vectors by

$$(\varrho'|T|\varrho)\{\varrho, \phi\} = \{\varrho', T\phi\} \tag{2.11}$$

and on fields by

$$(\varrho'|T|\varrho) \circ \{\varrho, A\} = \{\varrho', TA\} \ . \tag{2.12}$$

They commute with observables. Fields $\{\varrho, A\}$ are said to be localized in $O$ if they commute with all observables $\{\iota, B\}, B \in \mathscr{A}(O')$. By (2.8) this means

$$AB = \varrho(B)A \ , \quad B \in \mathscr{A}(O') \ . \tag{2.13}$$

Let $U$ be a unitary intertwiner from $\varrho$ to $\hat{\varrho}$, where $\hat{\varrho}$ is localized in $O$. Then $\varrho(B) = U^{-1}BU$ for $B \in \mathscr{A}(O')$, and thus from (2.13)

$$UAB = BUA \ , \tag{2.14}$$

which implies $U\,A \in \mathscr{A}(O)$ by duality. Thus fields localized in $O$ are of the form

$$F \equiv \{\varrho, A\} = \{\varrho, U^{-1}C\} \tag{2.15}$$

with $C \in \mathscr{A}(O)$ and a unitary intertwiner $U$ from $\varrho$ to some morphism $\hat{\varrho}$ which is localized in $O$.

We now analyze the commutation rules for fields which are localized in spacelike separated regions. Let $O_1 \subset O_2'$, and let $\{\varrho_i, A_i\} = F_i$ be localized in $O_i, i = 1, 2$. Choose as in (2.15) unitary intertwiners $U_i$ from $\varrho_i$ to "spectator" morphisms $\hat{\varrho}_i$ localized in $O_i$ and $C_i \in \mathscr{A}(O_i)$, $U_i^{-1}C_i = A_i$. Then

$$F_1F_2 = (\varrho_2\varrho_1|\varepsilon|\varrho_1\varrho_2) \circ F_2F_1 , \tag{2.16}$$

[3] This follows from Borchers' result that each projection $E \in \mathscr{A}(O)$, $E \neq 0$ can be written in the form $E = WW^*$ with an isometry $W \in \mathscr{A}(O + O_g)$, $O_g$ being an arbitrary neighbourhood of the origin [10]

where $\varepsilon=\varrho_2(U_1^{-1})U_2^{-1}U_1\varrho_1(U_2)$ is an intertwiner from $\varrho_1\varrho_2$ to $\varrho_2\varrho_1$. $\varepsilon$ does not depend on the choice of the unitary intertwiners $U_i, i=1,2$. This is obvious from (2.16) and may also be verified by direct computation. Moreover, $\varepsilon$ does not change if one replaces $O_1$ by $\hat{O}_1$ with $O_1\subset\hat{O}_1\subset O_2'$ since then one can use the same intertwiners in the formula for $\varepsilon$. The same holds if one replaces $O_2$ by $\hat{O}_2$ with $O_2\subset\hat{O}_2\subset O_1'$. By iteration one finds that $\varepsilon$ is independent of $O_1, O_2$ if the spacetime dimension is at least 3 (this is the situation analyzed in [2]) and that it depends only on the spatial order in a theory in two space time dimensions or in a theory on a one dimensional light cone. In the following we will pay attention to this spatial order. We use the notation

$$\varepsilon=\varepsilon(\varrho_1,\varrho_2) \tag{2.17}$$

if $O_1$ is to the right of $O_2(O_1>O_2)$. *If* $O_1$ is to the left of $O_2$ we have

$$\varepsilon=\varepsilon(\varrho_2,\varrho_1)^{-1}\ . \tag{2.18}$$

Using the elementary transposition $\varepsilon(\varrho_i,\varrho_j)$ one can permute the factors in any product of mutually spacelike localized fields $\{\varrho_i, A\}=F_i, i=1,\ldots,n$.

Let $n=3$ and let the localization regions be in the order $O_1>O_2>O_3$. Then one finds two representations of the operator $\varepsilon$ in

$$F_1F_2F_3=(\varrho_3\varrho_2\varrho_1|\varepsilon|\varrho_1\varrho_2\varrho_3)\circ F_3F_2F_1\ , \tag{2.19}$$

which implies

$$\varepsilon=\varrho_3(\varepsilon(\varrho_1,\varrho_2))\varepsilon(\varrho_1,\varrho_3)\varrho_1(\varepsilon(\varrho_2,\varrho_3))=\varepsilon(\varrho_2,\varrho_3)\varrho_2(\varepsilon(\varrho_1,\varrho_3))\varepsilon(\varrho_1,\varrho_2)\ . \tag{2.20}$$

Equation (2.20) corresponds to an equivalence relation for coloured braids (Fig. 1):

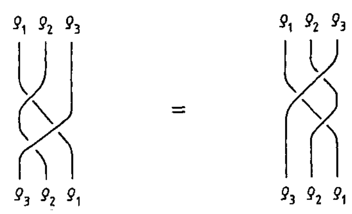


**Fig. 1.** Artin relation for coloured braids

We define a groupoid representation for coloured braids as follows:

For localized morphisms $\varrho_1,\ldots,\varrho_n$ we represent the generators $\sigma_i, i=1,\ldots,n-1$, of the braid group on $n$ threads $B_n$ [11] by

$$\varepsilon_{\sigma_i}(\varrho_1,\ldots,\varrho_n)=\varrho_1\cdots\varrho_{i-1}(\varepsilon(\varrho_i,\varrho_{i+1})) \tag{2.21}$$

which are unitary intertwiners from $\varrho_1\ldots\varrho_n$ to $\varrho_1\ldots\varrho_{i-1}\varrho_{i+1}\varrho_i\varrho_{i+2}\ldots\varrho_n$. These intertwining properties yield also the second equivalence relation for $i\geqq3$:

$$\varepsilon(\varrho_1,\varrho_2)\cdot\varrho_1\varrho_2\varrho_3\cdots\varrho_{i-1}(\varepsilon(\varrho_i,\varrho_{i+1}))=\varrho_2\varrho_1\varrho_3\cdots\varrho_{i-1}(\varepsilon(\varrho_i,\varrho_{i+1}))\varepsilon(\varrho_1,\varrho_2)\ . \tag{2.22}$$

Hence the multiplication law

$$\varepsilon_{b_1b_2}(\varrho_1,\ldots,\varrho_n)=\varepsilon_{b_1}(\varrho_{\pi_2^{-1}(1)},\ldots,\varrho_{\pi_2^{-1}(n)})\varepsilon_{b_2}(\varrho_1,\ldots,\varrho_n) \tag{2.23}$$

for $b_i \in B_n$, where $\pi_2 = \pi(b_2)$ is the image of $b_2$ under the natural homomorphism $\pi: B_n \to S_n$, respects all equivalence relations of the groupoid of coloured braids and induces therefore a unitary representation by intertwiners $\varepsilon_b(\varrho_1, \ldots, \varrho_n)$ from $\varrho_1 \cdots \varrho_n$ to $\varrho_{\pi(b)^{-1}(1)} \cdots \varrho_{\pi(b^{-1})(n)}, b \in B_n$.

For $\varrho_1 = \ldots = \varrho_n = \varrho$ this yields a homomorphism of $B_n$ into $\varrho^n(\mathscr{A})'$,

$$\sigma_i \mapsto \varepsilon_\varrho^{(n)}(\sigma_i) = \varrho^{i-1}(\varepsilon_\varrho) \ , \tag{2.24}$$

with $\varepsilon_\varrho = \varepsilon(\varrho, \varrho)$. The above equivalence relations read

$$\varrho(\varepsilon_\varrho)\varepsilon_\varrho\varrho(\varepsilon_\varrho) = \varepsilon_\varrho\varrho(\varepsilon_\varrho)\varepsilon_\varrho \ , \tag{2.25}$$

$$\varepsilon_\varrho\varrho^k(\varepsilon_\varrho) = \varrho^k(\varepsilon_\varrho)\varepsilon_\varrho \ , \quad (k \geqq 2) \ . \tag{2.26}$$

This representation characterizes the statistics associated with $\varrho$. Up to unitary equivalence it only depends on the equivalence class of $\varrho$. As in the case of the permutation group [2], positivity leads to restrictions. These will be discussed in Sect. 3, but the full range of admissible representations is not yet known.

## 3. Left Inverses of Localized Morphism and the Markov Trace of Algebraic QFT

In this chapter we discuss the theory of left-inverses of localized morphisms. Remarkably, left-inverses give rise to special Markov traces (or link-invariants) on the braid group, and the positivity of the latter restricts the values of the relevant physical parameters. We illustrate this fact by the explicit analysis of Markov traces on the Hecke algebra. Then we proceed by the presentation of general properties of left-inverses such as existence, uniqueness, and behaviour under composition and reduction of morphisms. We end the section with the definition of the quantum field theoretical Markov trace in the general case.

The localized morphisms $\varrho$ are isomorphisms of $\mathscr{A}$ into some subalgebra $\varrho(\mathscr{A})$ of $\mathscr{A}$ which in general does not coincide with $\mathscr{A}$.[4] In such a case $\varrho$ does not have an inverse on $\mathscr{A}$. There are, however, so called left-inverses, i.e. positive linear mappings $\phi$ from $\mathscr{A}$ to $\mathscr{A}$ with the properties

$$\begin{aligned} &\text{(i)} \qquad \phi(\varrho(A)B\varrho(C)) = A\phi(B)C \ , \quad A, B, C \in \mathscr{A} \ , \\ &\text{(ii)} \qquad \phi(1) = 1 \ . \end{aligned} \tag{3.1}$$

The existence of $\phi$ follows from certain compactness properties (see DHRI [2]). $\varrho \circ \phi$ is a conditional expectation from $\mathscr{A}$ onto $\varrho(\mathscr{A})$. In the case of normal statistics analyzed in [DHRI] the arising representation of the permutation group associated to an irreducible $\varrho$ can be characterized in terms of the so-called statistics parameter $\lambda_\varrho$,

$$\lambda_\varrho 1 = \phi(\varepsilon_\varrho) \ . \tag{3.2}$$

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[4] If $\varrho(\mathscr{A}) = \mathscr{A}$, $\varrho$ is an automorphism. This case is characterized by the equivalent conditions: (i) $\varrho^2$ is irreducible, (ii) $\varepsilon_\varrho = \lambda 1, \lambda \in C$, (iii) $\varrho(\mathscr{A}(O'))' = \varrho(\mathscr{A}(O))$ for all $O$ (see DHR). Sectors of this form are called abelian; in conformal field theories they have recently been studied by Buchholz et al. [6]

Also in the more general case considered here, $\phi(\varepsilon_\varrho)$ is a multiple of the identity (it commutes with $\varrho(\mathscr{A})$ since $\varepsilon_\varrho$ commutes with $\varrho^2(\mathscr{A})$); however the representation of the braid group is not determined by $\lambda_\varrho$ alone.

Nevertheless, we will show that in the case $\lambda_\varrho \neq 0$ the iterated left inverses $\phi^n$ converge to a trace state $\varphi$ on the braid group algebra possessing the so-called Markov property

$$\varphi(\varepsilon_\varrho \varrho(\varepsilon_\varrho(b))) = \lambda_\varrho \varphi(\varepsilon_\varrho(b)) \ , \quad b \in B_\infty \ . \tag{3.3}$$

There is an important special case which can be treated in essentially the same way as the normal statistics case. Assume that $\varrho^2$ has exactly two nonzero irreducible subrepresentations. The eigenprojections of $\varepsilon_\varrho$ reduce $\varrho^2$, hence $\varepsilon_\varrho$ can have at most two different eigenvalues $\lambda_1, \lambda_2$. On the other hand, $\varepsilon_\varrho$ is not a multiple of the identity since then $\varrho^2$ would be irreducible (cf. footnote 4). The equation

$$(\varepsilon_\varrho - \lambda_1 1)(\varepsilon_\varrho - \lambda_2 1) = 0 \tag{3.4}$$

implies that the operators $g_k = -\lambda_2^{-1} \varrho^{k-1}(\varepsilon_\varrho), k \in \mathbb{N}$, fulfill the defining relations of the generators of the Hecke algebra $H(t)$ [4],

$$\begin{aligned} &\text{(i)} && g_k g_{k+1} g_k = g_{k+1} g_k g_{k+1} \ , \\ &\text{(ii)} && g_k g_j = g_j g_k \ , \quad |j-k| \geqq 2 \ , \\ &\text{(iii)} && g_k^2 = (t-1) g_k + t \ , \end{aligned} \tag{3.5}$$

with $t = -\lambda_1 \lambda_2^{-1} \neq -1$. We have $\varrho(g_k) = g_{k+1}$ and

$$\phi(g_1) = -\lambda_\varrho \lambda_2^{-1} 1, \quad \phi(g_{k+1}) = g_k \ , \quad k \geqq 1 \ . \tag{3.6}$$

The positivity of $\phi$ leads to restrictions on the allowed parameter values of $t$ and $\lambda_\varrho$ which have been obtained by Ocneanu and Wenzl [12]. It is an amusing observation that essentially the same methods had been used in the 1971 paper of Doplicher et al. [2] where they proved that in the case $t = 1$ the only possible values of $\lambda_\varrho$ are $\pm d^{-1}, d \in \mathbb{N}$, and 0.

The idea is to evaluate the left inverse $\phi$ on the projections $E_i^{(n)}$onto the intersection of eigenspaces of $\varrho^k(\varepsilon_\varrho), k = o, \ldots, n-2$ with eigenvalue $\lambda_i$, i $= 1, 2$. In the case of the permutation group ($t = 1$) these are the projections corresponding to the totally symmetric and the totally antisymmetric representation of the permutation group.

The computation of $E_i^{(n)}$ is slightly more tedious in the general case than in the case of the permutation group. We start by noting that $E_i^{(1)} = 1$ and

$$E_i^{(2)} = (\lambda_i - \lambda_j)^{-1} (\varepsilon_\varrho - \lambda_j 1), \quad i \neq j \ . \tag{3.7}$$

The projections $E_i = E_i^{(2)}$ have the following property:

$$E_i \varrho(E_i) E_i - \tau E_i = \varrho(E_i) E_i \varrho(E_i) - \tau \varrho(E_i) \tag{3.8}$$

with $\tau = t(1+t)^{-2}$, as may be easily verified from (3.7), the definition of $t$ in the line before Eq. (3.6), and the braid relation (2.25). The projections $E_i^{(n)}$ satisfy the following recursion relation which is due to Wenzl [12].

**3.1. Proposition.** *Let* $t=e^{2i\alpha}$, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, *and let* $q=\inf\{n\in\mathbb{N}, n|\alpha|\geqq\pi\}$ *for* $\alpha\neq 0$ *and* $q=\infty$ *for* $\alpha=0$. *Then (with the convention* $\sin n\alpha/\sin(n+1)\alpha=n/n+1$ *for* $\alpha=0$*)*

$$E_i^{(n+1)}=\varrho(E_i^{(n)})-\frac{2\cos\alpha\sin n\alpha}{\sin(n+1)\alpha}\varrho(E_i^{(n)})E_j\varrho(E_i^{(n)})\ ,\quad i\neq j \tag{3.9}$$

*for* $n+1<q$ *and*

$$E_i^{(q)}=\varrho(E_i^{(q-1)})\ . \tag{3.10}$$

For the convenience of the reader we include a proof in Appendix A.

We now evaluate the left inverse $\phi$ on the projections $E_i^{(n)}$. From (3.1) and (3.9) we get the recursion relation

$$\phi(E_i^{(n+1)})=E_i^{(n)}\left(1-\frac{2\cos\alpha\sin n\alpha}{\sin(n+1)\alpha}\eta_j\right)\ ,\quad i\neq j \tag{3.11}$$

*with* $\eta_j=\phi(E_j)$, $0\leqq\eta_j\leqq 1$, $\eta_1+\eta_2=1$.

In the case $\alpha=0$, Doplicher et al. used (3.11) to prove that positivity of $\phi$ restricts the possible values of $\eta_j$ to $\frac{1}{2}$ and $\frac{1}{2}\left(1\pm\frac{1}{d}\right)$, $d\in\mathbb{N}$ [DHRI, Lemma 5.3]. In the case $\alpha\neq 0$ one first notes that from (3.10) and (3.1),

$$\eta_jE_i^{(q-1)}=\phi(E_j\varrho(E_i^{(q-1)}))=\phi(E_jE_i^{(q)})=0\ ,\quad i\neq j\ , \tag{3.12}$$

where the last equality comes from the definition of the projections $E_i^{(n)}$. Since $\eta_1+\eta_2=1$ we must have $E_i^{(q-1)}=0$ for $i=1$ or $i=2$. $E_i^{(q-1)}\neq 0$ would imply $\eta_j=0$ and $E_j^{(q-1)}=0$. But this leads to $E_j^{(2)}\equiv E_j=0$ in contradiction to the assumption that $\varepsilon_\varrho$ has two different eigenvalues. For $q=3$ this is obvious, and for $q>3$ we infer from (3.11),

$$\phi(E_j^{(3)})=\frac{\sin\alpha}{\sin 3\alpha}E_j^{(2)}\ , \tag{3.13}$$

so $E_j^{(2)}=0$ is required by positivity of $\phi$.

We therefore have $E_i^{(q-1)}=0$, $i=1,2$, $q\geqq 4$. Using (3.11) several times we conclude that there are $k_i\in\mathbb{N}$, $2\leqq k_i\leqq q-2$, such that

$$\eta_i=\frac{\sin(k_i+1)\alpha}{2\cos\alpha\sin k_i\alpha},\quad i=1,2\ . \tag{3.14}$$

Summing over $i$ we find the condition

$$\sin(k_1+k_2)\alpha=0\ . \tag{3.15}$$

The only solutions are $\alpha=\pm\frac{\pi}{q}$, $k_1=d$, $k_2=q-d$, $d\in\mathbb{N}$, $2\leqq d\leqq q-2$. For the statistics parameter $\lambda_\varrho$ we find

$$\lambda_\varrho=\sum_{i=1}^{2}\lambda_i\eta_i=-\lambda_2[(t+1)\eta_1-1]=-\lambda_2e^{\pm\pi i(d+1)/q}\frac{\sin\pi/q}{\sin d\pi/q}\ . \tag{3.16}$$

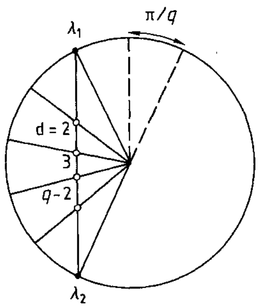


**Fig. 2.** The admissible values of the eigenvalues of $\varepsilon_\varrho$ and the statistics parameter $\lambda_\varrho$ (marked by ○) as points in the unit disc of the complex plane in the case $q=7$. The real axis is an arbitrary straight line through the center of the disc

We summarize the results in the following

**3.2. Theorem.** *Let $\varrho$ be an irreducible localized morphism such that $\varrho^2$ has exactly two irreducible subrepresentations.*
*Then*

(*i*) *$\varepsilon_\varrho$ has two different eigenvalues $\lambda_1$, $\lambda_2$ with ratio*

$$\frac{\lambda_1}{\lambda_2} = -e^{\pm 2\pi i/q} , \quad q \in \mathbb{N} \cup \{\infty\}, q \geqq 4 . \tag{3.17}$$

(ii) *The modulus of the statistics parameter $\lambda_\varrho = \phi(\varepsilon_\varrho)$ has the possible values*

$$|\lambda_\varrho| = \begin{cases} \dfrac{\sin \pi/q}{\sin d\pi/q}, & q < \infty \\ \dfrac{1}{d}, 0 & q = \infty \end{cases} \tag{3.18}$$

*with $d \in \mathbb{N}, 2 \leqq d \leqq q-2$.*

(iii) *The representation $\varepsilon_\varrho^{(n)}$ of the braid group $B_n$ which is generated by $\varrho^{k-1}(\varepsilon_\varrho), k=1, \ldots, n-1$ is an infinite multiple of the Ocneanu-Wenzl representation tensored with a one dimensional representation.*

(iv) *The projections $E_2^{(m)}, d < m \leqq n$ and $E_1^{(m)}, q-d < m \leqq n$ vanish.*

(v) *The iterated left inverse $\varphi = \phi^n$ defines a Markov trace* tr *on the braid group $B_n$*

$$\mathrm{tr}(b) = \varphi \circ \varepsilon_\varrho(b) . \tag{3.19}$$

The Markov trace property in (v) requires some comment. Whereas the Markov II property (3.3) is a straightforward consequence of the property (3.1) for $\phi$ (and its iteration $\varphi$), the trace property

$$\mathrm{MI} \quad \varphi(\varepsilon_\varrho(b_2)\varepsilon_\varrho(b_1)) = \varphi(\varepsilon_\varrho(b_1)\varepsilon_\varrho(b_2)) \tag{3.20}$$

follows in a purely algebraic manner by the use of the Hecke relation as shown by Ocneanu et al. [12]. We shall show at the end of this section that MI remains true in the general case. The Hecke relations allow to write a word in the braid group in a completely analogous manner as for the permutation group. In fact the representations in both cases can be labelled by Young tableaux [12]. The quantization (positivity) cuts off the height of the tableaux as in the case of the permutation group $\left(\text{where only the heights } \left(\frac{1}{d}\right)^{-1} \text{ occur}\right)$, but in addition for the Hecke case there is also a cutoff (analogous to the RSOS condition in statistical mechanics) in the horizontal direction. The unitary representation $\varepsilon_\varrho^{(n)}$ of QFT is quasi-equivalent to the representation defined via the Gelfand-Neumark-Segal construction from the faithful trace. This parallels the argument of DHR and means unitary equivalence up to multiplicities which in QFT as a result of the structure of $\mathscr{A}$ are always infinite. The algebra generated by the braid group representation is isomorphic to the Hecke algebra divided by the annihilator ideal of $\varphi$. In part II of this work, we will show that all the "quantized" traces originate from the Lie-Hopf[5] algebras associated to $su(n)$.

There exists a two-parametric algebra, the Birman-Wenzl algebra $\mathrm{BW}(\alpha, m)$ [13], in which $\varepsilon_\varrho$ has three instead of two eigenvalues which generalizes the Hecke algebra (it contains $H$ as a factor):

$$B_n \to BW_n \to H_n \ .$$

For this algebra no direct calculation of a positive Markov functional has been carried out. Using however our general analysis of the Markov functional of algebraic quantum field theory (see below) leading to the path representation in terms of $R$-matrices in Sect. 4, and combining this with the finite dimensional representation theory of the BW algebra by Murakami [14] (it is easy to extract the unitary representation on the restricted (RSOS) paths for $q = m^{th}$ unit root and to determine positive semidefinite Markov traces from Murakami's work), one can easily determine all quantum field theoretical Markov traces on the BW algebra. Again this trace is faithful on the BW algebra divided by the annihilator of $\varphi$. A more detailed discussion of the case of the BW algebra will appear in model discussion in part II where the relation of these "quantized" traces with Lie-Hopf algebras of the $B_n$, $D_n$ type is discussed.

We continue this section by collecting some general results on the left-inverses $\phi$. To a certain degree we can base our discussion on the work of Doplicher et al. [2], but there are also some modifications due to the more complicated structure of the braid group compared to the permutation group.

The existence of left inverses is tied to the existence of conjugate representations. Let $\bar{\varrho}$ be a localized morphism and $R$ an isometry such that

$$\text{(i)} \qquad \bar{\varrho}\varrho(A)R = RA \ , \quad A \in \mathscr{A} \tag{3.21}$$

(so $\bar{\varrho}\varrho$ contains a subrepresentation equivalent to the vacuum representation),

$$\text{(ii)} \qquad \bar{\varrho}(\mathscr{A})R\Omega \text{ is dense in } \mathscr{H} \ . \tag{3.22}$$

Then

$$\phi(A) = R^* \bar{\varrho}(A) R \ , \quad A \in \mathscr{A} \tag{3.23}$$

is a left inverse of $\varrho$, and, on the other hand, $\bar{\varrho}$ is the GNS representation induced by the state $\omega_0 \circ \phi$, $\omega_0$ denoting the vacuum, with cyclic vector $R\Omega$.

[5] See for example [28]

**3.3. Definition.** Let $\varrho$ be a localized morphism satisfying the spectrum condition. A left inverse $\phi$ of $\varrho$ is called regular if it is of the form (3.23) where $\bar{\varrho}$ satisfies the spectrum condition.

A criterion for the existence of regular left inverses and hence for the existence of a conjugate positive energy representation is provided, as in DHR by the non-vanishing of the statistics parameter.

**3.4. Theorem.** *Let $\varrho$ be an irreducible localized morphism satisfying the spectrum condition. Then either $\phi(\varepsilon_\varrho)=0$ for all left inverses $\phi$ of $\varrho$ or there exists a unique regular left inverse $\phi$ of $\varrho$, and $\phi(\varepsilon_\varrho)\neq 0$.*

The proof of this theorem will be given in Appendix B.

For representations containing massive one particle states the first alternative of the theorem can be excluded, hence antiparticles always exist [20] (in many models as boundstates of particles). We will consider in the following only those localized morphisms which satisfy the spectrum condition, are finite direct sums of irreducible representations and have regular left inverses. As in [21, Sect. 7] one can show that this set is stable under composition, taking subrepresentations and conjugates.

For reducible morphisms $\varrho$ the regular left inverses $\phi$ with $\phi(\varepsilon_\varrho)\neq 0$ are no longer unique. In analogy to [DHR I] we define a standard left-inverse $\phi$ of $\varrho$ to be a regular left-inverse with

$$\phi(\varepsilon_\varrho)^*\phi(\varepsilon_\varrho)=\phi(\varepsilon_\varrho)\phi(\varepsilon_\varrho)^*\in\mathbb{C}\cdot 1\ . \tag{3.24}$$

The following proposition shows that as in the DHR case [2] products of standard left inverses are standard. The rule of composition for the statistics parameters is, however, more complicated due to the nontrivial structure of the braid group (cf. DHR I, Lemma 6.7 [2]).

**3.5. Proposition.** *Let $\phi$ be a standard left-inverse of $\varrho$, and let $\phi_i$ be a standard left inverse of $\varrho_i$ with $\phi_i(\varepsilon_{\varrho_i})=\lambda_i 1$, $i=1\dots n$. Then*

(i) *$\phi\phi_1$ is a standard left-inverse of $\varrho_1\varrho$ satisfying*

$$\phi\phi_1(\varepsilon_{\varrho_1\varrho})=\lambda_1\varepsilon(\varrho,\varrho_1)\phi(\varepsilon_\varrho)\varepsilon(\varrho_1,\varrho)\ . \tag{3.25}$$

(ii) *$\phi_n\dots\phi_1$ is a standard left inverse of $\varrho_1\dots\varrho_n$ satisfying*

$$\phi_n\dots\phi_1(\varepsilon_{\varrho_1\dots\varrho_n})=\lambda_1\dots\lambda_n\varepsilon^{(n)}_{C_n}(\varrho_1,\dots,\varrho_n)\ , \tag{3.26}$$

*where $C_n=(\sigma_1\dots\sigma_{n-1})^n$ is the generator of the center of the braid group $B_n$.*

*Proof.* From the definition of statistics operators in Sect. 2,

$$\varepsilon_{\varrho_1\varrho}=\varepsilon^{(4)}_{\sigma_2\sigma_1\sigma_3\sigma_2}(\varrho_1,\varrho,\varrho_1,\varrho)=\varrho_1(\varepsilon(\varrho_1,\varrho))\varrho_1^2(\varepsilon_\varrho)\varepsilon_{\varrho_1}\varrho_1(\varepsilon(\varrho,\varrho_1))\ . \tag{3.27}$$

Applying $\phi_1$ and using (3.1) and the equivalence relation for coloured braids (2.20) yields

$$\phi_1(\varepsilon_{\varrho_1\varrho})=\lambda_1\varrho(\varepsilon(\varrho,\varrho_1))\varepsilon_\varrho\varrho(\varepsilon(\varrho_1,\varrho))\ . \tag{3.28}$$

Applying $\phi$ and using (3.1) yields (3.25). If $\phi$, $\phi_1$ are regular,

$$\phi(A)=R^*\bar{\varrho}(A)R\ ,\quad \phi_1(A)=R_1^*\bar{\varrho}_1(A)R_1\ ,\quad A\in\mathscr{A} \tag{3.29}$$

with conjugate morphisms $\bar{\varrho}$ and $\bar{\varrho}_1$ and isometries $R$, $R_1$ satisfying (3.21) for $\varrho$ and $\varrho_1$, respectively then

$$\phi\phi_1(A) = R^*\bar{\varrho}(R_1^*)\bar{\varrho}\bar{\varrho}_1(A)\bar{\varrho}(R_1)R \ , \tag{3.30}$$

hence $\phi\phi_1$ is regular with conjugate morphism $\bar{\varrho}\bar{\varrho}_1$ and isometry $\bar{\varrho}(R_1)R$. By iteration of (i) it is evident that $\phi_n \dots \phi_1$ is a standard left inverse of $\varrho_1 \dots \varrho_n$. Equation (3.26) follows inductively from (3.25) with $\varrho = \varrho_2 \dots \varrho_n$, $\phi = \phi_n \dots \phi_2$:

$$\phi_n \dots \phi_2 \phi_1(\varepsilon_{\varrho_1\varrho_2\dots\varrho_n}) = \lambda_1\lambda_2\dots\lambda_n \varepsilon(\varrho_2\dots\varrho_n, \varrho_1)\varepsilon^{(n-1)}_{C_{n-1}}(\varrho_2, \dots, \varrho_n)\varepsilon(\varrho_1, \varrho_2\dots\varrho_n) \tag{3.31}$$

if one inserts

$$\begin{aligned} \varepsilon(\varrho_2\dots\varrho_n, \varrho_1) &= \varepsilon^{(n)}_{\sigma_1\dots\sigma_{n-1}}(\varrho_2, \dots, \varrho_n, \varrho_1) \ , \\ \varepsilon(\varrho_1, \varrho_2\dots\varrho_n) &= \varepsilon^{(n)}_{\sigma_{n-1}\dots\sigma_1}(\varrho_1, \dots, \varrho_n) \ , \end{aligned} \tag{3.32}$$

and uses (2.23) with $C_n = \sigma_1 \dots \sigma_{n-1} C_{n-1} \sigma_{n-1} \dots \sigma_1$. q.e.d.

It is amusing to visualize the process of successive evaluation of $\phi_i$, $i = 1, \dots, n$, with (3.1) pictorially as some "interpolation" between braids and links. Starting from

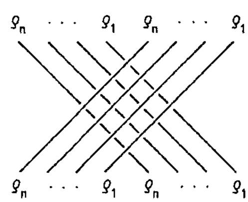


**Fig. 3**

$\varepsilon_{\varrho_1\dots\varrho_n}$ written as the $2n$-braid (Fig. 3) every step corresponds to short-circuiting the rightmost string $\phi_i$ and undoing the resulting loop at the price of a factor of $\lambda_i$, while the thus processed lines organize into $C_n$, see Fig. 4.

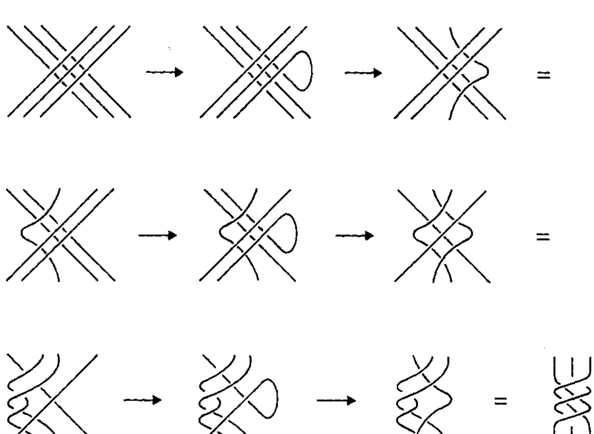

**Fig. 4**

Now let $\phi$ be a standard left inverse of $\varrho$. The norm of $\phi(\varepsilon_\varrho)$ can be interpreted as the inverse of the "statistical dimension" of $\varrho$

$$d(\varrho)=\|\phi(\varepsilon_\varrho)\|^{-1} . \tag{3.33}$$

From Proposition 3.5 we see that for irreducible representations $\varrho_1,\dots,\varrho_n$ the statistical dimension of $\varrho_1\dots\varrho_n$ is

$$d(\varrho_1\dots\varrho_n)=d(\varrho_1)\dots d(\varrho_n) . \tag{3.34}$$

We now decompose a reducible representation $\varrho$ into irreducible ones. Let $\phi$ be a standard left inverse and $\phi(\varepsilon_\varrho)\neq 0$. Then using (DHRI, Lemma 6.1 [2]) one gets for any projection $E\in\varrho(\mathscr{A})'$,

$$\phi(E)\geqq d(\varrho)^{-2}1 . \tag{3.35}$$

Since $\phi(1)=1$ (3.1), $\varrho$ is a direct sum of at most $d(\varrho)^2$ subrepresentations. Let

$$\varrho\cong\bigoplus_{i\in I}\varrho_i \tag{3.36}$$

be the decomposition of $\varrho$ into irreducible representations. There are isometric intertwiners $W_i$ from $\varrho_i$ to $\varrho$

$$\varrho(A)W_i=W_i\varrho_i(A) , \quad A\in\mathscr{A} , \quad i\in I \tag{3.37}$$

with $W_i^*W_j=\delta_{ij}1$ and $\Sigma W_iW_i^*=1$.

We may choose these intertwiners such that each projection $E_i=W_iW_i^*$ is bounded by some spectral projection of $\phi(\varepsilon_\varrho)$. Then $\phi(\varepsilon_\varrho)E_i=\mu_i d(\varrho)^{-1}E_i$ for some eigenvalue $\mu_i$ of $\phi(\varepsilon_\varrho)\|\phi(\varepsilon_\varrho)\|^{-1}$. A left-inverse $\phi_i$ of $\varrho_i$ can be defined by

$$\phi_i(A)=\phi(E_i)^{-1}\phi(W_iAW_i^*) . \tag{3.38}$$

Actually, $\phi_i$ is the unique regular left inverse of $\varrho_i$. Namely, $\omega_0\circ\phi_i$ is induced by the vector

$$\Omega_i=\bar{\varrho}(W_i^*)R\Omega\|\bar{\varrho}(W_i^*)R\Omega\|^{-1} \tag{3.39}$$

in the representation $\bar{\varrho}$. Let $\bar{E}_i$ denote the projection onto the closure of $\bar{\varrho}(\mathscr{A})\Omega_i$. According to the theorem of Borchers [10] (cf. footnote 3 in Sect. 2) there is an isometry $\bar{W}_i$ with $\bar{W}_i\bar{W}_i^*=\bar{E}_i$, and $\bar{W}_i$ can be found in $\mathscr{A}(O_1)$, where $O_1$ contains the localization region of $\bar{\varrho}$ in its interior. Then

$$\bar{\varrho}_i(A)=\bar{W}_i^*\bar{\varrho}(A)\bar{W}_i , \quad A\in\mathscr{A} \tag{3.40}$$

is conjugate to $\varrho_i$ with isometric intertwiner

$$R_i=\phi(E_i)^{-1/2}\bar{W}_i^*\bar{\varrho}(\bar{W}_i^*)R . \tag{3.41}$$

$\bar{\varrho}_i$ is equivalent to a subrepresentation of $\bar{\varrho}$ and therefore a positive energy representation. Since

$$\phi_i(A)=R_i^*\bar{\varrho}_i(A)R_i , \quad A\in\mathscr{A} , \tag{3.42}$$

$\phi_i$ is a regular left inverse. One finds (see DHRI [2], cf. also [21])

$$\lambda_{\varrho_i}=\phi_i(\varepsilon_{\varrho_i})=\phi(E_i)^{-1}W_i^*\phi(\varepsilon_\varrho)W_i=\phi(E_i)^{-1}\mu_i d(\varrho)^{-1} . \tag{3.43}$$

Since $\lambda_{\varrho_i}$ depends only on the equivalence class of $\varrho_i$, representations belonging to different eigenvalues of $\phi(\varepsilon_\varrho)$ must be inequivalent. Hence $\phi(\varepsilon_\varrho)$ belongs to the center of $\varrho(\mathscr{A})'$. Moreover

$$\phi(E_i)=\frac{d(\varrho_i)}{d(\varrho)} \tag{3.44}$$

is invariant under inner automorphisms of $\varrho(\mathscr{A})'$, hence $\phi$ defines a trace state on $\varrho(\mathscr{A})'$. Summing (3.44) over $i \in I$ and using $\Sigma E_i = 1$ and $\phi(1)=1$ yield the formula

$$d(\varrho)=\sum_{i\in I} d(\varrho_i) \tag{3.45}$$

for the statistical dimensions of $\varrho$ and $\varrho_i, i\in I$. If $\varrho$ is a product of the irreducible morphisms $\varrho_j, j\in J$, (3.45) and (3.34) yield the following "sum rule" for the statistical dimensions:

$$\prod_{j\in J} d_j=\sum_{i\in I} d_i \ , \qquad d_i=d(\varrho_i) \ . \tag{3.46}$$

We now define the quantum field theoretical Markov trace induced by the regular left-inverse of an irreducible localized morphism $\varrho$.

**3.6. Proposition.** *Let $\phi$ be a standard left-inverse of $\varrho$.*

(i) *Then*

$$\varphi(A)\underline{1}:=\phi^n(A) \ , \quad A\in\varrho^n(\mathscr{A})' \tag{3.47}$$

*defines a faithful trace state on $\cup_n \varrho^n(\mathscr{A})'$.*

(ii) *Let $\varrho$ be irreducible and $\phi(\varepsilon_\varrho)=\lambda\underline{1}\neq 0$. Then*

$$\mathrm{tr}\,[B_n=\varphi\circ\varepsilon_\varrho^{(n)} \tag{3.48}$$

*defines a nonnegative Markov trace on the group algebra of $B_\infty=\cup B_n$ (where $B_n\subset B_{n+1}$ in the natural way), satisfying*

$$\begin{aligned} &MI:\mathrm{tr}(ab)=\mathrm{tr}(ba) \ , \\ &MII:\mathrm{tr}(a\sigma_n)=\lambda\,\mathrm{tr}(a) \ , \\ &\qquad \mathrm{tr}(a\sigma_n^{-1})=\bar{\lambda}\,\mathrm{tr}(a) \ , \\ &\qquad \mathrm{tr}(e)=1 \ . \end{aligned} \tag{3.49}$$

*In particular, $\alpha(b)=|\lambda|^{-(n-1)}\left(\frac{\bar{\lambda}}{|\lambda|}\right)^{n_+-n_-}\mathrm{tr}(b)$, $b\in B_n$, where $n_\pm$ count the generators $\sigma_i^\pm$ in $b$ as a word in $\sigma_i^\pm$, is a link invariant.*

*Proof.* (i) For $B\in\mathscr{A}$ and $A\in\varrho^n(\mathscr{A})'$ we have $\varrho^n(B)A=A\varrho^n(B)$. Equation (3.1) implies $B\phi^n(A)=\phi^n(A)B$, i.e. $\phi^n(A)$ is a multiple $\varphi(A)$ of 1. $\varphi$ is a state because $\phi^n$ is positive and satisfies $\phi^n(1)=1$. $\varphi$ is faithful because $\phi^n$ is a standard left-inverse (Proposition 3.5) and is therefore faithful [Eq. (3.35)]. Since $\phi^{n+1}(A)=\phi^n(A)$ for $A\in\varrho^n(\mathscr{A})'$, $\varphi$ is compatible with the inclusion $\varrho^n(\mathscr{A})'\subset\varrho^{n+1}(\mathscr{A})'$. The trace property finally follows from the remark after Eq. (3.44).

(ii) As an intertwiner from $\varrho^n$ to $\varrho^n$, $\varepsilon_\varrho^{(n)}(b)$ is an element of $\varrho^n(\mathscr{A})'$. $\varepsilon_\varrho^{(n)}$ is a unitary representation and $\varphi$ is a state, hence $\mathrm{tr}\upharpoonright B_n=\varphi\circ\varepsilon_\varrho^{(n)}$ is a function of positive type with $\mathrm{tr}(e)=1$. Moreover, for $b\in B_n$,

$$\varphi(\varepsilon_\varrho^{(n+1)}(b))=\varphi(\varepsilon_\varrho^{(n)}(b)) \ , \tag{3.50}$$

hence tr extends to $B_\infty$. Property MI follows from (i). Using MI and the fact that the automorphism $\sim: B_n\to B_n$, $\sigma_i\to\sigma_{n-i}$ is inner, the first relation in MII is equivalent to

$$\mathrm{tr}(b\sigma_1)=\lambda\,\mathrm{tr}\,(b) \tag{3.51}$$

for all $b\in B_{n+1}$ which are words in $\sigma_i^{\pm1}, i=2,\ldots n$. For such a $b$ we have

$$\varepsilon_\varrho^{(n+1)}(b)=\varrho(\varepsilon_\varrho^{(n)}(b')) \tag{3.52}$$

for some $b'\in B_n$, hence (3.51) follows from

$$\phi(\varepsilon_\varrho^{(n+1)}(b)\varepsilon_\varrho)=\phi(\varrho(\varepsilon_\varrho^{(n)}(b'))\varepsilon_\varrho)=\lambda\varepsilon_\varrho^{(n)}(b')$$

and

$$\phi(\varepsilon_\varrho^{(n+1)}(b))=\phi(\varrho(\varepsilon_\varrho^{(n)}(b')))=\varepsilon_\varrho^{(n)}(b') \ .$$

The second relation in MII follows by unitarity of $\varepsilon_\varrho^{(n)}$ and positivity of $\varphi$.

The functional $\alpha$ on the braid group $B_\infty$ is invariant under the Markov moves $ab\to ba$, $a\to a\sigma_n^{\pm1}$, $a\in B_n$, and is therefore, according to Markov's theorem [4], a link invariant. q.e.d.

With the present methods we cannot make similar predictions about the phases of $\lambda_\varrho$ unless there is some information about the phases in the central element (3.26). Such information is expected to arise from a generalized spin-statistics theorem (e.g. Proposition 2 in [5]) associated with the covariance properties of the theory. We shall come back to this point in part II of this paper.

## 4. Statistics Operators and *R*-Matrices

It is not obvious how the statistics operators of the preceeding sections might be calculated in a given model. We therefore present here an explicit matrix representation which can be compared with the *R*-matrices occurring in several models.

Let us choose from each equivalence class of irreducible localized morphisms one representative $\varrho_\alpha$. Then, for a given morphism $\varrho$ the composed representation $\varrho_\alpha\varrho$ may be decomposed into irreducible ones which are unitarily equivalent to localized morphisms $\varrho_\beta$. Thus, there are intertwiners $T_{\alpha\beta}$ from $\varrho_\beta$ to $\varrho_\alpha\varrho$,

$$\varrho_\alpha\varrho(A)T_{\alpha\beta}=T_{\alpha\beta}\varrho_\beta(A) \ . \tag{4.1}$$

Let $T_{\alpha\beta}$ and $T'_{\alpha\beta}$ be two such intertwiners. Then $T^*_{\alpha\beta}T'_{\alpha\beta}$ commutes with $\varrho_\beta(\mathscr{A})$, thus it is a multiple of the identity since $\varrho_\beta$ is irreducible. Following ideas of Doplicher and Roberts [13] one considers this multiple as a scalar product,

$$T^*_{\alpha\beta}T'_{\alpha\beta}=(T_{\alpha\beta},T'_{\alpha\beta})1 \ , \tag{4.2}$$

hence the set of intertwiners from $\varrho_\beta$ to $\varrho_\alpha\varrho$ gets the structure of a Hilbert space $\mathscr{H}_{\alpha\beta}$. Its dimension $N_{\alpha\beta}$ is the multiplicity of $\varrho_\beta$ in $\varrho_\alpha\varrho$,

$$[\varrho_\alpha\varrho]=\sum_\beta N_{\alpha\beta}[\varrho_\beta] \ . \tag{4.3}$$

$N=(N_{\alpha\beta})$ is called the fusion (or incidence) matrix of $\varrho$. We may now choose an orthonormal basis $\{T^{(i)}_{\alpha\beta}, i=1,\ldots,N_{\alpha\beta}\}$ in each Hilbert space $\mathscr{H}_{\alpha\beta}$. Then, we have the completeness relation

$$\sum_{\beta,i} T^{(i)}_{\alpha\beta}T^{(i)*}_{\alpha\beta}=1 \tag{4.4}$$

and the orthogonality relation

$$T^{(i)*}_{\alpha\beta}T^{(j)}_{\alpha\beta'}=\delta_{\beta\beta'}\delta_{ij} \ . \tag{4.5}$$

Iterating the intertwining relation (4.1) we find that the space of intertwiners from $\varrho_\alpha\varrho^2$ to $\varrho_\beta$ is

$$\mathscr{H}^{(2)}_{\alpha\beta}=\sum_\gamma \mathscr{H}_{\alpha\gamma}\mathscr{H}_{\gamma\beta} \ , \tag{4.6}$$

where product and sum of Hilbert spaces are the sets of products and sums of intertwiners as elements of $\mathscr{A}$. Actually by (4.4) and (4.5),

$$\mathscr{H}^{(2)}_{\alpha\beta}\cong\bigoplus_\gamma \mathscr{H}_{\alpha\gamma}\otimes\mathscr{H}_{\gamma\beta} \ . \tag{4.7}$$

In the same way we observe that the space of intertwiners from $\varrho_\beta$ to $\varrho_\alpha\varrho^n$ is

$$\mathscr{H}^{(n)}_{\alpha\beta}=\sum_{\gamma_1\ldots\gamma_{n-1}} \mathscr{H}_{\alpha\gamma_1}\ldots\mathscr{H}_{\gamma_{n-1}\beta} \ . \tag{4.8}$$

The statistics operators $\varrho_\alpha(\varepsilon^{(n)}_\varrho(b))$, $b\in B_n$, are intertwiners from $\varrho_\alpha\varrho^n$ to $\varrho_\alpha\varrho^n$. Therefore, the maps

$$T\to\varrho_\alpha(\varepsilon^{(n)}_\varrho(b))T \tag{4.9}$$

are unitary operators in $\mathscr{H}^{(n)}_{\alpha\beta}$. Thus one obtains in this way a unitary representation of $B_n$ in the finite dimensional Hilbert space $\mathscr{H}^{(n)}_{\alpha\beta}(\dim\mathscr{H}^{(n)}_{\alpha\beta}=(N^n)_{\alpha\beta})$.

A convenient orthonormal basis in $\mathscr{H}^{(n)}_{\alpha\beta}$ is the set of products of the distinguished basis elements in the spaces $\mathscr{H}_{\gamma\delta}$. Following ideas of Ocneanu [16] we describe this set in the following way. Let $G$ be a graph whose vertices are the labels of morphisms $\alpha$ and where $N_{\delta\gamma}$ directed edges $e$ go from $\delta$ to $\gamma$, each of which corresponds to an intertwiner

$$T_e=T^{(i)}_{\delta\gamma} \ , \tag{4.10}$$

$e=(\delta,i,\gamma)$ being the $i^{\text{th}}$ edge from $\delta$ to $\gamma$. A path $\xi$ of length $n$ from $\alpha$ to $\beta$ is a sequence

$$\xi=(e_1,\ldots,e_n) \tag{4.11}$$

of edges $e_i$ where $e_1$ starts at $\alpha$, the endpoint of $e_k$ is the initial point of $e_{k+1}$, $k=1\ldots n-1$, and $e_n$ ends at $\beta$. Let $\mathrm{Path}^{(n)}_{\alpha\beta}$ denotes the set of all these paths. Then the intertwiners

$$T(\xi)=T_{e_1}\ldots T_{e_n} \ , \qquad \xi\in\mathrm{Path}^{(n)}_{\alpha\beta} \tag{4.12}$$

are an orthonormal basis of $\mathscr{H}_{\alpha\beta}^{(n)}$. The matrix elements of $\varrho_\alpha(\varepsilon_\varrho^{(n)}(b))$ with respect to this basis are

$$R_{\xi\xi'}(b)1 = T(\xi)^*\varrho_\alpha(\varepsilon_\varrho^{(n)}(b))T(\xi') \qquad \xi, \xi' \in \mathrm{Path}_{\alpha\beta}^{(n)} \ . \tag{4.13}$$

These are the $R$-matrices in the so-called path language. They are unitary matrices and can be identified with the $R$-matrices occurring in the exchange algebras (see below). Using the orthogonality and completeness relations of the intertwiners we find

$$\varrho_\alpha(\varepsilon_\varrho^{(n)}(b)) = \sum_{(\xi,\xi')\in \mathrm{String}_\alpha^{(n)}} R_{\xi\xi'}(b)T(\xi)T(\xi')^* \ , \tag{4.14}$$

where

$$\mathrm{String}_\alpha^{(n)} = \{(\xi, \xi'),\ \xi, \xi' \in \mathrm{Path}_{\alpha\beta}^{(n)} \text{ for some } \beta\}$$

(hence $\mathrm{String}_\alpha^{(n)}$ consists of pairs of paths of length $n$ with source $\alpha$ and a common range). Note that the operators $T(\xi)T(\xi')^*$ have the multiplication law

$$T(\xi)T(\xi')^*T(\eta)T(\eta')^* = \delta_{\xi'\eta}T(\xi)T(\eta')^* \ , \tag{4.15}$$

which is a discrete form of Witten's product rule for strings [17].

Instead of multiplying $\varrho$ from the right in (4.1) we can equally well multiply from the left. We get intertwiners $S_{\alpha\beta}$ from $\varrho_\beta$ to $\varrho\varrho_\alpha$

$$\varrho\varrho_\alpha(A)S_{\alpha\beta} = S_{\alpha\beta}\varrho_\beta(A) \ , \tag{4.16}$$

which form a Hilbert space $\mathscr{H}'_{\alpha\beta}$. Iterating (4.16) we find

$$\varrho^2\varrho_\alpha(A)\varrho(S_{\alpha\gamma})S_{\gamma\beta} = \varrho(S_{\alpha\gamma})S_{\gamma\beta}\varrho_\beta(A) \ , \quad A \in \mathscr{A} \ , \tag{4.17}$$

and we finally get a space of intertwiners from $\varrho^n\varrho_\beta$ to $\varrho_\alpha$,

$${}^{(n)}\mathscr{H}_{\alpha\beta} = \sum_{\gamma_1\dots\gamma_{n-1}} \varrho^{n-1}(\mathscr{H}'_{\alpha\gamma_1})\dots\mathscr{H}'_{\gamma_{n-1}\beta} \tag{4.18}$$

with orthonormal basis

$$S(\xi) = \varrho^{n-1}(S_{e_1})\dots S_{e_n} \ , \qquad \xi = (e_1, \dots, e_n) \in \mathrm{Path}_{\alpha\beta}^{(n)} \ . \tag{4.19}$$

The operators $S(\xi)S(\xi')^*$, $(\xi, \xi') \in \mathrm{String}_\alpha^{(n)}$, satisfy again (4.15) and are a linear basis of the algebras $\varrho^n\varrho_\alpha(\mathscr{A})'$. They have a simple transformation property under $\varrho$ and $\phi$:

$$\varrho(S(\xi)S(\xi')^*) = \sum_e S(\xi\circ e)S(\xi'\circ e)^* \ , \tag{4.20}$$

where the sum is over all edges emanating from the endpoint of $\xi$, and by (3.1), (3.44) and (4.20) for $(\xi\circ e, \xi'\circ e') \in \mathrm{String}_\alpha^{(n)}$

$$\phi(S(\xi\circ e)S(\xi'\circ e')^*) = \delta_{ee'} \frac{d(\varrho_\beta)}{d(\varrho)d(\varrho_\gamma)} S(\xi)S(\xi')^* \ , \tag{4.21}$$

where $\beta$ is the endpoint and $\gamma$ is the starting point of $e$.

The bases (4.12) and (4.19) are related by the braid group. We are free to choose

$$S_{\alpha\beta}^{(i)} = \varepsilon(\varrho_\alpha, \varrho)T_{\alpha\beta}^{(i)} \ . \tag{4.22}$$

Then one finds the following formula:

$$S(\xi)=\varepsilon(\varrho_\alpha,\varrho^n)\varrho_\alpha(\varepsilon_\varrho^{(n)}(b_n))T(\xi)\ ,\qquad \xi\in\mathrm{Path}_{\alpha\beta}^{(n)}\ ,\tag{4.23}$$

where $b_n$ is the following element of the braid group

$$b_n=(\sigma_1\sigma_2\ldots\sigma_{n-1})(\sigma_1\ldots\sigma_{n-2})\ldots(\sigma_1\sigma_2)\sigma_1\ ,\tag{4.24}$$

$\sigma_i$ denoting the elementary transposition represented by $\varrho^{i-1}(\varepsilon_\varrho)$.

*Proof of (4.23).* For $n=1$ (4.23) reduces to (4.22) which is true by our choice of $S_{\alpha\beta}^{(i)}$. We may then assume that (4.23) holds for $\xi\in\mathrm{Path}_{\alpha\beta}^{(n-1)}$. We have for an edge $e$ from $\delta$ to $\alpha$

$$S(e\circ\xi)=\varrho^{n-1}(S_e)S(\xi)=\varrho^{n-1}(\varepsilon(\varrho_\delta,\varrho)T_e)\varepsilon(\varrho_\alpha,\varrho^{n-1})\varrho_\alpha(\varepsilon_\varrho^{(n-)}(b_{n-1}))T(\xi)\ .\tag{4.25}$$

We use the following formulae for commutation properties of an intertwiner $V_i$ from $\varrho^{(i)}$ to $\hat\varrho^{(i)}$ with the statistics operators,

$$\begin{aligned}\varrho^{(2)}(V_1)\varepsilon(\varrho^{(1)},\varrho^{(2)})&=\varepsilon(\hat\varrho^{(1)},\varrho^{(2)})V_1\ ,\\ V_2\varepsilon(\varrho^{(1)},\varrho^{(2)})&=\varepsilon(\varrho^{(1)},\hat\varrho^{(2)})\varrho^{(1)}(V_2)\ ,\end{aligned}\tag{4.26}$$

which may be checked by inserting the definition of $\varepsilon$ and using the properties of intertwiners (see [DHR] and [21] where these calculations are performed. Note, however, that Theorem 4.3 of DHR I does not remain valid in our more general context). Inserting (4.26) with $V=T_e$, $\varrho^{(2)}=\varrho^{n-1}$, $\varrho^{(1)}=\varrho_\alpha$ and $\hat\varrho^{(1)}=\varrho_\delta\varrho$ we find

$$S(e\circ\xi)=\varrho^{n-1}(\varepsilon(\varrho_\delta,\varrho))\varepsilon(\varrho_\delta\varrho,\varrho^{n-1})\varrho_\delta\varrho(\varepsilon_\varrho^{(n-1)}(b_{n-1}))T(e\circ\xi)\ ,\tag{4.27}$$

where we used the intertwining property of $T_e$ to move it to the right. It remains to check the form of the braid group element occurring in (4.27) which amounts to simple manipulations in the braid group. q.e.d.

So the matrices $\tilde R$ in the $S$-basis are related to the $R$-matrices in the $T$-basis by the formula

$$\begin{aligned}\tilde R_{\xi\xi'}(b)&=S(\xi)^*\varepsilon_\varrho^{(n)}(b)S(\xi')\\ &=T(\xi)^*\varrho_\alpha(\varepsilon_\varrho^{(n)}(b_n))^*\varepsilon(\varrho_\alpha,\varrho^n)^*\varepsilon_\varrho^{(n)}(b)\varepsilon(\varrho_\alpha,\varrho^n)\varrho_\alpha(\varepsilon_\varrho^{(n)}(b_n))T(\xi')\ .\end{aligned}\tag{4.28}$$

From (4.26)

$$\varepsilon_\varrho^{(n)}(b)\varepsilon(\varrho_\alpha,\varrho^n)=\varepsilon(\varrho_\alpha,\varrho^n)\varrho_\alpha(\varepsilon_\varrho(b))\ ,\tag{4.29}$$

hence

$$\tilde R_{\xi\xi'}(b)=R_{\xi\xi'}(b_n^{-1}bb_n)=R_{\xi\xi'}(\tilde b)\ ,\tag{4.30}$$

where $\sim:B_n\to B_n$ denotes the isomorphism $\sigma_i\to\sigma_{n-i}$. Formulae (4.30) and (4.21) are very convenient for the computation of the Markov trace. We have

**4.1. Proposition.** *The Markov trace associated with an irreducible morphism $\varrho$ is a weighted sum over characters of finite dimensional representations of $B_n$:*

$$\mathrm{tr}\,(b)=\frac{1}{d(\varrho)^n d(\varrho_\alpha)}\sum_\beta d(\varrho_\beta)\sum_{\xi\in\mathrm{Path}_{\alpha\beta}^{(n)}}R_{\xi\xi}(b)\tag{4.31}$$

*independently of the choice of $\varrho_\alpha$. The vector $(d(\varrho_\beta))$ is the Frobenius vector of the fusion matrix $N_{\alpha\beta}$ with eigenvalue $d(\varrho)$.*

*Proof.* From the completeness of the basis $S(\xi)$ we have

$$\varepsilon_\varrho^{(n)}(b)=\sum_{(\xi,\xi')\in\mathrm{String}_\alpha^{(n)}}\tilde{R}_{\xi\xi'}(b)S(\xi)S(\xi')^*$$

for any choice of $\alpha$. Evaluating $\phi^n$ with the help of (4.21) yields (4.31) with $\tilde{R}_{\xi\xi}(b)$ instead of $R_{\xi\xi}(b)$. By virtue of (4.20), $\tilde{R}_{\xi\xi}(b)=R_{\xi\xi}(\tilde{b})$, and by virtue of Proposition 3.6 (MI), $\tilde{b}$ may be replaced by $b$. The last statement is just a reformulation of (3.46), (3.47) for the reducible morphism $\varrho_\alpha\varrho\cong\oplus_\beta N_{\alpha\beta}\varrho_\beta$. q.e.d.

We conjecture that the particular form (4.31) of quantum field theoretical Markov traces is crucial for their classification beyond the Hecke case; cf.[14].

With the help of the path formalism part of the redundancy of the field bundle may be removed. On the Hilbert space

$$\mathscr{H}=\bigoplus_\alpha\{\varrho_\alpha,\mathscr{H}_o\}\equiv\bigoplus_\alpha\mathscr{H}_\alpha \tag{4.32}$$

with a choice of one representative $\varrho_\alpha$ from each equivalence class of irreducible morphisms, observables act by

$$A\{\varrho_\alpha,\Phi\}=\{\varrho_\alpha(A),\,\Phi\}\,. \tag{4.33}$$

One now can introduce reduced field bundle elements

$$F_e,\ F=\{\varrho,A\},\ A\in\mathscr{A}, \tag{4.34}$$

with $e=(\alpha,i,\beta)$ as before. $F_e$ annihilates $\mathscr{H}_\gamma$ for $\gamma\neq\alpha$ and acts on $\mathscr{H}_\alpha$ as

$$F_e\{\varrho_\alpha,\Phi\}=\{\varrho_\beta,T_e^*\varrho_\alpha(A)\Phi\}\,. \tag{4.35}$$

The translation of (2.16) to the reduced fields $F_{e_i}^{(i)}$ localized in $O_i$, $O_1>O_2$ yields the exchange algebra [5] with numerical $R$-matrices

$$F_{e_1}^{(1)}F_{e_2}^{(2)}=\sum_{\xi'\in\mathrm{Path}_{\alpha\gamma}^{(2)}}R_{\xi\xi'}(\sigma_1)F_{e_2'}^{(2)}F_{e_1'}^{(1)}, \tag{4.36}$$

where $\xi=e_2\circ e_1\in\mathrm{Path}_{\alpha\gamma}^{(2)}$ and $\xi'=e_1'\circ e_2'$. Hence the exchange algebra [5] which has first been observed as the algebraic structure underlying the eigenspaces of the center of the conformal covering group [18], in more recent times known under the name of conformal blocks[18], is identical to the reduced field bundle described above. In the case of permutation group statistics, one can go a step further and construct a field algebra where the commutation rules of fields are simply of Bose- or Fermi type, and where the statistical dimensions occur as multiplicities of the corresponding representations [1]. It is an interesting question whether a similar construction can also be conceived for noninteger statistical dimensions $d$.

Another application of the path formalism is the study of the sequence of inclusions of algebras,

$$\varrho: M_n\to M_{n+1}, \tag{4.37}$$

where $M_n = \varrho^n(\mathscr{A})'$. According to DHR I [cf. (3.35)] each algebra $M_n$ is a finite direct sum of full matrix algebras. Using the Markov trace

$$\varphi = \lim_{n\to\infty} \phi^n \tag{4.38}$$

we can enlarge $\cup M_n$ to a hyperfinite type $\mathrm{II}_1$ v. Neumann algebra $M$. The inclusion

$$\varrho(M) \subset M \tag{4.39}$$

turns out to have index $d(\varrho)^2$ in the sense of Jones [4]:

$$[M:\varrho(M)] = \lim_{n\to\infty} \frac{\dim M_{n+1}}{\dim M_n} = \lim_{n\to\infty} \frac{\sum_\alpha (N^{n+1})^2_{\beta\alpha}}{\sum_\alpha (N^n)^2_{\beta\alpha}} = d(\varrho)^2 \ , \tag{4.40}$$

where $\varrho \cong \varrho_\beta$ since $d(\varrho)$ is the Frobenius eigenvalue of the fusion matrix $N$.

Actually, there is also a sequence of projections $E_n$ satisfying the Temperley-Lieb-Jones relations. They occur naturally in a sequence of algebras where also the conjugate morphism is used.

Let $\varrho$ be irreducible and let $\bar{\varrho}$ be a conjugate of $\varrho$ with

$$\bar{\varrho}\varrho(A)R = RA \ , \quad A \in \mathscr{A} \tag{4.41}$$

for some isometry $R$. Then $\bar{R} = \varepsilon(\bar{\varrho}, \varrho)R$ satisfies

$$\varrho\bar{\varrho}(A)\bar{R} = \bar{R}A \ , \quad A \in \mathscr{A} \ . \tag{4.42}$$

**4.2. Proposition.** *Let $\varrho$ be irreducible and let $\bar{\varrho}$, $R$ and $\bar{R}$ be as above. Then*

(i) *The statistics parameters $\lambda_\varrho$ and $\lambda_{\bar{\varrho}}$ coincide,*

(ii) $\bar{\varrho}(\bar{R})^* R = \lambda_\varrho 1 = \bar{R}^* \varrho(R)$,

(iii) *Let $E_{2i-1} = (\bar{\varrho}\varrho)^{i-1}(RR^*)$, $E_{2i} = (\bar{\varrho}\varrho)^{i-1}\bar{\varrho}(\bar{R}\bar{R}^*)$, $i \in \mathbb{N}$. Then the sequence $\{E_n, n \in \mathbb{N}\}$ satisfies the Temperley-Lieb-Jones relations*

$$\begin{aligned} E_n E_m &= E_m E_n \ , \quad |n-m| \geqq 2 \ , \\ E_n E_{n\pm1} E_n &= d(\varrho)^{-2} E_n \ , \end{aligned} \tag{4.43}$$

*Proof.* (i) and (ii) are proven in DHR II, Theorem 3.3 where, however, the intertwiners $R$ and $\bar{R}$ are differently normalized. A direct proof may be found in [21, Proposition 6.4]. (iii) is an easy consequence of (ii). q.e.d.

Using the sequence of projections $\{E_n\}$ constructed in Proposition 4.2 we can apply the result of Jones [4] and find that either $d(\varrho) \geqq 2$ or $d(\varrho) = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$; $q \geqq 3$.

## Concluding Remarks

The main topic of this work is the generalization and application of the DHR framework of superselection sectors to low dimensional quantum field theories. In addition to those statements about exchange algebras which were obtained previously [5] within the standard framework of correlation functions (the Wightman framework [22]), the algebraic approach shows that the exchange algebra is a necessary consequence of the Einstein causality of local observables. Furthermore the $R$-matrices are necessarily unitary as a consequence of the quantum (positivity) aspect of the observable algebras. Finally one obtains a consistent picture about the composition of "charge" sectors which, using the braid-group terminology, amounts to "strand-formation" [5]. In order to apply these ideas to the problem of classification of two-dimensional conformal field theories, we only have to add the requirement of global conformal invariance which then leads to the previously derived decomposition theory in terms of light-cone fields obeying two "one-dimensional" exchange algebras [5]. This discussion including the role of the "quantization" of the field theoretic Markov trace as a generalization of the Friedan-Qiu-Shenker quantization [23] (which was limited to $c<1$) will be the subject of the second part II. The main difference to other approaches, which emphasize more the analytic aspects of correlation functions, lies in the physical interpretation. For us a quantum field theory is characterized by physical principles as discussed in the second section of this paper, possibly enlarged by covariance properties (as in the case of conformal field theories). Braid and "strand" identities (e.g. the pentagon identity [24]) are *not* part of the physical characterization, they are rather derived by mathematical arguments using additional definitions suggested from the physical postulates. We deliberately separated the presentation of the general framework from two dimensional conformal field theory (to be discussed in part II) in order to avoid the impression that the "new structures" come from conformal invariance. However the simplest explicit illustrations are certainly given by massless conformal exchange algebras. In fact, since the local observables (e.g. the energy momentum tensor or currents) of conformal $QFT_2$ contain no algebraic interactions (i.e. the commutators are, as in Huygens' principle localized solely on the light cone), the field algebra in this case is given by a new sort of "sophisticated free field." Only for the abelian case of "exotic spin" or "anyon" statistics there are conjectured Lagrangian scenarios [25] of this "new structure." The exchange algebras are also expected to appear in two dimensional massive theories for which the localization regions (i.e. the arguments of point-like localized fields) can be ordered for space-like separations. Formally such situations can be imagined as arising from the conformal $QFT_2$ by a "relevant perturbation" around the Kadanoff-Wilson fixed point. The perturbation destroys the conformal invariance, but a subgroup of the center $(Z_+Z_-)$ of the universal covering of the left-right Möbius group:

$$Z_+Z_-^{-1}$$

may remain unbroken and hence serve as a label for massive superselection sectors (this picture has been checked for the massive Ising field theory [26]).[6] $2+1$ dimensional scenarios for the "new structure" are more subtle. In this case the elements of the field algebra (which applied to the vacuum create finite energy states), if they are not localized in bounded regions such that permutation group

[6] However such a picture must also explain the occurence of "shadow operators" in the massive theory [26]

statistics applies, yet are at least localized in thin space-like cones ("strings") [21] and the localization in the formulation of the exchange algebra refers to directions rather than small regions or points. Again Lagrangian scenarios seem to be restricted to the special case where $R$-matrices are just numerical phases, i.e. "anyons" [25]. The explanation for the extraordinary power and potentiality of algebraic quantum field theory (say compared with the Lagrangian approach) should be seen in the economic separation of the full problem into two steps:

(a) The specification of the observable algebra which encodes the space-time (Einstein causality) as well as the quantum aspects of ($C^*$ algebra, positivity) quantum field theory.
(b) The construction of representations of "physical interest" by encoding representation theory into "localized morphism of the observable algebra."

This dichotomy (or duality) is best pictured by associating "charge measuring" operators as ideal i.e. limiting elements with an enlarged observable algebra. The carrier-fields for these charges are related to the localized morphisms. This space-time aspect is already contained in (a), whereas their additional discrete composition structure is best described in terms of the path-space formalism of Sect. 4. It is in this discrete superselecting structure (b), that non-perturbative deep ideas as those of Yang Baxter and Faddeev [27] which hitherto appeared in a different and more special physical context are naturally incorporated. The principles of algebraic quantum field theory point clearly in the direction of a generalization of these ideas. In the simpler case of charge superselection in higher dimensional QFT for which permutation group statistics makes its appearance, Doplicher and Roberts succeeded to prove that the charge measuring operators are belonging to compact Lie groups. In order to achieve this, they had to go beyond the Tanaka-Krein duality theory and develop a new theory (adapted more to the requirements of algebraic field theory) which used the universal $C^*$ Cuntz algebra as the relevant mathematical tool [1]. In a forthcoming paper we will take up this discussion of the analogous duality problem for the braid group statistics. In the language of Doplicher and Roberts this amounts to the question of which is the dual structure to the "monoidal braid-like category."

## Appendix A

*Proof of Proposition 3.1.* We follow Wenzl [12] who proved (3.9) under the more restrictive condition that both sides in Eq. (3.8) vanish separately. (In this case the projections $\varrho^k(E_i)$, $k=0,1,\ldots$ satisfy the defining relations of the Temperley-Lieb algebra [18].)

For $n=1$ Eq. (3.9) means

$$E_i = 1 - E_j \ , \quad i \neq j \ , \quad i, j = 1, 2 \tag{A.1}$$

which is evidently true. We now proceed by induction and assume that Eq. (3.9) holds for $n-1$, $n \geqq 2$. Let

$$F = E_j \varrho(E_i^{(n)}) \ . \tag{A.2}$$

In the computation of $F^2$ we replace the first factor $\varrho(E_i^{(n)})$ according to the induction hypothesis by

$$\varrho(E_i^{(n)}) = \varrho^2(E_i^{(n-1)}) - \frac{2\cos\alpha\sin(n-1)\alpha}{\sin n\alpha}\,\varrho^2(E_i^{(n-1)})\varrho(E_j)\varrho^2(E_i^{(n-1)}) \ . \quad \text{(A.3)}$$

$\varrho^2(E_i^{(n-1)})$ commutes with $E_j \in \varrho^2(\mathscr{A})'$ and its range contains the range of $\varrho(E_i^{(n)})$, i.e.

$$\varrho^2(E_i^{(n-1)})\varrho(E_i^{(n)}) = \varrho(E_i^{(n)}) \ , \quad \text{(A.4)}$$

hence

$$F^2 = E_j\varrho(E_i^{(n)}) - \frac{2\cos\sin(n-1)\alpha}{\sin n\alpha}\,\varrho^2(E_i^{(n-1)})E_j\varrho(E_j)E_j\varrho(E_i^{(n)}) \ . \quad \text{(A.5)}$$

In the second term on the right-hand side we now can apply relation (3.8) with $\tau = (2\cos\alpha)^{-2}$. Since $\varrho(E_j)$ is orthogonal to $\varrho(E_i^{(n-1)})$ by definition, the terms corresponding to the right-hand side of (3.8) vanish, and one gets as in the Temperley-Lieb case,

$$F^2 = E_j\varrho(E_i^{(n)}) - \frac{\sin(n-1)\alpha}{2\cos\alpha\sin n\alpha}\,\varrho^2(E_i^{(n-1)})E_j\varrho(E_i^{(n)}) \ . \quad \text{(A.6)}$$

Using again $E_j \in \varrho^2(\mathscr{A})'$ and (A.4) we finally obtain

$$F^2 = \frac{\sin(n+1)\alpha}{2\cos\alpha\sin n\alpha}\,F \ , \quad \text{(A.7)}$$

where we used the relation

$$2\cos\alpha\sin n\alpha - \sin(n-1)\alpha = \sin(n+1)\alpha \ . \quad \text{(A.8)}$$

For $n = q-1$ the factor multiplying $F$ in (A.7) is nonpositive. Since $F^2E_j = (FF^*)^2$ and $FE_j = FF^*$ are positive, this implies $F = 0$, hence $E_j$ is orthogonal to $\varrho(E_j^{(q-1)})$. This proves (3.10).

For $n < q-1$ we consider the operator $E$ on the right-hand side of Eq. (3.9). $E$ is selfadjoint, and Eq. (A.2) and (A.7) imply $E^2 = E$, hence $E$ is an orthogonal projection. Clearly $E$ is orthogonal to $\varrho^k(E_j)$, $k = 1, \ldots, n-2$, and from (A.7),

$$E_jE = 0 \ , \quad \text{(A.9)}$$

i.e. $E$ is also orthogonal to $E_j$. Moreover, $E$ is the largest projection with this property. Namely, let $\chi \in \bigcap_{k=0}^{n-2} \varrho^k(E_i)\mathscr{H}$. Then by definition of $E_i^{(n)}$,

$$\varrho(E_i^{(n)})\psi = \psi \ , \quad \text{(A.10)}$$

and $E_j\psi = 0$, since $E_jE_i = 0$, thus we obtain $E\psi = \psi$. Hence $E = E_i^{(n+1)}$ by definition of $E_i^{(n+1)}$.

## Appendix B

*Proof of Theorem 3.4.* Assume that $\varrho$ has a left inverse which does not vanish on $\varepsilon_\varrho$. The set of left inverses of $\varrho$ is convex, and it is a compact subset of the space $\mathscr{M}$ of all bounded linear mappings from $\mathscr{A}$ to $B(\mathscr{H})$ equipped with the pointwise weak topology, i.e. the topology induced by the family of seminorms

$$\|\Lambda\|_{A,\Phi,\Psi}=(\Phi,\Lambda(A)\Psi)\ ,\qquad A\in\mathscr{A},\Phi,\Psi\in\mathscr{H}\ . \tag{B.1}$$

According to the Krein-Milman Theorem a compact convex set contains extremal points, so there are extremal left inverses, and at least for one of them, say $\phi$, is $\phi(\varepsilon_\varrho)=\lambda 1\neq 0$. Let $\beta_x^\varrho$ denote the automorphism of $\mathscr{A}$ which is implemented by the unitary representation $U_\varrho$ of the translation group in the representation $\varrho$ of $\mathscr{A}$ (hence $\beta_x^\varrho\varrho=\varrho\alpha_x$). Then $\alpha_{-x}\beta_x^\varrho\in\mathscr{M}$, and the weak limit points for $x$ tending to spacelike infinity are left inverses of $\varrho$. From DHRI,

$$\phi\geqq|\lambda|^2\phi_0 \tag{B.2}$$

for any limit point $\phi_0$ of $(\alpha_{-x}\beta_x^\varrho)$. Since $\phi$ was assumed to be extremal this means $\phi=\phi_o$, so in particular all limit points of $(\alpha_{-x}\beta_x^\varrho)$ coincide, hence

$$\lim\alpha_{-x}\beta_x^\varrho=\phi\ . \tag{B.3}$$

Moreover, $\phi$ is locally normal [i.e. normal on each subalgebra $\mathscr{A}(O)$] as limit of a sequence of locally normal maps.

We now can construct the conjugate sector. The state $\omega_0\circ\phi$ is invariant under $\beta_x^\varrho$, hence in the GNS representation $(\pi,\mathscr{H}_\pi,\Omega_\pi)$,

$$\omega_0\circ\phi(A)=(\Omega_\pi,\pi(A)\Omega_\pi)\ ,\qquad A\in\mathscr{A} \tag{B.4}$$

$$\pi(\mathscr{A})\Omega_\pi\ \text{dense in}\ \mathscr{H}_\pi\ , \tag{B.5}$$

the automorphism $\beta_x^\varrho$ can be unitarily implemented by

$$U_{\pi\varrho}(x)\pi(A)\Omega_\pi=\pi\beta_x^\varrho(A)\Omega_\pi\ . \tag{B.6}$$

$U_{\pi\varrho}$ is strongly continuous since $\pi$ is locally normal, and it satisfies the spectrum condition since $U_\varrho$ does. One then can show as in [20] that $\pi$ is equivalent to a localized morphism $\bar\varrho$. The automorphism $\alpha_x$ is implemented by

$$U_\pi(x)=\pi(U(x)U_\varrho(-x))U_{\pi\varrho}(x) \tag{B.7}$$

and $U_\pi$ satisfies again the spectrum condition since

$$sp\ U_\pi+sp\ U_\varrho\subset sp\ U_{\pi\varrho} \tag{B.8}$$

[20, 21]. Let $R$ be defined by

$$RA\Omega=\pi\varrho(A)\Omega_\pi\ . \tag{B.9}$$

Then $R$ is an isometric intertwiner from $\pi\varrho$ to the identity

$$\pi\varrho(A)R=RA\ , \tag{B.10}$$

hence $\pi$ is a conjugate representation. Finally we find

$$\phi(A)=R^*\pi(A)R\ , \tag{B.11}$$

so $\phi$ is a regular left inverse, and $\phi(\varepsilon_\varrho)\neq 0$.

Actually, by the spectrum condition, any regular left inverse cannot vanish on $\varepsilon_\varrho$ [20]. Now let $\phi_1$ be some regular left inverse. Since $\phi$ is the only extremal left inverse which does not vanish on $\varepsilon_\varrho$, we have

$$\phi_1(\varepsilon_\varrho)=\mu\lambda 1 \ , \qquad 0<\mu\leqq 1 \ . \tag{B.12}$$

If $\mu<1$, $\phi_\infty=(1-\mu)^{-1}(\phi_1-\mu\phi)$ is also a left inverse, and one can show that $\phi_\infty$ is regular in contradiction to the fact that $\phi_\infty(\varepsilon_\varrho)=0$. Thus $\mu=1$ and $\phi_1=\phi$. q.e.d.

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**Note added in proof.** After submitting this paper we received a preprint of Longo [29] which also treats the superselection structure of 2-dimensional quantum field theories. Among other results he proved the quantization of the statistical dimension $d(\varrho)$ (cf. the remark after Proposition 4.2) by identifying it with the square root of the index of the inclusion $\varrho(\mathfrak{A}(\mathcal{O}))\subset\mathfrak{A}(\mathcal{O})$, where $\varrho$ is localized in $\mathcal{O}$. Moreover, his new results and ideas on the space of conditional expectations improve and simplify our discussion of left inverses. The statement of Theorem 3.4, e.g., can be sharpened in the following way: if $\phi(\varepsilon_\varrho)\neq 0$ for some left inverse $\phi$, then $\phi$ is the only left inverse (hence the hypothetical nonregular left inverses do not exist). Actually, this follows directly from the equality (cf. Proposition 4.2)

$$A=\varrho(R^*)\varrho\bar{\varrho}(A)F\varrho(R)d(\varrho)^R$$

[i.e. $\mathfrak{A}$ is generated by $\varrho(\mathfrak{A})$ and $F$ ($F$ is the "Jones projection" for the inclusion $\varrho\bar{\varrho}(\mathfrak{A})\subset\varrho(\mathfrak{A})$)]. Applying a left inverse $\phi'$ of $\varrho$ yields

$$\phi'(A)=R^*\bar{\varrho}(A)R\phi'(F)d(\varrho)^2 \ ,$$

where $\phi'(F)\in\bar{\varrho}(\mathfrak{A})'=\mathbb{C}1$. $\phi'(1)=1$ implies $\phi'(F)=d(\varrho)^{-2}$, hence

$$\phi'(A)=R^*\bar{\varrho}(A)R\equiv\phi(A) \ .$$