Publicacions Matemàtiques, Vol 33 (1989), 369–381.

SUPERSOLUTIONS AND STABILIZATION OF THE SOLUTION OF A NONLINEAR PARABOLIC SYSTEM

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Abstract

Let us consider a nonlinear parabolic system of the following type:

$$(S)\ \begin{array}{l}\frac{\partial u}{\partial t} - \operatorname{div}\,(|\nabla u|^{p-2}\,\nabla u) = \frac{\partial H}{\partial u}(x,u,v)\\ \frac{\partial v}{\partial t} - \operatorname{div}(|\nabla v|^{p-2}\,\nabla v) = \frac{\partial H}{\partial v}(x,u,v)\end{array}$$

with Dirichlet boundary conditions and initial data.

In this paper, we construct sub-supersolutions of (S), and by use of them, we prove that, for $t_n \to +\infty$, the solution of (S) converges to some solution of the elliptic system associated with (S).

0. Introduction

This paper concerns the existence and asymptotic behaviour of bounded, non negative solutions of the following system of nonlinear equations:

$$(\mathbb{S})\begin{cases}\frac{\partial u}{\partial t} - \Delta_p u = f(x,u,v) & \text{in } \Omega_x \mathbf{R}_+\\ \frac{\partial v}{\partial t} - \Delta_q v = g(x,u,v) & \text{in } \Omega_x \mathbf{R}_+\\ u(x,t) = v(x,t) = 0 & \text{in } \partial\Omega_x \mathbf{R}_+\\ u(x,o) = \varphi_0(x),\ v(x,0) = \psi_0(x) & \text{in } \Omega\end{cases}$$

where $p \geq 2$, $q \geq 2$, $\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\,\nabla u)$ and Ω is a bounded regular open subset of $\mathbf{R}^N$.

For $p = q = 2$, Problem $(\mathbb{S})$ has been investigated by many authors [5, 6, 12].

$(\mathbb{S})$ is an example of a nonlinear parabolic system arising from non-Newtonian fluid mechanics. NAKAO [9] studies a similar system in which $p = q > 2$ and the right hand side is $f, -\lambda f, \lambda$ constant. The case of a single equation of the type $(\mathbb{S})$ is studied in [2, 3, 8, 11]. The purpose of this paper is to extend the results of [3] to the system $(\mathbb{S})$.

First, using sub-supersolutions, we show that $(\mathbb{S})$ has a solution. Moreover, supposing that there exist $\lambda > 0$, $\mu > 0$ and a function $H(x,u,v)$ such that

$f = \lambda \frac{\partial H}{\partial u}$ and $g = \mu \frac{\partial H}{\partial v}$, we prove that the solution of (S) converges to a solution of the Dirichlet problem for the elliptic system.

We obtain regularizing effects such that:

$$\frac{\partial}{\partial t}\left(|\nabla u|^{p-2}\nabla u\right) \in L^2(t_0,+\infty\,;\,L^{p^*}(\Omega)) \text{ and}$$
$$\frac{\partial}{\partial t}\left(|\nabla v|^{q-2}\nabla v\right) \in L^2(t_0,+\infty\,;\,L^{q^*}(\Omega)).$$

Our method is closely related to the paper of LANGLAIS and PHILLIPS [7], and also to the paper of ELHACHIMI and DE THELIN [3] who study the stabilization of the solution of a single equation. Some examples are discussed in part IV, and include:

$$H(x,u,v) = k(x)\,uv + \lambda\, u^{\gamma_1+1} + \mu\, v^{\gamma_2+1}$$
$$H(x,u,v) = -u^m v^n + \lambda\, u^{\gamma_1+1} + \mu\, v^{\gamma_2+1}$$

Some numerical results related to the system (S) are given in[4].

All Theorems are written in the case $p > 2$, $q > 2$; obvious modifications (for Theorem 6) give the case $p = 2$ or $q = 2$.

1. Preliminaries and sub-supersolutions

Throughout this paper, Ω stands for a regular bounded open subset of $\mathbb{R}^N$. Left f and g be some functions from $\mathbb{R}^{N+2}$ to $\mathbb{R}$ such that:

$$(1.1)\quad \begin{cases} f, g \in C^1(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}) \\ \text{and for any } x \in \Omega,\ u \in \mathbb{R}_+,\ v \in \mathbb{R}_+ : f(x,0,v) \geq 0, g(x,u,0) \geq 0 \end{cases}$$

and

$$(1.2)\quad \begin{cases} \text{For any } M > 0,\ N > 0, \text{ there exist } k^1_{M,N} > 0,\ k^2_{M,N} > 0 \\ \text{such that :} \\ a) f(x,u,v) - f(x,w,v) \leq k^1_{M,N}(u-w),\ \forall x \in \Omega, \\ \quad \forall u,v,w : 0 \leq w \leq u \leq M, v \in [0,N] \\ b) g(x,u,v) - g(x,u,w) \leq k^2_{M,N}(v-w),\ \forall x \in \Omega \\ \quad \forall u,v,w : 0 \leq w \leq v \leq N,\ u \in [0,M]. \end{cases}$$

Remark 1: the condition 1.2. a) is satisfied if $u \to f(x,u,v)$ is a non increasing function on $\mathbb{R}_+$.

We shall also use the following notations:

$$\text{for } T > 0,\ Q_T = \Omega\times]0,T[,\ S_T = \partial\Omega \times [0,T],$$
$$F(\nabla u) = |\nabla u|^{p-2}\nabla u, G(\nabla v) = |\nabla v|^{q-2}\nabla v, \text{with } p > 2 \text{ and } q > 2.$$
$$\Delta_p u = \operatorname{div}(F(\nabla u)),\ \Delta_q v = \operatorname{div}(G(\nabla v)).$$

Let φ_0, ψ_0 be given such that:

$$\begin{cases} \varphi_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ \varphi_0 \geq 0 \\ \psi_0 \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega),\ \psi_0 \geq 0, \end{cases} \tag{1.3}$$

We say that (u, v) is a solution of (S) in Q_T (resp: $(\hat{u}, \hat{v})$ is a supersolution of (S) in Q_T) iff

$$\begin{cases} u(\text{ resp } \hat{u}) \in L^\infty(0, T\,;\, W^{1,p}(\Omega) \cap L^\infty(\Omega)) \\ v(\text{ resp } \hat{v}) \in L^\infty(0, T\,;\, W^{1,q}(\Omega) \cap L^\infty(\Omega)) \end{cases} \tag{1.4}$$

$$\frac{\partial u}{\partial t}\left(\text{resp}\frac{\partial \hat{u}}{\partial t}\right) \in L^2(Q_T), \frac{\partial v}{\partial t}\left(\text{resp}\frac{\partial \hat{v}}{\partial t}\right) \in L^2(Q_T). \tag{1.5}$$

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u - f(x, u, v) = 0 \\ \frac{\partial v}{\partial t} - \Delta_q v - g(x, u, v) = 0 \end{cases} \quad \text{in } Q_T \tag{1.6}$$
$$(\text{resp } \cdots \geq 0 \quad \text{in } Q_T)$$

$$u = v = 0\ (\text{resp}:\ \hat{u} \geq 0,\ \hat{v} \geq 0) \text{ in } S_T. \tag{1.7}$$

$$\begin{cases} u(.,0) = \varphi_0\ (\text{resp}:\ \hat{u}(.,0) \geq \varphi_0) \\ v(.,0) = \psi_0\ (\text{resp}:\ \hat{v}(.,0) \geq \psi_0) \end{cases} \text{in } \Omega \tag{1.8}$$

Our method is based upon a comparison principale for the system (S); but the usual notion of supersolution does not work; so, following Hernandez [**6**], we set:

Definition 1. $[(0,0), (\hat{u}, \hat{v})]$ is said to be a sub-supersolution of (S) in Q_T if it satisfies the following conditions:

$$\hat{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega),\ \hat{v} \in W^{1,q}(\Omega) \cap L^\infty(\Omega) \tag{1.9}$$

$$\forall x \in \Omega : 0 \leq \varphi_0(x) \leq \hat{u}(x) \leq M_1,\ 0 \leq \psi_0(x) \leq \hat{v}(x) \leq N_1 \tag{1.10}$$

$$\begin{cases} \forall x \in \Omega,\ \forall v \in [0, \hat{v}] : -f(x, 0, v) \leq 0 \leq -\Delta_p \hat{u} - f(x, \hat{u}, v) \\ \forall x \in \Omega,\ \forall u \in [0, \hat{u}] : -g(x, u, 0) \leq 0 \leq -\Delta_q \hat{v} - g(x, u, \hat{v}) \end{cases} \tag{1.11}$$

Remark 2: if we suppose that $v \to f(x, u, v) \uparrow$ and $u \to g(x, u, v) \uparrow$ any supersolution of (S) gives a sub-supersolution of (S).

Our first results are sufficient conditions for the existence of sub-supersolutions of (S).

It is well known (cf.[3]) that the problem:

$$\begin{cases} -\Delta_p u = k_0 + k_1 u^\gamma, & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

has a supersolution if $\gamma \in]0, p-1[$.

Theorem 1. *Assume that $v \to f(x,u,v)$ and $u \to g(x,u,v)$ are monotone non–increasing functions, and that there exist*

$$\lambda_0 \geq 0,\ \mu_0 \geq 0,\ \lambda_1 > 0,\ \mu_1 > 0 \text{ and } \gamma_1 \in]0, p-1[,\ \lambda_2 \in]0, q-1[$$

such that

$$\left\{ \begin{array}{l} f(x,u,0) \leq \lambda_0 + \lambda_1 u^{\gamma_1},\ \forall x \in \Omega,\ \forall u \in \mathbf{R}_+ \\ g(x,0,v) \leq \mu_0 + \mu_1 v^{\gamma_2},\ \forall x \in \Omega,\ \forall v \in \mathbf{R}_+ \end{array} \right. \tag{1.12}$$

Then (**S**) *has a sub-supersolution.*

Proof: By [**3**] and (1.12), the equations

$$\begin{aligned} -\Delta_p u &= f(x,u,0) \text{ and} \\ -\Delta_q v &= g(x,0,v) \end{aligned}$$

have supersolutions $\bar{u}$ and $\bar{v}$. In fact, the monotonicity assumptions on f and g prove that $[(0,0), (\bar{u},\bar{v})]$ is a sub-supersolution of (**S**). ■

Theorem 2. *Let $v \to f(x,u,v)$ be a non-decreasing function and $u \to g(x,u,v)$ be a non-increasing function.*

Assume that there exist constants

$$\mu_0 \geq 0,\ \mu_1 > 0,\ \gamma_2 \in]0, q-1[$$

and for any N:

$$\lambda_0 \geq 0,\ \lambda_1 > 0,\ \gamma_1 \in]0, p-1[$$

such that:

$$\left\{ \begin{array}{l} f(x,u,N) \leq \lambda_0 + \lambda_1 u^{\gamma_1} \\ g(x,0,v) \leq \mu_0 + \mu_1 v^{\gamma_2} \end{array} \right. \tag{1.14}$$

Then (**S**) *has a sub-supersolution.*

Proof: By [**3**] there exits $\bar{v}$ such that:

$$-\Delta_q \bar{v} \geq \mu_0 + \mu_1 \bar{v}^{\gamma_2}$$

Let $N \leq \bar{v}$ and $\bar{u}$ be such that:

$$-\Delta_p \bar{u} \leq \lambda_0 + \lambda_1 \bar{u}^{\gamma_1}$$

Then:

$$\begin{aligned} -\Delta_p \bar{u} &\geq f(x,u,N) \geq f(x,\bar{u},\bar{v}) \\ -\Delta_q \bar{v} &\geq g(x,0,\bar{v}) \end{aligned}$$

whence the result. ■

Theorem 3. *Assume that there exist $k_i > 0$, $] = 0, 1$ such that:*

$$\begin{cases} f(x,u,v) = k_0\, v + \varphi(x,u) \\ g(x,u,v) = k_1\, u + \psi(x,v) \end{cases}$$

and that there exist $\mu_1 \geq 0$, $\lambda_1 \geq 0$, $\mu_2 > 0$, $\lambda_2 > 0, \gamma_1 \in [1, p-1[$ and $\gamma_2 \in [1, q-1[$ such that:

$$\varphi(x,u) \leq \mu_1 + \mu_2\, u^{\gamma_1},\ \psi(x,v) \leq \lambda_1 + \lambda_2\, v^{\gamma_2}, \forall x \in \Omega,\ \forall u,\, v \in \mathbb{R}_+$$

Then (S) *has a sub-supersolution.*

Proof: By Remark 2, it is sufficient to show the existence of a supersolution. Let $M_0 = ||\psi_0||_{L^\infty(\Omega)}$, $N_0 = ||\varphi_0||_{L^\infty(\Omega)}$ and R be such that $\Omega \subset B(0,R)$
We seek $\bar{u}$ and $\bar{v}$ of the following type:

$$\bar{u}(x) = \alpha\, r^{p*} + \beta,\ \bar{v}(x) = \gamma\, r^{q*} + \delta,\ \text{where } r = |x|,$$
$$\alpha < 0,\ \beta > 0,\ \gamma < 0,\ \delta > 0.$$

(1.7) and (1.8) are satisfied if:

(1.16)
$$\begin{cases} \alpha\, R^{p*} + \beta = M_0 \\ \gamma\, R^{q+} + \delta = N_0 \end{cases}$$

We want

(1.17)
$$\begin{cases} -\text{div}(|\nabla \bar{u}|^{p-2}\, \nabla \bar{u}) = N|\alpha p^*|^{p-1} \geq k_0\, \bar{v} + \mu_1 + \mu_2\, \bar{u}^{\gamma_1} \\ -\text{div}(|\nabla \bar{v}|^{q-2}\, \nabla \bar{v}) = N|\gamma q^*|^{q-1} \geq k_1\, \bar{u} + \lambda_1 + \lambda_2\, \bar{v}^{\gamma_2} \end{cases}$$

Set $\beta = \delta$. Then, if β is sufficiently large, using the fact that $\gamma_1 \in [1, p-1[$ and $\gamma_2 \in\in [1, q-1[$, we can obtain

(1.18)
$$\begin{cases} \frac{N(p^*)^{p-1}}{R^p}\, \frac{(\beta - M_0)^{p-1}}{\mu_1 + \mu_2\, \beta^{\lambda_1}} - 1 - \frac{k_0 \beta}{\mu_1 + \mu_2\, \beta^{\gamma_1}} \geq 0 \text{ and} \\ \frac{N(q^*)^{q-1}}{R^q}\, \frac{(\delta - N_0)^{q-1}}{\lambda_1 + \lambda_2\, \delta^{\gamma_2}} - 1 - \frac{k_1 \delta}{\lambda_1 + \lambda_2\, \delta^{\gamma_2}} \geq 0 \end{cases}$$

So (1.17) is satisfied and we check α and γ with (1.16) ■

2. Existence results

Our main result is the following one:

Theorem 4. *Let $p > 2$, $q > 2$ and $\varphi_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi_0 \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $\varphi_0 \geq 0$, $\psi_0 \geq 0$ be given.*

Suppose that f and g verify (1.1) and (1.2) and that (S) *has a sub-super-solution [(0, 0), ($\bar{u}\,\bar{v}$)] in Q_T.*

Then (S) *has a unique solution (u, v) in Q_T satisfying:*

$$\begin{cases} 0 \leq u \leq \bar{u} \\ 0 \leq v \leq \bar{v} \end{cases} \text{in } Q_T$$

Proof:: By Theorem II.2[3], we can choose $u_0 \in L^\infty(0, T; W^{1,p}(\Omega) \cap L^\infty(\Omega))$ $v_0 \in L^\infty(0, T; W^{1,q}(\Omega) \cap L^\infty(\Omega))$ satisfiying $0 \leq u_0 \leq \bar{u}$ and $0 \leq v_0 \leq \bar{v}$, such that:

$$\begin{cases} \frac{\partial u_0}{\partial t} - \Delta_p u_o = f(x, u_o, 0) & \text{in } Q_T \\ u_0(x,t) = 0 & \text{in } S_T \\ v_0(x, 0) = \varphi_0(x) & \text{in } \Omega \end{cases}$$

$$\begin{cases} \frac{\partial v_0}{\partial t} - \Delta_q v_0 = g(x, 0, v_0) & \text{in } Q_T \\ v_0(x,t) = 0 & \text{in } S_T \\ v_0(x, 0) = \psi_0(x) & \text{in } \Omega \end{cases}$$

By the existence theorem of Meïke ([9], p 1024) we construct two sequences of functions, (u_n) and (v_n), such that:

$$\begin{array}{l} (2.1) \\ (2.2) \\ (2.3) \end{array} \begin{cases} \frac{\partial u_{n+1}}{\partial t} - \Delta_p u_{n+1} = f(x, u_{n+1}, v_n) & \text{in } Q_T \\ u_{n+1}(x,t) = 0 & \text{in } S_T \\ u_{n+1}(x, 0) = \varphi_0(x) & \text{in } \Omega \end{cases}$$

and

$$\begin{array}{l} (2.4) \\ (2.5) \\ (2.6) \end{array} \begin{cases} \frac{\partial v_{n+1}}{\partial t} - \Delta_q v_{n+1} = g(x, u_n, v_{n+1}) & \text{in } Q_T \\ v_{n+1}(x,t) = 0 & \text{in } S_T \\ u_{n+1}(x, 0) = \psi_0(x) & \text{in } \Omega \end{cases}$$

We need several lemmas to complete the proof of Theorem 4:

Lemma 1. *For any $n \in \mathbb{N}$, the relations $0 \leq u_n \leq \bar{u}$, $0 \leq v_n \leq \bar{v}$ imply that $0 \leq u_{n+1} \leq \bar{u}$ and $0 \leq v_{n+1} \leq \bar{v}$*

Proof of lemma 1: By (1.10), (1.11) and the above assumptions, we have:

$$(2.7) \qquad \frac{\partial}{\partial t}(u_{n+1} - u) - (\Delta_p u_{n+1} - \Delta_p \bar{u}) \leq f(x, u_{n+1}, v_n) - f(x, \bar{u}, v_n)$$

Multiplying (2.7) by $(u_{n+1} - \bar{u})_+$, the monoticity of Δ_p implies:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_{n+1} - \bar{u})_+^2 \leq \lambda \int_\Omega (f(x, u_{n+1}, v_n) - f(x, \bar{u}, v_n))(u_{n+1} - \bar{u})_+$$

By the Lipschitz condition (1.2), the initial condition and Gronwall's Lemma, we obtain : $u_{n+1} \leq \bar{u}$.

The hypothesis $f(x, 0, v_n) \geq 0$, gives $u_{n+1} \geq 0$; similarly, we get $0 \leq v_{n+1} \leq \bar{v}$. ■

Lemma 2. *There exists $C = C(M_1, N_1, T)$such that:*

(2.8) $$\|u_{n+1}\|_{L^\infty(Q_T)} \leq C$$

(2.9) $$\|u_{n+1}\|_{L^\infty(0,T;W_0^{1,p})} \leq C$$

(2.10) $$\|\frac{\partial u_{n+1}}{\partial t}\|_{L^2(Q_T)} \leq C$$

The same estimates hold for v_{n+1} with p replaced by q.

Proof of Lemma 2: By lemma 1, for any $n \in N, u_n$ and v_n are bounded; whence (2.8). The properties of the functions f and g, then imply that $f(x, u_{n+1}, v_n)$ is bounded.

We therefore obtain:

$$\int_\Omega f(x, u_{n+1}, v_n)\frac{\partial u_{n+1}}{\partial t} \leq \frac{1}{2}\int_\Omega (f(x, u_{n+1}, v_n))^2 +$$

$$+\frac{1}{2}\int_\Omega \left(\frac{\partial u_{n+1}}{\partial t}\right)^2 \leq C_0 + \frac{1}{2}\int_\Omega \left(\frac{\partial u_{n+1}}{\partial t}\right)^2 dx$$

Multiplying (2.1) by $\frac{\partial u_{n+1}}{\partial t}$, we get:

$$\frac{1}{2}\int_0^T \int_\Omega \left(\frac{\partial u_{n+1}}{\partial t}\right)^2 dx\, dt + \frac{1}{p}\int_\Omega |\nabla u_{n+1}(\cdot, T)|^p dx \leq C_0 T + \frac{1}{p}\int_\Omega |\nabla \varphi_0|^p dx$$

It is the same for v_{n+1}. ■

Proof of theorem 4: By (2.8), (2.9), (2.10), there is a subsequence (u_n, v_n) with the following properties:

u_n converges to u in the weak $*$ sense in $L^\infty(0, T; W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$ and u_n converges weakly in $L^p(0, T; W_0^{1,p}(\Omega))$; u_n is such that $\frac{\partial u_n}{\partial t}$ converges to $\frac{\partial u}{\partial t}$ in weak $L^2(Q_T)$; the same holds also for v_n with p replaced by q.

By standard monotonicity argument [8], $\Delta_p u_{n+1}$, converges to $\Delta_p u$ in weak $L^{p*}(0, T; W^{-1,p*}(\Omega)), \Delta_q v_{n+1}$ converges to $\Delta_q v$ in weak $L^{q*}(0, T; W^{-1,q*}(\Omega))$. u_n converges almost everywhere to u and v_n converges almost everywhere to v.

By Lebesgue's theorem:

$$f(\cdot, u_{n+1}, v_n) \text{ converges to } f(\cdot, u, v)$$
$$g(\cdot, u_n, v_{n+1}) \text{ converges to } g(\cdot, u, v)$$

whence (u, v) is a solution of (S) in Q_T.

Applying lemma 1, we have $0 \leq u \leq \bar{u}$, $0 \leq v \leq \bar{v}$. ■

Remark 3: Uniqueness follows from the Lipschitz condition on f and g.

3. Asymptotic behaviour

Hereafter, we assume that there exist positive constants $\lambda > 0$ and $\mu > 0$ and a function H from $\mathbf{R}^{N+2}$ to $\mathbf{R}$ such that

$$(1.1') - (1.2')\begin{cases} f = \lambda \frac{\partial H}{\partial u}, \quad g = \mu \frac{\partial H}{\partial v} \\ f \text{ and } g \text{ satisfy (1.1) and (1.2)} \end{cases}$$

For a solution (u,v) of (**S**), we define the ω-limit set by:

$$\omega(\varphi_0,\psi_0) = \{w = (w_1,w_2) : w_1 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ w_2 \in w_0^{1,q}(\Omega) \cap L^\infty(\Omega)|$$

$$\exists\, t_n \to +\infty : u(\cdot,t_n) \to w_1 \text{ in } W_0^{1,p}(\Omega)$$
$$v(\cdot,t_n) \to w_2 \text{ in } W_0^{1,q}(\Omega)\}$$

Let $\mathcal{E}$ be the set of non negative solutions $w = (w_1,w_2)$ of the elliptic problem:

$$\begin{cases} -\Delta_p w_1 = \lambda \frac{\partial H}{\partial u}(x,w_1,w_2) & \text{in } \Omega \\ -\Delta_q w_2 = \mu \frac{\partial H}{\partial v}(x,w_1,w_2) & \text{in } \Omega \\ w_1 = w_2 = 0 \end{cases}$$

Our main result is the following:

Theorem 5. *Let $p > 2$, $q > 2$ and $\varphi_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi_0 \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $\varphi_0 \geq 0$, $\psi_0 \geq 0$.*
Suppose that H satisfies (1.1'), (1.2') and that (**S**) *has a sub-supersolution. Then $\omega(\varphi_0,\psi_0) \neq \phi$ and $\omega(\varphi_0,\psi_0) \subset \mathcal{E}$.*

To prove this Theorem, we need the following lemmas:

Lemma 3. *Under the assumptions of Theorem 5, there exists a constant $C = C(M_1,N_1)$ such that for any $T > 0$:*

$$\|u\|_{L^\infty(Q_T)} \leq C < +\infty,\ \|v\|_{L^\infty(Q_T)} \leq C < +\infty \tag{3.1}$$
$$\|u\|_{L^\infty(0,T;W_0^{1,p}(\Omega))} \leq C < +\infty,\ \|v\|_{L^\infty(0,T;W_0^{1,q}(\Omega))} \leq C < +\infty \tag{3.2}$$
$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2(Q_T)} \leq C < +\infty, \left\|\frac{\partial v}{\partial t}\right\|_{L^2(Q_T)} \leq C < +\infty \tag{3.3}$$

Proof of lemma 3: By Theorem 4 we have (3.1).

Multiplying the first equation (1.6) by $\frac{1}{\lambda}\frac{\partial u}{\partial t}$ and the second equation by $\frac{1}{\mu}\frac{\partial v}{\partial t}$, we obtain:

$$\begin{aligned}
(3.4)\qquad & \frac{1}{\lambda}\int_{Q_T}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt + \frac{1}{\mu}\int_{Q_T}\left(\frac{\partial v}{\partial t}\right)^2 dx\,dt \\
& + \frac{1}{\lambda p}\int_\Omega |\nabla u(\cdot,T)|^p\,dx + \frac{1}{\mu q}\int_\Omega |\nabla v(\cdot,T)|^q\,dx \\
& = \int_{Q_T}\left(\frac{\partial H}{\partial u}(\cdot,u,v)\frac{\partial u}{\partial t} + \frac{\partial H}{\partial v}(\cdot,u,v)\frac{\partial v}{\partial t}\right) dx\,dt \\
& + \frac{1}{\lambda p}\int_\Omega |\nabla\varphi_0|^p\,dx + \frac{1}{\mu q}\int_\Omega |\nabla\psi_0|^q\,dx \\
& = \int_\Omega (H(\cdot,u(T),v(T)) - H(\cdot,\varphi_0,\psi_0))\,dx \\
& + \frac{1}{\lambda p}\int_\Omega |\nabla\varphi_0|^p\,dx + \frac{1}{\mu q}\int_\Omega |\nabla\psi_0|^q
\end{aligned}$$

H is continuous and (u,v) is bounded; we then obtain:

$$\underbrace{\frac{1}{\lambda}\int_{Q_T}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{\mu}\int_{Q_T}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{\lambda p}\int_\Omega |\nabla u(\cdot,T)|^p\,dx + \frac{1}{\mu q}\int_\Omega |\nabla v(\cdot,T)|^q\,dx}_{\leq C(M_1,N_1)}$$

whence (3.2) and (3.3). ■

Lemma 4. *Let $t_0 \in]0,1[$. Under the assumptions of Theorem 5, there exists $C = C(t_0) > 0$ such that for any $T \geq t_0$:*

$$(3.5)\qquad \left\|\tfrac{\partial u}{\partial t}\right\|_{L^\infty(t_0,+\infty;L^2(\Omega))} \leq C,\ \left\|\tfrac{\partial v}{\partial t}\right\|_{L^\infty(t_0,+\infty;L^2(\Omega))} \leq C$$

$$(3.6)\qquad \left\|\tfrac{\partial}{\partial t}F(\nabla u)\right\|_{L^2(t_0,T;L^{p*}(\Omega))} \leq C,\ \left\|\tfrac{\partial}{\partial t}G(\nabla v)\right\|_{L^2(t_0,T;L^{q*}(\Omega))} \leq C$$

Proof of Lemma 4: Let $E_p(\nabla u) = |\nabla u|^{\frac{p-2}{2}}\nabla u$. Calculations, **[cf.3]**, give:

$$(3.7)\qquad \left|\tfrac{\partial}{\partial t}E_p(\nabla u)\right|^2 \leq \tfrac{p+2}{4}\tfrac{\partial}{\partial t}F(\nabla u)\cdot\tfrac{\partial}{\partial t}\nabla u$$

$$(3.8)\qquad \left|\tfrac{\partial}{\partial t}F(\nabla u)\right| \leq \left(\tfrac{4p}{p+2}\right)^{1/2}|\nabla u|^{\frac{p-2}{2}}\left|\tfrac{\partial}{\partial t}E_p(\nabla u)\right|$$

Similary for $E_q(\nabla v) = |\nabla v|^{\frac{q-2}{2}}\nabla v$.

By formal derivation of the first equation of (1.6), we get

$$\begin{aligned}
(3.9)\qquad & \frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(\frac{\partial}{\partial t}F(\nabla u)\right) = \lambda\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial u}\right) = \\
& = \lambda\frac{\partial^2 H}{\partial u^2}\cdot\frac{\partial u}{\partial t} + \lambda\frac{\partial^2 H}{\partial u\partial v}\cdot\frac{\partial v}{\partial t}
\end{aligned}$$

Multiplying (3.9) by $\frac{\partial u}{\partial t}$, we get with (3.7) and (1.1')

(3.10)

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 + \frac{4}{p+2}\int_\Omega \left|\frac{\partial}{\partial t}E_p(\nabla u)\right|^2 dx \le k_0 \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 + k_1 \int_\Omega \left(\frac{\partial v}{\partial t}\right)^2$$

Similarly, we have:

(3.11)

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \left(\frac{\partial v}{\partial t}\right)^2 + \frac{4}{q+2}\int_\Omega \left|\frac{\partial}{\partial t}E_q(\nabla v)\right|^2 dx \le k_0' \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 + k_1' \int_\Omega \left(\frac{\partial v}{\partial t}\right)^2$$

By (3.3), there exists $t_1 \in]0, t_0[$ such that:

$$\left\|\frac{\partial u}{\partial t}(\cdot,t_1)\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial t}(\cdot,t_1)\right\|^2_{L^2(\Omega)} = \tag{3.12}$$

$$= \int_0^{t_0}\left(\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial t}(\cdot,t)\right\|^2_{L^2(\Omega)}\right) dt \le C < +\infty$$

and integrating (3.10) + (3.11) on (t_1, T), we obtain with (3.3):

(3.13)

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u}{\partial t}(\cdot,T)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial v}{\partial t}(\cdot,T)\right\|^2_{L^2(\Omega)} \\
&+ \frac{4}{p+2}\int_{t_0}^T\int_\Omega \left|\frac{\partial}{\partial t}E_p(\nabla u)\right|^2 dx\,dt + \frac{4}{q+2}\int_{t_0}^T\int_\Omega \left|\frac{\partial}{\partial t}E_q(\nabla v)\right|^2 dx\,dt \\
&\le K\left[\int_{t_1}^T\int_\Omega\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2\right)\right] dx\,dt + \frac{1}{2}\left\|\frac{\partial u}{\partial t}(\cdot,t_1)\right\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2}\left\|\frac{\partial v}{\partial t}(\cdot,t_1)\right\|^2_{L^2(\Omega)} \le C < +\infty
\end{aligned}$$

By (3.8) and Holder's inequality, we obtain with (3.13) and (3.2):

(3.14)

$$\begin{aligned}
&\left\|\frac{\partial u}{\partial t}(\cdot,T)\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial t}(\cdot,T)\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial}{\partial t}F(\nabla u)\right\|^2_{L^2(t_0,T;L^{p*}(\Omega))} \\
&+ \left\|\frac{\partial}{\partial t}G(\nabla v)\right\|^2_{L^2(t_0,T,L^{q*}(\Omega))} \le C
\end{aligned}$$

(3.14) gives the estimates (3.5) and (3.6).

This formal proof of (3.14) can be made rigorous by means of the finite dimensional problems associated with (S).

The details are in [4, p 35] and are omitted.

Theorem 6. *Let $p > 2$, $q > 2$, $\varphi_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi_0 \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, and $\varphi_0 \geq 0$, $\psi_0 \geq 0$.*

Suppose that H satisfies (1.1'), (1.2') and that (S) has a sub-supersolution $[(0,0),(\bar{u},\bar{v})]$. Then, for any $t_0 \in]0,1[$, the solution (u,v) of (S) satisifies the following regularizing estimates:

$$u \in L^\infty\left(t_0,+\infty;\ B_\infty^{1+1/(p-1)^2,p}(\Omega)\right) \tag{3.15}$$
$$\tfrac{\partial u}{\partial t} \in L^2(t_0,+\infty;\ L^2(\Omega)) \cap L^\infty(t_0,+\infty;\ L^2(\Omega)) \tag{3.16}$$
$$\tfrac{\partial}{\partial t} F(\nabla u) \in L^2(t_0,+\infty;\ L^{p*}(\Omega)), \tag{3.17}$$

where $B_\infty^{1+1/(p-1)^2,p}(\Omega)$ is a BESOV space defined by the real interpolation method (cf.[1], [13]).

The same estimates hold for v provided p and F are replaced by q and G respectively.

Proof of Theorem 6: By (3.3), (3.5) and (3.6) we have (3.16) and (3.17), whence $\frac{\partial u}{\partial t} \in L^\infty(t_0,+\infty;\ L^{p*}(\Omega))$ and $\frac{\partial v}{\partial t} \in L^\infty(t_0,+\infty;\ L^{q*}(\Omega))$.

By SIMON'S regularity results [13], we have:

$$\|u(\cdot,t)\|_{B_\infty^{1+1/p-1)^2,p}(\Omega)} \leq C \left\| \frac{\partial H}{\partial u}(\cdot,u,v) - \frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^{p*}(\Omega)} + C'$$

whence (3.15). The proof is the same for v. ■

Proof of theorem 5:

a) $\omega(\varphi_0,\psi_0) \neq \phi$ because $B_\infty^{1+1/(r-1)^2,r}(\Omega)$ is Compactly inbedded in $W^{1,r}(\Omega)$ for $r = p$ and q[1]. By Theorem 6, letting
$$w_1 = \lim_{n\to+\infty} u(\cdot,t_n), w_2 = \lim_{n\to+\infty} v(\cdot,t_n) \text{ and}$$

$w = (w_1,w_2) \in \omega(\varphi_0,\psi_0)$, we get that $w = (w,w_2) \in \mathcal{E}$. The proof is analogous to EL HACHIMI and DE THELIN [3] and is omitted ■

4. Examples

Example 1. Let $H(x,u,v) = K(x)uv + \lambda u^{\gamma_1+1} + \mu v^{\gamma_2+1}$, where $K \in C(\bar{\Omega})$, $K(x) > 0$, $\lambda > 0$, $\mu > 0$, $\gamma_1 \in [1,p-1[$ and $\gamma_2 \in [1,q-1[$.

Then we can apply Theorem 3, 4, 5, 6.

Example 2. Let $H(x,u,v) = -u^m v^n + \lambda^{\gamma_1+1} + \mu v^{\gamma_2+1}$, where $m \geq 1$, $n \geq 1$, $\lambda > 0$, $\mu > 0$, $\gamma_1 \in [1,p-1[$ and $\gamma_2 \in [1,q-1[$. Then we can apply theorems 1, 4, 5, 6.

13. J. SIMON, Régularité de solution d'un problème aux limites Non linéaires, *Annales Faculté Sciences, Toulouse série 5, t 3* (1981), 247–274.

Keywords. Sub–supersolutions, comparison principle, nonlinear parabolic systems, asymptotic behaviour, ω–limit set.

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Rebut el 12 d'Abril de 1989