Superselection structures for C*–algebras with nontrivial center

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Abstract. We present and prove some results within the framework of Hilbert C*–systems $\{\mathcal{F}, \mathcal{G}\}$ with a compact group $\mathcal{G}$. We assume that the fixed point algebra $\mathcal{A} \subset \mathcal{F}$ of $\mathcal{G}$ has a nontrivial center $\mathcal{Z}$ and its relative commutant w.r.t. $\mathcal{F}$ coincides with $\mathcal{Z}$, i.e. we have $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z} \supset \mathbb{C}\mathbb{1}$. In this context we propose a generalization of the notion of an irreducible endomorphism and study the behaviour of such irreducibles w.r.t. $\mathcal{Z}$. Finally, we give several characterizations of the stabilizer of $\mathcal{A}$.

1 Introduction

The Doplicher–Roberts superselection theory [1, 2, 3, 4, 5, 6, 7, 8] starts with a C*–algebra $\mathcal{A}$ with trivial center, i.e. $\mathcal{Z}(\mathcal{A}) := \mathcal{Z} = \mathbb{C}\mathbb{1}$. $\mathcal{A}$ is interpreted as the algebra of quasilocal observables. The field algebra $\mathcal{F} \supset \mathcal{A}$, together with the gauge group $\mathcal{G}$ is then constructed as a special C*–dynamical system $\{\mathcal{F}, \mathcal{G}\}$ (cf. [9]), namely as a crossed product [6], also called a Hilbert C*–system in [10, Chapter 10]. It satisfies the condition that the relative commutant is trivial, i.e. $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$.

The paper by Fredenhagen, Rehren and Schroer [12], where conformal theories in 1+1 dimensions are studied, suggests that a nontrivial center (containing for example (global) 'Casimir operators') of the universal algebra may also appear in physically relevant examples, and this situation is related to the superselection theory of the model (see also [13]). Furthermore, there are good mathematical reasons for considering the center of $\mathcal{A}$ to be nontrivial, and indeed this case has been treated in the past. For example, in the framework of strict symmetric moinidal C*–categories with conjugates Doplicher and Roberts [7, Sections 2 and 3] present some results where (ι, ι) is not necessarily trivial. Further, Longo and Roberts [11, Section 2] also study the notion of conjugation in the more general setting of strict monoidal C*–categories without assuming that (ι, ι) is trivial. They also present a result for the case that (ι, ι) is finite dimensional.

One of the problems of dealing with a nontrivial center of $\mathcal{A}$ is mentioned in [7, Introduction]: "There is, however, no known analogue of Theorem 4.1 of [6] for a C*-algebra with a non-trivial center and hence nothing resembling a "duality" in this more general setting." The theorem mentioned before guarantees the existence of a C*–algebra containing an algebraic Hilbert space

that satisfies the usual nice properties, once $\mathcal{A}$ with $\mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$ and a suitable endomorphism are given. Contrarily, the present paper deals with Hilbert C*–systems $\{\mathcal{F}, \mathcal{G}\}$ for compact groups $\mathcal{G}$, where the fixed point algebra $\mathcal{A}$ has nontrivial center $\mathcal{Z}$ and satisfies the condition

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z} \supset \mathbb{C}\mathbb{1} \,. \tag{1}$$

We adopt a pure mathematical point of view and we will use the words field algebra $\mathcal{F}$ and observable algebra $\mathcal{A}$ only in a 'metaphorical' sense, not claiming any relation to QFT.

We remind that the condition $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (which implies $\mathcal{Z} = \mathbb{C}\mathbb{1}$) leads to the property $\pi(\mathcal{A})'' = U(\mathcal{G})'$, where π denotes a so–called regular representation of $\{\mathcal{F}, \mathcal{G}\}$, i.e. π is the GNS–representation of the state $\omega(F) := \omega_0(\Pi_\iota F)$ and the GNS–representation of $\mathcal{A}$ w.r.t. ω_0 is faithful (see, for example, [13, p. 18 ff.]). In the general case given by (1) it can be shown that for a regular representation π the equation

$$\pi(\mathcal{A}) \cap U(\mathcal{G})'' = \mathbb{C}\mathbb{1}$$

holds. In this case we have that the condition $\pi(\mathcal{A})'' = U(\mathcal{G})'$ implies $\mathcal{Z} = \mathbb{C}\mathbb{1}$ and, therefore, if we assume $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$, then the proper inclusion $\pi(\mathcal{A})'' \subset U(\mathcal{G})'$ must hold (note that $\pi(\mathcal{A})'' \subseteq U(\mathcal{G})'$ is always true). Roughly speaking we can say that the group $\mathcal{G}$ does not determine the 'observables' completely.

We hope that this (mathematical) model will serve to get familiar with certain structures (e.g. Hilbert $\mathcal{Z}$–modules) that may possibly appear when trying to construct $\mathcal{G}$ and $\mathcal{F}$ starting from the Doplicher–Roberts analysis in [7], for example in the special case that the "statistical dimensions" $d(\rho)$ are scalars. However, at the present we have no convincing argument that this could be even possible.

The paper is structured in 8 sections: In Section 2 we collect standard results concerning Hilbert C*–systems that will be used later on. In Section 3 the notion of a Hilbert $\mathcal{Z}$–module is introduced. It is a natural generalization of the usual notion of an algebraic Hilbert space [1, Section 2] when the center of the observable algebra $\mathcal{Z}$ is nontrivial and Eq. (1) is satisfied. In Section 4 the bijection between the set of right Hilbert $\mathcal{Z}$–modules and the set of canonical endomorphism is extended to a functor between the corresponding categories.

From very general arguments it is easy to see that the original notion of irreducible endomorphism ρ, i.e. $(\rho, \rho) = \mathbb{C}\mathbb{1}$, is not meaningful anymore when $\mathcal{Z}$ is nontrivial. Section 5 proposes a generalization of this concept. A first justification of this new notion is given by the observation that the action of the inverse of an 'irreducible' endomorphism restricted to the center can be described by certain continuous function acting on $\operatorname{spec} \mathcal{Z}$ (cf. with Remark 5.5). Section 6 deals with the decomposition theory of a general Hilbert $\mathcal{Z}$–module $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, with $\mathcal{H}$ a group invariant algebraic Hilbert space, in terms of $\mathfrak{H}_D = \mathcal{H}_D\mathcal{Z}$, where $D \in \widehat{\mathcal{G}}$. The main results of the article are presented in Section 7, where different statements and characterizations concerning the stabilizer of $\mathcal{A}$, $\operatorname{stab} \mathcal{A}$, are proved. For example, $\operatorname{stab} \mathcal{A}$ is identified as a certain subgroup of the group of all continuous functions from $\operatorname{spec} \mathcal{Z}$ into $\mathcal{G}$, and the description of this subgroup uses the functions associated to irreducibles mentioned above.

2 Basic material on Hilbert C*–system

We start introducing some notation and results concerning Hilbert C*–systems. General references are [10, Chapter 10], [14, 13].

A Hilbert C*–system is denoted by $\{\mathcal{F}, \mathcal{G}\}$, where $\mathcal{G} \subset \operatorname{aut} \mathcal{F}$ is a compact group w.r.t. the pointwise norm topology. Π_D, $D \in \widehat{\mathcal{G}}$ (the dual of $\mathcal{G}$), are the spectral projections that satisfy the orthonormality relation

$$\Pi_{D_1} \circ \Pi_{D_2} = \delta_{D_1 D_2} \Pi_{D_1} \,, \qquad D_1, D_2 \in \widehat{\mathcal{G}} \,.$$

For the trivial representation class $\iota \in \widehat{\mathcal{G}}$, we put

$$\mathcal{A} := \Pi_\iota \mathcal{F} = \left\{ F \in \mathcal{F} \colon g(F) = F, \quad g \in \mathcal{G} \right\},$$

i.e. $\mathcal{A}$ is the fixed point algebra in $\mathcal{F}$ w.r.t. $\mathcal{G}$. Further, the spectrum of $\mathcal{G}$,

$$\operatorname{spec} \mathcal{G} := \{ D \in \widehat{\mathcal{G}} \colon \Pi_D \neq 0 \},$$

can be defined equivalently as the "Arveson spectrum" (cf. [13]). According to the definition of a Hilbert C*–system we have $\operatorname{spec} \mathcal{G} = \widehat{\mathcal{G}}$ and to each $D \in \widehat{\mathcal{G}}$ there corresponds an algebraic Hilbert space $\mathcal{H}_D \subset \Pi_D \mathcal{F}$, $\dim \mathcal{H}_D = \dim D = d$, such that $\operatorname{supp} \mathcal{H}_D = 1$ and $\mathcal{G}$ acts irreducibly on $\mathcal{H}_D$. Further, the unitary representation $\mathcal{G} \!\restriction\! \mathcal{H}_D$ is an element of of the equivalence class D. Recall that if $\{\Phi_{D,i}\}_{i=1}^d$ is an orthonormal basis in $\mathcal{H}_D$, i.e. the basis elements satisfy

$$\Phi_{D,i}^* \, \Phi_{D,i'} = \delta_{ii'} \, \mathbb{1} \,,$$

then

$$\operatorname{supp} \mathcal{H}_D := \sum_{i=1}^d \Phi_{D,i} \, \Phi_{D,i}^* = \mathbb{1} \,.$$

In terms of the matrix elements we have

$$g \left(\Phi_{D,i} \right) = \sum_{i'=1}^d \Phi_{D,i'} \, U_{D,i'i}(g), \quad g \in \mathcal{G} \,.$$

We denote by $\overline{D} \in \widehat{\mathcal{G}}$, the conjugated representation of $D \in \widehat{\mathcal{G}}$. If D is related to the matrix $\left(U_{i'i}(g) \right)_{i',i} \in \operatorname{Mat}_d(\mathbb{C})$ as above, then $\overline{D}$ is realized by the complex conjugated matrix $\left(\overline{U_{i'i}(g)} \right)_{i',i} \in \operatorname{Mat}_d(\mathbb{C})$ w.r.t. the conjugated orthonormal basis $\{\Phi_{\overline{D},i}\}_{i=1}^d$ of $\mathcal{H}_{\overline{D}}$.

The following equations will be useful for later on [15, 16], [17, p. 182]:

$$\mathcal{F} = \operatorname{clo}_{\|\cdot\|} \left(\operatorname{span} \left\{ \Pi_D \mathcal{F} \colon D \in \widehat{\mathcal{G}} \right\} \right) \tag{2}$$

$$\Pi_D \mathcal{F} = \operatorname{span}\{\mathcal{A} \cdot \mathcal{H}_D\}, \qquad D \in \widehat{\mathcal{G}}, \tag{3}$$

where $\|\cdot\| = \|\cdot\|_{\mathcal{F}}$ is the C*–norm in $\mathcal{F}$.

By $\operatorname{end} \mathcal{A}$ we denote the set of all unital endomorphisms of $\mathcal{A}$. Further for $\lambda, \mu \in \operatorname{end} \mathcal{A}$ we consider, as usual, the intertwiner space

$$(\lambda, \mu) := \{ A \in \mathcal{A} \colon A \, \lambda(X) = \mu(X) \, A, \quad X \in \mathcal{A} \}.$$

If $\mathcal{H}$ denotes an arbitrary $\mathcal{G}$–invariant algebraic Hilbert space with support $\mathbb{1}$ in $\mathcal{F}$, then the canonical endomorphism associated to $\mathcal{H}$ is

$$\lambda_{\mathcal{H}}(F) := \sum_{i=1}^h \Psi_i \, F \, \Psi_i^*, \quad F \in \mathcal{F} \,, \tag{4}$$

where $\{\Psi_i\}_{i=1}^h$ is an orthonormal basis of $\mathcal{H}$. $\lambda_{\mathcal{H}}$ is unital and since $\mathcal{G}$ leaves $\mathcal{H}$ invariant we have $\lambda_{\mathcal{H}}(\mathcal{A}) \subseteq \mathcal{A}$, i.e. $\lambda_{\mathcal{H}} \in \operatorname{end} \mathcal{A}$. If $\mathcal{H} := \mathcal{H}_D$, then $\rho_{\mathcal{H}_D}$ is briefly denoted by ρ_D.

From Eqs. (2) und (3) (see also [10, Subsection 10.1.3]) we obtain the relation

$$\mathcal{A}' \cap \mathcal{F} = \operatorname{clo}_{\|\cdot\|} \left(\sum_{D \in \widehat{\mathcal{G}}} (\rho_D, \iota) \cdot \mathcal{H}_D \right). \tag{5}$$

This implies that, $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ iff $(\rho_D, \iota) = \{0\}$, for all $D \neq \iota$ and therefore, from our fundamental assumption,

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z} \supset \mathbb{C}\mathbb{1}, \tag{6}$$

we get the following disjointness relation between the canonical endomorphisms ρ_D, $D \in \widehat{\mathcal{G}}$:

$$(\rho_{D_1}, \rho_{D_2}) = \{0\}, \quad D_1 \neq D_2, \quad D_1, D_2 \in \widehat{\mathcal{G}}. \tag{7}$$

Note that Eq. (7) implies also Eq. (6).

For a general $\lambda \in \operatorname{end}\mathcal{A}$ and, in particular, for $\lambda := \rho_{\mathcal{H}}$ we get

$$\mathcal{Z} \subseteq (\lambda, \lambda). \tag{8}$$

On the other hand we also obtain the inclusions,

$$\mathbb{C}\mathbb{1} \subseteq \lambda(\mathcal{Z}) \subseteq (\lambda, \lambda), \tag{9}$$

because from the relation, $Z A = A Z$, $A \in \mathcal{A}$, $Z \in \mathcal{Z}$, it follows that,

$$\lambda(Z)\, \lambda(A) = \lambda(A)\, \lambda(Z),$$

and therefore, $\lambda(Z) \in (\lambda, \lambda)$. Note that since λ is not surjective in general we can not assure that $\lambda(Z) \in \mathcal{Z}$ for all $Z \in \mathcal{Z}$. Therefore a typical feature of the present case is expressed by the fact that in general,

$$\lambda(\mathcal{Z}) \not\subset \mathcal{Z}. \tag{10}$$

Eq. (8) implies that the usual "intrinsic" (i.e. group independent) notion of irreducible endomorphism λ, namely $(\lambda, \lambda) = \mathbb{C}\mathbb{1}$, is meaningless in our situation (cf. nevertheless with Section 5 and with [10, Subsection 10.1.3] for further details concerning this point).

Next we introduce the *–subalgebra $\mathcal{F}_0 \subset \mathcal{F}$ and the $\mathcal{A}$–valued scalar product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $\mathcal{F}$. We select first, a family of algebraic Hilbert spaces $\{\mathcal{H}_D\}_{D \in \widehat{\mathcal{G}}}$ corresponding to $\{\mathcal{F}, \mathcal{G}\}$ and, second, a corresponding family $\{\Phi_{D,i}\}_{i=1}^d$, $D \in \widehat{\mathcal{G}}$, of orthonormal basis. Then,

$$\mathcal{F}_0 := \left\{ \sum_{\substack{D \\ \text{finite sum}}} \left(\sum_{i=1}^d A_{D,i}\, \Phi_{D,i} \right) : A_{D,i} \in \mathcal{A} \right\} \subset \mathcal{F}. \tag{11}$$

$\mathcal{F}_0$ is a dense *–subalgebra of $\mathcal{F}$. On $\mathcal{F}$ we can define the following $\mathcal{A}$–valued scalar product,

$$\langle F_1, F_2 \rangle_{\mathcal{A}} := \Pi_\iota \left(F_1\, F_2^* \right), \quad F_1, F_2 \in \mathcal{F}.$$

It satisfies the equations:

$$\langle \Phi_{D_1,i_1}, \Phi_{D_2,i_2} \rangle_{\mathcal{A}} = \frac{1}{d_1}\, \delta_{D_1 D_2}\, \delta_{i_1 i_2}\, \mathbb{1}, \quad D_1, D_2 \in \widehat{\mathcal{G}}, \ i_1 = 1, \ldots, d_1, \ i_2 = 1, \ldots, d_2.$$

$$\langle A_1 F_1, A_2 F_2 \rangle_{\mathcal{A}} = A_1\, \langle F_1, F_2 \rangle_{\mathcal{A}}\, A_2^*, \quad A_1, A_2 \in \mathcal{A}, \ F_1, F_2 \in \mathcal{F}.$$

$$\langle F, F \rangle_{\mathcal{A}} \geq 0 \quad \text{and} \quad \langle F, F \rangle_{\mathcal{A}} = 0 \ \text{iff} \ F = 0, \quad F \in \mathcal{F}_0.$$

From this we obtain for $F_1 := \sum_{i,D} A_{D,i}\, \Phi_{D,i} \in \mathcal{F}_0$ and $F_2 := \sum_{i,D} B_{D,i}\, \Phi_{D,i} \in \mathcal{F}_0$, the equation

$$\langle F_1, F_2 \rangle_{\mathcal{A}} = \sum_{i,D} \frac{1}{d}\, A_{D,i}\, B_{D,i}^*.$$

Consider $\mathcal{H}$, $\rho_{\mathcal{H}}$ and $\{\Psi_{\mathcal{H},i}\}_{i=1}^h$ as specified in (4). Denote by $\overline{\mathcal{H}}$ a conjugated algebraic Hilbert space (carrying the conjugated representation) with orthonormal basis given by $\{\Psi_{\overline{\mathcal{H}},i}\}_{i=1}^h$. Putting

$$R_{\mathcal{H}} \quad := \quad \sum_{i=1}^h \Psi_{\overline{\mathcal{H}},i}\,\Psi_{\mathcal{H},i} \in \mathcal{A} \qquad \text{and} \tag{12}$$

$$\varepsilon(\mathcal{H}_1,\mathcal{H}_2) \quad := \quad \sum_{i,j} \Psi_{\mathcal{H}_2,i}\,\Psi_{\mathcal{H}_1,j}\,\Psi_{\mathcal{H}_2,i}^*\,\Psi_{\mathcal{H}_1,j}^*\,, \tag{13}$$

we get the following relations:

$$R_{\mathcal{H}} \quad \in \quad (\iota,\,\rho_{\overline{\mathcal{H}}} \circ \rho_{\mathcal{H}}). \tag{14}$$

$$\varepsilon(\mathcal{H}_1,\mathcal{H}_2) \quad \in \quad (\rho_{\mathcal{H}_1} \circ \rho_{\mathcal{H}_2},\, \rho_{\mathcal{H}_2} \circ \rho_{\mathcal{H}_1}). \tag{15}$$

$$\Psi_{\mathcal{H},i}^* \quad = \quad R_{\mathcal{H}}^*\,\Psi_{\overline{\mathcal{H}},i}\,,\ i=1,\dots,h \tag{16}$$

$$R_{\overline{\mathcal{H}}} \quad = \quad \varepsilon(\overline{\mathcal{H}},\mathcal{H})R_{\mathcal{H}}\,. \tag{17}$$

$$R_{\mathcal{H}}^*\,R_{\mathcal{H}} \quad = \quad R_{\overline{\mathcal{H}}}^*\,R_{\overline{\mathcal{H}}} \quad = \quad h\mathbb{1}\,. \tag{18}$$

$$\mathbb{1} \quad = \quad \varepsilon(\mathcal{H}_1,\mathcal{H}_2)\,\varepsilon(\mathcal{H}_2,\mathcal{H}_1). \tag{19}$$

The so–called standard left inverse is given by

$$\phi_{\mathcal{H}}(A) := \frac{1}{h}\,R_{\mathcal{H}}^*\,\rho_{\overline{\mathcal{H}}}(A)\,R_{\mathcal{H}}\,, \quad A \in \mathcal{A}\,. \tag{20}$$

3   Hilbert $\mathcal{Z}$–modules

The stabilizer, $\operatorname{stab}\mathcal{A}$, is a subgroup of $\operatorname{aut}\mathcal{F}$ defined by

$$\operatorname{stab}\mathcal{A} := \{\beta \in \operatorname{aut}\mathcal{F}\colon\ \beta(A) = A \quad \text{for all} \quad A \in \mathcal{A}\}.$$

The study of $\operatorname{stab}\mathcal{A}$ in the present situation leads in a natural way to the notion of a right Hilbert $\mathcal{Z}$–module [18, Chapter 15] (see the following two propositions). Some of the results of this and the next section may be compared by putting $\mathcal{Z} = \mathbb{C}\mathbb{1}$ with standard results in e.g. [1]. Note that we work with finite–dimensional algebraic Hilbert spaces.

Let $\mathcal{H}$ be an $\mathcal{G}$–invariant algebraic Hilbert space in $\mathcal{F}$ of finite dimension d. Then we define the free right $\mathcal{Z}$–module $\mathfrak{H}$ by extension

$$\mathfrak{H} := \mathcal{H}\mathcal{Z} = \left\{\sum_{i=1}^d \Phi_i\,Z_i\colon\ Z_i \in \mathcal{Z}\right\},$$

where $\{\Phi_i\}_{i=1}^d$ is an orthonormal basis in $\mathcal{H}$. In other words, the set $\{\Phi_i\}_{i=1}^d$ becomes a module basis of $\mathfrak{H}$ and $\dim \mathfrak{H} = d$. For $H_1, H_2 \in \mathfrak{H}$ put

$$\langle H_1,\, H_2 \rangle_{\mathfrak{H}} := H_1^* H_2 \in \mathcal{Z}\,.$$

Then, $\{\mathfrak{H},\, \langle\cdot,\,\cdot\rangle_{\mathfrak{H}}\}$ is a Hilbert (right) $\mathcal{Z}$–module or a Hilbert $\mathcal{Z}$–module, for short.

A system of d elements $\{\Psi_i\}_{i=1}^d \subset \mathfrak{H}$ with

$$\Psi_i = \sum_{i'=1}^d \Phi_{i'}\,Z_{i'i}\,, \quad Z_{i'i} \in \mathcal{Z}\,, i',i=1,\dots,d\,, \tag{21}$$

is an orthonormal basis of $\mathfrak{H}$, i.e. $\langle \Psi_i, \Psi_k \rangle = \delta_{ik}$, if the matrix $\mathfrak{Z} := \left(Z_{i'i} \right)_{i',i=1}^{d} \in \mathrm{Mat}_d(\mathcal{Z})$ satisfies

$$\mathfrak{Z}^* \mathfrak{Z} = \mathbb{1}_d \,. \tag{22}$$

Using Gelfand's Theorem we denote the values of the corresponding matrix–valued function on $\mathrm{spec}\, \mathcal{Z}$ by $\mathfrak{Z}(\varphi)$, $\varphi \in \mathrm{spec}\, \mathcal{Z}$:

$$\mathrm{spec}\, \mathcal{Z} \ni \varphi \longmapsto \mathfrak{Z}(\varphi) := \left(Z_{i'i}(\varphi) \right)_{i',i} \,.$$

Then Eq. (22) is equivalent to

$$\mathfrak{Z}(\varphi)^* \mathfrak{Z}(\varphi) = \mathbb{1}_d \,, \quad \varphi \in \mathrm{spec}\, \mathcal{Z} \,. \tag{23}$$

Recall that in the finite–dimensional case, Eq. (23) implies

$$\mathfrak{Z}(\varphi) \mathfrak{Z}(\varphi)^* = \mathbb{1}_d \,, \quad \varphi \in \mathrm{spec}\, \mathcal{Z} \,.$$

and again by Gelfand's Theorem we obtain

$$\mathfrak{Z} \mathfrak{Z}^* = \mathbb{1}_d \,. \tag{24}$$

The last equation implies that the canonical endomorphism $\rho_{\mathcal{H}}$ can now be associated to $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, because

$$\rho_{\mathcal{H}}(A) = \sum_{i=1}^{d} \Psi_i \, A \, \Psi_i^* \,, \quad A \in \mathcal{A} \,,$$

where now $\{\Psi_i\}_{i=1}^{d}$ can be *any* orthonormal basis in $\mathfrak{H}$, is independent of the choice of the orthonormal basis in $\mathfrak{H}$:

$$
\begin{aligned}
\sum_{i=1}^{d} \Psi_i \, A \, \Psi_i^* &= \sum_{i,i',i''} \Phi_{i'} \, Z_{i'i} \, A \, Z_{i''i}^* \, \Phi_{i''}^* \\
&= \sum_{i',i''} \Phi_{i'} \left(\sum_i Z_{i'i} \, A \, Z_{i''i}^* \right) \Phi_{i''}^* \\
&= \sum_{i',i''} \Phi_{i'} \left(\sum_i Z_{i'i} \, Z_{i''i}^* \right) A \, \Phi_{i''}^* \\
&= \sum_{i=1}^{d} \Phi_i \, A \, \Phi_i^*
\end{aligned}
$$

For this reason we use the notation $\rho_{\mathcal{H}} = \rho_{\mathfrak{H}}$, $\mathfrak{H} = \mathcal{H}\mathcal{Z}$.

We emphasize that $\rho_{\mathfrak{H}}$ does not characterize anymore the algebraic Hilbert space $\mathcal{H}$ (as in the case where $\mathcal{Z} = \mathbb{C}\mathbb{1}$). However we have

3.1 Proposition *Let $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ be a Hilbert $\mathcal{Z}$–module as above and let $\rho_{\mathfrak{H}}$ be the corresponding canonical endomorphism. Then the relation*

$$H \in \mathfrak{H} \quad \textit{iff} \quad H \, A = \rho_{\mathfrak{H}}(A) \, H \,, \quad A \in \mathcal{A} \,,$$

holds. With other words, $\rho_{\mathfrak{H}}$ characterizes the Hilbert $\mathcal{Z}$–module $\mathfrak{H}$ uniquely.

Proof: Let $H \in \mathfrak{H}$. Then we can write $H = \sum\limits_{i=1}^{d} \Phi_i \, Z_i$ for certain $Z_i \in \mathcal{Z}$ and $\{\Phi_i\}_{i=1}^{d}$ an orthonormal basis of $\mathcal{H}$. We compute directly

$$H A = \sum_{i=1}^{d} \Phi_i \, Z_i \, A = \sum_{i=1}^{d} \Phi_i \, A \, Z_i = \rho_{\mathfrak{H}}(A) \sum_{i=1}^{d} \Phi_i \, Z_i = \rho_{\mathfrak{H}}(A) \, H \, .$$

Conversely, suppose that $H \in \mathcal{F}$ satisfies the equation $H A = \rho_{\mathfrak{H}}(A) \, H$, $A \in \mathcal{A}$. Then, since $\Phi_i \, A = \rho_{\mathfrak{H}}(A) \, \Phi_i$, and $A \, \Phi_i^* = \Phi_i^* \, \rho_{\mathfrak{H}}(A)$ for all $A \in \mathcal{A}$, $i = 1, \ldots, d$, we have $\Phi_i^* \, H \, A = A \, \Phi_i^* \, H$, $A \in \mathcal{A}$, i.e. we get $\Phi_i^* \, H \in \mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$. Putting now $Z_i := \Phi_i^* \, H \in \mathcal{Z}$, we finally obtain, $H = \sum\limits_{i=1}^{d} \Phi_i \, Z_i \in \mathfrak{H}$. ∎

3.2 Proposition *Let $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ be a Hilbert $\mathcal{Z}$–module as above. Then, $\mathfrak{H}$ is* stab $\mathcal{A}$*–invariant, i.e. $\beta(\mathfrak{H}) \subset \mathfrak{H}$ for all $\beta \in$ stab $\mathcal{A}$.*

Proof: Let $\beta \in$ stab $\mathcal{A}$ and $\Phi, \Psi \in \mathcal{H}$. Since $\rho_{\mathfrak{H}}(A) \in \mathcal{A}$ for $A \in \mathcal{A}$, we have

$$\Phi^* \, \beta(\Psi) \, A = \Phi^* \, \beta(\Psi A) = \Phi^* \, \beta\big(\rho_{\mathfrak{H}}(A)\Psi\big) = \Phi^* \, \rho_{\mathfrak{H}}(A) \, \beta(\Psi) = A \, \Phi^* \, \beta(\Psi) \, ,$$

for all $A \in \mathcal{A}$, and therefore $\Phi^* \, \beta(\Psi) \in \mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$. In particular, putting as in the preceding proposition $Z_i := \Phi_i^* \, \beta(\Psi) \in \mathcal{Z}$, where $\{\Phi_i\}_{i=1}^{d}$ is an orthonormal basis of $\mathcal{H}$, we obtain $\beta(\Psi) = \sum\limits_{i=1}^{d} \Phi_i \, Z_i \in \mathfrak{H}$. ∎

Next we ask the question of how to characterize $\mathcal{G}$–invariant algebraic Hilbert spaces that are contained in a given $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, with $\mathcal{H}$ itself a $\mathcal{G}$–invariant algebraic Hilbert space. By $U_{\mathcal{H}}(g) \in \mathrm{Mat}_d(\mathbb{C})$, $g \in \mathcal{G}$, we denote the unitary matrix representation of $\mathcal{G}$ given on $\mathcal{H}$, w.r.t. an orthonormal basis $\{\Phi_i\}_{i=1}^{d}$ specified in $\mathcal{H}$. If we choose another orthonormal basis $\{\Psi_i\}_{i=1}^{d}$ in $\mathfrak{H}$, related to $\{\Phi_i\}_{i=1}^{d}$ by means of the unitary matrix $\mathfrak{Z} \in \mathrm{Mat}_d(\mathcal{Z})$ of Eq. (21), then the representation of $\mathcal{G}$ w.r.t. the new basis is given by the matrices $V(g) \in \mathrm{Mat}_d(\mathcal{Z})$, defined by

$$V(g) := \mathfrak{Z}^* \, U_{\mathcal{H}}(g) \, \mathfrak{Z} \, , \qquad g \in \mathcal{G} \, . \tag{25}$$

In contrast to $U_{\mathcal{H}}(g)$, the matrix $V(g)$ cannot in general be associated to a *constant* matrix–valued function on spec $\mathcal{Z}$. The condition "$V(g)$, $g \in \mathcal{G}$, is a constant matrix–valued function on spec $\mathcal{Z}$" reads

$$\mathfrak{Z}(\varphi_1)^* \, U_{\mathcal{H}}(g) \, \mathfrak{Z}(\varphi_1) = \mathfrak{Z}(\varphi_2)^* \, U_{\mathcal{H}}(g) \, \mathfrak{Z}(\varphi_2) \, , \quad \varphi_1, \varphi_2 \in \mathrm{spec}\, \mathcal{Z} \, , \; g \in \mathcal{G} \, ,$$

or

$$\mathfrak{Z}(\varphi_2) \mathfrak{Z}(\varphi_1)^* \, U_{\mathcal{H}}(g) = U_{\mathcal{H}}(g) \, \mathfrak{Z}(\varphi_2)\mathfrak{Z}(\varphi_1)^* \, , \quad \varphi_1, \varphi_2 \in \mathrm{spec}\, \mathcal{Z} \, , \; g \in \mathcal{G} \, , \tag{26}$$

i.e. $\mathfrak{Z}(\varphi_2)\mathfrak{Z}(\varphi_1)^*$ is an intertwiner of $U_{\mathcal{H}}(\mathcal{G})$.

Now consider the special case that $\mathcal{G}$ acts irreducibly on $\mathcal{H}$. Then Eq. (26) is equivalent to

$$\mathfrak{Z}(\varphi_2) = \mu(\varphi_1, \varphi_2) \, \mathfrak{Z}(\varphi_1) \, ,$$

where $|\mu(\varphi_1, \varphi_2)| = 1$, $\varphi_1, \varphi_2 \in \mathrm{spec}\, \mathcal{Z}$. Let $\varphi_1 := \varphi_0$ be a fixed point of spec $\mathcal{Z}$ and put $W := \mathfrak{Z}(\varphi_0)$. Then we get the condition

$$\mathfrak{Z}(\varphi) = \mu(\varphi) \, W \, , \quad |\mu(\varphi)| = 1 \, , \; \varphi \in \mathrm{spec}\, \mathcal{Z} \, ,$$

where $\mu(\cdot)$ is a continuous scalar function. Let $W := \left(W_{i'i}\right)_{i',i=1}^{d} \in \mathrm{Mat}_d(\mathbb{C})$ and $U \in \mathcal{Z}$ with $U(\varphi) = \mu(\varphi)$. Then $U^*U = UU^* = \mathbb{1}$ and

$$\mathfrak{Z} = \left(W_{i'i} \, U\right)_{i',i=1}^{d} \, .$$

In other words, we obtain that $V(g)$, $g \in \mathcal{G}$, is a constant matrix–valued function w.r.t. $\{\Psi_i\}_{i=1}^d$ iff

$$\Psi_i = \sum_{i'=1}^d \Phi_{i'} W_{i'i} U \,, \quad U \in \mathcal{Z} \text{ unitary.}$$

Putting $\widetilde{\Phi}_i = \sum_{i'} \Phi_{i'} W_{i'i}$, then $\Psi_i = \widetilde{\Phi}_i U$ and we obtain from Eq. (25)

$$V(g) = \mathfrak{Z}^* U_{\mathcal{H}}(g) \, \mathfrak{Z} = \operatorname{diag} U^* \cdot \widetilde{U_{\mathcal{H}}}(g) \cdot \operatorname{diag} U = \left(U^* \widetilde{U}_{i'i} U \right)_{i',i=1}^d = \widetilde{U_{\mathcal{H}}}(g) \,, \quad g \in \mathcal{G} \,,$$

where the matrix $\widetilde{U_{\mathcal{H}}}(g)$ corresponds to the orthonormal basis $\{\widetilde{\Phi}_i\}_{i=1}^d$ of $\mathcal{H}$. We have obtained from the preceding considerations the following result:

3.3 Lemma *If $\mathcal{G}$ acts irreducibly on $\mathcal{H}$ and if $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, then $\mathcal{H}' \subset \mathfrak{H}$ is a $\mathcal{G}$–invariant algebraic Hilbert space iff $\mathcal{H}' = \mathcal{H}Z$, where $Z \in \mathcal{Z}$ is unitary.*

3.4 Remark Using the right Hilbert $\mathcal{Z}$–module $\{\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}}\}$ one can construct canonically a continuous field of Hilbert spaces, a so–called "Dixmier field" [19, Chapter 10]. Recall that according to Gelfand's Theorem $\mathcal{Z} \cong \mathrm{C}(\operatorname{spec} \mathcal{Z})$. For $\varphi \in \operatorname{spec} \mathcal{Z}$ we put

$$\mathcal{N}(\varphi) := \left\{ H \in \mathfrak{H} \colon \langle H, H \rangle_{\mathfrak{H}}(\varphi) = 0 \right\}.$$

Denoting by $\widetilde{H}(\varphi)$ the coset in $\mathfrak{H} / \mathcal{N}(\varphi)$ associated to $H \in \mathfrak{H}$, we consider for a fixed $\varphi \in \operatorname{spec} \mathcal{Z}$ the space

$$\widetilde{\mathfrak{H}}(\varphi) := \left\{ \widetilde{H}(\varphi) \in \mathfrak{H} / \mathcal{N}(\varphi) \colon H \in \mathfrak{H} \right\},$$

as a pre Hilbert space with scalar product given by

$$\left\langle \widetilde{H_1}(\varphi), \widetilde{H_2}(\varphi) \right\rangle_{\varphi} := \langle H_1, H_2 \rangle_{\mathfrak{H}}(\varphi) \,, \quad H_1, H_2 \in \mathfrak{H} \,.$$

Denote by $\mathbb{H}(\varphi)$ the completion of $\widetilde{\mathfrak{H}}(\varphi)$, i.e. $\mathbb{H}(\varphi) := \operatorname{clo}_{\|\cdot\|_{\varphi}} \left(\widetilde{\mathfrak{H}}(\varphi) \right)$. Then it can be shown that the pair $\left(\prod\limits_{\varphi \in \operatorname{spec} \mathcal{Z}} \mathbb{H}(\varphi), \prod\limits_{\varphi \in \operatorname{spec} \mathcal{Z}} \widetilde{\mathfrak{H}}(\varphi) \right)$, is a continuous field of Hilbert spaces.

4 The canonical functor

In this section we will show that the bijection between $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, $\mathcal{H}$ being $\mathcal{G}$–invariant, and $\rho_{\mathfrak{H}}$ established in Proposition 3.1 can be extended to a functor from the category of the right Hilbert $\mathcal{Z}$–modules into the category of unital endomorphisms of $\mathcal{A}$.

The first part of this section is concerned with Hilbert $\mathcal{Z}$–modules, $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, where $\mathcal{H}$ is a finite–dimensional algebraic Hilbert space, but not necessarily $\mathcal{G}$–invariant.

Let $\mathfrak{H}_1, \mathfrak{H}_2$ be two such modules. By $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$ we denote the set of all $\mathcal{Z}$–module morphisms form $\mathfrak{H}_1$ into $\mathfrak{H}_2$, i.e. if $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$, then T is linear and satisfies

$$T(H_1 Z) = T(H_1) \, Z \,, \quad H_1 \in \mathfrak{H}_1 \,, Z \in \mathcal{Z} \,.$$

For $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$ there is always an adjoint $T^* \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_2 \to \mathfrak{H}_1)$ such that

$$\langle H_2, T H_1 \rangle_{\mathfrak{H}_2} = \langle T^* H_2, H_1 \rangle_{\mathfrak{H}_1} \,, \quad H_i \in \mathfrak{H}_i \,, i = 1, 2 \,.$$

Indeed, given orthonormal basis $\{\Phi_i\}_{i=1}^{d_1} \subset \mathfrak{H}_1$ and $\{\Psi_j\}_{j=1}^{d_2} \subset \mathfrak{H}_2$, then T is characterized by $\mathfrak{Z} = \left(Z_{ji}\right)_{j,i} \in \mathrm{Mat}_{d_2 \times d_1}(\mathcal{Z})$, via the equation $T(\Phi_i) = \sum_{j=1}^{d_2} \Psi_j\, Z_{ji}$, $i = 1, \ldots, d_1$. In this case, T^* is given by $T^*(\Psi_j) := \sum_{i=1}^{d_1} \Phi_i\, Z_{ji}^*$ and $\mathfrak{Z}^* := \left(Z_{ji}^*\right)_{i,j} \in \mathrm{Mat}_{d_1 \times d_2}(\mathcal{Z})$.

4.1 Definition *Let $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$ be characterized by $\mathfrak{Z} = \left(Z_{ji}\right)_{j,i} \in \mathrm{Mat}_{d_2 \times d_1}(\mathcal{Z})$ as above. Then we define*

$$\widehat{T} := \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \Psi_j\, Z_{ji}\, \Phi_i^* \in \mathcal{F}.$$

4.2 Proposition *The assignment, $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2) \ni T \longmapsto \widehat{T} \in \mathcal{F}$, with $\widehat{T}$ given in the preceding definition, satisfies the following properties:*

(i) $T(H_1) = \widehat{T} H_1$, $\quad H_1 \in \mathfrak{H}_1$.

(ii) $\widehat{T} = 0$ *implies* $T = 0$ *(injectivity)*.

(iii) $\widehat{T^*} = \left(\widehat{T}\right)^*$.

(iv) *If $T_{12} \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_2 \to \mathfrak{H}_1)$ and $T_{23} \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_3 \to \mathfrak{H}_2)$, then we have $T_{12} \circ T_{23} \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_3 \to \mathfrak{H}_1)$ and $\widehat{T_{12} \circ T_{23}} = \widehat{T_{12}} \cdot \widehat{T_{23}}$.*

(v) $\|T\|_{\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)} = \|\widehat{T}\|_{\mathcal{F}}$.

Proof: (i) Denote by $\{\Phi_i\}_{i=1}^{d_1}$ and $\{\Psi_j\}_{j=1}^{d_2}$ orthonormal basis of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then the equation

$$\widehat{T} \cdot \Phi_{i_0} = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \Psi_j\, Z_{ji}\, \Phi_i^* \Phi_{i_0} = T(\Phi_{i_0}),$$

holds for all $i_0 = 1, \ldots, d_1$.

(ii) Suppose that $\widehat{T} := \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \Psi_j\, Z_{ji}\, \Phi_i^* = 0$. Then, multiplying from the left with $\Psi_{j_0}^*$ and form the right with Φ_{i_0} we get $Z_{j_0 i_0} = 0$ for all $j_0 = 1, \ldots, d_2$, $i_0 = 1, \ldots, d_1$, and therefore $T = 0$.

(iii) Recall that if T is realized by the matrix $\mathfrak{Z} = \left(Z_{j,i}\right)_{j,i}$, then T^* is given by the matrix $\mathfrak{Z}^* = \left(Z_{ji}^*\right)_{i,j}$, so that

$$\left(\widehat{T}\right)^* = \left(\sum_{i,j} \Psi_j\, Z_{ji}\, \Phi_i^*\right)^* = \sum_{i,j} \Phi_i\, Z_{ji}^*\, \Psi_j^* = \widehat{T^*}.$$

(iv) Add to the orthonormal basis introduced in (i) the orthonormal basis $\{\Omega_k\}_{k=1}^{d_3}$ of $\mathfrak{H}_3$. Denoting the matrices of T_{12} and T_{23} by $\mathfrak{Z}_{12} = \left(Z_{ij}^{(12)}\right)_{i,j}$ and $\mathfrak{Z}_{23} = \left(Z_{jk}^{(23)}\right)_{j,k}$, respectively, we get that the matrix of $T_{12} \circ T_{23}$ is given by

$$\mathfrak{Z}_{12}\, \mathfrak{Z}_{23} = \left(\sum_{j=1}^{d_2} Z_{ij}^{(12)} Z_{jk}^{(23)}\right)_{i,k} \in \mathrm{Mat}_{d_1 \times d_3}(\mathcal{Z}).$$

Then we calculate

$$
\begin{aligned}
\widehat{T_{12}} \cdot \widehat{T_{23}} &= \left(\sum_{i,j} \Phi_i \, Z_{ij}^{(12)} \, \Psi_j^* \right) \cdot \left(\sum_{j',k} \Psi_{j'} \, Z_{j'k}^{(23)} \, \Omega_k^* \right) \\
&= \sum_{i,k} \Phi_i \left(\sum_j Z_{ij}^{(12)} Z_{jk}^{(23)} \right) \Omega_k^* \\
&= \widehat{T_{12} \circ T_{23}} \, .
\end{aligned}
$$

(v) Put $\mathfrak{H}_1 = \mathfrak{H}_2 =: \mathfrak{H}$. Then from (ii)–(iv) it follows that $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H})$ and $\widehat{\mathcal{L}_{\mathcal{Z}}(\mathfrak{H})} \subset \mathcal{F}$ are *–isomorphic *–algebras. Both algebras are C*–algebras with C*–norms $\| \cdot \|_{\mathcal{L}_{\mathcal{Z}}(\mathfrak{H})}$ and $\| \cdot \|_{\mathcal{F}}$, respectively. Therefore the isomorphy implies that the C*–norms coincide, i.e. $\|T\|_{\mathcal{L}_{\mathcal{Z}}(\mathfrak{H})} = \|\widehat{T}\|_{\mathcal{F}}$, $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H})$. In the general case we have that if $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$, then $T^*T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1)$ and $\widehat{T^*T} = \left(\widehat{T} \right)^* \widehat{T}$. Therefore,

$$
\|T\|^2_{\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)} = \|T^*T\|_{\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1)} = \|\widehat{T}^*\widehat{T}\|_{\mathcal{F}} = \|\widehat{T}\|^2_{\mathcal{F}}
$$

and the proof is concluded. ∎

In the following we restrict again to the case where $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, with $\mathcal{H}$ an $\mathcal{G}$–invariant algebraic Hilbert space. From Proposition 3.2 we know that, in this case, $\mathfrak{H}$ is stab $\mathcal{A}$–invariant.

Recall that $g \in \mathcal{G}$ acts on $\mathcal{H}$ as a unitary operator $U_{\mathcal{H}}(g)$, i.e. $g {\restriction} \mathcal{H} = U_{\mathcal{H}}(g) \in \mathcal{L}(\mathcal{H})$ and if an orthonormal basis $\{\Phi_i\}_{i=1}^d$ is given, then the representation $U_{\mathcal{H}}$ of $\mathcal{G}$ in $\mathcal{H}$ is specified by a scalar unitary $d \times d$–matrix, $\left(U_{i'i}(g) \right)_{i',i} \in \mathrm{Mat}_d(\mathbb{C})$:

$$
g(\Phi_i) = \sum_{i'=1}^d \Phi_{i'} \, U_{i'i}(g) \, .
$$

In analogy we consider next those $\mathcal{Z}$–module morphisms which are $\mathcal{G}$–invariant.

4.3 Definition *Let $\mathfrak{H}_i = \mathcal{H}_i \mathcal{Z}$, $i = 1, 2$, be Hilbert $\mathcal{Z}$–modules, where the associated algebraic Hilbert spaces $\mathcal{H}_i$ are $\mathcal{G}$–invariant. Denote by $U_{\mathcal{H}_i}(g)$, $g \in \mathcal{G}$, the corresponding unitary representations on $\mathcal{H}_i$. Then the subset $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2 ; \mathcal{G}) \subset \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$ is defined as the set of all intertwining operators:*

$$
\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2 ; \mathcal{G}) := \{T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2) \colon U_{\mathcal{H}_2}(g) \circ T = T \circ U_{\mathcal{H}_1}(g) \, , \quad g \in \mathcal{G}\}.
$$

4.4 Lemma *Let $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$. Then*

$$
T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2 ; \mathcal{G}) \quad \textit{iff} \quad g\Big(T(H_1) \Big) = T\Big(g(H_1) \Big) \qquad g \in \mathcal{G} \, , \, H_1 \in \mathfrak{H}_1 \, .
$$

Proof: Obvious, since the following equations,

$$
g\Big(T(H_1) \Big) = U_{\mathcal{H}_2}(g)\Big(T(H_1) \Big) = \Big(U_{\mathcal{H}_2}(g) \circ T \Big)(H_1)
$$

and

$$
T\Big(g(H_1) \Big) = T\Big(U_{\mathcal{H}_1}(g)(H_1) \Big) = \Big(T \circ U_{\mathcal{H}_1}(g) \Big)(H_1),
$$

hold for all $H_1 \in \mathcal{H}_1$. ∎

4.5 Proposition *Let $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$. Then,*

$$T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2\,;\,\mathcal{G}) \quad iff \quad \widehat{T} \in \mathcal{A}. \tag{27}$$

Even more we have

$$\left\{ \widehat{T} \colon\, T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2\,;\,\mathcal{G}) \right\} = (\rho_{\mathfrak{H}_1},\, \rho_{\mathfrak{H}_2}), \tag{28}$$

i.e. the mapping $T \longmapsto \widehat{T}$ exhausts the whole intertwiner space $(\rho_{\mathfrak{H}_1},\, \rho_{\mathfrak{H}_2})$.

Proof: First, let $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2\,;\,\mathcal{G})$. Then, according to Lemma 4.4 we have $T\big(g(H_1)\big) = g\big(T(H_1)\big)$ for all $g \in \mathcal{G}$, $H_1 \in \mathfrak{H}_1$. From Proposition 4.2 (i) we get further

$$\widehat{T} \cdot g(H_1) = T\big(g(H_1)\big) = g\big(T(H_1)\big) = g\big(\widehat{T} \cdot H_1\big) = g\big(\widehat{T}\big) \cdot g(H_1)\,,$$

so that $g\big(\widehat{T}\big) = \widehat{T}$ and, therefore, $\widehat{T} \in \mathcal{A}$. Second, if $g\big(\widehat{T}\big) = \widehat{T}$ for all $g \in \mathcal{G}$ we have

$$g\big(T(H_1)\big) = g\big(\widehat{T} \cdot H_1\big) = g\big(\widehat{T}\big) \cdot g(H_1) = \widehat{T} \cdot g(H_1) = T\big(g(H_1)\big)\,,$$

which by Lemma 4.4 implies that $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2\,;\,\mathcal{G})$.

To prove Eq. (28) let $\{\Phi_i\}_{i=1}^{d_1}$ and $\{\Psi_j\}_{j=1}^{d_2}$ orthonormal basis of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then

$$\widehat{T} := \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \Psi_j\, Z_{ji}\, \Phi_i^* \in \mathcal{F}\,,$$

where $\mathfrak{Z} = \big(Z_{ji}\big)_{j,i} \in \mathrm{Mat}_{d_2 \times d_1}(\mathcal{Z})$ denotes the matrix corresponding to $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2\,;\,\mathcal{G})$. But from the definition of the canonical endomorphisms we have for $A \in \mathcal{A}$:

$$
\begin{aligned}
\widehat{T}\,\rho_{\mathfrak{H}_1}(A) &= \left(\sum_{i,j} \Psi_j\, Z_{ji}\, \Phi_i^*\right)\left(\sum_{i'} \Phi_{i'}\, A\, \Phi_{i'}^*\right) \\
&= \left(\sum_{i,j} \Psi_j\, Z_{ji}\, A\, \Phi_i^*\right) \\
&= \left(\sum_{i,j} \Psi_j\, A \left(\sum_{j'} \Psi_j^* \Psi_{j'}\right) Z_{j'i}\, \Phi_i^*\right) \\
&= \left(\sum_j \Psi_j\, A\, \Psi_j^*\right)\left(\sum_{i,j'} \Psi_{j'}\, Z_{j'i}\, \Phi_i^*\right) \\
&= \rho_{\mathfrak{H}_2}(A)\, \widehat{T}\,.
\end{aligned}
$$

On the other hand if $B \in (\rho_{\mathfrak{H}_1},\, \rho_{\mathfrak{H}_2})$, for all $A \in \mathcal{A}$, $i = 1, \ldots, d_1$, $j = 1, \ldots, d_2$ we have that

$$\Psi_j^*\, B\, \Phi_i\, A = \Psi_j^*\, B\, \rho_{\mathfrak{H}_1}(A)\, \Phi_i = \Psi_j^*\, \rho_{\mathfrak{H}_2}(A)\, B\, \Phi_i = A\, \Psi_j^*\, B\, \Phi_i\,,$$

and $\Psi_j^*\, B\, \Phi_i \in \mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$. Putting $Z_{ji} := \Psi_j^*\, B\, \Phi_i \in \mathcal{Z}$ and using the support properties of the spaces generated by $\{\Phi_i\}_{i=1}^{d_1}$ and $\{\Psi_j\}_{j=1}^{d_2}$ we obtain

$$B := \sum_{i,j} \Psi_j\, Z_{ji}\, \Phi_i^*\,.$$

Therefore, denoting by T the operator in $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2)$ which is characterized by the matrix $\mathfrak{Z} = \left(Z_{ji} \right)_{j,i} \in \mathrm{Mat}_{d_2 \times d_1}(\mathcal{Z})$ we have from the last equation that $\widehat{T} = B$. Since $B \in \mathcal{A}$ we even know from the first part of the proposition that $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2 \,;\, \mathcal{G})$. ∎

We have the following direct consequence of the preceding result

4.6 Corollary *If $T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H})$, then*

$$T \circ U_{\mathcal{H}}(g) = U_{\mathcal{H}}(g) \circ T \,, \ g \in \mathcal{G} \quad \textit{iff} \quad \widehat{T} \in (\rho_{\mathfrak{H}},\, \rho_{\mathfrak{H}}),$$

i.e. T is an intertwiner of the representation $U_{\mathcal{H}}(g)$ on $\mathcal{H}$ iff $\widehat{T}$ is an intertwiner of the canonical endomorphism, $\rho_{\mathfrak{H}}$.

As it was announced at the beginning of this section we can now extend the mapping

$$\rho_{\mathfrak{H}} \longmapsto \mathfrak{H}, \quad \mathfrak{F}(\rho_{\mathfrak{H}}) := \mathfrak{H}$$

to a functor by means of the assignment

$$(\rho_{\mathfrak{H}_1},\, \rho_{\mathfrak{H}_2}) \ni A \longmapsto \mathfrak{F}(A) \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1 \to \mathfrak{H}_2 \,;\, \mathcal{G}),$$

where

$$\big(\mathfrak{F}(A) \big)(H_1) := A \cdot H_1 \,.$$

In other words, $\mathfrak{F}(A)$ can be characterized, once the orthonormal basis are chosen in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, by the matrix

$$\mathfrak{Z} = \left(Z_{ji} \right)_{j,i} \in \mathrm{Mat}_{d_2 \times d_1}(\mathcal{Z}) \,,$$

that satisfies the equation

$$\mathfrak{Z} \circ U_{\mathcal{H}_1}(g) = U_{\mathcal{H}_2}(g) \circ \mathfrak{Z} \,, \quad g \in \mathcal{G} \,.$$

In particular, since $\mathcal{Z} \subseteq (\rho_{\mathfrak{H}},\, \rho_{\mathfrak{H}})$ (cf. with Eq. (8)), for each $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, where $\mathcal{H}$ is $\mathcal{G}$–invariant, we have that $ZH \in \mathfrak{H}$ for all $H \in \mathfrak{H}$. This means that for each $\widetilde{Z} \in \mathcal{Z}$ there corresponds a matrix $\widetilde{\mathfrak{Z}} = \left(\widetilde{Z}_{i'i} \right)_{i',i} \in \mathrm{Mat}_d(\mathcal{Z})$, such that

$$\widetilde{Z}\, \Phi_i = \sum_{i'=1}^{d} \Phi_{i'} \widetilde{Z}_{i'i} \,. \tag{29}$$

Therefore, the tensor product of two Hilbert $\mathcal{Z}$–modules $\mathfrak{H}_1$ and $\mathfrak{H}_2$,

$$\mathfrak{H}_1\, \mathfrak{H}_2 := \mathrm{span}_{\mathcal{Z}} \left(\mathfrak{H}_1 \cdot \mathfrak{H}_2 \right) = \mathrm{span}_{\mathcal{Z}} \left\{ H_1 H_2 \colon\ H_k \in \mathfrak{H}_k,\ k = 1, 2 \right\},$$

is again a Hilbert $\mathcal{Z}$–module. Indeed, this follows from the computation:

$$(H_1 H_2)^* H_1' H_2' = H_2^* \left\langle H_1,\, H_1' \right\rangle_{\mathfrak{H}_1} H_2' = \left\langle H_2,\, \left\langle H_1,\, H_1' \right\rangle_{\mathfrak{H}_1} H_2' \right\rangle_{\mathfrak{H}_2} \in \mathcal{Z} \,,$$

where $H_k, H_k' \in \mathfrak{H}_k,\ k = 1, 2$. With orther words $\mathfrak{H}_1\, \mathfrak{H}_2$ is the inner tensor product of the Hilbert $\mathcal{Z}$–modules $\mathfrak{H}_1$ and $\mathfrak{H}_2$ w.r.t. *–homomorphism $\mathcal{Z} \to \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_2)$ defined in Eq. (29) (see also [20]). Obviously, we have $\mathfrak{H}_1\, \mathfrak{H}_2 = (\mathcal{H}_1 \mathcal{H}_2)\mathcal{Z}$, where $\mathcal{H}_1 \mathcal{H}_2$ denotes the $\mathbb{C}$–tensor product, $\mathrm{span}_{\mathbb{C}}(\mathcal{H}_1 \cdot \mathcal{H}_2)$, of $\mathcal{H}_1$ and $\mathcal{H}_2$.

5 Irreducible endomorphisms

In the present section we will determine the intertwiner space $(\rho_\mathfrak{H}, \rho_\mathfrak{H})$, where $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ and the algebraic Hilbert space $\mathcal{H}$ is invariant and irreducible w.r.t. $\mathcal{G}$.

5.1 Theorem *Let $\mathfrak{H}$ be as described above. Then the equation*

$$(\rho_\mathfrak{H}, \rho_\mathfrak{H}) = \rho_\mathfrak{H}(\mathcal{Z}), \tag{30}$$

holds.

Proof: The inclusion $\rho_\mathfrak{H}(\mathcal{Z}) \subseteq (\rho_\mathfrak{H}, \rho_\mathfrak{H})$ follows from Eq. (9).

To prove the other inclusion suppose that $A \in (\rho_\mathfrak{H}, \rho_\mathfrak{H})$. Then from the relation (28), there exists $T \in \mathcal{L}_\mathcal{Z}(\mathfrak{H} \to \mathfrak{H}; \mathcal{G})$, such that $A = \widehat{T}$. According to Corollary 4.6 this means that

$$T \circ U_\mathcal{H}(g) = U_\mathcal{H}(g) \circ T, \quad g \in \mathcal{G},$$

where $U_\mathcal{H}$ is the unitary representation of $\mathcal{G}$ on $\mathcal{H}$ (cf. with the paragraph before Definition 4.3). Choosing an orthonormal basis $\{\Phi_i\}_{i=1}^d$ of $\mathcal{H}$, we can rewrite the preceding equation as

$$\sum_{i'=1}^d Z_{ii'} U_{\mathcal{H}, i'i''}(g) = \sum_{i'=1}^d U_{\mathcal{H}, ii'}(g) Z_{i'i''}, \quad g \in \mathcal{G}, \ i, i'' = 1, \dots, d, \tag{31}$$

where $\left(Z_{ii'}\right)_{i,i'} \in \mathrm{Mat}_d(\mathcal{Z})$ is the matrix characterizing T and $\left(U_{\mathcal{H}, ii'}(g)\right)_{i,i'}$ corresponds to $U_\mathcal{H}(g)$. Therefore, by Gelfand's Theorem, we can associate to each $Z_{ii'} \in \mathcal{Z}$ a continuous function $Z_{ii'}(\cdot) \in \mathrm{C}(\mathrm{spec}\,\mathcal{Z})$, satisfying $Z_{ii'}(\varphi) = \varphi(Z_{ii'})$, $\varphi \in \mathrm{spec}\,\mathcal{Z}$. From Eq. (31) we obtain

$$\sum_{i'=1}^d Z_{ii'}(\varphi) U_{\mathcal{H}, i'i''}(g) = \sum_{i'=1}^d U_{\mathcal{H}, ii'}(g) Z_{i'i''}(\varphi), \quad \varphi \in \mathrm{spec}\,\mathcal{Z}, g \in \mathcal{G}, \ i, i'' = 1, \dots, d.$$

Denoting by $T(\varphi) \in \mathcal{L}(\mathcal{H})$ the operator whose scalar matrix w.r.t. the orthonormal basis $\{\Phi_i\}_{i=1}^d$ is $\left(Z_{ii'}(\varphi)\right)_{i,i'}$, we get from the preceding equation

$$T(\varphi) \circ U_\mathcal{H}(g) = U_\mathcal{H}(g) \circ T(\varphi), \quad \varphi \in \mathrm{spec}\,\mathcal{Z}, g \in \mathcal{G}.$$

But $U_\mathcal{H}(\mathcal{G})$ is irreducible, hence

$$T(\varphi) = c(\varphi)\mathbb{1}_\mathcal{H},$$

follows, where $c(\cdot) \in \mathrm{C}(\mathrm{spec}\,\mathcal{Z})$. Again, by Gelfand's Theorem, the function $c(\cdot)$ can be associated to an element $Z_0 \in \mathcal{Z}$, such that $c(\varphi) = Z_0(\varphi)$. We obtain from this $Z_{ii'} = Z_0\,\delta_{ii'}$ or

$$T = Z_0\,\mathbb{1}_\mathcal{H}.$$

But, since $A = \widehat{T}$ we get from the last equation

$$A = \sum_{i,i'=1}^d \Phi_i\,Z_{ii'}\,\Phi_{i'}^* = \sum_{i,i'=1}^d \Phi_i\,Z_0\,\delta_{ii'}\,\Phi_{i'}^* = \sum_{i=1}^d \Phi_i\,Z_0\,\Phi_i^* = \rho_\mathfrak{H}(Z_0)$$

and the proof is concluded. ∎

5.2 Corollary *If $\mathcal{H}$ is irreducible, then the inclusion*

$$\mathcal{Z} \subseteq \rho_\mathcal{H}(\mathcal{Z}),$$

holds.

Proof: Use Eq. (8) and the preceding theorem. $\blacksquare$

We have therefore the following relations for the canonical endomorphism, $\rho_D \equiv \rho_{\mathfrak{H}_D}$, with $\mathfrak{H}_D = \mathcal{H}_D \mathcal{Z}$, $D \in \widehat{\mathcal{G}}$:

$$
\begin{aligned}
(\rho_D, \rho_{D'}) &= \{0\}, \quad D \neq D', \quad D, D' \in \widehat{\mathcal{G}} \\
(\rho_D, \rho_D) &= \rho_D(\mathcal{Z}).
\end{aligned}
$$

Motivated by Theorem 5.1 we introduce the notion of an irreducible endomorphism, which is independent of the group $\mathcal{G}$.

5.3 Definition *An arbitrary endomorphism $\lambda \in \operatorname{end} \mathcal{A}$ is said to be irreducible if $(\lambda, \lambda) = \lambda(\mathcal{Z})$.*

Note that this definition coincides with the usual notion of irreduciblility in the case where $\mathcal{Z} = \mathbb{C} \mathbb{1}$ (see for instance [17]).

5.4 Proposition *If $\rho_{\mathfrak{H}} \in \operatorname{end} \mathcal{A}$ is irreducible, then $\rho_{\mathfrak{H}}^{-1}$ can be considered on $\mathcal{Z}$ and the inclusion*

$$
\rho_{\mathfrak{H}}^{-1}(\mathcal{Z}) \subseteq \mathcal{Z},
$$

holds, i.e. $\rho_{\mathfrak{H}}^{-1} \!\restriction\! \mathcal{Z} \in \operatorname{end} \mathcal{Z}$.

Proof: From the existence of a left inverse (cf. Eq. (20)) it follows that $\rho_{\mathfrak{H}}$ is injective. Then, Corollary 5.2 ends the proof. $\blacksquare$

5.5 Remark According to the preceding Proposition we have that if the endomorphism $\lambda := \rho_{\mathfrak{H}}$ is irreducible, then $\lambda^{-1} \!\restriction\! \mathcal{Z}$ is a unital injective endomorphism of $\mathcal{Z}$. But according to Gelfand's Theorem the category of unital abelian C*–algebras and their *–homomorphisms and the category that is opposite to the category of compact topological spaces and their continuous maps are isomorphic (see e.g. [21, Chapter IV]).

This means that to any unital endomorphism $\lambda^{-1} \!\restriction\! \mathcal{Z}$ there corresponds a continuous mapping

$$
f_\lambda : \operatorname{spec} \mathcal{Z} \longrightarrow \operatorname{spec} \mathcal{Z},
$$

such that

$$
\left(\lambda^{-1}(Z)\right)(\varphi) = Z\left(f_\lambda(\varphi)\right), \quad Z \in \mathcal{Z}, \ \varphi \in \operatorname{spec} \mathcal{Z},
$$

where f_λ is surjective in our case.

$$
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\ \lambda^{-1}\ } & \mathcal{Z} \\
\downarrow & & \downarrow \\
\mathrm{C}(\operatorname{spec} \mathcal{Z}) & & \mathrm{C}(\operatorname{spec} \mathcal{Z}) \\
\uparrow & & \uparrow \\
\operatorname{spec} \mathcal{Z} & \xleftarrow{\ f_\lambda\ } & \operatorname{spec} \mathcal{Z}
\end{array}
$$

From the preceding comments we can divide the irreducible endomorphisms $\lambda := \rho_{\mathfrak{H}}$ into two different families: the first one is characterized by the fact that $\lambda^{-1} \!\restriction\! \mathcal{Z}$ is also a surjective mapping. In this case the equations

$$
\mathcal{Z} = \lambda(\mathcal{Z}) = (\lambda, \lambda)
$$

hold. For the second family $\lambda^{-1}{\restriction}\mathcal{Z}$ is not surjective and the following chain of proper inclusions can be easily established

$$\ldots \subset \left(\lambda^{-1}\right)^n (\mathcal{Z}) \subset \left(\lambda^{-1}\right)^{(n-1)} (\mathcal{Z}) \subset \ldots \subset \lambda^{-1}(\mathcal{Z}) \subset \mathcal{Z} \,.$$

If $\mathcal{Z}$ is finite–dimensional only the first type of endomorphisms will appear.

The continuous mappings $f_\lambda \equiv f_{\rho_{\mathfrak{H}}}$, $\mathfrak{H}$ irreducible, are essential ingredients of the Hilbert C*–system $\{\mathcal{F}, \mathcal{G}\}$. Roughly, the function f_λ reflects, at the level of the spectrum, the action of an irreducible endomorphism λ on $\mathcal{Z}$.

5.6 Example A simple example that illustrates the present situation is constructed as follows: let Ω be a compact topological space and $\mathcal{B} := \mathrm{CAR}(h, \Gamma)$ the C*–algebra of the canonical anticommutation relations over an infinite–dimensional Hilbert space h with an antiunitary involution Γ [22]. Define the C*–algebra

$$\mathcal{A} := \mathrm{C}(\Omega, \mathcal{B}) = \{f\colon \Omega \longrightarrow \mathcal{B}\colon f \text{ is continuous}\}$$

with the natural operations and C*–norm. From $\mathcal{Z}(\mathcal{B}) = \mathbb{C}\mathbb{1}$ we obtain that

$$\mathcal{Z}(\mathcal{A}) = \mathrm{C}(\Omega, \mathbb{C}) \supset \mathbb{C}\mathbb{1} \,.$$

Define next the automorphism $\gamma \in \mathrm{aut}\,\mathcal{A}$ as $(\gamma f)(p) = -f(p)$, $p \in \Omega$, $f \in \mathcal{A}$. The field algebra $\mathcal{F}$ is constructed using the automorphism γ as in [17, Section 3.6] and since $(\mathrm{id}, \gamma) = \{0\}$ we obtain $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}(\mathcal{A})$. The automorphism γ is irreducible and satisfies

$$\gamma\big(\mathcal{Z}(\mathcal{A})\big) = \mathcal{Z}(\mathcal{A}) = (\gamma, \gamma) \,.$$

The group in this example is $\mathcal{G} = \mathbb{Z}_2 = \{\mathrm{id}, \alpha\}$ ($\alpha \in \mathrm{aut}\,\mathcal{F}$ satisfies $\alpha^2 = \mathrm{id}$) and $\widehat{\mathcal{G}} = \{\iota, \chi\} \cong \mathbb{Z}_2$.

6 Decomposition of $\mathfrak{H}$ in terms of $\mathfrak{H}_D$, $D \in \widehat{\mathcal{G}}$

As at the beginning of Section 3, we consider a Hilbert $\mathcal{Z}$–module $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, where the algebraic Hilbert space $\mathcal{H}$ is $\mathcal{G}$–invariant. Denote the associated canonical endomorphism by $\lambda := \rho_{\mathfrak{H}}$. Further, we will need the quantities $\mathfrak{H}_D = \mathcal{H}_D\mathcal{Z}$ and ρ_D associated to irreducible $D \in \widehat{\mathcal{G}}$ and defined in Section 2. As before, $U_{\mathcal{H}}(\mathcal{G})$ and $U_D(\mathcal{G}) \equiv U_{\mathcal{H}_D}(\mathcal{G})$ are the unitary representations of $\mathcal{G}$ on $\mathcal{H}$ and on $\mathcal{H}_D$, respectively. U_D is irreducible.

6.1 Proposition *With the notation introduced above the following properties are true:*

(i) *The intertwiner space $(\rho_D\,, \lambda)$ is a Hilbert $\rho_D(\mathcal{Z})$–module, $D \in \widehat{\mathcal{G}}$.*

(ii) *For $D, D' \in \widehat{\mathcal{G}}$ and $D \neq D'$ the Hilbert modules $(\rho_D\,, \lambda)$ and $(\rho_{D'}\,, \lambda)$ are mutually orthogonal.*

(iii) *The Hilbert $\rho_D(\mathcal{Z})$–module $(\rho_D\,, \lambda)$ is a free module. There exists an orthonormal basis $\{C_{D,l}\}_{l=1}^{m(D)}$, where $m(D)$ denotes the multiplicity of $D \in \widehat{\mathcal{G}}$ in the decomposition of $U_{\mathcal{H}}$ as a direct sum of irreducible representations. Further, the following equation holds:*

$$\mathrm{supp}\,(\rho_D\,, \lambda) = \sum_{l=1}^{m(D)} C_{D,l}\, C_{D,l}^* = P_D \,,$$

where P_D is the uniquely determined isotypic projection belonging to D in the mentioned decomposition.

Proof: (i) Let $A, B \in (\rho_D, \lambda)$. Since the endomorphisms ρ_D are irreducible, we have that

$$A^* B \in (\rho_D, \rho_D) = \rho_D(\mathcal{Z}).$$

(ii) Let $A \in (\rho_D, \lambda)$ and $B \in (\rho_{D'}, \lambda)$ with $D, D' \in \widehat{\mathcal{G}}$ and $D \neq D'$. From Eq. (7) we get

$$A^* B \in (\rho_{D'}, \rho_D) = \{0\}.$$

(iii) First we decompose $U_{\mathcal{H}}$ on $\mathcal{H}$ into a direct sum of irreducible components. For $D \in \widehat{\mathcal{G}}$ we write explicitly

$$P_D = \sum_{l=1}^{m(D)} E_{D,l}, \tag{32}$$

where the orthonormal family of projections $\{E_{D,l}\}_{l=1}^{m(D)}$ satisfy $E_{D,l} \circ U_{\mathcal{H}}(g) = U_{\mathcal{H}}(g) \circ E_{D,l}$, $g \in \mathcal{G}$, and the subspaces $\{E_{D,l} \mathcal{H}\}_l$ are irreducible. Then, according to Corollary 4.6 we have that $\widehat{E_{D,l}} \in (\lambda, \lambda)$. We denote by

$$\{\Psi_{D,i,l}: D \in \widehat{\mathcal{G}}, \; i = 1, \dots, d, \; l = 1, \dots, m(D)\}$$

an adapted orthonormal basis of $\mathcal{H}$ w.r.t the decomposition specified in (32). Further, choose an orthonormal basis $\{\Phi_{D,i}\}_{i=1}^{d}$ of $\mathcal{H}_D$ and put

$$C_{D,l} := \sum_{i=1}^{d} \Psi_{D,i,l} \, \Phi_{D,i}^*. \tag{33}$$

We will show that $C_{D,l} \in (\rho_D, \lambda)$, $D \in \widehat{\mathcal{G}}$, $l = 1, \dots, m(D)$. Indeed, note first that $g(C_{D,l}) = C_{D,l}$, $g \in \mathcal{G}$, so that $C_{D,l} \in \mathcal{A}$. Moreover, we obtain for all $A \in \mathcal{A}$

$$
\begin{aligned}
C_{D,l}\, \rho_D(A) &= \sum_{i=1}^{d} \Psi_{D,i,l} \, \Phi_{D,i}^* \, \rho_D(A) \\
&= \sum_{i=1}^{d} \Psi_{D,i,l} \, A \, \Phi_{D,i}^* \\
&= \lambda(A) \sum_{i=1}^{d} \Psi_{D,i,l} \, \Phi_{D,i}^* \\
&= \lambda(A)\, C_{D,l},
\end{aligned}
$$

where for the third equation we have used Proposition 3.1 and the fact that $\Psi_{D,i,l} \in \mathcal{H}$.

Next, we obtain by a direct computation that $C_{D,l}^* C_{D,l'} = 0$, for $l \neq l'$:

$$
\begin{aligned}
C_{D,l}^* C_{D,l'} &= \left(\sum_{i=1}^{d} \Phi_{D,i} \, \Psi_{D,i,l}^* \right) \cdot \left(\sum_{i'=1}^{d} \Psi_{D,i',l'} \, \Phi_{D,i'}^* \right) \\
&= \sum_{i,i'} \Phi_{D,i} \, \Psi_{D,i,l}^* \Psi_{D,i',l'} \, \Phi_{D,i'}^* \\
&= 0,
\end{aligned}
$$

because the family of projections $\{E_{D,l}\}_l$ are mutually orthogonal and, therefore, the equation $\Psi_{D,i,l}^* \Psi_{D,i',l'} = 0$ holds for $l \neq l'$. Note that from Eq. (33) we get

$$\Psi_{D,i,l} = C_{D,l} \Phi_{D,i}. \tag{34}$$

Now we have to prove that $\{C_{D,l}\}_{l=1}^{m(D)}$ is a module basis of $(\rho_D\,,\lambda)$. Let $B \in (\rho_D\,,\lambda)$, $D \in \widehat{\mathcal{G}}$. Then we have

$$C_{D,l}^* \, B \, \rho_D(A) = C_{D,l}^* \, \lambda(A) \, B = \rho_D(A) \, C_{D,l}^* \, B\,,$$

i.e. $C_{D,l}^* \, B \in (\rho_D\,,\rho_D) = \rho_D(\mathcal{Z})$. We put

$$C_{D,l}^* \, B =: \rho_D(Z_{D,l})\,, \qquad (35)$$

for some $Z_{D,l} \in \mathcal{Z}$. Furthermore, we have

$$C_{D',l}^* \, B = 0\,, \quad D' \neq D\,, \qquad (36)$$

because $C_{D',l}^* \, B \in (\rho_D\,,\rho_{D'}) = \{0\}$. Moreover, we calculate using Eq. (33)

$$
\begin{aligned}
\sum_{l=1}^{m(D)} C_{D,l} \, C_{D,l}^* &= \sum_{l} \left(\sum_{i=1}^{d} \Psi_{D,i,l} \, \Phi_{D,i}^* \right) \cdot \left(\sum_{i'=1}^{d} \Phi_{D,i'} \, \Psi_{D,i',l}^* \right) \\
&= \sum_{l} \sum_{i,i'} \Psi_{D,i,l} \, \Phi_{D,i}^* \, \Phi_{D,i'} \, \Psi_{D,i',l}^* \\
&= \sum_{l,i} \Psi_{D,i,l} \, \Psi_{D,i,l}^* \\
&= P_D,
\end{aligned}
$$

where we have used the fact that $\{\Psi_{D,i,l}\}_{D,i,l}$ is an adapted basis w.r.t. the decomposition specified in Eq. (32). Note that $\mathrm{supp}\,\mathcal{H} = \sum_{D,l,i} \Psi_{D,i,l} \, \Psi_{D,i',l}^* = 1$. From Eq. (35) we get

$$P_D \, B = \left(\sum_{l=1}^{m(D)} C_{D,l} \, C_{D,l}^* \right) B = \sum_{l} C_{D,l} \, \rho_D(Z_{D,l})$$

and from Eq. (36) we obtain

$$P_{D'} \, B = \left(\sum_{l=1}^{m(D')} C_{D',l} \, C_{D',l}^* \right) B = 0\,,$$

for all $D' \neq D$. From this we finally have

$$B = \sum_{l=1}^{m(D)} C_{D,l} \, \rho_D(Z_{D,l})$$

and the proof is concluded. ∎

6.2 Theorem *With the notation of the beginning of this section, let $D \in \widehat{\mathcal{G}}$. Then $(\rho_D, \lambda)\,\mathfrak{H}_D$ is a Hilbert $\mathcal{Z}$–module. Further, $\mathfrak{H}$ can be decomposed into the following orthogonal direct sum:*

$$\mathfrak{H} = \underset{D}{\oplus}\, (\rho_D,\, \lambda)\, \mathfrak{H}_D\,.$$

Proof: Let $A_k \in (\rho_D, \lambda)$, $X_k \in \mathcal{H}_D$, $Z_k \in \mathcal{Z}$, $k = 1,2$, so that $A_k \, X_k \, Z_k \in (\rho_D, \lambda)\,\mathfrak{H}_D$. According to Proposition 6.1 (i) we have that $A_1^* A_2 = \rho_D(Z)$ for some $Z \in \mathcal{Z}$ and, therefore,

$$
\begin{aligned}
\left(A_1 \, X_1 \, Z_1 \right)^* \left(A_2 \, X_2 \, Z_2 \right) &= Z_1^* \, X_1^* \, A_1^* \, A_2 \, X_2 \, Z_2 \\
&= Z_1^* \, X_1^* \, \rho_D(Z) \, X_2 \, Z_2 \\
&= Z_1^* \, X_1^* \, X_2 \, Z \, Z_2 \\
&= Z_1^* \, \langle X_1, X_2 \rangle_{\mathfrak{H}_D} \, Z \, Z_2 \in \mathcal{Z}\,.
\end{aligned}
$$

The mutual orthogonality of the Hilbert $\mathcal{Z}$–modules $(\rho_D,\,\lambda)\,\mathfrak{H}_D$, $D \in \widehat{\mathcal{G}}$, follows from the mutual orthogonality of the $\rho_D(\mathcal{Z})$–module $(\rho_D,\,\lambda)$ (cf. Proposition 6.1 (ii)).

It remains to show that $\widehat{P_D} \cdot \left(\mathcal{H}\mathcal{Z}\right) = (\rho_D,\,\lambda)\,\mathcal{H}_D\mathcal{Z}$ where, as before, $\mathcal{G}\!\upharpoonright\!\mathcal{H}$ acts by the unitary representation $U_{\mathcal{H}}(\mathcal{G})$ and $P_D \in \mathcal{L}(\mathcal{H})$ denotes the isotypical projection w.r.t. $D \in \widehat{\mathcal{G}}$. Recall that $P_D \circ U_{\mathcal{H}}(g) = U_{\mathcal{H}}(g) \circ P_D$, $g \in \mathcal{G}$, implies $\widehat{P_D} \in (\lambda,\,\lambda) \subset \mathcal{A}$ (cf. Eq. (28)). The family $\{\Phi_{D,i}\}_{i=1}^{d}$ denotes an orthonormal basis of $\mathcal{H}_D$.

First, we prove $\widehat{P_D} \cdot \left(\mathcal{H}\mathcal{Z}\right) \subseteq (\rho_D,\,\lambda)\,\mathcal{H}_D\mathcal{Z}$. For $H \in \mathcal{H}\mathcal{Z}$ we have

$$\widehat{P_D}\,H \in \operatorname{span}\{\mathcal{A}\,\mathcal{H}_D\} = \Pi_D\mathcal{F}$$

and therefore

$$\widehat{P_D}\,H = \sum_{i=1}^{d} A_i\,\Phi_{D,i}\,, \quad A_i \in \mathcal{A}\,.$$

To prove that $A_i \in (\rho_D,\,\lambda)$, $i = 1,\dots,d$, we take $B \in \mathcal{A}$ and put

$$\widehat{P_D}\,H\,B = \sum_{i=1}^{d} A_i\,\Phi_{D,i}\,B = \sum_{i=1}^{d} A_i\,\rho_D(B)\,\Phi_{D,i}\,.$$

On the other hand we get

$$\widehat{P_D}\,H\,B = \widehat{P_D}\,\lambda(B)\,H = \lambda(B)\,\widehat{P_D}\,H = \sum_{i=1}^{d} \lambda(B)\,A_i\,\Phi_{D,i}\,.$$

Recall that $\{\Phi_{D,i}\}_{i=1}^{d}$ is also an $\mathcal{A}$–module basis of $\mathcal{F}_0$. Therefore $\lambda(B)\,A_i = A_i\,\rho_D(B)$ for all $B \in \mathcal{A}$, i.e. $A_i \in (\rho_D,\,\lambda)$.

Second, to prove the other inclusion take $A \in (\rho_D,\,\lambda)$. Then the inclusion $A\,\mathcal{H}_D \subseteq \mathcal{H}\mathcal{Z}$ follows from

$$(A\,\Phi_{D,i})\,B = A\,\rho_D(B)\,\Phi_{D,i} = \lambda(B)\,(A\,\Phi_{D,i})\,, \quad B \in \mathcal{A}\,,$$

so that $A\,\Phi_{D,i} \in \mathcal{H}\mathcal{Z}$ according to Proposition 3.1. Finally, for $g \in \mathcal{G}$ we have

$$
\begin{aligned}
g(A\,\Phi_{D,i}) &= A\,g(\Phi_{D,i}) \\
&= A\left(U_{\mathcal{H}_D}(\Phi_{D,i})\right) \\
&= \sum_{i'=1}^{d} (A\,\Phi_{D,i'})\,U_{D,i'i}(g)\,,
\end{aligned}
$$

so that $A\,\Phi_{D,i}$ transforms according to $D \in \widehat{\mathcal{G}}$ and therefore $A\,\mathcal{H}_D \subseteq \widehat{P_D}\,\mathcal{H}\mathcal{Z}$. ∎

6.3 Remark According to Proposition 3.2 the Hilbert $\mathcal{Z}$–module $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ is $\operatorname{stab}\mathcal{A}$–invariant. This means that for each $\beta \in \operatorname{stab}\mathcal{A}$ and for an orthonormal basis $\{\Psi_i\}_{i=1}^{d}$ of $\mathcal{H}$ we obtain

$$\beta(\Psi_i) = \sum_{i'=1}^{d} \Psi_{i'}\,Z_{i'i}(\beta)\,, \quad Z_{i'i}(\beta) \in \mathcal{Z}\,,$$

i.e. to each $\beta \in \operatorname{stab}\mathcal{A}$ there corresponds a matrix $\mathfrak{Z}(\beta) = \left(Z_{i'i}(\beta)\right)_{i',i} \in \operatorname{Mat}_d(\mathcal{Z})$. From the orthonormality of the basis $\{\Psi_i\}_{i=1}^{d}$ we get immediately that (cf. also with Lemma 7.2)

$$\mathfrak{Z}(\beta)^{*}\mathfrak{Z}(\beta) = \mathbb{1}_d\,,$$

where $\mathfrak{Z}(\beta)^* = \left(Z^*_{i'i}(\beta)\right)_{i,i'}$, $\mathbb{1}_d \in \operatorname{Mat}_d(\mathcal{Z})$. We also have

$$\mathfrak{Z}(\beta_1 \circ \beta_2) = \mathfrak{Z}(\beta_1)\mathfrak{Z}(\beta_2) \quad \text{and} \quad \mathfrak{Z}(\iota) = \mathbb{1}_d \, ,$$

where $\beta_1, \beta_2 \in \operatorname{stab}\mathcal{A}$ and ι denotes the identical automorphism. In particlular, choosing an orthonormal basis $\{\Phi_{D,i}\}_{i=1}^d$ of $\mathcal{H}_D$ we denote the corresponding matrices by

$$\mathfrak{Z}_D = \left(Z_{D,\,i'i}\right)_{i',i} \in \operatorname{Mat}_d(\mathcal{Z}) \, .$$

Choose $D_1, D_2 \in \widehat{\mathcal{G}}$ and consider $\mathcal{H}_{D_1}\mathcal{H}_{D_2}$ as well as the Hilbert $\mathcal{Z}$–module $\mathfrak{H}_{\rho_{D_1} \circ \rho_{D_2}} = \mathfrak{H}_{D_1}\mathfrak{H}_{D_2}$, $\mathfrak{H}_{D_k} = \mathcal{H}_{D_k}\mathcal{Z}$, $k = 1,2$. Recall that the representation $U_{\mathcal{H}_{D_1}\mathcal{H}_{D_2}}(\mathcal{G})$ on $\mathcal{H}_{D_1}\mathcal{H}_{D_2}$ belongs to the class $D_1 D_2$. Let

$$\mathfrak{Z}_{D_1 D_2} = \left(Z_{D_1 D_2,\, i'_1 i'_2 i_1 i_2}\right)_{i'_1,\, i'_2,\, i_1,\, i_2}$$

be the matrix associated to $\beta\big\vert \mathfrak{H}_{D_1}\mathfrak{H}_{D_2}$ w.r.t. orthonormal basis $\{\Phi_{D_1, i_1}\,\Phi_{D_2, i_2}\}_{i_1, i_2}$. According to Theorem 6.2 we have the following decomposition of $\mathfrak{H}_{D_1}\mathfrak{H}_{D_2}$:

$$\mathfrak{H}_{\rho_{D_1} \circ \rho_{D_2}} = \mathfrak{H}_{D_1}\mathfrak{H}_{D_2} = \bigoplus_D \left(\rho_D, \, \rho_{D_1} \circ \rho_{D_2}\right) \mathfrak{H}_D \, .$$

This means that there is an orthonormal basis in $\mathfrak{H}_{D_1}\mathfrak{H}_{D_2}$ of the form $\{C_{D,l}\,\Phi_{D,i}\}_{l,i}$, $l = 1, \ldots, m(D)$, $i = 1, \ldots, d$, $m(D)$ being the multiplicity of $D \in \widehat{\mathcal{G}}$ in the decomposition of $U_{\mathcal{H}_{D_1}\mathcal{H}_{D_2}}(\mathcal{G})$ (cf. with Proposition 6.1 and with Eq. (34)), where $\sum_D d \cdot m(D) = d_1 d_2$. Denote by $\Gamma^{D_1 D_2} = \left(\Gamma^{D_1 D_2}_{i_1 i_2,\, D\, l\, i}\right)_{i_1 i_2,\, D\, l\, i} \in \operatorname{Mat}_{d_1 d_2}(\mathbb{C})$ the corresponding scalar unitary transformation matrix (Clebsch–Gordan matrix)

$$\Phi_{D_1, i_1}\,\Phi_{D_2, i_2} = \sum_D \sum_{l=1}^{m(D)} \sum_{i=1}^{d} \Gamma^{D_1 D_2}_{i_1 i_2,\, D\, l\, i}\, C_{D,l}\,\Phi_{D,i} \, .$$

Then we have

$$\mathfrak{Z}_{D_1 D_2}(\beta) = \Gamma^{D_1 D_2}\left(\operatorname*{diag}_D m(D)\,\mathfrak{Z}_D\right)\left(\Gamma^{D_1 D_2}\right)^{-1} , \tag{37}$$

where $\operatorname{diag}\left(m(D)\,\mathfrak{Z}_D\right) := \operatorname{diag}\underbrace{\left(\mathfrak{Z}_D, \ldots, \mathfrak{Z}_D\right)}_{m(D)\text{–times}}$.

6.4 Remark Note that the expression (13) is *not* independent of the choice of the orthonormal module basis in $\mathfrak{H}$. There is, however, the possibility to define an ε associated to the Hilbert $\mathcal{Z}$–module: let $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ with $\mathcal{H}$ irreducible, i.e. $\rho_{\mathfrak{H}}$ is irreducible. Then, according to Lemma 3.3, the set of all $\mathcal{G}$–invariant algebraic Hilbert spaces $\mathcal{H}' \subset \mathfrak{H}$ is given by $\mathcal{H}' = \mathcal{H}Z$ with $Z \in \mathcal{Z}$ unitary. The first step is to select from each class $\{\mathcal{H}Z\}_{Z \in \mathcal{U}(\mathcal{Z})}$ exactly one representant $\mathcal{H} \subset \mathfrak{H}$.

Now, according to Lemma 6.2 we have for an arbitrary $\mathfrak{H}_\lambda$ with $\lambda = \rho_{\mathfrak{H}}$ the unique decomposition

$$\mathfrak{H}_\lambda = \bigoplus_{D \in \widehat{\mathcal{G}}} \left(\rho_D, \lambda\right) \mathfrak{H}_D \, .$$

Thus, by

$$\mathcal{H}_\lambda := \bigoplus_{D \in \widehat{\mathcal{G}}} \left(\rho_D, \lambda\right) \mathcal{H}_D \, ,$$

where $\mathcal{H}_D$ now denotes the representant in the corresponding class $\{\mathcal{H}_D Z\}_{Z \in \mathcal{U}(\mathcal{Z})}$, we obtain a *unique* $\mathcal{G}$–invariant algebraic Hilbert space and on the basis of this choice we define

$$\varepsilon(\mathfrak{H}_1, \mathfrak{H}_2) := \varepsilon(\mathcal{H}_1, \mathcal{H}_2) \, ,$$

where $\varepsilon(\mathcal{H}_1, \mathcal{H}_2)$ is given by the expression (13). Unfortunately, this definition of $\varepsilon(\mathfrak{H}_1, \mathfrak{H}_2)$ depends on the initial choice of the representants from each class $\{\mathcal{H}Z\}_{Z \in \mathcal{U}(\mathcal{Z})}$.

7 The stabilizer, $\operatorname{stab} \mathcal{A}$

We start with a first characterization of the elements of $\operatorname{stab} \mathcal{A}$. Recall the notions of spectral projection, Π_D, $D \in \widehat{\mathcal{G}}$, and $\mathcal{A}$–scalar product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ introduced in Section 2.

7.1 Lemma *If $\beta \in \operatorname{stab} \mathcal{A}$, then the equation*

$$\beta \circ \Pi_D = \Pi_D \circ \beta, \quad D \in \widehat{\mathcal{G}}, \tag{38}$$

holds and, further, we have

$$\beta \in \operatorname{stab} \mathcal{A} \quad \text{iff} \quad \langle \beta(F_1), \beta(F_2) \rangle_{\mathcal{A}} = \langle F_1, F_2 \rangle_{\mathcal{A}}, \qquad F_1, F_2 \in \mathcal{F}. \tag{39}$$

Proof: We prove first Eq. (38). Let $\beta \in \operatorname{stab} \mathcal{A}$ and take $F \in \Pi_D \mathcal{F}$. Then from Proposition 3.1, Proposition 3.2 and Eq. (3) we have

$$\beta(F) \in \mathcal{A} \mathcal{H}_D \mathcal{Z} = \mathcal{A} \rho_D(\mathcal{Z}) \mathcal{H}_D \subseteq \mathcal{A} \mathcal{H}_D = \Pi_D \mathcal{F}$$

and, therefore, $\left(\Pi_D \circ \beta \right)(F) = \beta(F)$ for all $F \in \Pi_D \mathcal{F}$. This implies

$$\Pi_D \circ \beta \circ \Pi_D = \beta \circ \Pi_D$$

and

$$\Pi_{D'} \circ \beta \circ \Pi_D = 0, \quad D' \neq D, \ D', D \in \widehat{\mathcal{G}}.$$

For $F \in \mathcal{F}_0$ we have

$$F = \sum_{D' \in \widehat{\mathcal{G}}} \Pi_{D'} F,$$

(formal infinite sum) and using this expression we obtain

$$\left(\Pi_D \circ \beta \right)(F) = \left(\Pi_D \circ \beta \circ \left(\sum_{D'} \Pi_{D'} \right) \right)(F) = \left(\beta \circ \Pi_D \right)(F),$$

so that Eq. (38) holds.

Now, if $\beta \in \operatorname{stab} \mathcal{A}$, then $\beta(A) = A$ for all $A \in \mathcal{A}$ or $\beta \circ \Pi_\iota = \Pi_\iota$. But from Eq. (38) we have also the relation $\Pi_\iota = \Pi_\iota \circ \beta$ and since

$$\langle \beta(F_1), \beta(F_2) \rangle_{\mathcal{A}} = \Pi_\iota \left(\left(\beta(F_1) \right) \left(\beta(F_2) \right)^* \right) = \left(\Pi_\iota \circ \beta \right)(F_1 F_2^*),$$

we obtain

$$\langle \beta(F_1), \beta(F_2) \rangle_{\mathcal{A}} = \langle F_1, F_2 \rangle_{\mathcal{A}},$$

for all $F_1, F_2 \in \mathcal{F}$.

To prove the other implication suppose that $\langle \beta(F_1), \beta(F_2) \rangle_{\mathcal{A}} = \langle F_1, F_2 \rangle_{\mathcal{A}}, F_1, F_2 \in \mathcal{F}$, i.e. $\Pi_\iota \circ \beta = \Pi_\iota$. Since $\Pi_\iota \circ \Pi_\iota = \Pi_\iota$ we get

$$\Pi_\iota \circ \beta \circ \Pi_\iota = \Pi_\iota \tag{40}$$

and

$$\Pi_\iota \circ \beta \circ \Pi_D = 0, \quad D \neq \iota, \ D \in \widehat{\mathcal{G}}. \tag{41}$$

From this we have for $D \neq \iota$, $A \in \mathcal{A}$ and $\{\Phi_{D,i}\}_{i=1}^d$ an orthonormal basis of $\mathcal{H}_D$

$$\left(\beta(A) \right)_{D,i} = \langle \beta(A), \Phi_{D,i} \rangle_{\mathcal{A}} = \left\langle A, \beta^{-1}\left(\Phi_{D,i} \right) \right\rangle_{\mathcal{A}} = A \left(\beta^{-1}\left(\Phi_{D,i} \right) \right)_{\iota} = 0,$$

because

$$\left(\beta^{-1}\left(\Phi_{D,i}\right)\right)_{\iota} = \Pi_{\iota}\left(\left(\beta^{-1}\left(\Phi_{D,i}\right)\right)\right) = \left(\Pi_{\iota} \circ \beta^{-1} \circ \Pi_{D}\right)(\Phi_{D,i}) = 0 \,.$$

This implies

$$\beta(A) = \left(\beta(A)\right)_{\iota} \quad \text{or} \quad \left(\beta \circ \Pi_{\iota}\right)(F) = \left(\Pi_{\iota} \circ \beta \circ \Pi_{\iota}\right)(F) \,, \quad F \in \mathcal{F} \,.$$

Using Eq. (40) we obtain finally

$$\beta \circ \Pi_{\iota} = \Pi_{\iota} \circ \beta \circ \Pi_{\iota} = \Pi_{\iota}$$

and $\beta(A) = A$ for all $A \in \mathcal{A} = \Pi_{\iota}\mathcal{F}$, i.e. $\beta \in \operatorname{stab}\mathcal{A}$. ∎

7.2 Lemma *Let* $\mathfrak{Z} = \left(Z_{i'i}\right)_{i',i} \in \operatorname{Mat}_d(\mathcal{Z})$ *be the matrix corresponding to* $\beta \in \operatorname{stab}\mathcal{A}$ *restricted to the Hilbert* $\mathcal{Z}$*–module* $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, *where the orthonormal basis* $\{\Psi_i\}_{i=1}^d$ *is fixed. Then* $\mathfrak{Z}$ *is unitary, i.e. the equations*

$$\mathfrak{Z}^*\mathfrak{Z} = \mathfrak{Z}\mathfrak{Z}^* = \mathbb{1}_d$$

hold. In terms of the entries we can equivalently write

$$\sum_{i'=1}^d Z_{i'i}^* \, Z_{i'j} = \delta_{ij} = \sum_{i'=1}^d Z_{ii'} \, Z_{ji'}^* \,.$$

Proof: Write $H, H' \in \mathfrak{H}$ as $H = \sum\limits_{i=1}^d \Psi_i \, X_i$, $H' = \sum\limits_{j=1}^d \Psi_j \, Y_j$, where $X_i, Y_i \in \mathcal{Z}$, $i = 1, \ldots, d$. Then we have on the one hand

$$H^*H' = \sum_{i=1}^d X_i^* \, Y_i \in \mathcal{Z}$$

and on the other hand

$$H^*H' = \beta(H^*H') = \beta(H)^*\beta(H') = \sum_{i,j,i',j'} X_i^* \, Z_{i'i}^* \, \Psi_{i'}^* \, \Psi_{j'} \, Z_{j'j} \, Y_j = \sum_{i,j,i'} X_i^* \, Z_{i'i}^* \, Z_{i'j} \, Y_j \,.$$

Therefore the equation, $\sum\limits_{i'=1}^d Z_{i'i}^* \, Z_{i'j} = \delta_{ij}$, holds. The second equation, $\mathfrak{Z}\mathfrak{Z}^* = \mathbb{1}_d$, follows from Gelfand's Theorem and from the fact that the scalar matrices $\left(Z_{i'i}(\varphi)\right)_{i',i} \in \operatorname{Mat}_d(\mathbb{C})$, $\varphi \in \operatorname{spec}\mathcal{Z}$, are finite–dimensional (cf. with Section 3). ∎

The preceding lemma says that we can associate to each $\beta \in \operatorname{stab}\mathcal{A}$ and each Hilbert $\mathcal{Z}$–module $\mathfrak{H}$, a unitary module morphism

$$\operatorname{stab}\mathcal{A} \ni \beta \longmapsto U_{\mathfrak{H}}(\beta) \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}) \,, \tag{42}$$

because $\beta(HZ) = \beta(H)\, Z$, $H \in \mathfrak{H}$, $Z \in \mathcal{Z}$.

From the definition of the functor $\mathfrak{F}$, given after Corollary 4.6, and from Eq. (28) we know that the elements $A \in (\rho_{\mathfrak{H}_1}, \rho_{\mathfrak{H}_2})$ determine via $\mathfrak{F}(A)$, the set of intertwining operators between $U_{\mathfrak{H}_1}(\mathcal{G})$ and $U_{\mathfrak{H}_2}(\mathcal{G})$. Next we prove that this intertwining property is still valid for $U_{\mathfrak{H}}(\beta)$.

7.3 Proposition *Let* $\beta \in \operatorname{stab}\mathcal{A}$, $\mathfrak{H}_1$, $\mathfrak{H}_2$ *be Hilbert* $\mathcal{Z}$–*modules and* $U_{\mathfrak{H}_1}(\beta)$, $U_{\mathfrak{H}_2}(\beta)$ *the corresponding unitary module morphisms given in Eq. (42). If* $A \in (\rho_{\mathfrak{H}_1}, \rho_{\mathfrak{H}_2})$, *then the following intertwining relation holds:*

$$U_{\mathfrak{H}_2}(\beta) \circ \mathfrak{F}(A) = \mathfrak{F}(A) \circ U_{\mathfrak{H}_1}(\beta) \,, \quad \beta \in \operatorname{stab}\mathcal{A} \,, \tag{43}$$

where $\mathfrak{F}$ *is the functor defined after Corollary (4.6)*

Proof: First note that Eq. (43) can be rewritten as

$$\left(U_{\mathfrak{H}_2}(\beta)\right)(AH_1) = A\left(U_{\mathfrak{H}_1}(\beta)\right)(H_1)\,, \quad A \in (\rho_{\mathfrak{H}_1},\, \rho_{\mathfrak{H}_2})\,,\ H_1 \in \mathfrak{H}_1\,.$$

According to Eq. (42), $\beta\!\restriction\!\mathfrak{H}_k$ acts via $U_{\mathfrak{H}_k}(\beta) \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_k)$ as $\beta(H_k) = U_{\mathfrak{H}_k}(\beta)(H_k)$, $k = 1, 2$. Further we have $\beta(AF) = A\,\beta(F)$, $F \in \mathcal{F}$. Therefore we obtain

$$\left(U_{\mathfrak{H}_2}(\beta)\right)(AH_1) = \beta(AH_1) = A\,\beta(H_1) = A\left(U_{\mathfrak{H}_1}(\beta)\right)(H_1)\,,$$

and the proof is concluded. ∎

From Definition 4.1 we can associate to a unitary module morphism $U_{\mathfrak{H}}(\beta) \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H})$ an element of the field algebra $\widehat{U_{\mathfrak{H}}(\beta)} \in \mathcal{F}$. Obviously the assignment

$$\mathrm{stab}\,\mathcal{A} \ni \beta \longmapsto \widehat{U_{\mathfrak{H}}(\beta)}\,,$$

is a unitary representation of $\mathrm{stab}\,\mathcal{A}$ in $\mathcal{F}$.

7.4 Lemma *The representation*
$$\mathrm{stab}\,\mathcal{A} \ni \beta \longmapsto \widehat{U_{\mathfrak{H}}(\beta)}\,,$$

is continuous, where in $\mathrm{stab}\,\mathcal{A}$ *we use the topology of pointwise norm convergence and in* $\mathcal{F}$ *the topology given by the* C**–norm.*

Proof: Suppose that $(\beta_n)_n \longrightarrow \beta$, i.e. $\|\beta_n(F) - \beta(F)\|_{\mathcal{F}} \longrightarrow 0$ for all $F \in \mathcal{F}$. Now if $\{\Psi_i\}_{i=1}^d$ is an orthonormal basis in $\mathfrak{H}$, and using the support property, $\mathrm{supp}\,\mathfrak{H} = \mathbb{1}$, we obtain

$$
\begin{aligned}
\left\|\widehat{U_{\mathfrak{H}}(\beta_n)} - \widehat{U_{\mathfrak{H}}(\beta)}\right\|_{\mathcal{F}} &= \left\|\widehat{U_{\mathfrak{H}}(\beta_n)}\left(\sum_{i=1}^d \Psi_i\Psi_i^*\right) - \widehat{U_{\mathfrak{H}}(\beta)}\left(\sum_{i=1}^d \Psi_i\Psi_i^*\right)\right\|_{\mathcal{F}} \\
&\leq \sum_{i=1}^d \left\|\widehat{U_{\mathfrak{H}}(\beta_n)}\Psi_i - \widehat{U_{\mathfrak{H}}(\beta)}\Psi_i\right\|_{\mathcal{F}} \\
&= \|\beta_n(\Psi_i) - \beta(\Psi_i)\|_{\mathcal{F}}\,,
\end{aligned}
$$

which proves the assertion. ∎

Next we will show that the unitary operators $U_{\mathfrak{H}}(\beta)$ are "generated" by the elements $U_{\mathfrak{H}_D}(\beta)$, $D \in \widehat{\mathcal{G}}$ (recall Theorem 6.2).

7.5 Proposition *Let* $\beta \in \mathrm{stab}\,\mathcal{A}$. *Then each unitary module morphism* $U_{\mathfrak{H}}(\beta)$ *is uniquely determined by the family* $\left(U_{\mathfrak{H}_D}(\beta)\right)_{D \in \widehat{\mathcal{G}}}$, *where* $\mathfrak{H}_D = \mathcal{H}_D\mathcal{Z}$ *and* $\mathcal{H}_D$ *is the algebraic Hilbert space corresponding the irreducible* $D \in \widehat{\mathcal{G}}$. *Precisely, if* $\mathfrak{H}$ *is the Hilbert* $\mathcal{Z}$*–module associated to the endomorphism* $\lambda \equiv \rho_{\mathfrak{H}} \in \mathrm{end}\,\mathcal{A}$, *if* $\{\Phi_{D,i}\}_{i=1}^d$ *is an orthonormal basis of* $\mathcal{H}_D$ *and if*

$$\mathfrak{H} \ni H = \sum_D \sum_{i=1}^d A_{D,i}\,\Phi_{D,i}\,, \quad A_{D,i} \in (\rho_D,\, \lambda)\,,$$

is the orthogonal decomposition of H *according to Theorem* 6.2, *then we get*

$$\left(U_{\mathfrak{H}}(\beta)\right)(H) = \sum_D \sum_i A_{D,i}\left(U_{\mathfrak{H}_D}(\beta)\right)(\Phi_{D,i}).$$

Proof: Since $\beta \in \operatorname{stab} \mathcal{A}$, we have

$$\beta(H) = \sum_D \sum_i A_{D,i}\, \beta(\Phi_{D,i})\,,$$

and according to Proposition 7.3 the equation

$$\left(U_{\mathfrak{H}}(\beta)\right)(H) = \sum_D \sum_i A_{D,i}\, \left(U_{\mathfrak{H}_D}(\beta)\right)(\Phi_{D,i})\,,$$

finishes the proof. ∎

Proposition 7.5 justifies that we restrict to the study of $U_{\mathfrak{H}_D}(\beta)$, $D \in \widehat{\mathcal{G}}$, which determine completely the morphisms $U_{\mathfrak{H}}(\beta)$ for a general Hilbert $\mathcal{Z}$–module $\mathfrak{H}$.

Denote by $\mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)$ the set of all unitary module morphisms in $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_D)$.

7.6 Proposition *The mapping,*

$$\operatorname{stab} \mathcal{A} \ni \beta \longmapsto V(\beta) := \prod_D U_{\mathfrak{H}_D}(\beta) \in \prod_D \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)\,,$$

is a group monomorphism and a homeomorphism, where in $\operatorname{stab} \mathcal{A}$ *we take the same topology as in Lemma 7.4 and in* $\prod_D \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)$ *the Tychonoff product topology generated by the operator norm topology in* $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_D)$.

Proof: First note that if $V(\beta_1) = V(\beta_2)$, then $U_{\mathfrak{H}_D}(\beta_1) = U_{\mathfrak{H}_D}(\beta_2)$, $D \in \widehat{\mathcal{G}}$, and, therefore, $\beta_1(\Phi_{D,i}) = \beta_2(\Phi_{D,i})$ for all elements of the orthonormal basis $\Phi_{D,i} \in \mathfrak{H}_D$, $D \in \widehat{\mathcal{G}}$, hence $\beta_1(F) = \beta_2(F)$, for all $F \in \mathcal{F}_0$. Since $\mathcal{F} = \operatorname{clo}_{\|\cdot\|}\mathcal{F}_0$ we get $\beta_1 = \beta_2$. Further, from $U_{\mathfrak{H}_D}(\beta_1 \circ \beta_2) = U_{\mathfrak{H}_D}(\beta_1) \circ U_{\mathfrak{H}_D}(\beta_2)$, $\beta_1, \beta_2 \in \operatorname{stab} \mathcal{A}$, we obtain $V(\beta_1 \circ \beta_2) = V(\beta_1) \circ V(\beta_2)$ and $V(\iota) = \prod_D \mathbb{1}_{\mathfrak{H}_D} = \mathbb{1}$.

Second, to prove the homeomorphism property, note that the continuity already follows from Proposition 4.2 (v) and Lemma 7.4. For the rest of the proof we follow arguments given in [5, Lemma 3.2]. Suppose that we have a sequence $\{\beta_n\}_n \subset \operatorname{stab} \mathcal{A}$ such that

$$\prod_D \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D) \ni V(\beta_n) \longrightarrow V \in \prod_D \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)\,.$$

From the equation $\beta_n(\Psi_D Z) = \beta_n(\Psi_D) Z = \left(U_{\mathfrak{H}_D}(\beta_n)\right)(\Psi_D) Z$, $\Psi_D \in \mathcal{H}_D$, $Z \in \mathcal{Z}$, we get that

$$\mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D) \ni U_{\mathfrak{H}_D}(\beta_n) =: U_n \longrightarrow U_{\mathfrak{H}_D} =: U \in \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)$$

w.r.t. the operator norm topology. But from

$$U_n^{-1} - U^{-1} = U_n^{-1}(U - U_n)U^{-1}\,,$$

we also have

$$\mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D) \ni U_n^{-1} \longrightarrow U^{-1} \in \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)\,.$$

With other words, we have

$$\begin{aligned}
\beta_n(\Psi_D Z) &\longrightarrow \left(U_{\mathfrak{H}_D}\right)(\Psi_D) Z \quad \text{and} \\
\beta_n^{-1}(\Psi_D Z) &\longrightarrow \left(U_{\mathfrak{H}_D}^{-1}\right)(\Psi_D) Z\,.
\end{aligned}$$

Therefore, we can define for $F \in \mathcal{F}_0$

$$\begin{aligned}
\beta(F) &:= \lim_{\|\cdot\|_{\mathcal{F}}} \beta_n(F)\,, \quad F \in \mathcal{F}_0 \\
\gamma(F) &:= \lim_{\|\cdot\|_{\mathcal{F}}} \beta_n^{-1}(F)\,, \quad F \in \mathcal{F}_0\,.
\end{aligned}$$

Since β_n and β_n^{-1} are automorphisms of $\mathcal{F}$ the limits β and γ can be extended by continuity to $\mathcal{F} = \mathrm{clo}_{\|\cdot\|_{\mathcal{F}}} \mathcal{F}_0$, respectively. So we have

$$F = \big(\gamma \circ \beta\big)(F) = \big(\beta \circ \gamma\big)(F), \quad F \in \mathcal{F},$$

i.e. $\beta \in \mathrm{aut}\,\mathcal{F}$, $\gamma = \beta^{-1}$ and even more $\beta \in \mathrm{stab}\,\mathcal{A}$, so that $U = U_{\mathfrak{H}_D} = U_{\mathfrak{H}_D}(\beta)$ and $V = V(\beta)$. Thus $V(\beta_n) \longrightarrow V$ implies $\beta_n \longrightarrow \beta$, i.e. $V(\mathrm{stab}\,\mathcal{A})$ is closed and finally we have that the assignment $\beta \longmapsto V(\beta)$ is homeomorphism. $\blacksquare$

From the preceding result it follows also that if $\mathcal{Z}$ is finite–dimensional, then $\mathrm{stab}\,\mathcal{A}$ is compact as in the case where $\mathcal{Z} = \mathbb{C}\mathbb{1}$.

Recall that if we consider a fixed orthonormal basis $\{\Phi_{D,i}\}_{i=1}^d$ of $\mathcal{H}_D$, $D \in \widehat{\mathcal{G}}$, then $U_{\mathfrak{H}_D}(\beta)$ corresponds to a matrix

$$\mathfrak{Z}_D(\beta) = \Big(Z_{D,i'i}(\beta)\Big)_{i',i} \in \mathrm{Mat}_d(\mathcal{Z}), \quad \mathfrak{H}_D = \mathcal{H}_D\mathcal{Z},$$

by means of

$$\Big(U_{\mathfrak{H}_D}(\beta)\Big)(\Phi_{D,i}) = \sum_{i'=1}^d \Phi_{D,i'}\, Z_{D,i'i}(\beta)\,.$$

Using Gelfand's Theorem we can also interpret $\mathfrak{Z}_D(\beta)$ as a continuous matrix–valued function on $\mathrm{spec}\,\mathcal{Z}$, i.e. for each $\varphi \in \mathrm{spec}\,\mathcal{Z}$ we get a unitary scalar matrix $\big(\mathfrak{Z}_D(\beta)\big)(\varphi) \in \mathrm{Mat}_d(\mathbb{C})$.

In the next theorem we will characterize the subgroup $\mathcal{G} \subset \mathrm{stab}\,\mathcal{A}$.

7.7 Theorem *Let $\beta \in \mathrm{stab}\,\mathcal{A}$ and $\mathfrak{Z}_D(\beta)$ be the corresponding matrix from $\mathrm{Mat}_d(\mathcal{Z})$, where the orthonormal basis $\{\Phi_{D,i}\}_{i=1}^d$ of $\mathcal{H}_D$, $D \in \widehat{\mathcal{G}}$, are fixed. Then,*

$$\beta \in \mathcal{G} \quad \text{iff} \quad \mathfrak{Z}_D(\beta) \in \mathrm{Mat}_d(\mathbb{C}), \quad \text{for all} \quad D \in \widehat{\mathcal{G}}\,.$$

In other words $\beta \in \mathcal{G}$ iff the corresponding functions $\big(\mathfrak{Z}_D(\beta)\big)(\cdot)$ are constant unitary matrix functions on $\mathrm{spec}\,\mathcal{Z}$.

Proof: Define first the set

$$\mathcal{S} := \{\beta \in \mathrm{stab}\,\mathcal{A} \colon \mathfrak{Z}_D(\beta) \in \mathrm{Mat}_d(\mathbb{C}),\ D \in \widehat{\mathcal{G}}\}$$

and note that $\mathcal{S}$ is a subgroup of $\mathrm{stab}\,\mathcal{A}$. Further we have that $\mathcal{G} \subseteq \mathcal{S}$ (cf. with the remark before Proposition 3.1). We prove the other inclusion $\mathcal{G} \supseteq \mathcal{S}$. First note that for $\beta \in \mathcal{S} \subseteq \mathrm{stab}\,\mathcal{A}$ we have

$$\beta(\mathcal{H}) \subseteq \mathcal{H}\,,$$

for all $\mathcal{G}$–invariant algebraic Hilbert spaces $\mathcal{H} \subset \mathcal{F}$. Now we consider the set $\mathcal{C}$ of all functions

$$\mathcal{G} \ni g \longmapsto f_{H_2,H_1}(g) := \langle H_1, g(H_2)\rangle_{\mathcal{H}} = H_1^* g(H_2) \in \mathbb{C}\mathbb{1}, \quad H_1, H_2 \in \mathcal{H}\,,$$

where $\mathcal{H}$ runs through all finite–dimensional and $\mathcal{G}$–invariant algebraic Hilbert spaces in $\mathcal{F}$. Obviously, these functions are continuous on $\mathcal{G}$, i.e. $\mathcal{C} \subseteq \mathrm{C}(\mathcal{G})$. Further

(i) $\mathcal{C}$ is closed w.r.t. multiplication, because

$$(H_1 H_1')^* g(H_2 H_2') = H_1'^* H_1^* g(H_2)\, g(H_2') = H_1^* g(H_2)\, H_1'^* g(H_2')\,.$$

(ii) $\mathcal{C}$ is closed w.r.t. linear combinations. Let $\mathcal{H}_1, \mathcal{H}_2$ be given. Then $\mathcal{H} := W_1\mathcal{H}_1 + W_2\mathcal{H}_2$, with $W_1, W_2 \in \mathcal{A}$, $W_1^*W_1 = W_2^*W_2 = \mathbb{1}$, $W_1^*W_2 = W_2^*W_1 = 0$, $W_1W_1^* + W_2W_2^* = \mathbb{1}$, is also of the required type, and

$$
\begin{aligned}
(W_1H_1 + W_2H_2)^* & g(W_1H_1 + W_2H_2) \\
= \ & H_1^*W_1^* g(W_1)g(H_1) + H_2^*W_2^* g(W_1)g(H_1) \\
& + H_1^*W_1^* g(W_2)g(H_2) + H_2^*W_2^* g(W_2)g(H_2) \\
= \ & H_1^* g(H_1) + H_2^* g(H_2) \,,
\end{aligned}
$$

where we have used that $g(W_k) = W_k$, since $W_k \in \mathcal{A}$, $k = 1, 2$.

(iii) The function $c(g) \equiv 1$ belongs to $\mathcal{C}$, because $\mathbb{C}\mathbb{1}$ is an (irreducible) invariant subspace.

(iv) $\mathcal{G}$ is separated by $\mathcal{C}$, because from $g_1 \!\restriction\! \mathcal{H} = g_2 \!\restriction\! \mathcal{H}$ for all admissible $\mathcal{H}$ we obtain immediately $g_1 = g_2$.

(v) The complex–conjugated functions of elements in $\mathcal{C}$ belong also to $\mathcal{C}$. Namely, let $\{\Psi_i\}_{i=1}^d$ be an orthonormal basis of $\mathcal{H}$ and $\{\overline{\Psi}_i\}_{i=1}^d$ an orthonormal basis of a conjugated space $\overline{\mathcal{H}}$ (w.r.t. $\mathcal{G}$). Then according to Section 2 we have $\Psi_i^* = R_{\mathcal{H}}^* \overline{\Psi}_i$ and $g(\Psi_i) = \sum\limits_{i'=1}^d \Psi_{i'} \, U_{i'i}(g)$, where $\left(U_{i'i}(g) \right)_{i'i}$ is a scalar matrix. Then

$$
\begin{aligned}
\left(g(\Psi_i) \right)^* & = \sum_{i'=1}^d \Psi_{i'}^* \, \overline{U_{i'i}(g)} \\
& = g(\Psi_i^*) \quad = \quad g\left(R_{\mathcal{H}}^* \overline{\Psi}_i \right) \\
& = R_{\mathcal{H}}^* \, g(\overline{\Psi}_i) \quad = \quad R_{\mathcal{H}}^* \sum_{i'=1}^d \overline{\Psi}_{i'} \, V_{i'i}(g) \\
& = \sum_{i'=1}^d \Psi_{i'}^* \, V_{i'i}(g) \,,
\end{aligned}
$$

so that

$$
V_{i'i}(g) = \overline{U_{i'i}(g)}.
$$

Therefore, according to the Stone–Weierstraß Theorem the *–algebra $\mathcal{C}$ is dense in $C(\mathcal{G})$ and therefore also dense in $L^2(\mathcal{G})$.

Now let $\beta \in \mathcal{S}$, then $\beta(X) \in \mathcal{H}$ for $X \in \mathcal{H}$. As in [23, pp. 206-207] we define an operator U on $\mathcal{C}$ by

$$
U_\beta \left(f_{X,Y} \right) := f_{X, \beta(Y)} \,.
$$

We calculate for $X_k \in \mathcal{H}_k$, $k = 1, 2$:

$$
\begin{aligned}
\int\limits_{\mathcal{G}} \left(U_\beta \left(f_{X_1, Y_1} \right) \right)(g) \, \overline{\left(U_\beta \left(f_{X_2, Y_2} \right) \right)(g)} \, \mathrm{d}g & = \int\limits_{\mathcal{G}} f_{X_1, \beta(Y_1)}(g) \, \overline{f_{X_2, \beta(Y_2)}(g)} \, \mathrm{d}g \\
& = \int\limits_{\mathcal{G}} (\beta Y_1)^* g(X_1) \left((\beta Y_2)^* g(X_2) \right)^* \, \mathrm{d}g \\
& = (\beta Y_1)^* \left(\int\limits_{\mathcal{G}} g(X_1 X_2^*) \, \mathrm{d}g \right) (\beta Y_2)
\end{aligned}
$$

$$= \beta\left(Y_1^* \int_{\mathcal{G}} g(X_1 X_2^*) \, \mathrm{d}g \, Y_2\right)$$

$$= \beta\left(\int_{\mathcal{G}} f_{X_1,Y_1}(g) \, \overline{f_{X_2,Y_2}(g)} \, \mathrm{d}g\right)$$

$$= \int_{\mathcal{G}} f_{X_1,Y_1}(g) \, \overline{f_{X_2,Y_2}(g)} \, \mathrm{d}g \,,$$

where for the fourth equation we have used the the relation $\int_{\mathcal{G}} g(X_1 X_2^*) \, \mathrm{d}g \in \mathcal{A}$. This equation expresses the uniqueness of the definition of U as an operator on $\mathcal{C}$ and, simultaneously, its isometry property w.r.t. the scalar product in $\mathrm{L}^2(\mathcal{G})$. By continuous extension of U_β to the whole $\mathrm{L}^2(\mathcal{G})$ we obtain a unitary operator on $\mathrm{L}^2(\mathcal{G})$, which is also denoted by U_β. Moreover, we have

$$\begin{aligned}
U_\beta(f_{X_1,Y_1} \, f_{X_2,Y_2}) &= U_\beta(f_{X_1 X_2, Y_1 Y_2}) &= f_{X_1 X_2, \beta(Y_1 Y_2)} \\
&= f_{X_1 X_2, \beta(Y_1)\beta(Y_2)} &= f_{X_1, \beta(Y_1)} \, f_{X_2, \beta(Y_2)} \\
&= U_\beta(f_{X_1,Y_1}) \cdot U_\beta(f_{X_2,Y_2}),
\end{aligned}$$

i.e. $U_\beta \in \mathrm{aut}\,\mathcal{C}$, hence $U_\beta \in \mathrm{aut}\,\mathrm{C}(\mathcal{G})$. But according to Gelfand's Theorem the automorphisms of $\mathrm{C}(\mathcal{G})$ correspond bijectively to the homeomorphisms of $\mathcal{G}$. Therefore, there is a $g_\beta \in \mathcal{G}$ such that if e is the unit element in $\mathcal{G}$ the equation

$$\left(U_\beta f\right)(e) = f(g_\beta^{-1}), \quad f \in \mathrm{C}(\mathcal{G}), \ g_\beta \in \mathcal{G}\,.$$

Hence we obtain for $X, Y \in \mathcal{H}$

$$\begin{aligned}
\langle \beta(Y), X \rangle_{\mathcal{H}} &= f_{X, \beta(Y)}(e) &= \left(U_\beta(f_{X,Y})\right)(e) &= f_{X,Y}\left(g_\beta^{-1}\right) &= \langle Y, g_\beta^{-1}(X) \rangle_{\mathcal{H}} \\
&= \langle Y, U(g_\beta^{-1})(X) \rangle_{\mathcal{H}} &= \langle Y, U(g_\beta)^*(X) \rangle_{\mathcal{H}} &= \langle U(g_\beta)Y, X \rangle_{\mathcal{H}} \\
&= \langle g_\beta(Y), X \rangle_{\mathcal{H}},
\end{aligned}$$

where $g(X) = U_{\mathcal{H}}(g)X$ and $U_{\mathcal{H}}(g)$ is unitary for $g \in \mathcal{G}$. Recall that $\mathcal{F} = \mathrm{C}^*\left(\mathcal{A}, \{\mathcal{H}\}\right)$, which implies $\beta = g_\beta$. ∎

For the next theorem recall Proposition 5.4, Remark 5.5, Remark 6.3, Proposition 7.5 and Proposition 7.6.

7.8 Theorem *Let*

$$\mathrm{stab}\,\mathcal{A} \ni \beta \longmapsto V(\beta) := \prod_D U_{\mathfrak{H}_D}(\beta) \in \prod_D \mathcal{U}_{\mathcal{Z}}(\mathfrak{H}_D)\,,$$

be the assignment specified in Proposition 7.6 and choose a fixed orthonormal basis in $\mathcal{H}_D$, $\{\Phi_{D,i}\}_{i=1}^d$. The $d \times d$–matrices in $\mathrm{Mat}_d(\mathcal{Z})$ associated to $U_{\mathfrak{H}_D}(\beta)$ are denoted by $\mathfrak{Z}_D = \left(Z_{D,i'i}\right)_{i',i}$. Then the following conditions hold

$$\mathfrak{Z}_{D_1 D_2} = \rho_{D_2}^{-1}(\mathfrak{Z}_{D_1}) \otimes \mathfrak{Z}_{D_2}, \quad D_1, D_2 \in \widehat{\mathcal{G}} \tag{44}$$

$$\mathbb{1} = \rho_{\overline{D}}^{-1}(\mathfrak{Z}_D)\,\mathfrak{Z}_{\overline{D}}^t, \quad D, \overline{D} \in \widehat{\mathcal{G}}, \tag{45}$$

where $\mathfrak{Z}_{D_1 D_2}$ is given in Remark 6.3 by formula (37) and $\left(\rho_{D_2}^{-1}(\mathfrak{Z}_{D_1})\right)_{i'i} = \rho_{D_2}^{-1}\left(Z_{D_1, i'i}\right)$, with $i, i' = 1, \ldots, d$. The superindex t denotes the transposed matrix.

Proof: Since $\beta \in \operatorname{stab} \mathcal{A}$ is an automorphism, we have on the one hand for $i_1 = 1, \ldots, d_1$, $i_2 = 1, \ldots, d_2$,

$$
\begin{aligned}
\beta\left(\Phi_{D_1, i_1} \Phi_{D_2, i_2}\right) &= \sum_{i'_1} \sum_{i'_2} \Phi_{D_1, i'_1} \Phi_{D_2, i'_2} Z_{D_1 D_2, i'_1 i'_2 i_1 i_2} \\
&= \sum_{i'_1} \sum_{i'_2} \left(\rho_{D_1} \circ \rho_{D_2}\right)(Z_{D_1 D_2, i'_1 i'_2 i_1 i_2}) \Phi_{D_1, i'_1} \Phi_{D_2, i'_2}
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
\beta\left(\Phi_{D_1, i_1}\right) \beta\left(\Phi_{D_2, i_2}\right) &= \sum_{i'_1} \sum_{i'_2} \Phi_{D_1, i'_1} Z_{D_1, i'_1 i_1} \Phi_{D_2, i'_2} Z_{D_2, i'_2 i_2} \\
&= \sum_{i'_1} \sum_{i'_2} \rho_{D_1}(Z_{D_1, i'_1 i_1}) \left(\rho_{D_1} \circ \rho_{D_2}\right)(Z_{D_2, i'_2 i_2}) \Phi_{D_1, i'_1} \Phi_{D_2, i'_2},
\end{aligned}
$$

so that we obtain

$$
\rho_{D_2}\left(Z_{D_1 D_2, i'_1 i'_2 i_1 i_2}\right) = Z_{D_1, i'_1 i_1} \rho_{D_2}\left(Z_{D_2, i'_2 i_2}\right).
$$

This equation implies (44).

Further, from Equations (12) and (16) we have

$$
\beta(\Phi_{D, i})^* = \sum_{i'} Z^*_{D, i'i} \Phi^*_{D, i'} = \sum_{i'} Z^*_{D, i'i} R^*_D \Phi_{\overline{D}, i'} = R^*_D \sum_{i'} \Phi_{\overline{D}, i'} \rho_{\overline{D}}^{-1}\left(Z^*_{D, i'i}\right)
$$

and also

$$
\beta\left(\Phi^*_{D, i}\right) = \beta\left(R^*_D \Phi_{\overline{D}, i}\right) = R^*_D \sum_{i'} \Phi_{\overline{D}, i'} Z_{\overline{D}, i'i}.
$$

From these equations we obtain

$$
Z_{\overline{D}, i'i} = \rho_{\overline{D}}^{-1}\left(Z^*_{D, i'i}\right),
$$

which implies (45). $\blacksquare$

Taking again into account Remark 6.3 we can formulate the following counterpart of Theorem 7.8.

7.9 Theorem *Suppose that the matrices*

$$
\mathfrak{Z}_D = \left(Z_{D, i'i}\right)_{i', i} \in \operatorname{Mat}_d(\mathcal{Z}), \quad D \in \widehat{\mathcal{G}},
$$

satisfy the properties (44) and (45) of the preceding theorem, where $\mathfrak{Z}_{D_1 D_2}$ is defined in Eq. (37). Then the linear mapping $\gamma \colon \mathcal{F}_0 \longrightarrow \mathcal{F}_0$ defined by

$$
\begin{aligned}
\gamma(A) &:= A, \quad A \in \mathcal{A}, \\
\gamma(\Phi_{D, i}) &:= \sum_{i'} \Phi_{D, i'} Z_{D, i'i},
\end{aligned}
$$

*is an automorphism of the *-algebra $\mathcal{F}_0$ which can be uniquely extended to an automorphism γ of $\mathcal{F}$, with $\gamma \in \operatorname{stab} \mathcal{A}$.*

Proof: The properties $\gamma(\Phi_{D_1, i_1} \Phi_{D_2, i_2}) = \gamma(\Phi_{D_1, i_1})\gamma(\Phi_{D_2, i_2})$ and $\gamma(\Phi_{D, i})^* = \gamma(\Phi^*_{D, i})$, $D_1, D_2, D \in \widehat{\mathcal{G}}$, $i_k = 1, \ldots, d_k$, $k = 1, 2$, $i = 1, \ldots, d$, follow directly from Equations (44) and

(45) (cf. the proof of the preceding theorem). Furthermore, γ^{-1} exists on $\mathcal{F}_0$ and is defined by (cf. Lemma 7.2)

$$
\begin{aligned}
\gamma^{-1}(A) &:= A, \quad A \in \mathcal{A}, \\
\gamma^{-1}(\Phi_{D,i}) &:= \sum_{i'} \Phi_{D,i'} \, Z_{D,ii'}^*,
\end{aligned}
$$

so that γ is an automorphism of $\mathcal{F}_0$. It remains to show that γ is isometric on $\mathcal{F}_0$ and, therefore, it can be uniquely and isometrically extended to an automorphism of $\mathcal{F}$, also denoted by γ. In this case $\gamma \in \operatorname{stab}\mathcal{A}$.

In order to prove the isometry of γ we use the $\mathcal{A}$–valued scalar product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and its properties stated in Section 2. First we prove the equation

$$
\langle \gamma(F_1), \gamma(F_2) \rangle_{\mathcal{A}} = \langle F_1, F_2 \rangle_{\mathcal{A}}, \quad F_1, F_2 \in \mathcal{F}_0.
$$

Indeed, put $F_1 = \sum_{D_1, i_1} A_{D_1, i_1} \Phi_{D_1, i_1}$ and $F_1 = \sum_{D_2, i_2} B_{D_2, i_2} \Phi_{D_2, i_2}$. Then we compute

$$
\begin{aligned}
&\langle \gamma(F_1), \gamma(F_2) \rangle_{\mathcal{A}} \\
&= \sum_{D_1, i_1} \sum_{D_2, i_2} A_{D_1, i_1} \langle \gamma(\Phi_{D_1, i_1}), \gamma(\Phi_{D_2, i_2}) \rangle_{\mathcal{A}} B_{D_2, i_2}^* \\
&= \sum_{D_1, i_1} \sum_{D_2, i_2} \sum_{i_1', i_2'} A_{D_1, i_1} \, \rho_{D_1}\left(Z_{D_1, i_1' i_1} \right) \left\langle \Phi_{D_1, i_1'}, \Phi_{D_2, i_2'} \right\rangle_{\mathcal{A}} \rho_{D_2}\left(Z_{D_2, i_2' i_2} \right)^* B_{D_2, i_2}^* \\
&= \sum_{D_1, i_1} \sum_{i_1'} \sum_{i_1''} \frac{1}{d_1} A_{D_1, i_1} \, \rho_{D_1}\left(Z_{D_1, i_1' i_1''}^* \, Z_{D_1, i_1' i_1} \right) B_{D_1, i_1''}^* \\
&= \sum_{D_1, i_1} \frac{1}{d_1} A_{D_1, i_1} B_{D_1, i_1}^* \\
&= \langle F_1, F_2 \rangle_{\mathcal{A}}.
\end{aligned}
$$

Next recall (see e.g. [10, pp. 201-203]) that the C*–norm $\|\cdot\|_{\mathcal{F}}$ in the Hilbert C*–system $\{\mathcal{F}, \mathcal{G}\}$ can be written as

$$
\|F\|_{\mathcal{F}} = \|\pi(F)\|_{\mathrm{op}}, \quad F \in \mathcal{F},
$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm w.r.t. the norm $|F| := \left\| \langle F, F \rangle_{\mathcal{A}} \right\|^{\frac{1}{2}}$ of the operator $\pi(F)$ on $\mathcal{F}$, defined by

$$
\left(\pi(F) \right)(X) := X F^*, \quad F, X \in \mathcal{F}.
$$

Note that we have $|\gamma(F)| = |F|$ for all $F \in \mathcal{F}_0$. Further, we get

$$
\left| \pi\left(\gamma(F) \right)(X) \right| = \left| X \left(\gamma(F) \right)^* \right| = \left| \gamma\left(\gamma^{-1}(X) \, F^* \right) \right| = \left| \gamma^{-1}(X) \, F^* \right|, \quad F \in \mathcal{F}_0.
$$

This equation implies

$$
\left\| \pi\left(\gamma(F) \right) \right\|_{\mathrm{op}} = \|\pi(F)\|_{\mathrm{op}}, \quad F \in \mathcal{F}_0,
$$

or, equivalently,

$$
\left\| \gamma(F) \right\|_{\mathcal{F}} = \|F\|_{\mathcal{F}}, \quad F \in \mathcal{F}_0,
$$

and the proof is concluded. $\blacksquare$

Finally, we are able to give a characterization of $\operatorname{stab}\mathcal{A}$ in terms of $\operatorname{spec}\mathcal{Z}$, $\mathcal{G}$ and the irreducible endomorphisms, ρ_D, $D \in \widehat{\mathcal{G}}$, more precisely in terms of the continuous mappings

$$
f_D = f_{\rho_D} \colon \operatorname{spec}\mathcal{Z} \longrightarrow \operatorname{spec}\mathcal{Z}
$$

which correspond to ρ_D^{-1} (see Remark 5.5).

First we consider an element of the group $\mathrm{C}(\mathrm{spec}\,\mathcal{Z} \to \mathcal{G})$. Then to each $\varphi \in \mathrm{spec}\,\mathcal{Z}$ there corresponds $g(\varphi) \in \mathcal{G}$, such that the assignment $\varphi \mapsto g(\varphi)$ is continuous. Recall that $g(\varphi)$ acts unitarily on $\mathcal{H}_D$, i.e.

$$g(\varphi)(\Phi_{D,i}) = \sum_{i'=1}^{d} \Phi_{D,i'}\, U_{D,i'i}\big(g(\varphi)\big)\,,$$

where $\Big(U_{D,i'i}\big(g(\varphi)\big)\Big)_{i',i}$ is a continuous unitary matrix–valued function on $\mathrm{spec}\,\mathcal{Z}$ which in its turn determine the elements $Z_{D,i'i} \in \mathcal{Z}$, $i, i' = 1, \dots, d$ via the relation

$$Z_{D,i'i}(\varphi) := U_{D,i'i}\big(g(\varphi)\big)\,, \quad \varphi \in \mathrm{spec}\,\mathcal{Z}\,.$$

The matrix $\mathfrak{Z}_D = \Big(Z_{D,i'i}\Big)_{i',i} \in \mathrm{Mat}_d(\mathcal{Z})$ is unitary.

Next we define a closed subgroup $\mathcal{T} \subset \mathrm{C}(\mathrm{spec}\,\mathcal{Z} \to \mathcal{G})$.

7.10 Definition *The continuous function*

$$\mathrm{spec}\,\mathcal{Z} \ni \varphi \longmapsto g(\varphi) \in \mathcal{G}$$

is an element of $\mathcal{T} \subset \mathrm{C}(\mathrm{spec}\,\mathcal{Z} \to \mathcal{G})$ if the following two conditions are satisfied for all $\varphi \in \mathrm{spec}\,\mathcal{Z}$:

$$\mathfrak{Z}_{D_1 D_2}(\varphi) = \Big(\mathfrak{Z}_{D_1} \circ f_{D_2}\Big)(\varphi) \otimes \mathfrak{Z}_{D_2}(\varphi)\,, \quad D_1, D_2 \in \widehat{\mathcal{G}} \tag{46}$$

$$\mathbb{1} = \Big(\mathfrak{Z}_D \circ f_{\overline{D}}\Big)(\varphi)\, \mathfrak{Z}_{\overline{D}}(\varphi)^t\,, \quad D, \overline{D} \in \widehat{\mathcal{G}}\,, \tag{47}$$

where $\mathfrak{Z}_{D_1 D_2}(\varphi)$ is the matrix–valued function on $\mathrm{spec}\,\mathcal{Z}$ associated to $\mathfrak{Z}_{D_1 D_2}$ (see Eq. (37)), the superindex t means the transposed matrix and the functions f_D, $D \in \widehat{\mathcal{G}}$, are given in Remark 5.5.

Note that the fact that $g(\varphi) \in \mathcal{G}$ already implies that the scalar unitarities $g(\varphi)\!\restriction\!\mathcal{H}_{D_1}\mathcal{H}_{D_2}$ and $g(\varphi)\!\restriction\!\mathcal{H}_D$ satisfy the Eq. (37). Since Γ is a scalar (hence a constant) unitarity, the matrix $\mathfrak{Z}_{\mathfrak{H}}$ corresponding to a given continuous function $\varphi \mapsto g(\varphi)$ and to a Hilbert $\mathcal{Z}$–module $\mathfrak{H}$, satisfies also Eq. (37).

7.11 Theorem *The automorphism $\beta \in \mathrm{aut}\,\mathcal{F}$ satisfies $\beta \in \mathrm{stab}\,\mathcal{A}$ iff there is a continuous function*

$$\mathrm{spec}\,\mathcal{Z} \ni \varphi \longmapsto g(\varphi) \in \mathcal{G}\,,$$

such that the corresponding matrices $\mathfrak{Z}_D = \Big(Z_{D,i'i}\Big)_{i',i} \in \mathrm{Mat}_d(\mathcal{Z})$ satisfy the conditions (46) and (47). Moreover, $\mathrm{stab}\,\mathcal{A}$ is isomorphic and homeomorphic to the subgroup $\mathcal{T} \subset \mathrm{C}(\mathrm{spec}\,\mathcal{Z} \to \mathcal{G})$.

Proof: (i) Let $\beta \in \mathrm{stab}\,\mathcal{A}$, so that, according to Theorem 7.8, the associated matrices satisfy the conditions (44) and (45). Then the matrices $\mathfrak{Z}_D(\varphi)$, with $\varphi \in \mathrm{spec}\,\mathcal{Z}$ fixed, are constant matrix functions satisfying (46) and (47). Therefore, from Theorem 7.7 there is an automorphism $g(\varphi) \in \mathcal{G}$ associated to $\{\mathfrak{Z}_D(\varphi)\}_D$, $D \in \widehat{\mathcal{G}}$, and $\varphi \mapsto g(\varphi)$ is continuous (note Proposition (7.6)). Using the function f_D defined in Remark 5.5 the conditions (44) and (45) can be rewritten in the form of equations (46) and (47) of Definition 7.10. So the function $\varphi \mapsto g(\varphi)$ is an element of $\mathcal{T}$.

(ii) Conversely, if $\varphi \mapsto g(\varphi)$ is an element of $\mathcal{T}$, then, according to the remarks before Definition 7.10, we have the corresponding unitary matrices $\mathfrak{Z}_D$, that satisfy by assumption the conditions (46) and (47), which can be rewritten in the form (44) and (45) of Theorem 7.8. Then, by Theorem 7.9, they define an automorphism $\beta \in \mathrm{stab}\,\mathcal{A}$. The bijection between $\mathrm{stab}\,\mathcal{A}$ and $\mathcal{T}$, $\mathrm{stab}\,\mathcal{A} \leftrightarrow \mathcal{T}$, is an isomorphism and an homeomorphism by Theorem 7.7. $\blacksquare$

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